Introduction to Numerical Relativity

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COST NewCompStar School 2016

Exploring fundamental physics with compact stars

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Gravitational Waves: Ripples in spacetime

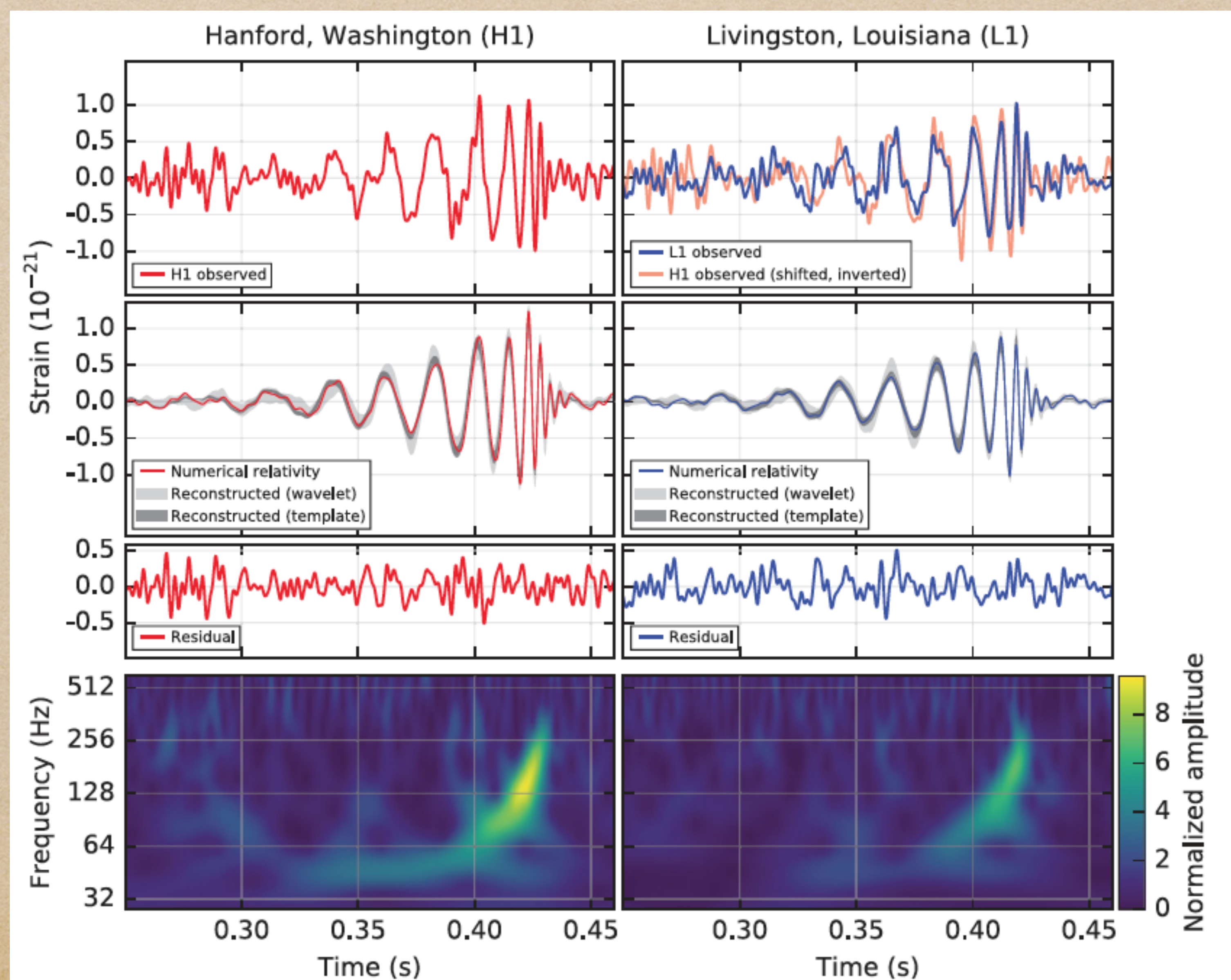
- Unusual news headlines on 11/12 February 2016
- First direct detection of gravitational waves: GW150914

So, what happened?

- Sep 14, 2015 at 09:50:45 UTC: SNR ~ 24

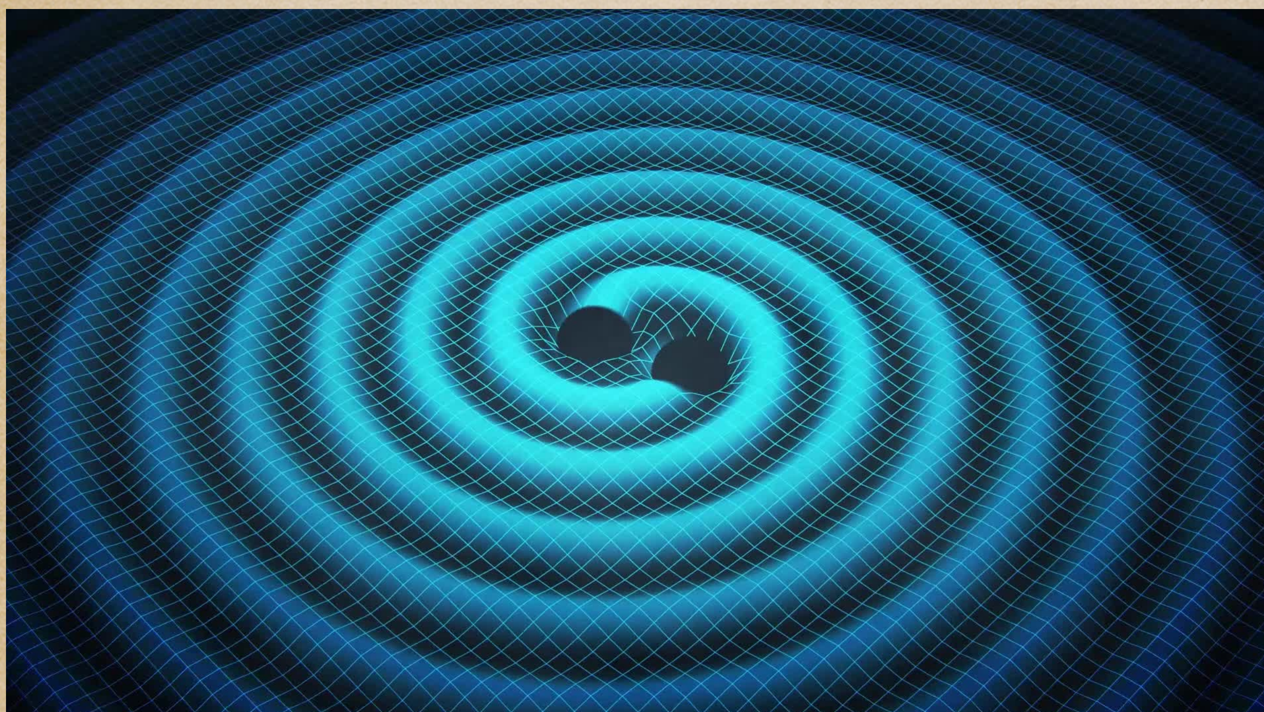
  Abbott et al. PRL 1602.03837, Abbott et al. 1606.01210

