Local quantum fields: their structure, their number theory

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Motivation

Basic algebraic properties of Feynman graphs: Hopf Algebras

Gauge symmetries

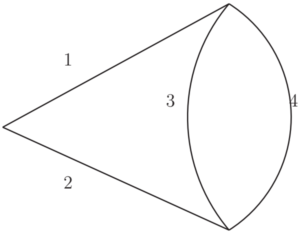
Feynman rules and their structure

Dyson Schwinger Equations in QED

Transcendentality, analytic strcuture of amplitudes

The fundamentals of fundamental processes

$$\sum_n \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$



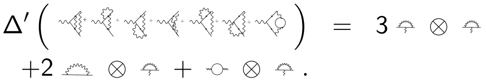
Try: integer$\to$graphs, power $^{-s}$ $\to$Feynman rules, $\zeta(s)$ $\to$Green function $G^R(\{g\}, \ln s, \{\Theta\})$

$$G^R(\{g\}, \ln s, \{\Theta\}) = 1 \pm \Phi^R_{\ln s, \{\Theta\}}(X^r(\{g\})) \qquad (1)$$

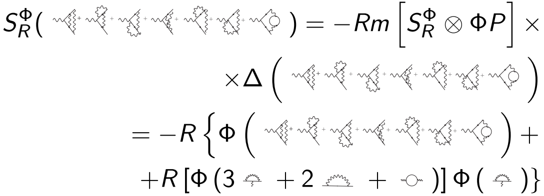
with $X^r = 1 \pm \sum_j g^j B_+^{r;j}(X^r Q^j(g))$, $bB_+^{r;j} = 0$.

An Example

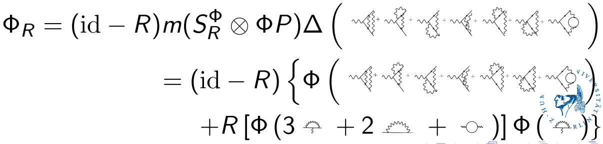
- The co-product



- The counterterm



- The renormalized result



sub-Hopf algebras

- summing order by order

$$c_k^r = \sum_{|\Gamma|=k,\mathrm{res}(\Gamma)=r} \frac{1}{|Aut(\Gamma)|}\Gamma \Rightarrow \Delta(c_k^r) = \sum_j \mathrm{Pol}_j(c_m^s) \otimes c_{k-j}^r. \quad (7)$$

- Hochschild closedness

$$X^r = 1 \pm \sum_j c_j^r \alpha^j = 1 \pm \sum_j \alpha^j B_+^{r;j}(X^r Q^j(\alpha)), \quad (8)$$

$Q^j = \frac{X^v}{\sqrt{\prod_{\text{edges e at v}} X^e}}$. Evaluates to invariant charge.

- $bB_+^{r;j} = 0$.

$$\Delta B_+^{r;j}(X) = B_+^{r;j}(X) \otimes 1 + (id \otimes B_+^{r;j})\Delta(X). \quad (9)$$

Implies locality of counterterms upon application of Feynman rules $\Phi B_+^{r;j}(X) = \int d\mu_{r;j}\Phi(X)$:

$$\bar{R}(\Gamma) = m(S_\Phi^R \otimes \Phi P))\Delta B_+^{r;j} = \int d\mu_{r;j}\Phi^R(X). \quad (10)$$

Symmetry

- Ward and Slavnov–Taylor ids

$$i_k := c_k^{\bar{\psi}\psi} + c_k^{\bar{\psi}\not{A}\psi} \tag{11}$$

span Hopf (co-)ideal I:

$$\Delta(I) \subseteq H \otimes I + I \otimes H. \tag{12}$$

$$\Delta(i_2) = i_2 \otimes 1 + 1 \otimes i_2 + (c_1^{\frac{1}{4}F^2} + c_1^{\bar{\psi}\not{A}\psi} + i_1) \otimes i_1 + i_1 \otimes c_1^{\bar{\psi}\not{A}\psi}.$$

- Feynman rules vanish on I $\Leftrightarrow$ Feynman rules respect quantized symmetry:
  $\Phi^R : H/I \to V$.
- Ideals for Slavnov–Taylor ids generated by equality of renormalized charges, also for the master equation in Batalin-Vilkovisky (see Walter van Suijlekom's work)
- Similar ideals for the core Hopf algebra are respected by the BCFW recursion, and fit naturally with the structure of perturbative quantum gravity

