PEPS & local (gauge) symmetries

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Based on a collaboration with

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Humboldt Kolleg "From the vacuum to the universe"

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HOW TO SET UP A PHYSICAL MODEL?

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1. CHICKEN FIRST, EGGS LATER

How to set up a physical model?

- Define the required
  - Physical degrees of freedom
  - Hilbert space
  - Required (reasonable) symmetries
- Based on which, write the most general local Hamiltonian, parametrized such that all the requirements are fulfilled
- Find the eigenstates of the Hamiltonian (might be difficult).

HOW TO SET UP A PHYSICAL MODEL?
2. EGGS FIRST, CHICKEN LATER.

How to set up a physical model?

- Define the required
  - Physical degrees of freedom
  - Hilbert space
  - Required (reasonable) symmetries
- Based on which, write the most general state, parametrized such that all the requirements are fulfilled
- Find a local parent Hamiltonian – a Hamiltonian whose ground state is the state in study (might be difficult).

Traditionally

- Hamiltonians are more fundamental than states.
- Symmetries are treated already on the Hamiltonian level.
- But, as we know, this is not enough, and important and interesting physics has to be derived from particular states, like the vacuum.
  - (Spontaneous symmetry breaking)

Without a fundamental Hamiltonian

- One can encode required symmetries already on the level of some state, living in a well defined Hilbert space with the physical degrees of freedom.
- Local parent Hamiltonians may be derived for such states (in principle).
- Phase structure may also be studied for a set of states: all is needed is a parameterization.

Today: States with a local symmetry

- We wish to describe a particular (to be defined) but yet general set of lattice states with **local gauge symmetry** as well as other physical symmetries.
- The states will be classified by a set of parameters.
- Upon changing the parameters, the physical interpretation of the states changes, and phase transitions among recognized physical phases take place.

Outline

- PEPS and global symmetries
- Gauging PEPS
- Example:

  Gaussian fermionic PEPS for U(1) lattice gauge theories

PEPS (PROJECTED ENTANGLED PAIR STATES) WITH GLOBAL SYMMETRIES

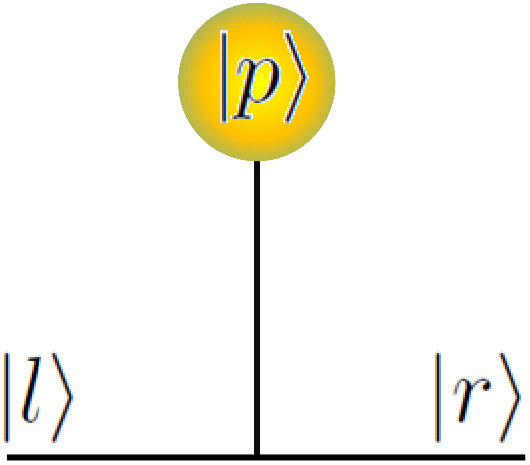
PEPS

- **P**rojected **E**ntangled **P**air **S**tates
  - A type of Tensor network states
- A type of quantum-many body states that
  - Allow to treat symmetries in a very natural way
  - Offer new approaches for the study of phase diagrams and other properties of many body systems
  - May be used as variational ansatze (not in this talk)

Constructing PEPS on a 1d lattice

- The state of a single site before construction consists of the state of a single physical "particle" and two virtual "particles":

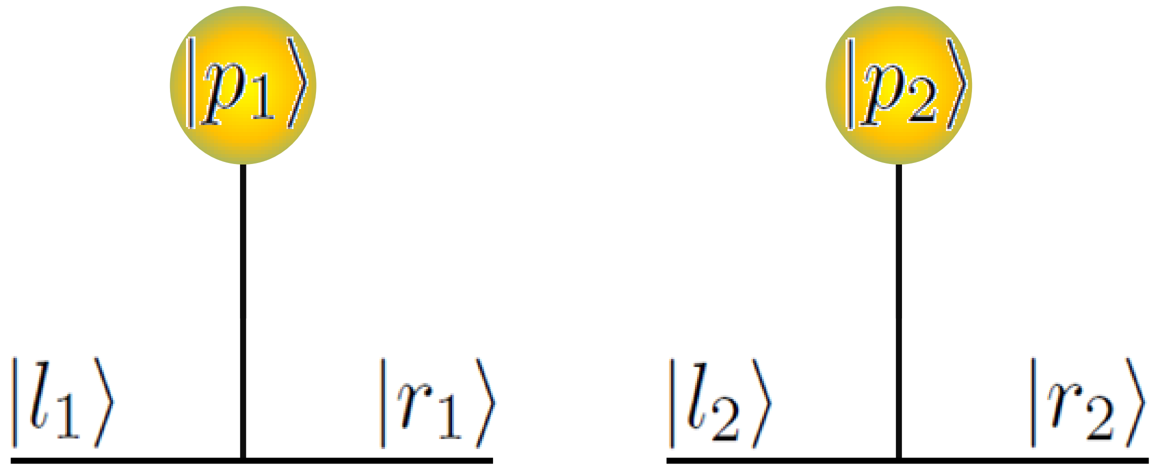
$$A^p_{lr} \left|p\right\rangle \left|l\right\rangle \left|r\right\rangle$$



Constructing PEPS on a 1d lattice

- Two neighboring sites may be connected by projecting the virtual particles on the bond connecting them to a maximally entangled state.

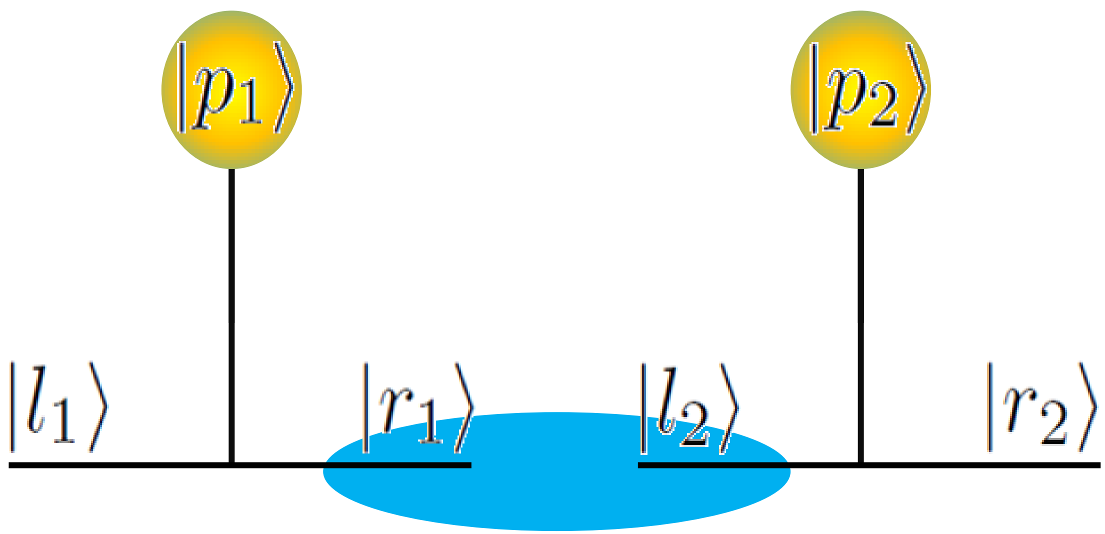
$$A^{p_1}_{l_1 r_1} |p_1\rangle |l_1\rangle |r_1\rangle \; A^{p_2}_{l_2 r_2} |p_2\rangle |l_2\rangle |r_2\rangle$$



Constructing PEPS on a 1d lattice

- For example, project $|r_1\rangle$ and $|l_2\rangle$ to $|\phi\rangle = \frac{1}{\sqrt{D}} \sum_n |nn\rangle$

$$\langle\phi| \, A^{p_1}_{l_1 r_1} \, |p_1\rangle \, |l_1\rangle \, |r_1\rangle \; A^{p_2}_{l_2 r_2} \, |p_2\rangle \, |l_2\rangle \, |r_2\rangle$$



Constructing PEPS on a 1d lattice

- For example, project $|r_1\rangle$ and $|l_2\rangle$ to $|\phi\rangle = \frac{1}{\sqrt{D}}\sum_n |nn\rangle$

$$A^{p_1}_{l_1 n} A^{p_2}_{n r_2} \, |p_1 p_2\rangle \, |l_1\rangle \, |r_2\rangle$$

Constructing PEPS on a 1d lattice

- Hence the name **P**rojected **E**ntangled **P**air **S**tates

$$A^{p_1}_{l_1 n} A^{p_2}_{n r_2} \left|p_1 p_2\right\rangle \left|l_1\right\rangle \left|r_2\right\rangle$$