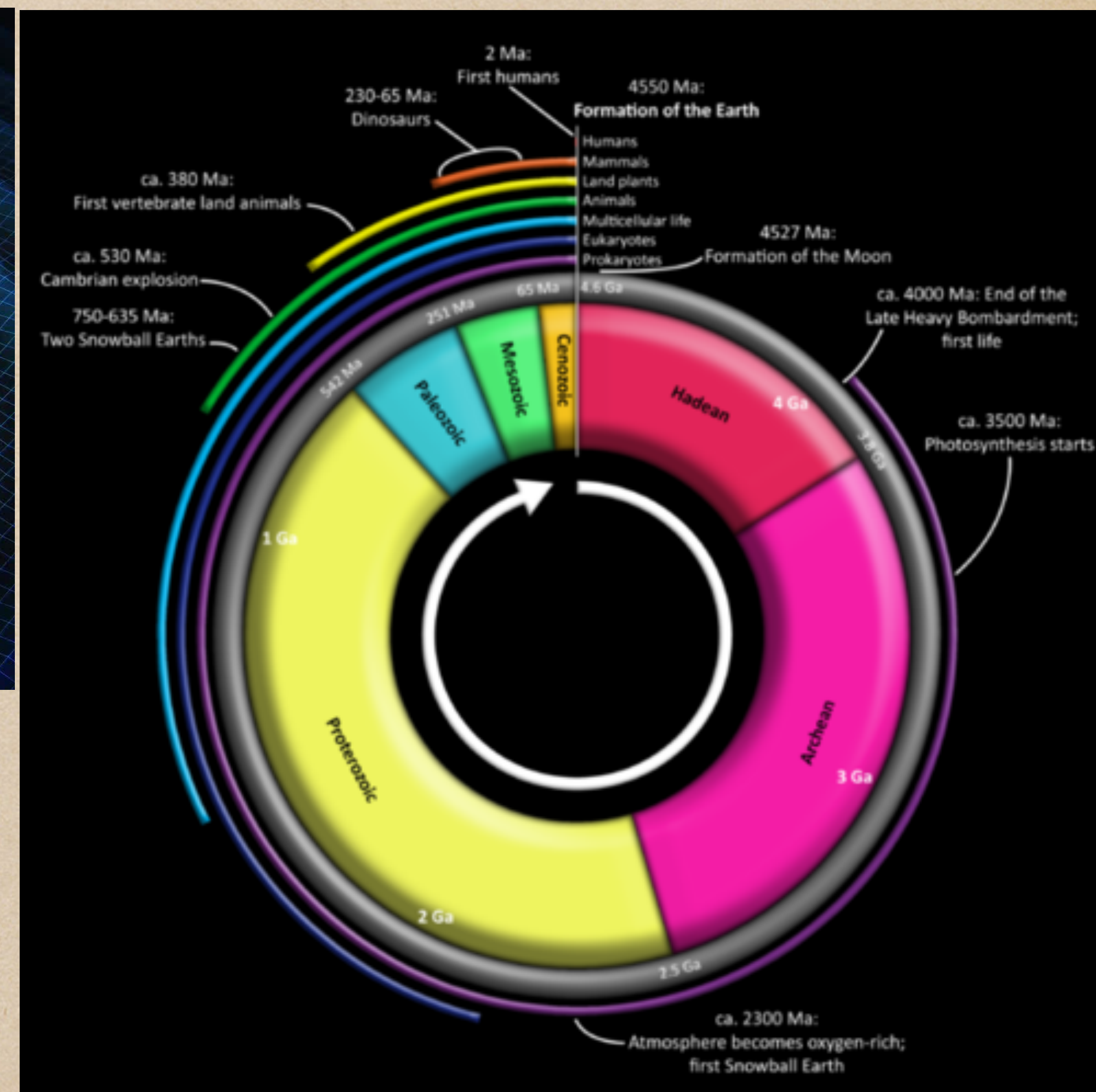
- BBH inspiral, merger and ringdown: $m_1 = 35^{+5}_{-3}\, m_\odot\,, \quad m_2 = 30^{+3}_{-4}\, M_\odot$



What really happened...

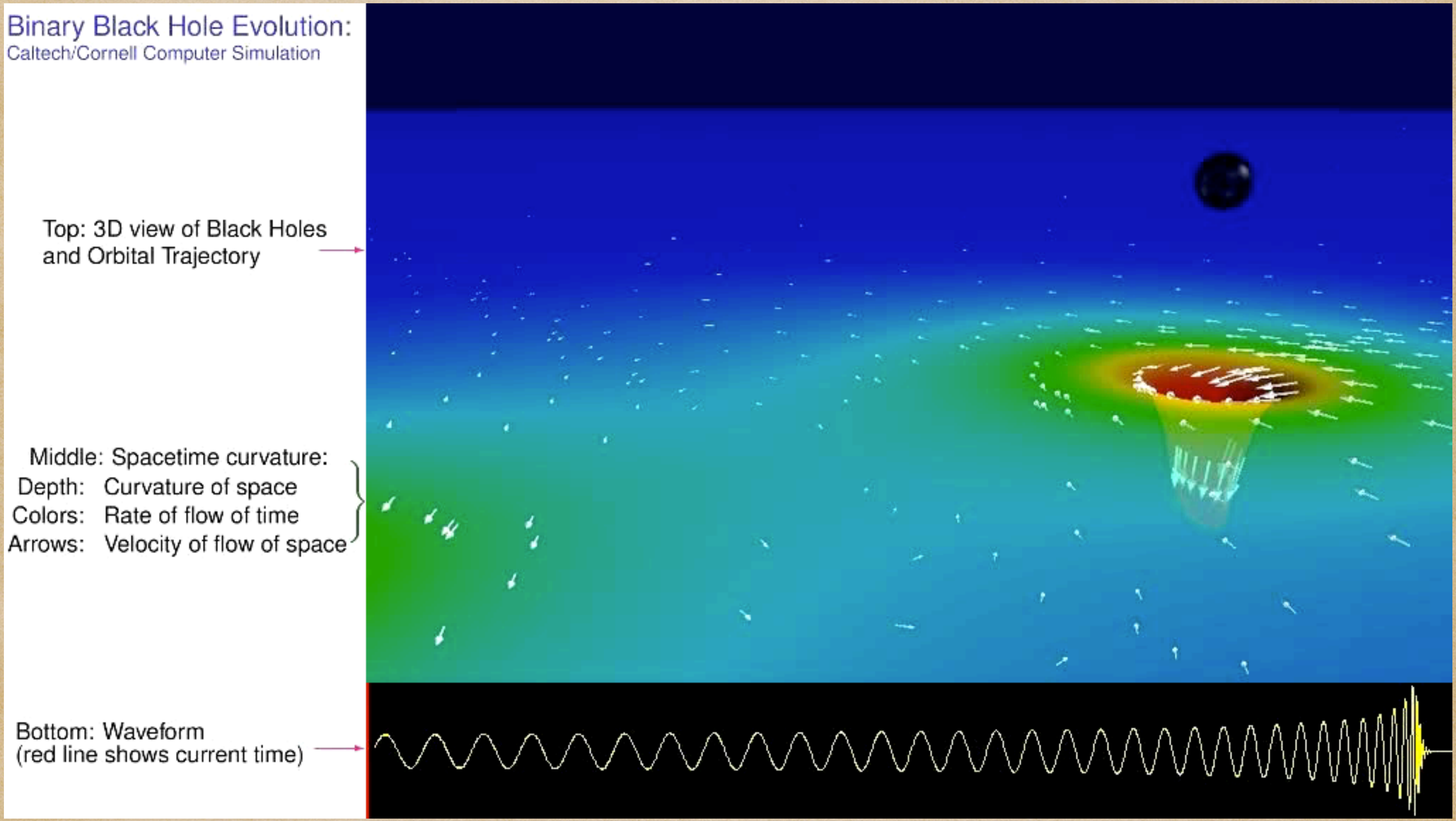
- Once upon a time: $1.34^{+0.52}_{-0.59}$ Gyr ago, somewhere in the universe





- Deep Precambrian

We can model this with NR



Thanks to Caltech-Cornell groups

Overview

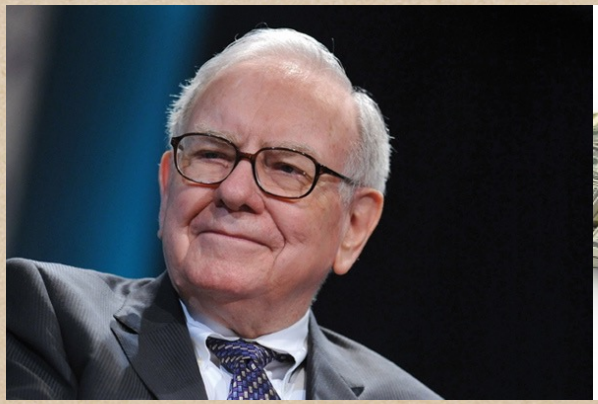
- Introduction, Motivation
- Foundations of numerical relativity
  - Formulations of Einstein's Eqs.: 3+1, BSSN, GHG, characteristic
  - Beyond 4D: Dimensional reduction
  - Initial data, gauge
  - Technical ingredients: Discretization, AMR, boundaries…
  - Diagnostics: Horizons, momenta, GWs,…
- Applications and selected results
  - Astrophysics
  - Gravitational wave physics
  - High-energy physics
  - Fundamental properties of gravity

1. Introduction, Motivation

Strong gravity = non linearity

- What is non-linearity? Think of the stock market

$\Rightarrow$ linear

$\Rightarrow$ NON-LINEAR!