Kinematics and Cohomology

- Exact co-cycles
$$[B_+^{r;j}] = B_+^{r;j} + b\phi^{r;j} \tag{22}$$
with $\phi^{r;j} : H \to \mathbb{C}$
- Variation of momenta
$$G^R(\{g\}, \ln s, \{\Theta\}) = 1 \pm \Phi^R_{\ln s, \{\Theta\}}(X^r(\{g\})) \tag{23}$$
with $X^r = 1 \pm \sum_j g^j B_+^{r;j}(X^r Q^j(g))$, $bB_+^{r;j} = 0$. Note:
$\beta(g) = 0 \Leftrightarrow Q(g) = \mathrm{constant}$.
Then, for kinematic renormalization schemes:
$\{\Theta\} \to \{\Theta'\} \Leftrightarrow B_+^{r;j} \to B_+^{r;j} + b\phi^{r;j}$.
$\Phi^R_{L_1+L_2,\{\Theta\}} = \Phi^R_{L_1,\{\Theta\}} \star \Phi^R_{L_2,\{\Theta\}}$.
$\Phi^R(\ln s, \{\Theta\}, \{\Theta_0\}) = \Phi^{-1}_{\mathrm{fin}}(\{\Theta_0\} \star \Phi^R_{1-\mathrm{scale}}(\ln s) \star \Phi_{\mathrm{fin}}(\{\Theta\})$.

The Feynman rules in projective space

First, $\phi_\Gamma \to \phi_\Gamma + \psi_\Gamma(\sum_e m_e^2 A_e)$.

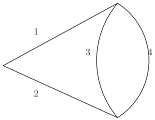
$$\Phi^R_\Gamma(S, S_0, \{\Theta, \Theta_0\}) = \int_{\mathbb{P}^{E-1}(\mathbb{R}_+)} \overbrace{\sum_f}^{\text{forestsum}} (-1)^{|f|}$$

$$\frac{\ln \frac{\frac{S}{S_0}\phi_{\Gamma/f}\psi_f + \phi^0_f\psi_{\Gamma/f}}{\phi^0_{\Gamma/f}\psi_f + \phi^0_f\psi_{\Gamma/f}}}{\psi^2_{\Gamma/f}\psi^2_f} \underbrace{\Omega_\Gamma}_{(E-1)-\text{form}}$$

Note: for 1-scale graphs, $\phi_\Gamma = \psi^\bullet_\Gamma$.

Example

$\Gamma =$



$$N_\Gamma = \begin{pmatrix} A_1 + A_2 + A_3 & A_1 + A_2 & A_1\mu_1 + A_2\mu_2 + A_3\mu_3 \\ A_1 + A_2 & A_1 + A_2 + A_4 & A_1\mu_1 + A_2\mu_2 + A_4\mu_4 \\ A_1\bar{\mu}_1 + A_2\bar{\mu}_2 + A_3\bar{\mu}_3 & A_1\bar{\mu}_1 + A_2\bar{\mu}_2 + A_4\bar{\mu}_4 & \sum_{i:=1}^4 A_i\bar{\mu}_i\mu_i \end{pmatrix}$$

$$\psi_\Gamma = (A_1 + A_2)(A_3 + A_4) + A_3A_4 = \sum_{\mathrm{sp.Tr.}\,T} \prod_{e \notin T} A_e$$

$$\phi_\Gamma = (A_3 + A_4)A_1A_2p_a^2 + A_2A_3A_4p_b^2 + A_1A_3A_4p_c^2 = \sum_{\mathrm{sp.2-Tr.}\,T_1 \cup T_2} Q(T_1) \cdot Q(T_2) \prod_{e \notin T_1 \cup T_2} A_e.$$

The renormalized result

Theorem

The unrenormalized Feynman integrand at n loops for the sum of all Feynman graphs contributing to the connected k-loop amplitude is $\Phi(\Gamma^k) =$

$$\sum_{|\Gamma|=n,|E_E(\Gamma)|=k} e^{-\sum_e \oint_{c_e}} \left(\prod_{e\in E^\Gamma} g_{\mu(v_1(e))\mu(v_2(e))} D^{\mathrm{gauge}}_{\mathrm{hom}}(\Gamma)\right) \frac{e^{-\frac{\phi_\Gamma}{\psi_\Gamma}}}{\psi_\Gamma^2} \prod_{e\in E^\Gamma} dA_e.$$

The renormalized result is obtained as

$$D^{\mathrm{gauge}}_{\mathrm{hom}} \sum_{f\in\mathcal{F}} (-1)^{|f|} \frac{e^{-\frac{\phi_{\Gamma/f}}{\psi_{\Gamma/f}}}}{\psi_{\Gamma/f}^2} \frac{e^{-\frac{\phi_f}{\psi_f}}}{\psi_f^2}$$

with the graph differential in front of the forest sum.

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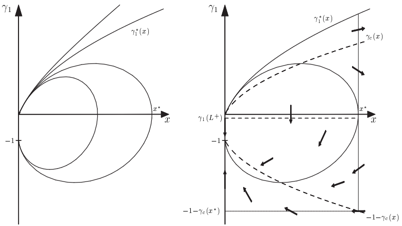


Figure 3: $P(x) = x$, $s = 1$ illustrating that, as a function of L, non-global solutions of (8) turn around and head to -1 as $L \to \infty$.