PEPS: Important properties

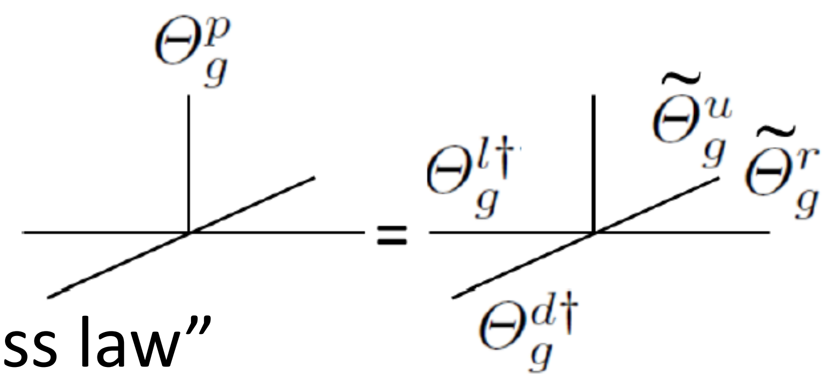
- **The state is in the center**: PEPS are states, and they are in the focus of interest. But for such states satisfying some reasonable physical requirements (e.g. translational invariance) one may construct a **local** Parent Hamiltonian – a Hamiltonian whose ground state is the PEPS.
- *In case you wish to (traditionally) find PEPS which are eigenstates of a given Hamiltonian, and not the other way around, there are some **variational Ansätze**.*

Symmetric PEPS

- Parametrize the local states such that acting on the physical level is equivalent to acting on the virtual level

$$|A(\mathbf{x})\rangle = A^{p}_{lrud} |p\rangle |l, r, u, d\rangle$$

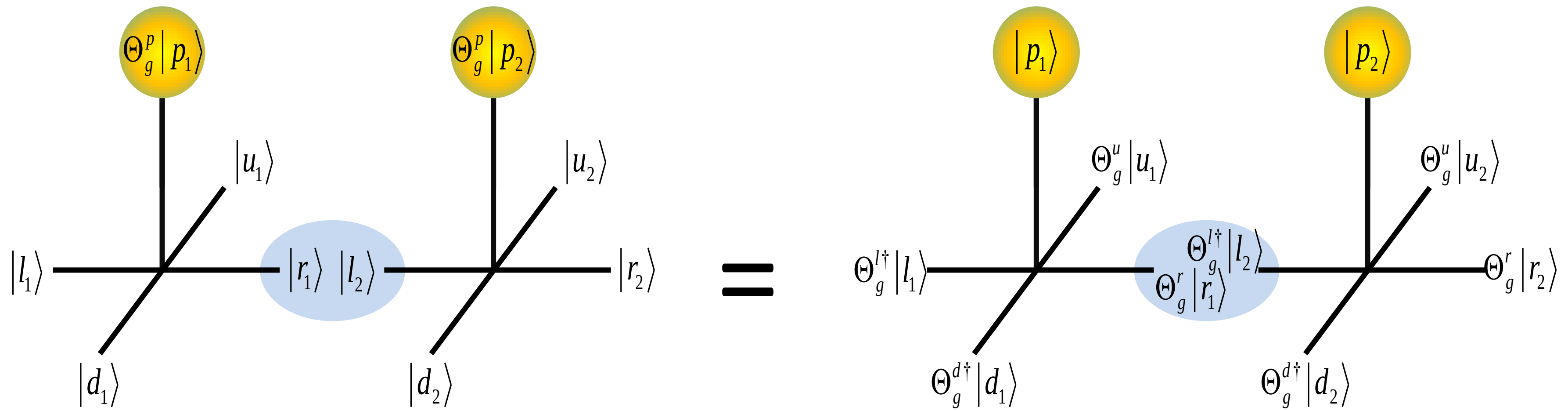
$$\Theta^{p}_{g}(\mathbf{x}) |A(\mathbf{x})\rangle = \Theta^{l\dagger}_{g}(\mathbf{x}) \tilde{\Theta}^{r}_{g}(\mathbf{x}) \tilde{\Theta}^{u}_{g}(\mathbf{x}) \Theta^{d\dagger}_{g}(\mathbf{x}) |A(\mathbf{x})\rangle$$



- "Virtual Gauss law"
- The physical information is stored at **a single tensor**

Symmetric PEPS

- A global symmetry (same *g* everywhere). $\prod_{sites} \Theta_g^p \left|\psi_{phys}\right\rangle = \left|\psi_{phys}\right\rangle$



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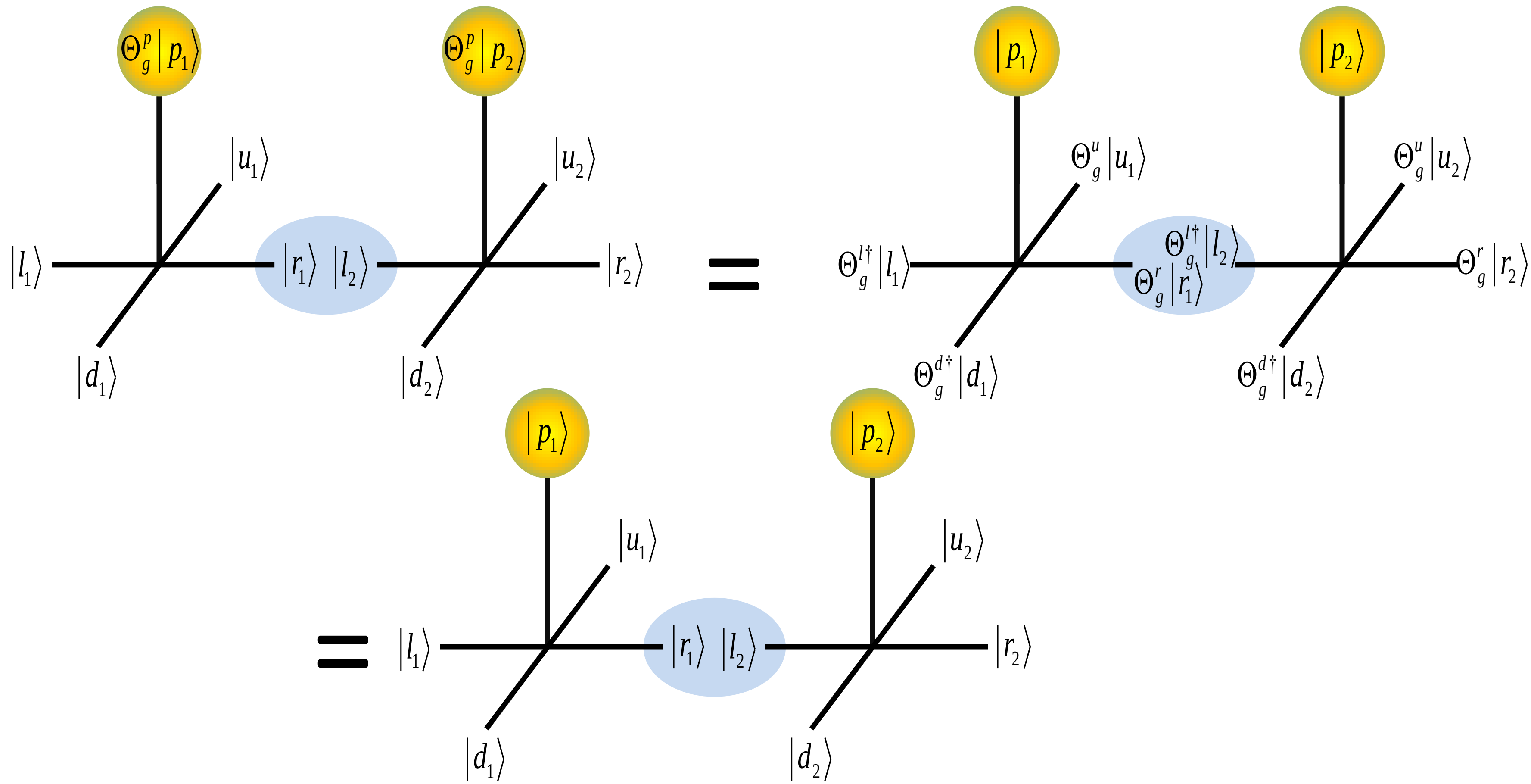
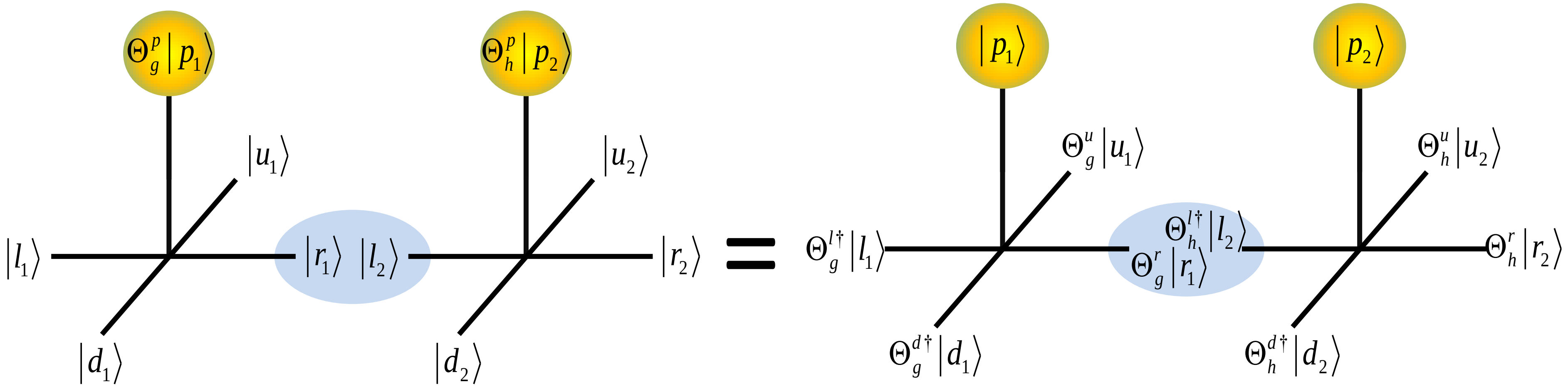


GAUGING THE SYMMETRY:
PEPS WITH LOCAL GAUGE INVARIANCE

Local symmetry

- Different group elements on different sites.