Strongest possible gravity: Black holes

- Einstein 1915: General Relativity; geometric theory of gravity
- Schwarzschild 1916: Solution to Einstein's equations

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 d\phi^2$$

- Singularities

  $r = 0$: physical

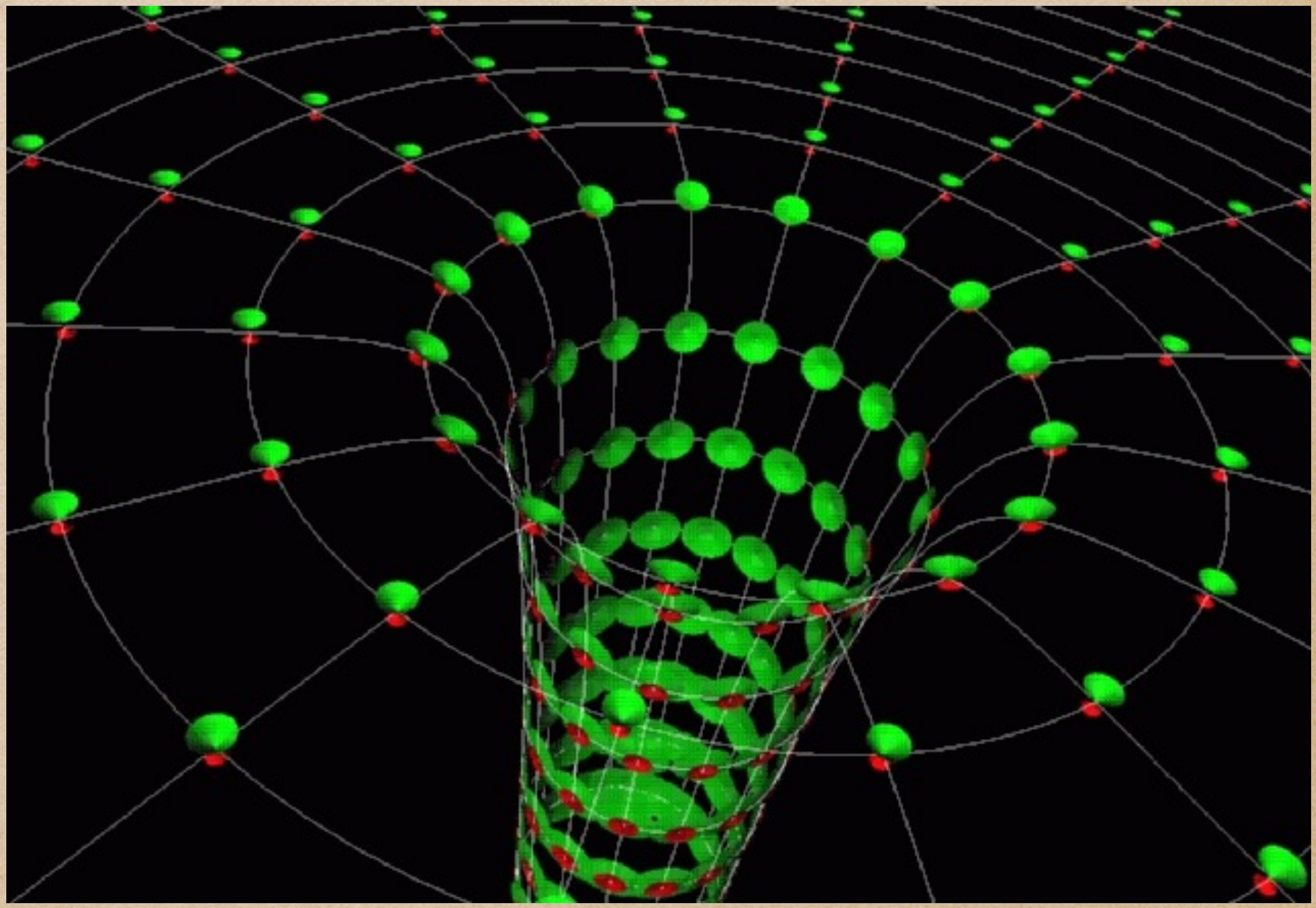
  $r = 2M$: singularity

- Horizon at $r = 2M$

  Light cones tilt over

- Newtonian escape velocity

$$v = \sqrt{\frac{2M}{r}}$$



Black-hole analogy

Evidence for astrophysical BHs

- LIGO observation of GWs (above)
- X-ray binaries

  e.g. Cygnus X-1 (1964)

  MS star + compact star

  $\Rightarrow$ Stellar Mass BH

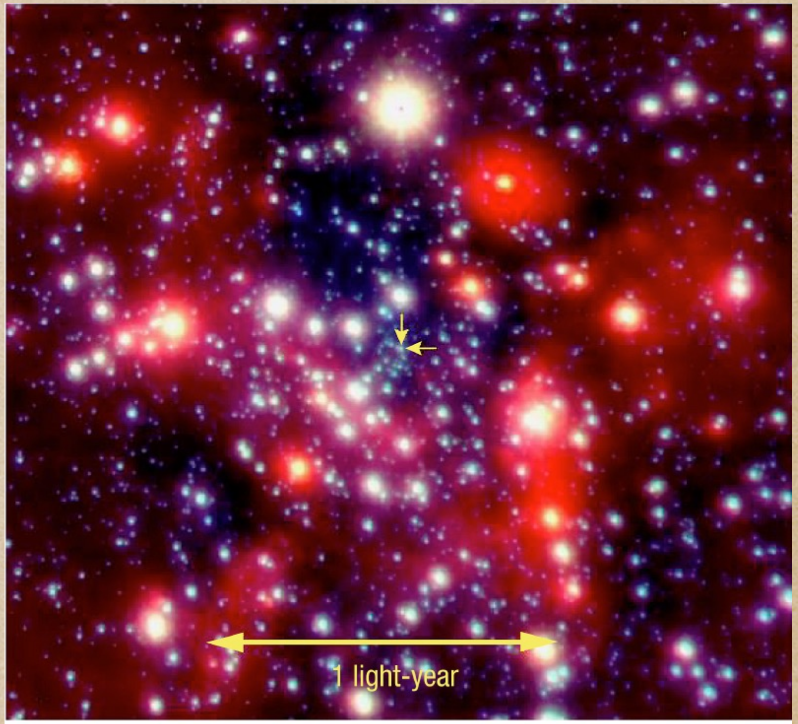
  $5 \ldots 50 \; M_\odot$
- Stellar dynamics near galactic center

  Iron emission line profiles

  $\Rightarrow$ Supermassive BHs

  $10^6 \ldots 10^{10} \; M_\odot$

  AGN engines



The Centre of the Milky Way
(VLT YEPUN + NACO)
ESO PR Photo 23a/02 (9 October 2002)
©European Southern Observatory