Our first step is to show that solutions that start below the nullcline $\gamma_c(x_0)$ cannot be continued as $x \to \infty$. Note that this does not follow directly from (20), since $\gamma_1(x)$ could a priori decrease indefinitely as $x \to \infty$ without ever reaching $\gamma_1 = 0$.

Lemma 4.1 *Let $\gamma_1(x_0) < \gamma_c(x_0)$ then the solution of (8) satisfies $\gamma_1(x_1) = 0$ for some finite $x_1 > x_0$.*

Proof. Let $\gamma_1(x_0) \equiv \gamma_c(x_0) - \epsilon$ for some $0 < \epsilon < \gamma_c(x_0)$. We first note that $\gamma_1(x) \le \gamma_1(x_0)$ for all $x \ge x_0$ such that the solution exists, otherwise there would be a local minimum at some $x^\star \in [x_0, x]$, which is precluded by (20). Since $P(x)$ is increasing, we find

$$\begin{aligned}\frac{\mathrm{d}\gamma_1(x)}{\mathrm{d}x} &\le \frac{\gamma_c(x_0) - \epsilon + (\gamma_c(x_0) - \epsilon)^2 - P(x_0)}{sx(\gamma_c(x_0) - \epsilon)} \\ &\le -\frac{\epsilon(1 + 2\gamma_c(x_0) - \epsilon)}{sx(\gamma_c(x_0) - \epsilon)} \equiv -\frac{R(x_0, \epsilon)}{x},\end{aligned} \tag{24}$$

for some $R(x_0, \epsilon) > 0$. Integrating (24) on $[x_0, x]$ gives

$$\gamma_1(x) \le \gamma_1(x_0) - R(x_0, \epsilon) \int_{x_0}^{x} \frac{\mathrm{d}z}{z} = \gamma_c(x_0) - \epsilon - R(x_0, \epsilon) \ln\left(\frac{x}{x_0}\right) ,$$

which shows that $\gamma_1(x_1) = 0$ for some $x_1 \le x_0 \exp\left(\frac{\gamma_c(x_0) - \epsilon}{R(x_0, \epsilon)}\right) < \infty$ as claimed. ■

The polylog as a Hodge structure

Iterated integrals: obvious Hopf algebra structure

$$\begin{pmatrix} 1 & 0 & 0 \\ -Li_1(z) & 2\pi i & 0 \\ -Li_2(z) & 2\pi i \ln(z) & (2\pi i)^2 \end{pmatrix} = (C_1, C_2, C_3) \qquad (24)$$

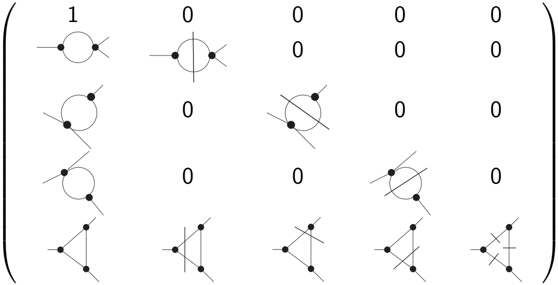
$$\mathrm{Var}(\Im Li_2(z) - \ln|z| \, \Im Li_1(z)) = 0 \qquad (25)$$

Hodge sructure from Hopf algebra structure: branch cut ambiguities columnwise

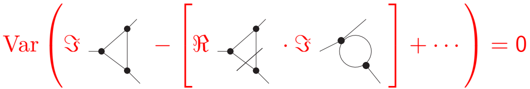
Griffith transversality $\Leftrightarrow$ differential equation

The Feynman graph as a Hodge structure

Hopf algebra structure as above



$= (C_1, C_2, C_3, C_4, C_5)$



Hodge structure: cut-reconstructability: from Hopf algebra structure: branch cut ambiguities columnwise

Griffith transversality ⇔ differential equation?

$$\zeta(s_1, \cdots, s_k) = \sum_{n_i < n_{i+1}} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}$$

- counting over $\mathbb{Q}$

$$1 - \frac{x^3 y}{1 - x^2} + \frac{x^{12} y^2 (1 - y^2)}{(1 - x^4)(1 - x^6)} = \prod_{n \geq 3} \prod_{k \geq 1} (1 - x^n y^k)^{D_{n,k}} \quad (26)$$

$\rightarrow$ first irreducible MZV from planar graphs at 7 loops in scalar field theory (integrability???)

- When is a graph redicible to MZVs? Francis Brown: when it has vertex width three.
- Caution! Non-MZVs at eight loops from non-planar graphs, at nine loops from planar graphs ('A $K3$ in ϕ^4', Brown and Schnetz). Proof from counting points $[X_\Gamma]$ on graph hypersurfaces X_Γ over $\mathbb{F}_q$, defined by vanishing of the first Symanzik polynomial. If the graph gives a MZV, $[X_\Gamma]$ better is polynomial in the prime power $q = p^n$. Alas, it is not in general, with counterexamples relating graphs to elliptic curves with complex multiplication, and point-counting function a modular form.