Local symmetry

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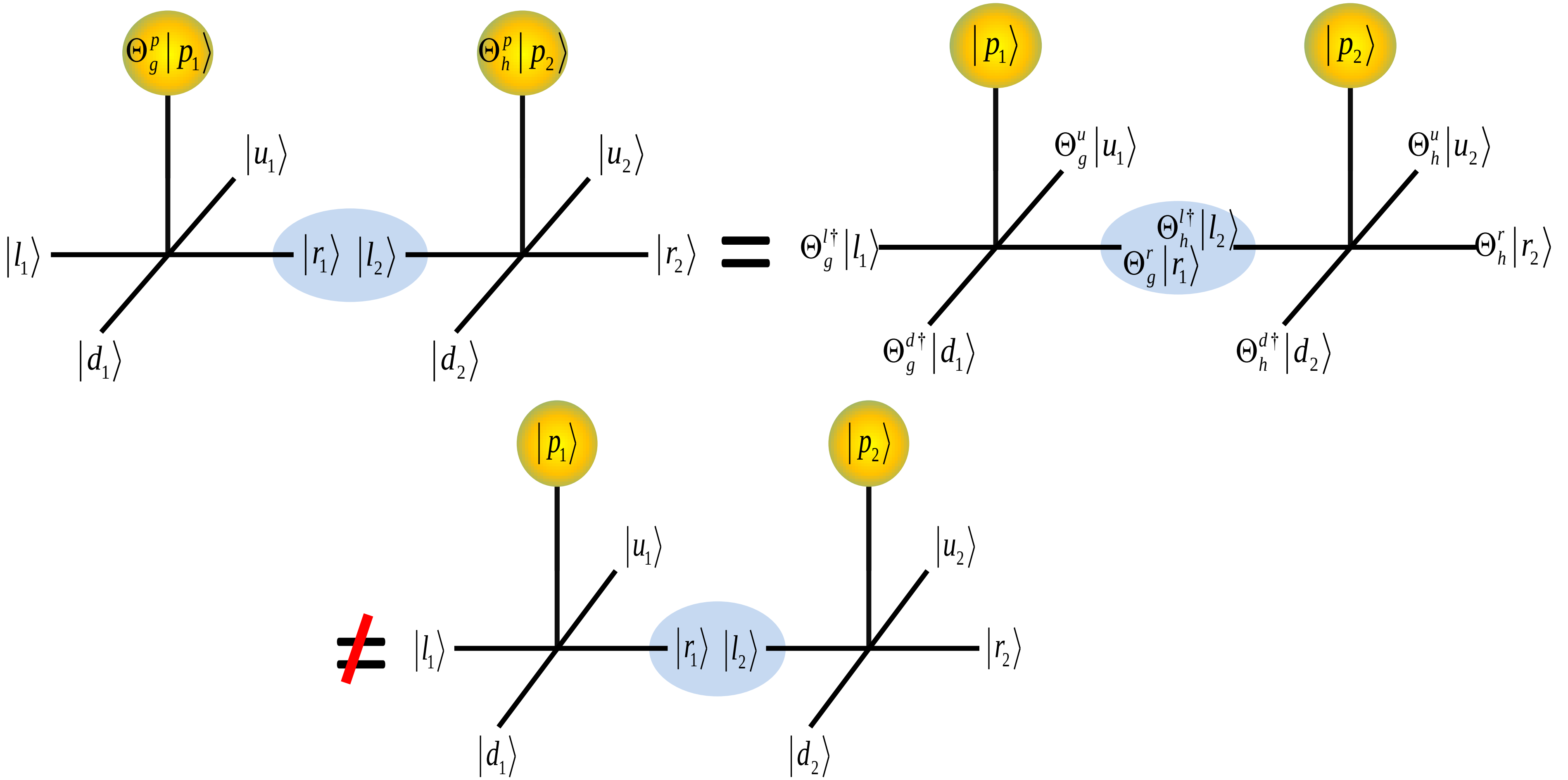
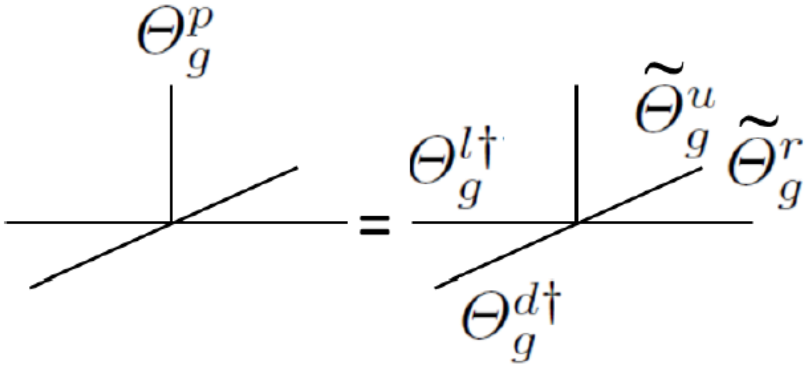


Matter tensors

- As before,



May be achieved by

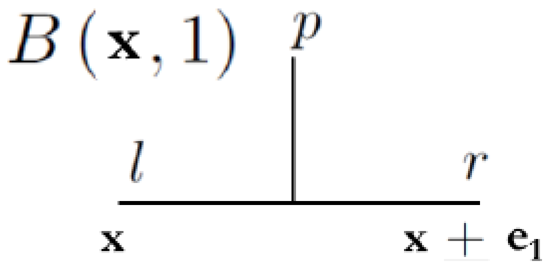
$$A^{j_p m_p}_{j_l m_l; j_r m_r; j_u m_u; j_d m_d} = \sum_{j_1 j_2} \alpha^{j_p j_1 j_2}_{j_l j_r j_u j_d} \left\langle j_l m_l j_d m_d | j_1 m_1 \right\rangle \left\langle j_1 m_1 j_p m_p | j_2 m_2 \right\rangle \left\langle j_2 m_2 | j_r m_r j_u m_u \right\rangle$$

(but it is not a single choice).

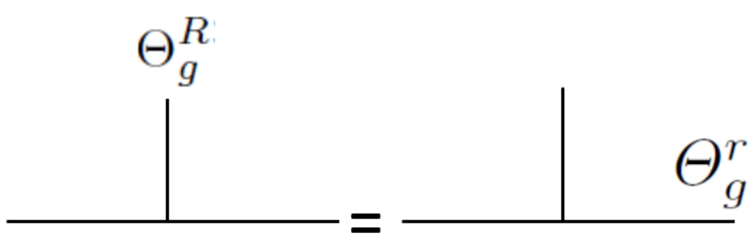
EZ and M. Burrello, New J. Phys. 18 043008 (2016)

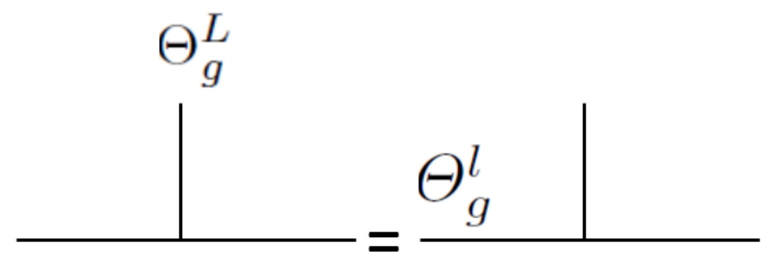
Gauge (connection) tensors

- But now, also



$$\tilde{\Theta}_g^{L/D}(\mathbf{x},k)\,|B(\mathbf{x},k)\rangle = \Theta_g^{l/d}(\mathbf{x},k)\,|B(\mathbf{x},k)\rangle$$
$$\Theta_g^{R/U}(\mathbf{x},k)\,|B(\mathbf{x},k)\rangle = \tilde{\Theta}_g^{r/u}(\mathbf{x},k)\,|B(\mathbf{x},k)\rangle$$