Conjectured BHs

- Intermediate BHs

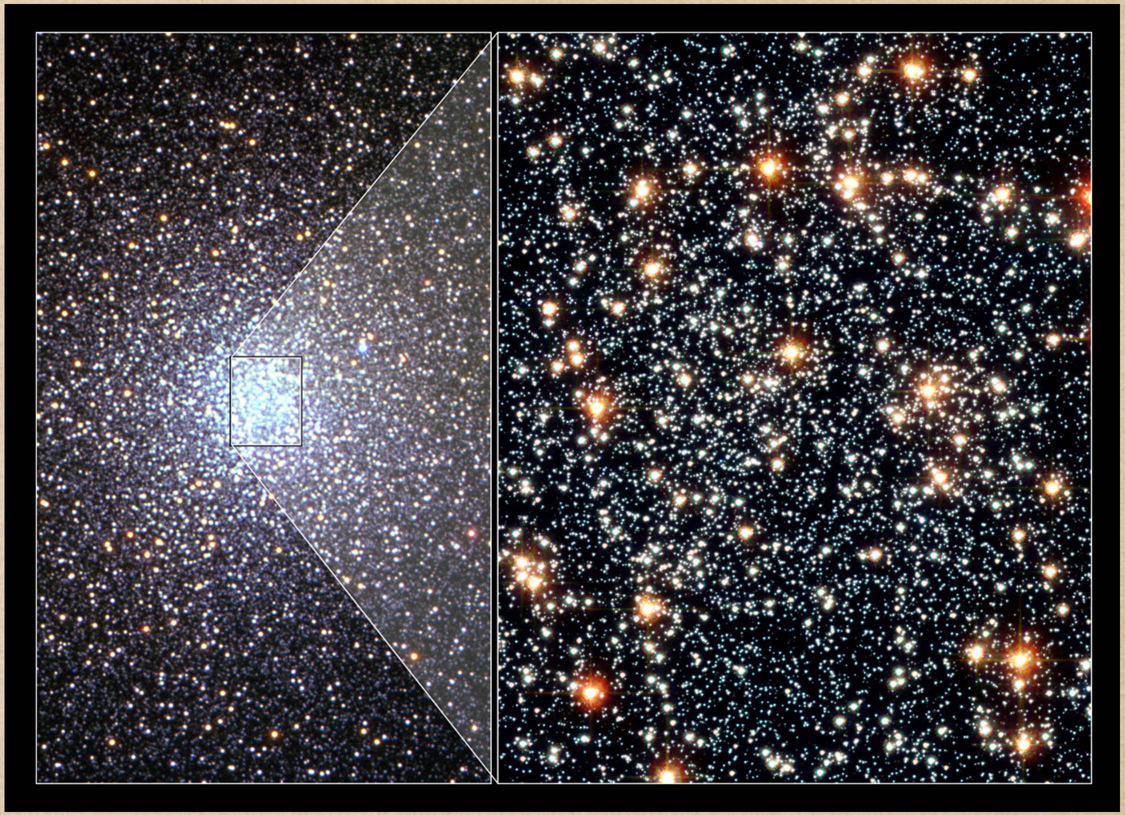
  $10^2 \ldots 10^5 M_\odot$

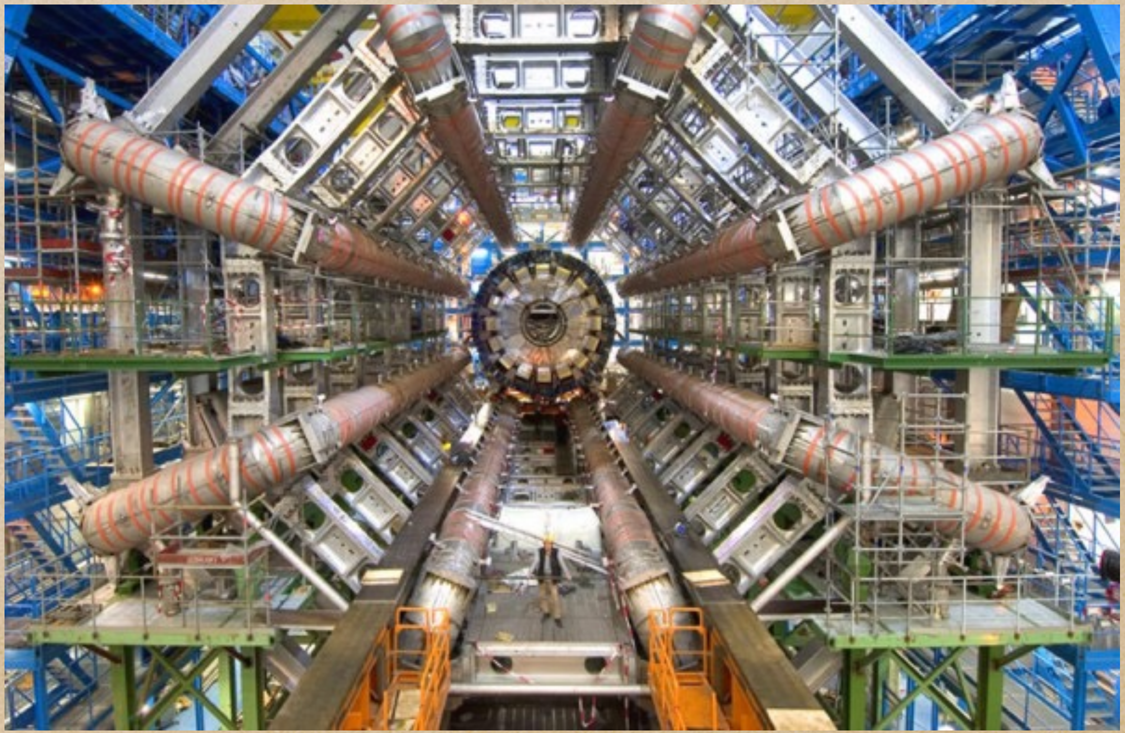
- Primordial BHs

  $\leq M_{\rm Earth}$

- Microscopic BHs, LHC

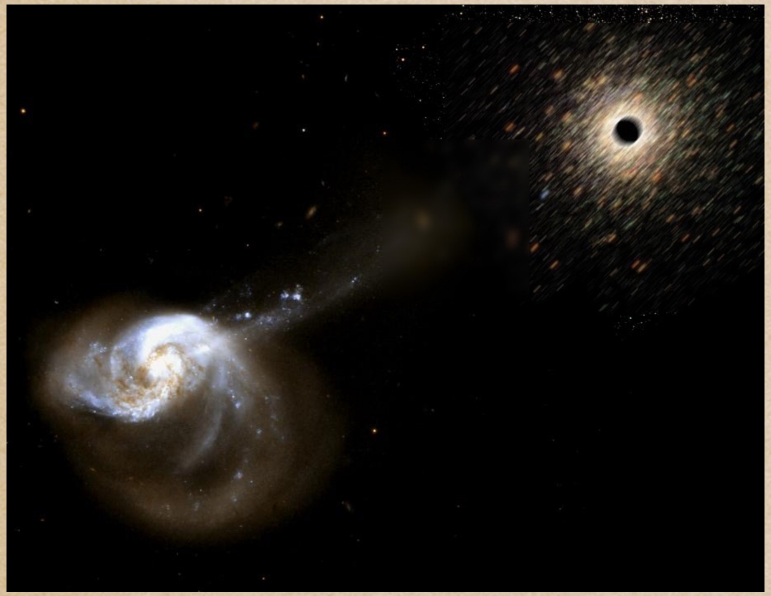
  $\sim \rm TeV$



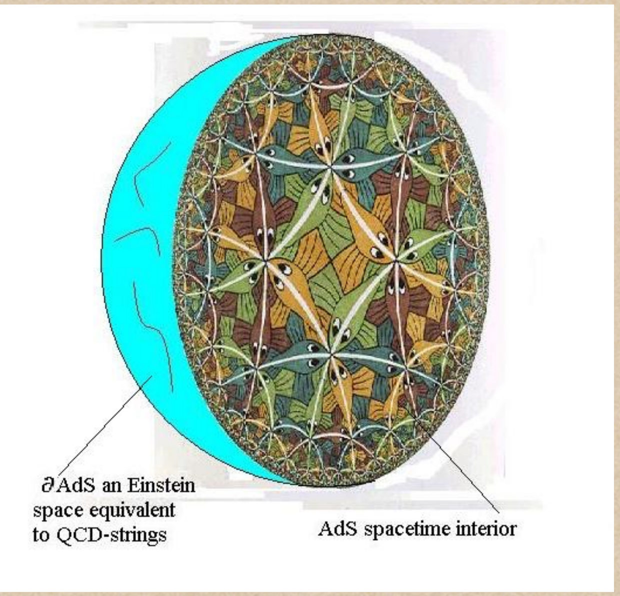


Research areas: BHs have come a long way

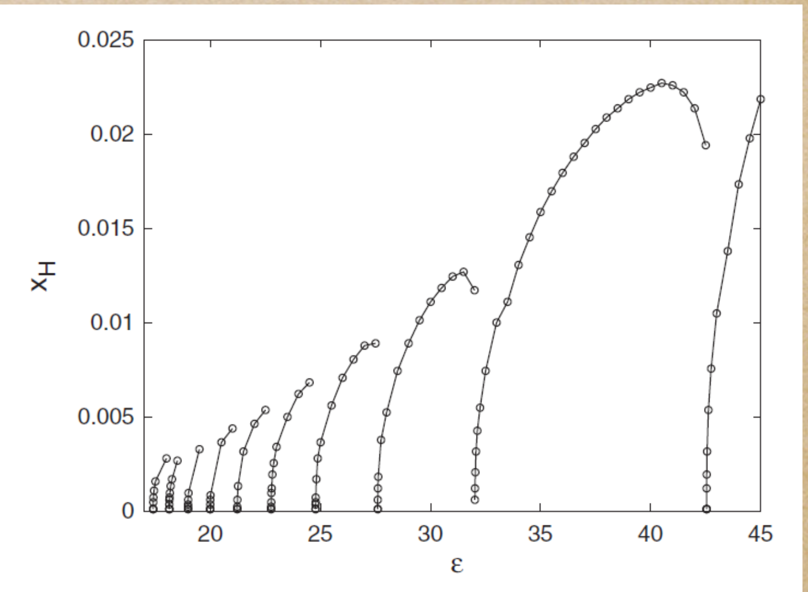
Astrophysics



Gauge gravity duality

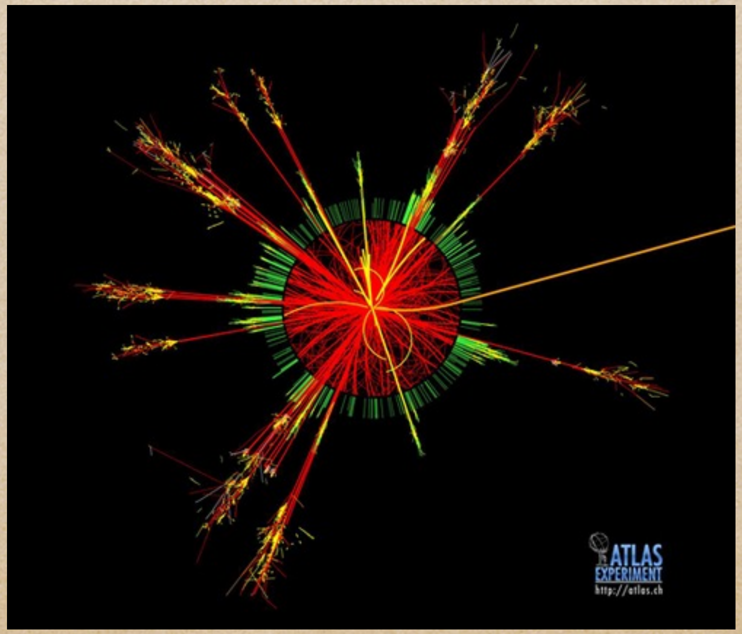


Fundamental studies

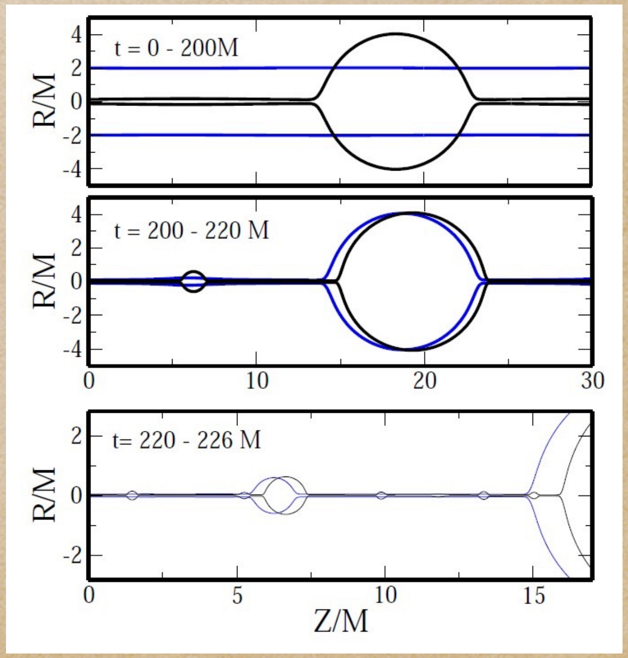


GW physics

High-energy physics



Fluid analogs



Vitor's talk in 30 seconds

- Spacetime as a curved manifold
- Key quantity: spacetime metric $g_{\alpha\beta}$
- Curvature, geodesics etc. all follow
- Einstein equations

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

10 non-linear PDEs for $g_{\alpha\beta}$

$T_{\alpha\beta} =$ Matter fields

- Conceptually simple,

  hard in practice
- E.g. Schwarzschild

$$g_{\mu\nu} = \begin{pmatrix} \left(1 - \frac{2GM}{rc^2}\right) & 0 & 0 & 0 \\ 0 & -\left(1 - \frac{2GM}{rc^2}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix}$$

$$ds^2 = c^2 dt^2 \left(1 - \frac{2GM}{rc^2}\right) - \frac{dr^2}{1 - 2GM/rc^2} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

How do we get the metric?

Solving this equation is our job

How do we get the metric?

- The metric must obey $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$
- Ricci tensor, Einstein tensor, matter tensor

$R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}$

$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R^{\mu}{}_{\mu}$ "Trace reverse Ricci"

$T_{\alpha\beta}$ "Matter"; see Talk by Luciano Rezzolla

Λ "Cosmological constant"

- Solutions: Easy!

Take metric $g_{\alpha\beta}$

$\Rightarrow$ Calculate $G_{\alpha\beta}$

$\Rightarrow$ Use that for $T_{\alpha\beta}$

- Physically meaningful solutions: That's the hard part!

Solving Einstein's Eqs.: The toolbox

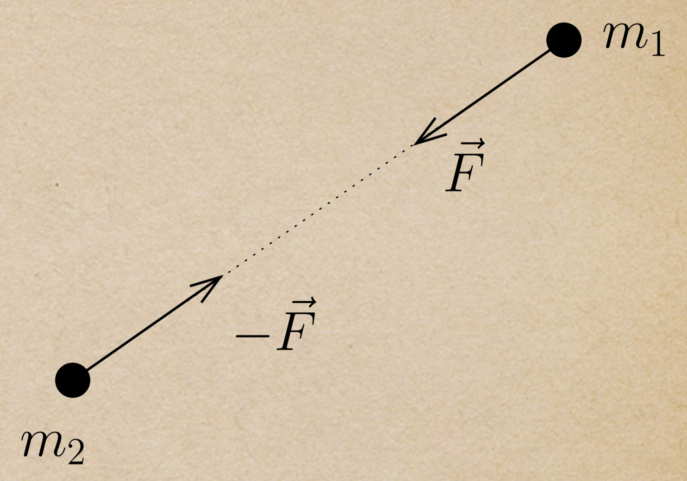
- **Analytic solutions**
  - Symmetry assumptions
  - Schwarzschild, Kerr, FLRW, Vaidya, Tangherlini, Myers-Perry, ...
- **Perturbation theory**
  - Assume solution is close to a known "background" $g^{(0)}_{\alpha\beta}$
  - Expand $g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + \ldots \quad \Rightarrow$ linear system
  - Regge-Wheeler-Zerilli-Moncrief, Teukolsky, QNMs, EOB, ...
- **Post-Newtonian theory**
  - Assume small velocities $\Rightarrow$ Expansion in $\frac{v}{c}$
  - N^{th} order expressions for GWs, momenta, orbits, ...
  - Blanchet, Buonanno, Damour, Kidder, Schäfer, Will, ...
- **Numerical Relativity**

2. Foundations of Numerical Relativity

The Newtonian 2-body problem

- Eqs. of motion

$$m_1 \frac{d^2 \vec{r}_1}{dt^2} = \vec{F} = -G \frac{m_1 m_2}{r^2} \hat{\vec{r}} = -m_2 \frac{d^2 \vec{r}_2}{dt^2}$$



- Solution: Keppler ellipses, parabolic, hyperolic

$$r = \frac{r_0}{1 + \epsilon \cos\theta}$$

e.g. Sperhake CQG 1411.3997

- What's different in GR?
  - No point particles in GR!
  - GR is non-linear
  - No "background" time and space
  - Systems typically are dissipative $\Rightarrow$ Gravitational waves
  - No obvious formulation as time evolution problem

A list of tasks in NR

- **Target:** Predict time evolution of a physical system in GR
- **Einstein eqs.:** 1) Cast as evolution system
  2) Choose a "good" formulation
  3) Discretize for a computer
- **Gauge:** Choose "good" coordinates
- **Technical aspects:** 1) Mesh refinement / spectral domains
  2) Singularity handling (excision)
  3) Parallelization
- **Initial data:** 1) Solve constraints
  2) Get "realistic" initial data
- **Diagnostics:** 1) GW extraction, kicks, …
  2) Horizon data, ADM mass,…

Notation

- Spacetime indices: Greek $\alpha,\, \beta,\, \ldots = 0,\, \ldots,\, D-1$
- Spatial indices: middle Latin $i,\, j,\, \ldots = 1,\, \ldots,\, D-1$
- Signature: $- + \ldots +$
- Christoffel symbols: $\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu}(-\partial_{\mu} g_{\beta\gamma} + \partial_{\beta} g_{\gamma\mu} + \partial\gamma g_{\mu\beta})$
- Riemann tensor $R^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\Gamma^{\mu}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\mu}_{\nu\rho} + \Gamma^{\tau}_{\nu\sigma}\Gamma^{\mu}_{\tau\rho} - \Gamma^{\tau}_{\nu\rho}\Gamma^{\mu}_{\tau\sigma}$
- Units: $c = 1 = G$
- Spatial metric γ_{ij}
- Spatial Riemann, Ricci tensor: $\mathcal{R}^{i}{}_{jkl}\,,\ \ \mathcal{R}_{ij}$
- We use Γ for the spatial and spacetime Christoffel symbols.

  Unlike for Riemann, it will always be clear from the context.

2.1 Formulations of Einstein's equations

The Einstein equations

- Recall: $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$

- In this form, the mathematical character is unclear!

  hyperbolic, elliptic, parabolic?