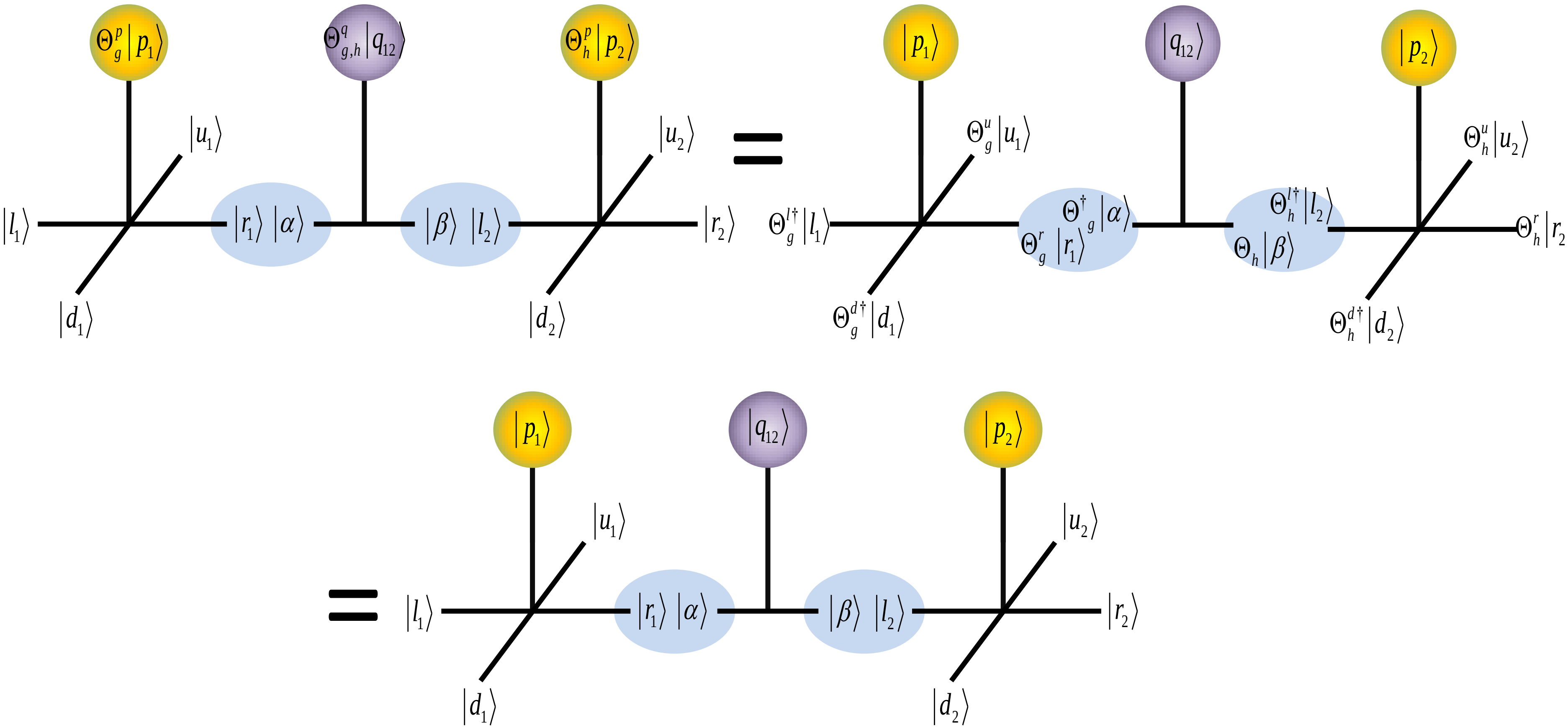


- The symmetry group is, in general, non-Abelian, and thus left and right transformations are different, and the physical state should be described by both left and right quantum numbers transforming differently.
- The requirement is met if the physical states are “identified” with the virtual ones.

EZ and M. Burrello, New J. Phys. 18 043008 (2016)

Local symmetry

- Add other tensors (gauge connections) on the links



FERMIONIC GAUSSIAN PEPS WITH LOCAL U(1) SYMMETRY

EZ, M. Burrello, T. B. Wahl and J. I. Cirac, Ann. Phys. **363**, 385 (2015).

Gaussian States

- Ground states of quadratic Hamiltonians.
- Completely described in terms of their covariance matrix Γ.
- For fermions, it is convenient to express everything in terms of Majorana modes

$$\Gamma_{ab}^{\mathbf{x}\mathbf{x}'} = \frac{i}{2}\langle[c_{\mathbf{x},a}, c_{\mathbf{x}',b}]\rangle$$

Fermionic PEPS

- Instead of states, use second-quantized fermionic operators to construct the PEPS (since a tensor product structure is not defined for fermions).
- The parent Hamiltonian of fermionic Gaussian PEPS may be easily obtained from its covariance matrix.
- Projecting to the maximally entangled bond states is very simple for Gaussian states (using Gaussian mapping).

The physical ingredients

- 2d spatial lattice (2+1d model); on every vertex (site) – a single fermionic mode $\psi_{\mathbf{x}}$.
- The fermions are staggered, with charge

$$Q_{\mathbf{x}} = s_{\mathbf{x}} \psi^{\dagger}_{\mathbf{x}} \psi_{\mathbf{x}}$$

$$s_{\mathbf{x}} \equiv (-1)^{x_1 + x_2}$$

  - Even vertices – particles;
  - Odd vertices – anti-particles;
  - The "real degrees of freedom" will be the result of a particle-hole transformation on the odd sublattice. Then a continuum spinor is formed out of blocking two neighboring sites.

The required symmetries

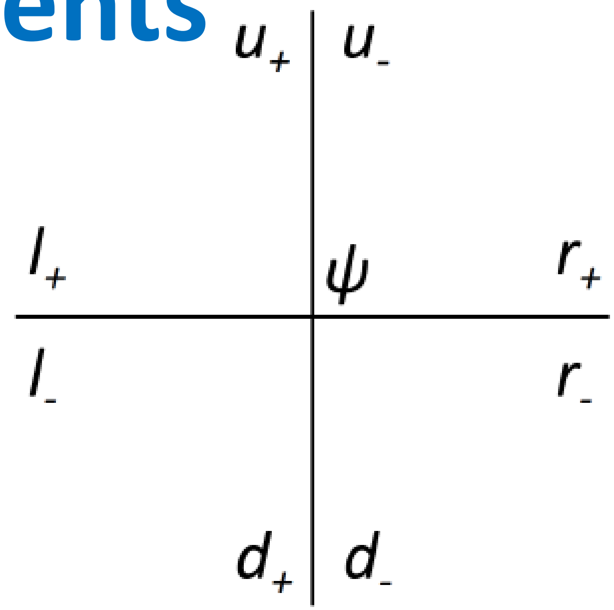
- Translation invariance → Charge conjugation
- Rotation (lattice) invariance
- Global U(1) invariance → Local gauge invariance

$$e^{i\phi\mathcal{G}_0}\left|\psi\left(\{t_i\}\right)\right\rangle = \left|\psi\left(\{t_i\}\right)\right\rangle$$

$$\mathcal{G}_0 = \sum_{\mathbf{x}} Q_{\mathbf{x}} = \sum_{\mathbf{x}} s_{\mathbf{x}} \psi^\dagger_{\mathbf{x}} \psi_{\mathbf{x}}$$

$$\psi^\dagger_{\mathbf{x}} \rightarrow e^{is_{\mathbf{x}}\phi} \psi^\dagger_{\mathbf{x}}$$

The PEPS ingredients



- One physical fermion per site
  - "Physical fermion"
- Eight virtual fermions
  - "Virtual electric fields"

$$G_0 = \mathrm{div}E - Q = E_r + E_u - E_l - E_d - Q =$$
$$r_+^\dagger r_+ - r_-^\dagger r_- + u_+^\dagger u_+ - u_-^\dagger u_- - l_+^\dagger l_+ + l_-^\dagger l_- - d_+^\dagger d_+ + d_-^\dagger d_- - s_{\mathbf{x}}\psi^\dagger\psi$$

- Created from the vacuum by

$$A = \exp\left(\sum_{ij}\hat{T}_{ij}\alpha_i^\dagger\alpha_j^\dagger\right)$$

$$T = \begin{pmatrix} t & \eta_p^2 t & \eta_p t & \eta_p^3 t \\ 0 & y & z/\sqrt{2} & z/\sqrt{2} \\ -y & 0 & -z/\sqrt{2} & z/\sqrt{2} \\ -z/\sqrt{2} & z/\sqrt{2} & 0 & y \\ -z/\sqrt{2} & -z/\sqrt{2} & -y & 0 \end{pmatrix}$$