- Coordinates x^α are on equal footing.

  Time singled out only through signature of the metric!

- Well-posedness of the equations? Suitable for numerics?

- There are various ways to address these questions

  $\rightarrow$ Formulations of the equations

2.1.1 ADM type (D−1)+1 formulations

The 3+1 decomposition

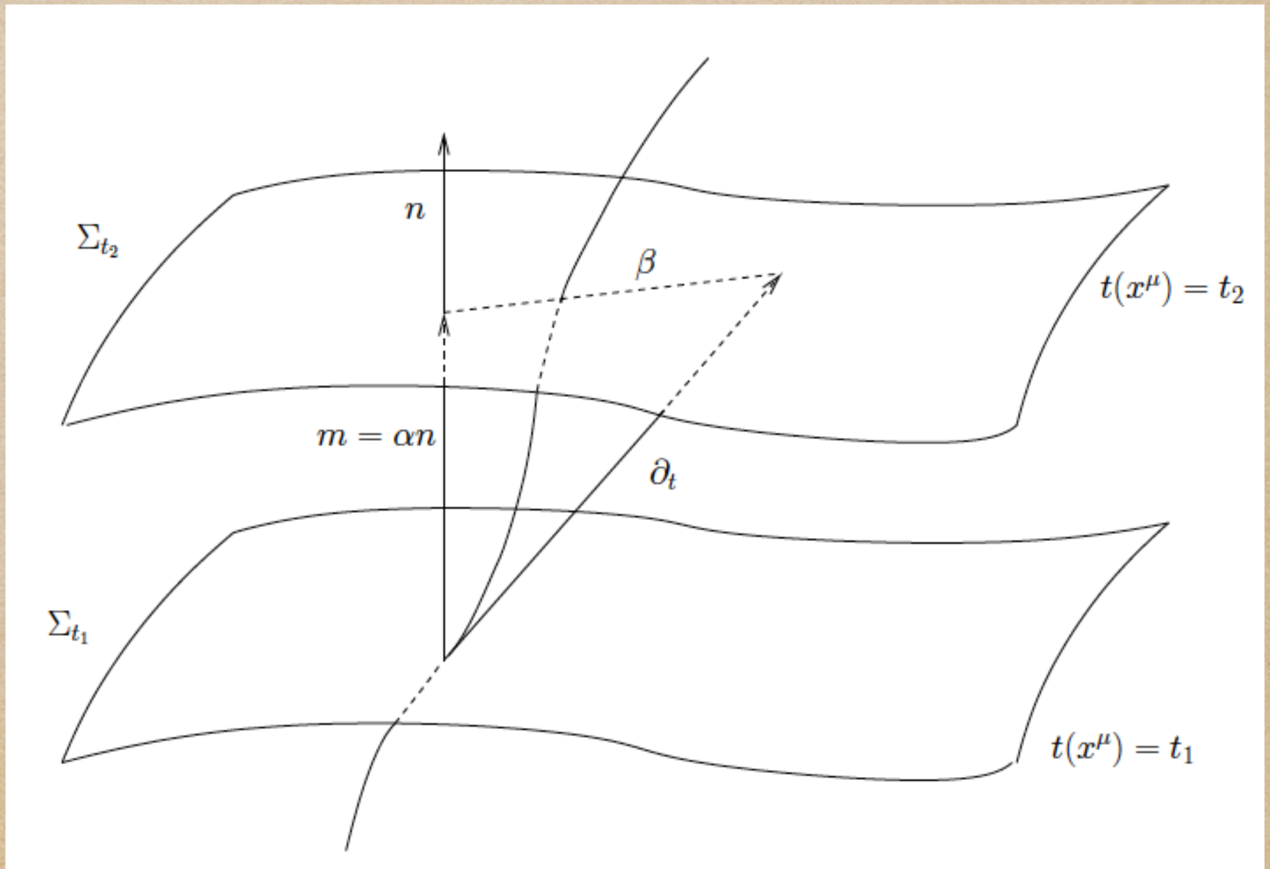
- ADM 3+1 split: Arnowitt, Deser & Misner 1962

  York 1979, Choquet-Bruhat & York 1980

Def.: Spacetime := $(\mathcal{M}, g)$

= Manifold with metric of signature $-+++$

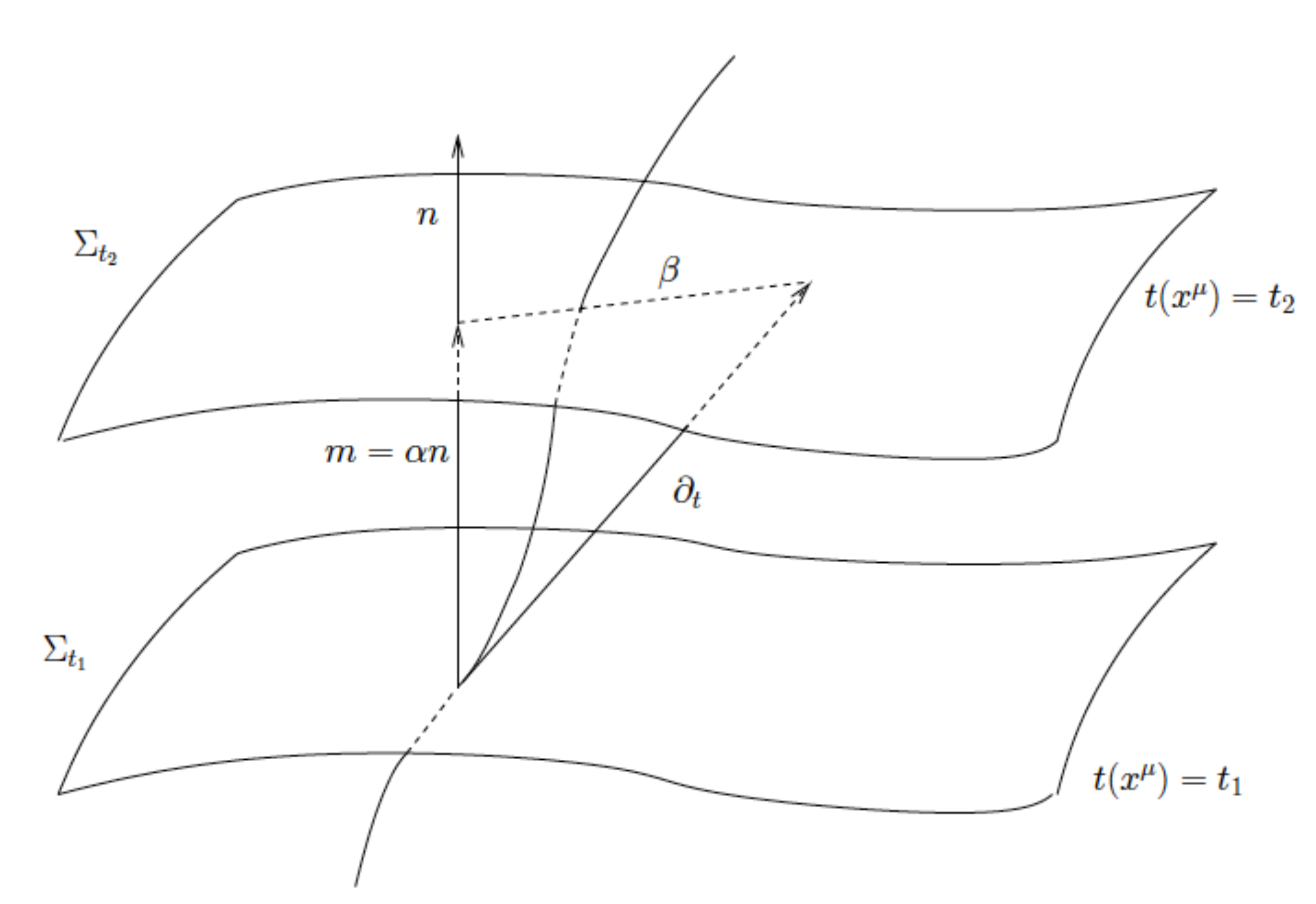
Def.: Cauchy surface := A spacelike hypersurface Σ in $\mathcal{M}$ such that each timelike or null curve without endpoint intersects Σ exactly once.



The (D-1)+1 decomposition

Def.: A spacetime is globally hyperbolic

$:\Leftrightarrow$ it admits a Cauchy surface



- From now on:

  Let $(\mathcal{M}, g)$ be glob.hyp.

Then one can show:

$\exists$ smooth $t : \mathcal{M} \mapsto \mathbb{R}$

such that

1) The gradient $\mathbf{d}t \neq 0$ everywhere

2) level surfaces $t = \text{const}$ are hypersurfaces:

$$\forall_{t_1 \in \mathbb{R}} \;\; \Sigma_{t_1} = \{p \in \mathcal{M} \;\; : \;\; t(p) = t_1\} \;, \;\; \Sigma_{t_1} \cap \Sigma_{t_2} = \emptyset \;\; \Leftrightarrow \;\; t_1 \neq t_2$$

The 3+1 decomposition

- 1-Form: $\mathbf{d}t$; vector: $\frac{\partial}{\partial t} =: \partial_t \quad \Rightarrow \quad \langle \mathbf{d}t, \partial_t \rangle = 1$

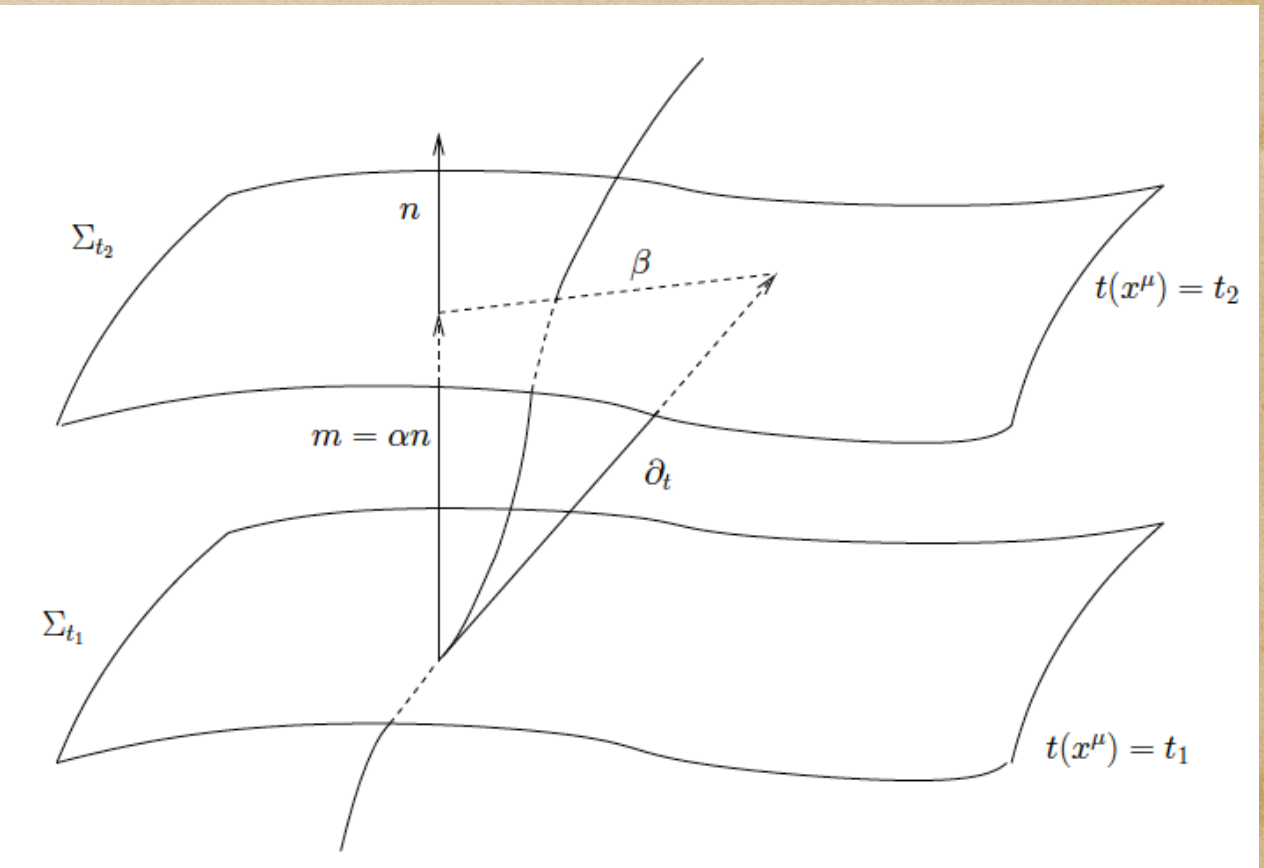
Def.: Time like unit field: $n_\mu := -\alpha(\mathbf{d}t)_\mu$

Lapse function: $\alpha := \frac{1}{||\mathbf{d}t||}$ Shift vector: $\beta^\mu := (\partial_t)^\mu - \alpha n^\mu$

Adapted coordinates: (t, x^i), x^i label points in Σ_t

Adapted coordinate basis:

$$\partial_t = \alpha n + \beta \,, \quad \partial_i := \frac{\partial}{\partial x^i}$$



The 3+1 decomposition

Def.: A vector $\boldsymbol{v}^\alpha$ is tangent to Σ_t $:\Leftrightarrow$ $\langle \mathbf{d}t, \boldsymbol{v}\rangle = (\mathbf{d}t)_\mu v^\mu = 0$

Def.: Projector $\perp^\alpha{}_\mu := \delta^\alpha{}_\mu + n^\alpha n_\mu$

- For a vector tangent to Σ_t one easily shows: $n_\mu v^\mu = 0$

  $$\perp^\alpha{}_\mu v^\mu = v^\alpha$$

- Projection of the metric

  $$\gamma_{\alpha\beta} := \perp^\mu{}_\alpha \perp^\nu{}_\beta g_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \quad \Rightarrow \quad \gamma_{\alpha\beta} = \perp_{\alpha\beta}$$

  For $\boldsymbol{v}^\alpha$ tangent to Σ_t : $g_{\mu\nu} v^\mu v^\nu = \gamma_{\mu\nu} v^\mu v^\nu$

- In adapted coordinates $(t,\, x^i)$:

  1) we can ignore t components for tensors tangential to Σ_t

  2) $\gamma_{ij}\,,\; i = 1,\,\ldots,\, D-1$ is the metric on Σ_t "First fundamental form"

(D-1)+1 decomposition of the metric

- In adapted coordinates, we write the spacetime metric

$$g_{\alpha\beta} = \left(\begin{array}{c|c} -\alpha^2 + \beta_m \beta^m & \beta_j \\ \hline \beta_i & \gamma_{ij} \end{array}\right)$$

$$\Leftrightarrow \qquad g^{\alpha\beta} = \left(\begin{array}{c|c} -\alpha^{-2} & \alpha^{-2}\beta^j \\ \hline \alpha^{-2}\beta^i & \gamma^{ij} - \alpha^{-2}\beta^i\beta^j \end{array}\right)$$

$$\Leftrightarrow \qquad ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

- Gauge variables: Lapse α , shift β^i

- For tensors tangent in all components to Σ_t we lower indices with γ_{ij} : $S^i{}_{jk} = \gamma_{jm} S^{im}{}_k$, etc.

Projections and spatial covariant derivative

Def.: Projections of an arbitrary tensor $\boldsymbol{S}$ of type $\binom{p}{q}$:

$$(\perp S)^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q} = \perp^{\alpha_1}{}_{\mu_1} \dots \perp^{\alpha_p}{}_{\mu_p} \perp^{\nu_1}{}_{\beta_1} \dots \perp^{\nu_q}{}_{\beta_q} S^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q}$$

"Project every free index"

Def.: Spatial covariant derivative of a tensor $\boldsymbol{S}$ tangential to Σ_t :

$$D\boldsymbol{S} := \perp(\nabla \boldsymbol{S})$$

$$D_\rho S^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q} := \perp^{\alpha_1}{}_{\mu_1} \dots \perp^{\alpha_p}{}_{\mu_p} \perp^{\nu_1}{}_{\beta_1} \dots \perp^{\nu_q}{}_{\beta_q} \perp^{\sigma}{}_{\rho} \nabla_\sigma S^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q}$$

Def.: One can show that

1) $D = \perp \nabla$ is torsion free on Σ_t if ∇ is on $\mathcal{M}$
2) $\nabla g_{\alpha\beta} = 0 \quad \Rightarrow \quad (D\gamma)_{ij} = 0$ "Metric compatible"
3) $D = \perp \nabla$ is unique in satisfying these properties