- All the physical info – at the level of a single site

$$z, y \in \mathbb{C} \text{ and } t > 0,\ \eta_p = e^{i\pi/4}$$

$$e^{i\phi G_0} A e^{-i\phi G_0} = A$$

Studying Gaussian fPEPS

- Before gauging, with the formalism of Gaussian mapping, one may obtain an exact form of the PEPS (BCS) and the parent Hamiltonian (BCS)

$$|\psi(\mathbf{k})\rangle = \left(\alpha(\mathbf{k}) + \beta(\mathbf{k})\,\psi^\dagger_{\mathbf{k}}\psi^\dagger_{-\mathbf{k}}\right)|\Omega_{\mathbf{k}}\rangle$$

$$R(\mathbf{k}) = \frac{|\alpha(\mathbf{k})|^2 - |\beta(\mathbf{k})|^2}{|\alpha(\mathbf{k})|^2 + |\beta(\mathbf{k})|^2}$$

$$\Delta(\mathbf{k}) = \frac{2\bar{\alpha}(\mathbf{k})\,\beta(\mathbf{k})}{|\alpha(\mathbf{k})|^2 + |\beta(\mathbf{k})|^2}$$

$$\begin{aligned}\alpha(k_x,k_y) &= A_0 + A_1\left[\cos(k_x+k_y) + \cos(k_x-k_y)\right] \\ &\quad + A_2\left[\cos(2k_x) + \cos(2k_y)\right] + A_3\left[\cos(2k_x+2k_y) + \cos(2k_x-2k_y)\right] \\ \beta(k_x,k_y) &= 2t^2 B_1\left(\sin k_x - i\sin k_y\right) + \\ &\quad 2t^2 B_2\left[\sin(k_x+2k_y) - i\sin(k_y-2k_x)\right] - 2t^2 B_2^*\left[\sin(2k_y-k_x) + i\sin(k_y+2k_x)\right]\end{aligned}$$

Phases of the fermionic theory

Gapless lines:

A. Strong Pairing

$\mathbf{k} \to 0 \qquad E(\mathbf{k}) \approx \alpha_1^2 k^4$

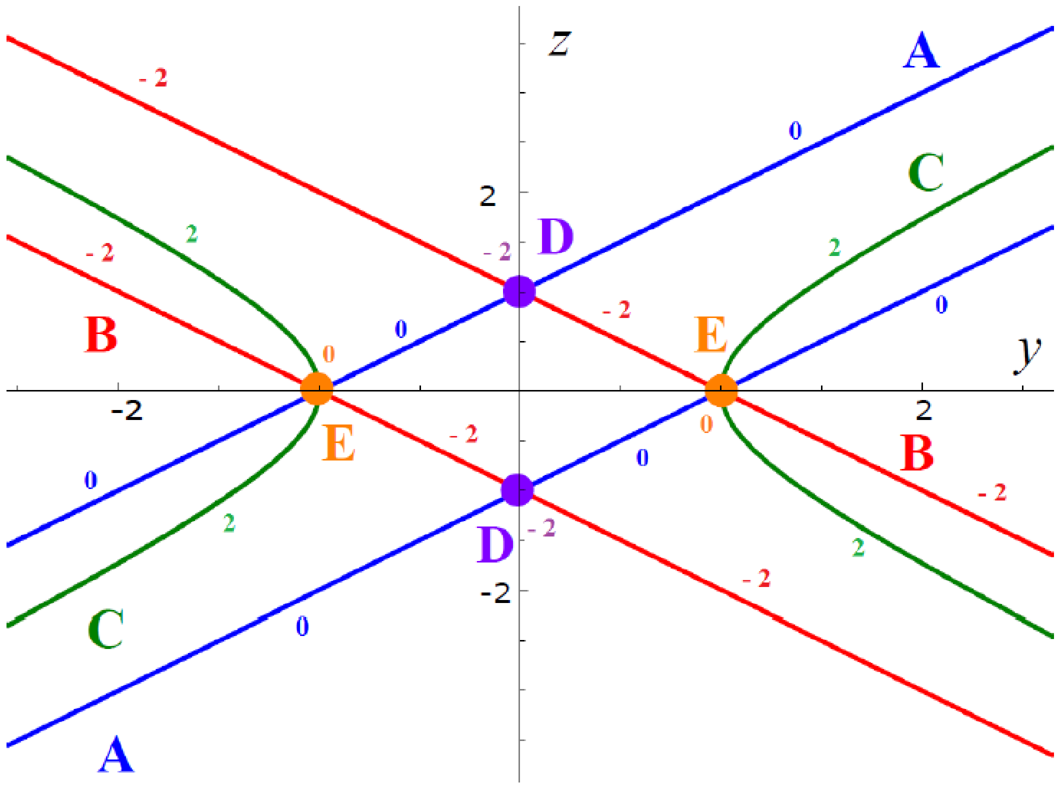
B. Weak Pairing

$\mathbf{k} \to 0 \qquad E(\mathbf{k}) \approx \beta_1^2 k^2$

C. Weak Pairing

$\mathbf{k} = (0,\pi),(\pi,0) \qquad k^6$

D. E. Weak Pairing, intersection points



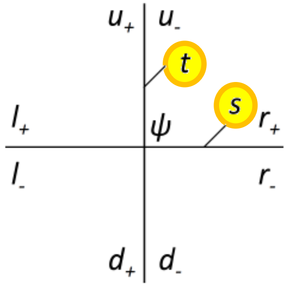
An example parent Hamiltonian in the gapped regime (y=z=0, after Particle-Hole trans. of the odd sites:

$$H = (1-4t^4)\sum_{\mathbf{x}}(-1)^{x_1+x_2}\,\psi^\dagger_{\mathbf{x}}\psi_{\mathbf{x}} + t^4\sum_{\mathbf{x}}(-1)^{x_1+x_2}\left(\psi^\dagger_{\mathbf{x}+2\hat{e}_1}\psi_{\mathbf{x}} + \psi^\dagger_{\mathbf{x}+2\hat{e}_2}\psi_{\mathbf{x}} + \mathrm{H.c.}\right)$$

Staggered mass

"Massless Dirac" → $-2t^2\sum_{\mathbf{x}}\left(i\psi^\dagger_{\mathbf{x}}\psi_{\mathbf{x}+\hat{e}_1} + (-1)^{x_1+x_2}\,\psi^\dagger_{\mathbf{x}}\psi_{\mathbf{x}+\hat{e}_2} + \mathrm{H.c.}\right)$

Gauging the PEPS



- The symmetry is local now

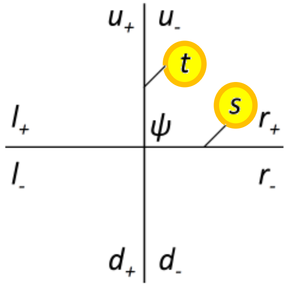
$$\psi_{\mathbf{x}}^{\dagger} \rightarrow e^{i s_{\mathbf{x}} \phi_{\mathbf{x}}} \psi_{\mathbf{x}}^{\dagger}$$

- Add "bosonic" Hilbert physical states on the links, spanned by $\{|m\rangle\}_{m=-\ell}^{\ell} \equiv \{|\ell m\rangle\}_{m=-\ell}^{\ell}$
- Electric field operator: $\Sigma = L_z$
- Raising and lowering operators ("Wilson operators")

$$\Sigma_{\pm} = \frac{L_{\pm}}{\sqrt{\ell(\ell+1)}} \qquad \Sigma_{\pm}|m\rangle \underset{\ell \rightarrow \infty}{\longrightarrow} e^{\pm i\theta}|m\rangle = |m+1\rangle$$

$$\exp\left(i \int_{\mathbf{x}}^{\mathbf{x}+\mathbf{e}_{\mu}} A_{\mu} dx^{\mu}\right) \equiv e^{i\theta^{s/t}(\mathbf{x})} \rightarrow \Sigma_{+}^{s/t}(\mathbf{x})$$

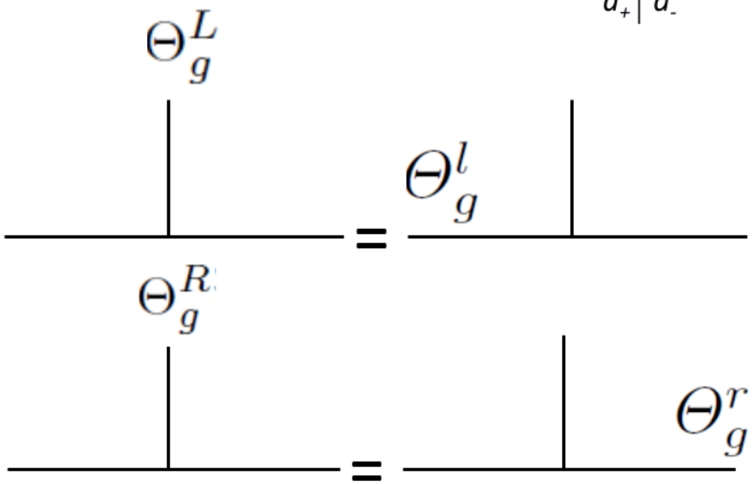
Gauging the PEPS



- Make the substitution

$$\begin{cases} r_+^\dagger \to \Sigma_+^s r_+^\dagger \\ r_-^\dagger \to \Sigma_-^s r_-^\dagger \\ u_+^\dagger \to \Sigma_+^t u_+^\dagger \\ u_-^\dagger \to \Sigma_-^t u_-^\dagger \end{cases}$$

in the fermionic local state.



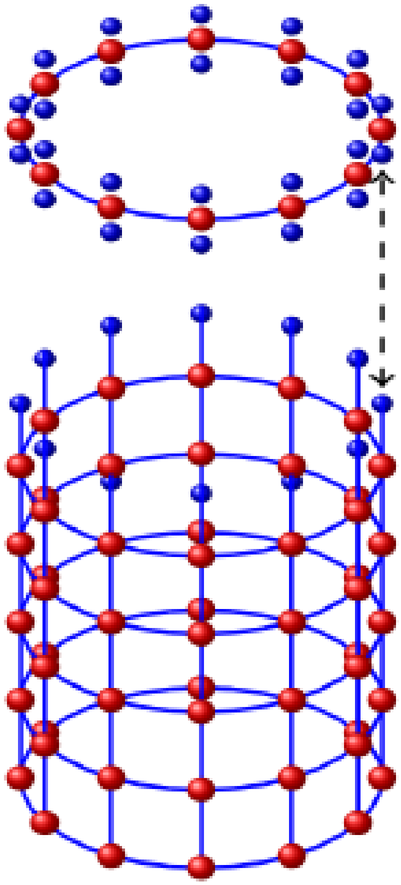
- That guarantees the symmetry condition for the "gauge tensor": the virtual gauge symmetry becomes a physical one

$$G_0 = \mathrm{div}E - Q = E_r + E_u - E_l - E_d - Q =$$
$$r_+^\dagger r_+ - r_-^\dagger r_- + u_+^\dagger u_+ - u_-^\dagger u_- - l_+^\dagger l_+ + l_-^\dagger l_- - d_+^\dagger d_+ + d_-^\dagger d_- - s_{\mathbf{x}} \psi^\dagger \psi$$

$$G_{\mathbf{x}} = \Sigma^s(\mathbf{x}) + \Sigma^t(\mathbf{x}) - \Sigma^{\bar{s}}(\mathbf{x}) - \Sigma^{\bar{t}}(\mathbf{x}) - Q_{\mathbf{x}}$$
$$= \Sigma^s(\mathbf{x}) + \Sigma^t(\mathbf{x}) - \Sigma^s(\mathbf{x} - \hat{\mathbf{e}}_1) - \Sigma^t(\mathbf{x} - \hat{\mathbf{e}}_2) - Q_{\mathbf{x}}$$

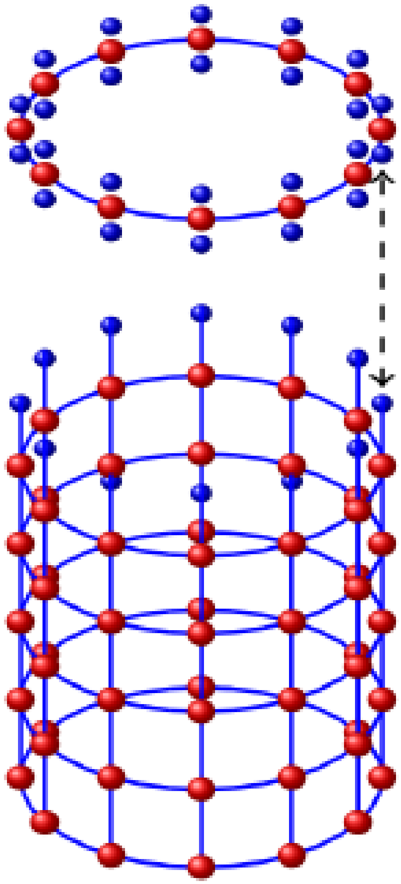
Studying the PEPS

- We use a cylindrical geometry: one dimension is periodic, the other is open.



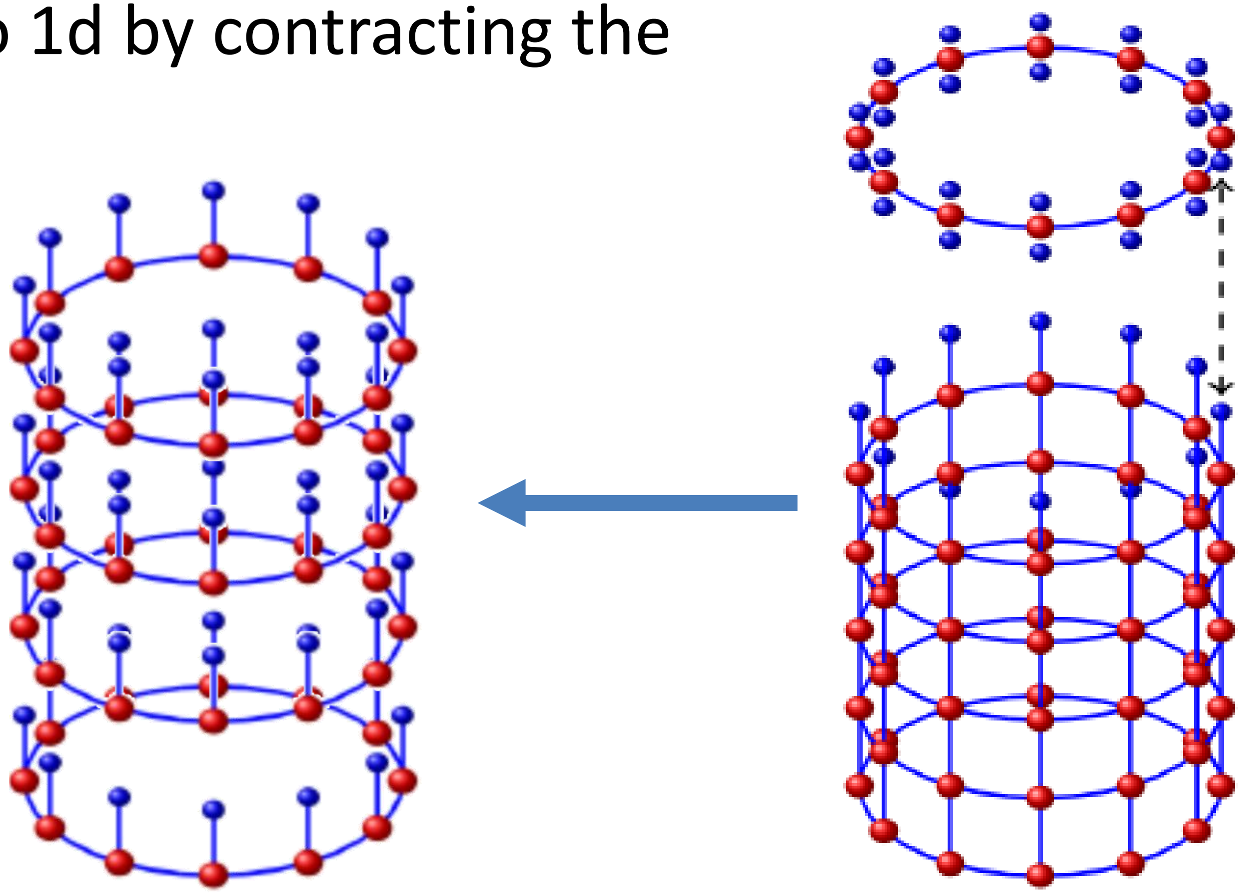
Studying the PEPS

- We use a cylindrical geometry: one dimension is periodic, the other is open.
- No external charges are put on the open edges (boundary condition).



Studying the PEPS

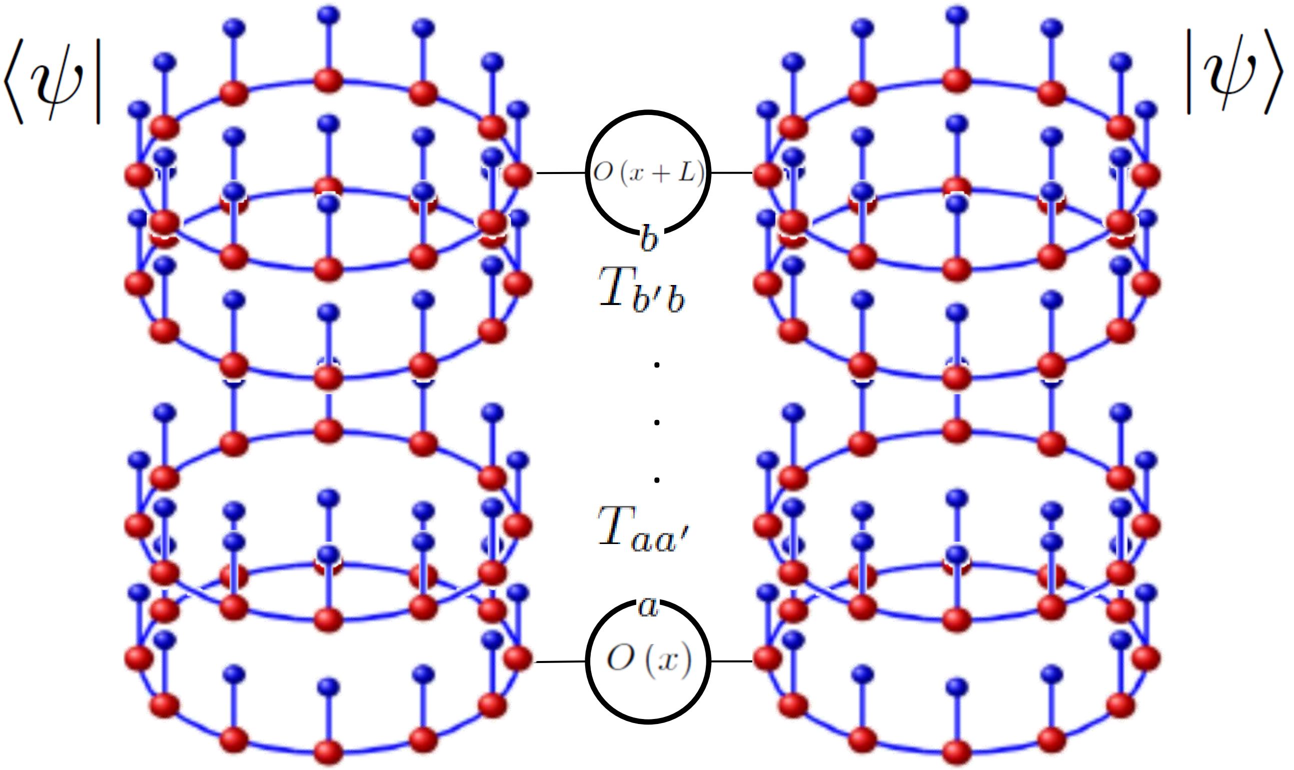
- The system may be converted effectively to 1d by contracting the rows.



Studying the PEPS

- A row-by-row transfer operator for calculation of correlations may be defined.

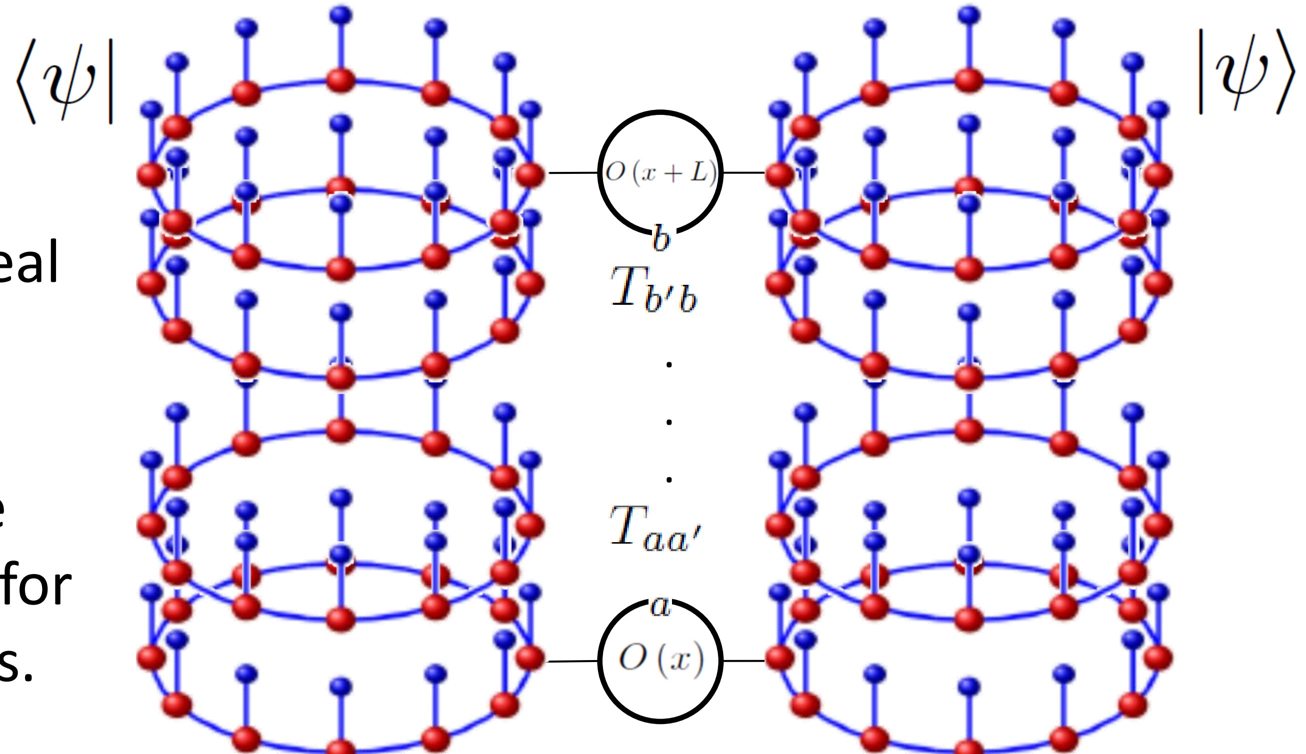
$$\langle O(x)\, O(x+L)\rangle \sim \overbrace{T_{aa'} T_{a'a''} \cdots T_{b'b}}^{\sim L}$$



Studying the PEPS

- A gap between the two highest eigenvalues of the transfer operator corresponds to exponentially decaying correlations in real space.
- The transfer operator may be used as a probe for phase transitions.

$$\langle O(x)\, O(x+L)\rangle \sim \overbrace{T_{aa'} T_{a'a''} \cdots T_{b'b}}^{\sim L}$$



What are the phases?

- Finding the phase boundaries is not enough, one should still identify the physical behavior of the various gapped phases.
- Several physical observables (order parameters) may be evaluated for the PEPS and help to give physical meanings to the phases found numerically (as well as another probe for the positions of the boundaries).

Phase diagrams from the transfer operator

Width - 6 | Width - 8

Pure gauge *t*=0

Interacting *t*=1

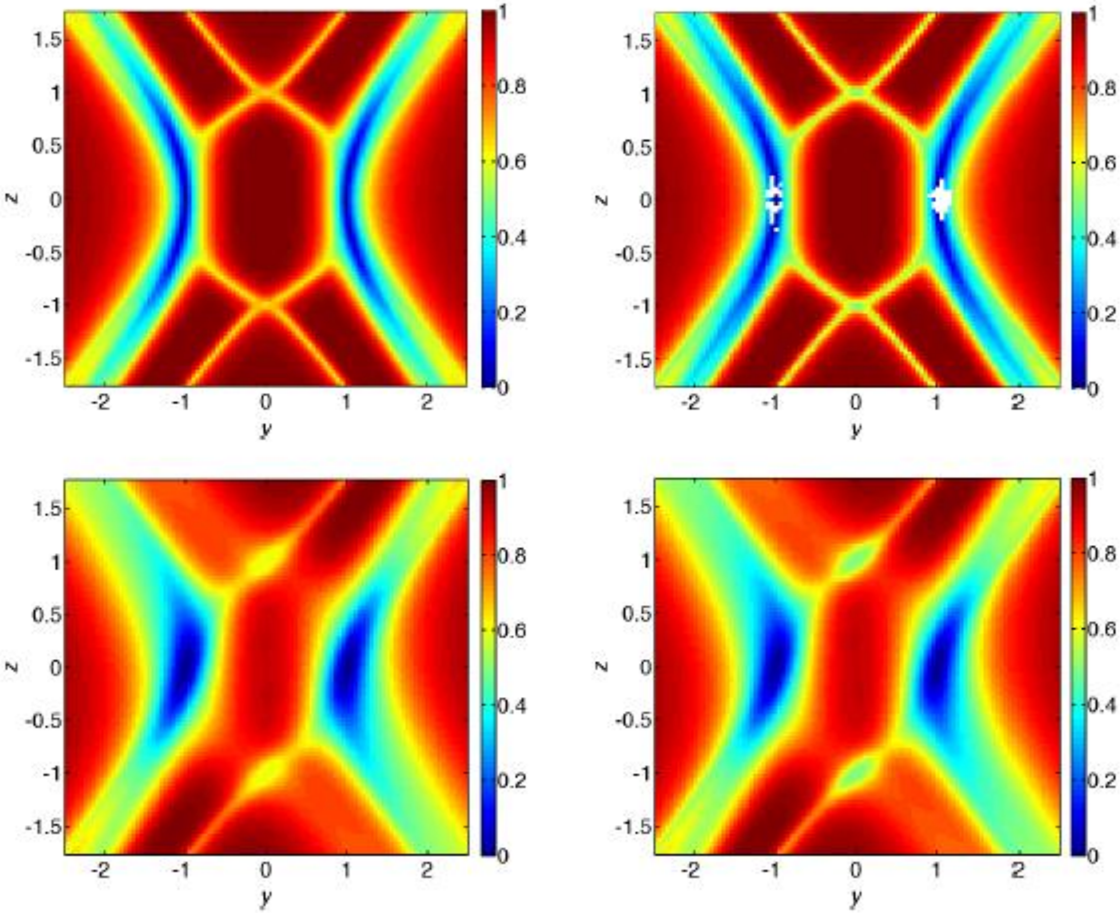
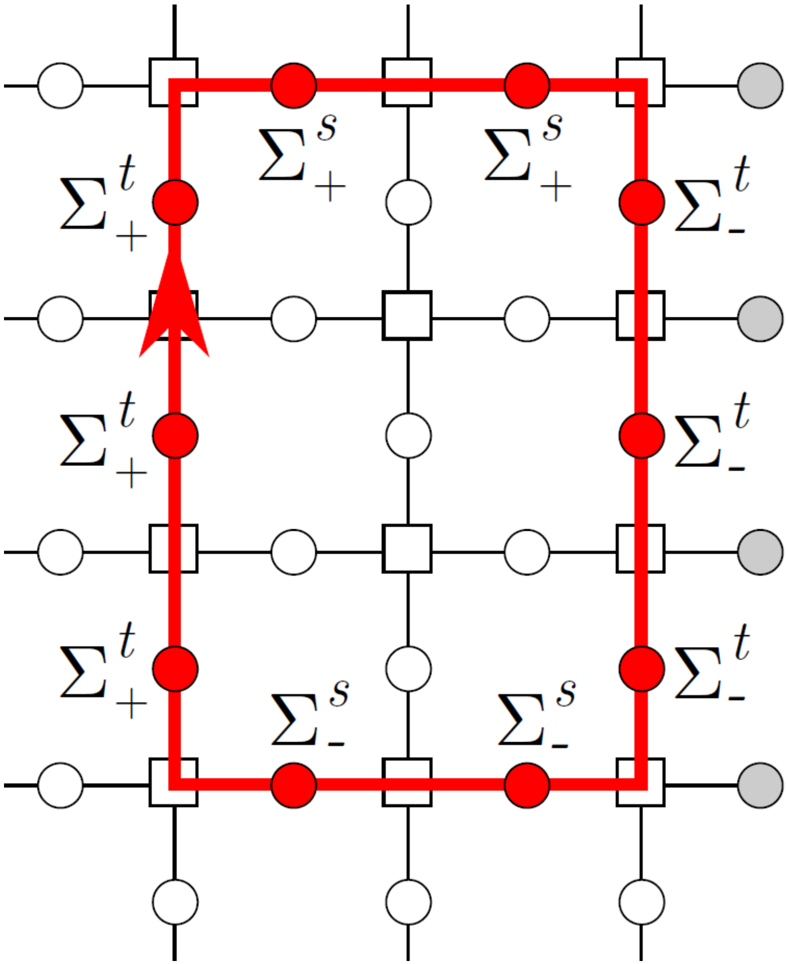


Figure 8: Left: Gap between the highest eigenvalue $\lambda_1 = 1$ and the second highest eigenvalue of the transfer matrix for $t = 0$ (top) and $t = 1$ (bottom) for a cylinder of circumference $L_1 = 6$ as a function of $y, z \in \mathbb{R}$. Right: Same plot for $L_1 = 8$. The lines with low values of the gap indeed seem to be gapless lines in the thermodynamic limit, as the corresponding values of the gap are significantly lower for $L_1 = 8$ than for $L_1 = 6$. Points were the Lanczos algorithm for calculating the eigenvalues of the transfer operator did not converge are marked in white.

Pure gauge theory – Wilson Loops

- Confined static charges – area law
- Deconfined static charges – perimeter law

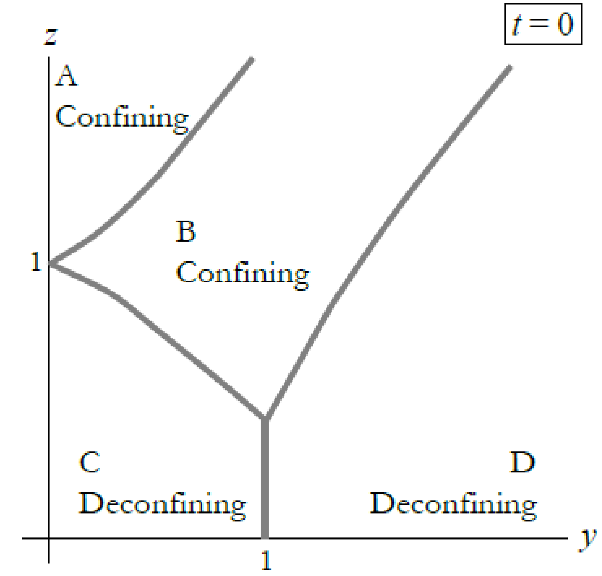
$$W_{\mathcal{C}} = \prod_{\mathbf{b}\in\mathcal{C}} \Sigma_{\pm}(\mathbf{b})$$

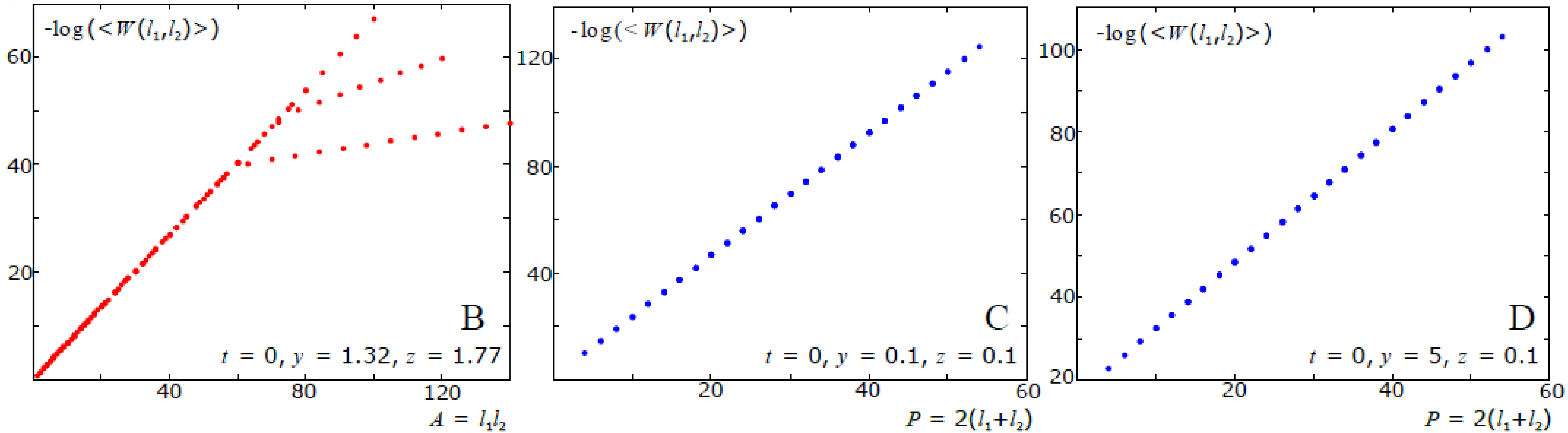


The phases of the pure gauge theory

B,C,D – clear results from the Wilson loops (also from other computations, such as the Creutz parameter)

A,D – also some analytical results from $1/z$ or $1/y$ expansions.





Summary

- PEPS are very useful for the study of many body systems with symmetries – even when the symmetries are local.
- With a given set of symmetries, one may parameterize a set of states and study its phase structure, using various methods such as transfer operator and direct computation of order parameters.
- Our proof-of-principle study of the U(1) case has resulted with the known phases of the theory.

Thank you!

- The talk is based on
  – Fermionic projected entangled pair states and local U(1) gauge theories. EZ, M. Burrello, T. B. Wahl and J. I. Cirac, Ann. Phys. **363**, 385 (2015).
  – Building Projected Entangled Pair States with a Local Gauge Symmetry. EZ and M. Burrello, New J. Phys. 18 043008 (2016)
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- **See also other works on the topic, from MPQ, ICFO, Vienna-Ghent, Innsbruck-Ulm, including variational approaches for LGTs as well.**