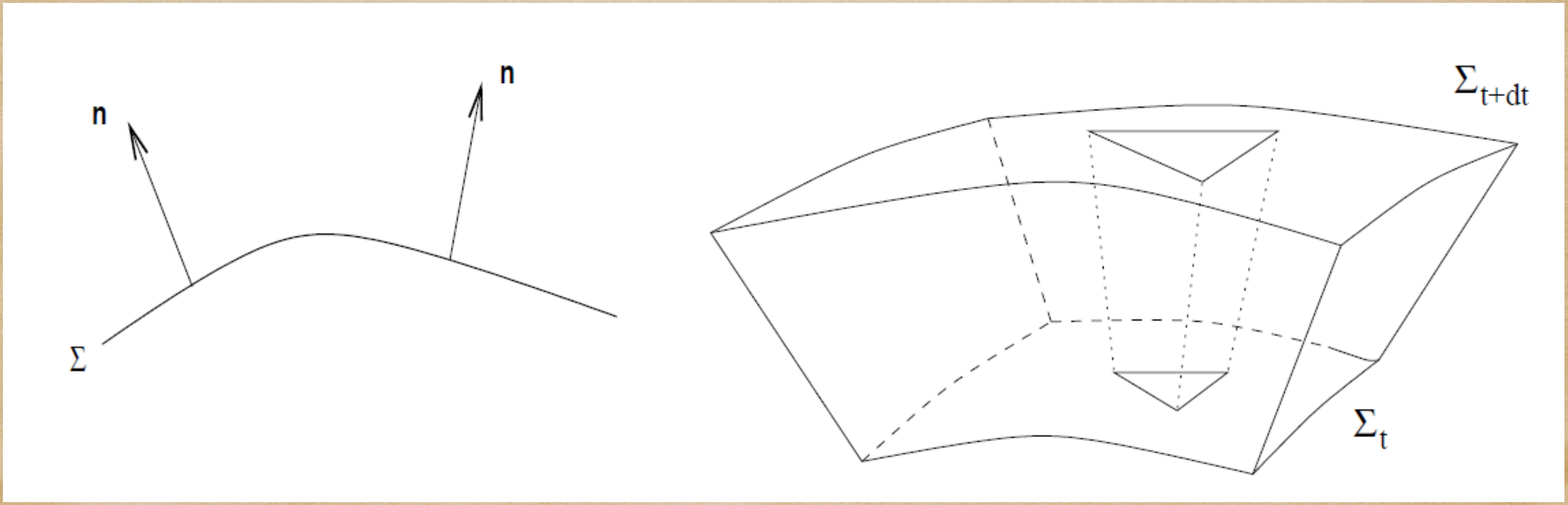
Extrinsic curvature

Def.: Extrinsic curvature: $K_{\alpha\beta} := -\perp\nabla_\beta n_\alpha$

- $\nabla_\beta n_\alpha$ is not symmetric, but $\perp\nabla_\beta n_\alpha$ is!
- The minus sign is a non-universal convention
- One can show that

$$\mathcal{L}_n \gamma_{\alpha\beta} = n^\mu \nabla_\mu \gamma_{\alpha\beta} + \gamma_{\mu\beta}\nabla_\alpha n^\mu + \gamma_{\alpha\mu}\nabla_\beta n^\mu = -2K_{\alpha\beta}$$

- Interpretation of $K_{\alpha\beta}$ $\rightarrow$ Embedding of Σ_t in $\mathcal{M}$



The projections of the Riemann tensor

- Projections of Riemann: $\perp R^{\alpha}{}_{\beta\gamma\delta}\,, \quad \perp R^{\alpha}{}_{\beta\gamma\mu} n^{\mu}\,, \quad \perp R^{\alpha}{}_{\mu\gamma\nu} n^{\mu} n^{\nu}$
- Starting point: Ricci identity $(\nabla_\gamma \nabla_\delta - \nabla_\delta \nabla_\gamma) Z^{\alpha} = R^{\alpha}{}_{\beta\gamma\delta} Z^{\beta}$

  Then a lengthy calculation yields   Gourgoulhon gr-qc/0703035

$$\perp R^{\alpha}{}_{\beta\gamma\delta} = \mathcal{R}^{\alpha}{}_{\beta\gamma\delta} + 2K^{\alpha}{}_{[\gamma} K_{\delta]\beta} \qquad \text{Gauss}$$

$$\perp R_{\alpha\beta} + \perp(R_{\alpha\delta\beta\nu}\, n^{\delta} n^{\nu}) = \mathcal{R}_{\alpha\beta} + K K_{\alpha\beta} - K_{\alpha\gamma} K^{\gamma}{}_{\beta} \qquad \text{contracted Gauss}$$

$$R + 2R_{\gamma\delta}\, n^{\gamma} n^{\delta} = \mathcal{R} + K^2 - K_{\gamma\delta} K^{\gamma\delta} \qquad \text{scalar Gauss}$$

$$\perp(R_{\alpha\beta\gamma\lambda}\, n^{\lambda}) = -D_\alpha K_{\beta\gamma} + D_\beta K_{\alpha\gamma} \qquad \text{Codazzi}$$

$$\perp(R_{\beta\delta}\, n^{\delta}) = -D_\alpha K^{\alpha}{}_{\beta} + D_\beta K \qquad \text{contracted Codazzi}$$

$$\perp(R_{\alpha\nu\beta\mu}\, n^{\mu} n^{\nu}) = \frac{1}{\alpha} \mathcal{L}_{\boldsymbol{m}} K_{\alpha\beta} + K_{\alpha\gamma} K^{\gamma}{}_{\beta} + \frac{1}{\alpha} D_\alpha D_\beta \alpha$$

$$\perp R_{\alpha\beta} = -\frac{1}{\alpha} \mathcal{L}_{\boldsymbol{m}} K_{\alpha\beta} - \frac{1}{\alpha} D_\alpha D_\beta \alpha + \mathcal{R}_\alpha \beta + K\, K_{\alpha\beta} - 2K_{\alpha\gamma} K^{\gamma}{}_{\beta}$$

$$R = \frac{2}{\alpha} \mathcal{L}_{\boldsymbol{m}} K - \frac{2}{\alpha} D_\gamma D^{\gamma} \alpha + \mathcal{R} + K^2 + K_{\gamma\delta} K^{\gamma\delta}$$

- Here: $\mathcal{L}$ is the Lie derivative and $m^{\mu} = \alpha n^{\mu}$
- Summation over spatial tensors: Can ignore time components

Decomposition of the Einstein eqs.

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

$$\Leftrightarrow R_{\alpha\beta} = 8\pi \left(T_{\alpha\beta} - \frac{1}{D-2} g_{\alpha\beta} T \right) + \frac{2}{D-2} \Lambda g_{\alpha\beta}$$

- Energy momentum tensor

$$\rho := T_{\mu\nu} n^\mu n^\nu \,,$$

$$j_\alpha := -\perp^\mu{}_\alpha T_{\mu\nu} n^\nu \,,$$

$$S_{\alpha\beta} := \perp T_{\alpha\beta} \,, \quad S = \gamma^{\mu\nu} S_{\mu\nu} \,,$$

$$T_{\alpha\beta} = S_{\alpha\beta} + n_\alpha j_\beta + n_\beta j_\alpha + \rho n_\alpha n_\beta \,, \quad T = S - \rho \,.$$

- Lie derivative

$$\mathcal{L}_{\boldsymbol{m}} K_{ij} = \partial_t K_{ij} - \beta^m \partial_m K_{ij} - K_{mj} \partial_i \beta^m - K_{im} \partial_j \beta^m$$

$$\mathcal{L}_{\boldsymbol{m}} \gamma_{ij} = \partial_t \gamma_{ij} - \beta^m \partial_m \gamma_{ij} - \gamma_{mj} \partial_i \beta^m - \gamma_{im} \partial_j \beta^m$$

The ADM version of the Einstein eqs.

- Introduction of the extrinsic curvature:

$$\mathcal{L}_{\boldsymbol{m}}\gamma_{ij} = -2\alpha K_{ij}$$

- $\perp^{\mu}{}_{\alpha}\perp^{\nu}_{\beta}$ projection

$$\mathcal{L}_{\boldsymbol{m}}K_{ij} = -D_iD_j\alpha + \alpha(\mathcal{R}_{ij} + KK_{ij} - 2K_{im}K^{m}{}_{j}) + 8\pi\alpha\left(\frac{S-\rho}{D-2}\gamma_{ij} - S_{ij}\right) - \frac{2}{D-2}\Lambda\gamma_{ij}$$

"Evolution equations"

- $n^{\mu}n^{\nu}$ projection

$$\mathcal{R} + K^2 - K^{mn}K_{mn} = 2\Lambda + 16\pi\rho$$

"Hamiltonian constraint"

- $\perp^{\mu}{}_{\alpha}n^{\nu}$ projection

$$D_iK - D_mK^m_i = -8\pi j_i$$

"Momentum constraints"

Well-posedness in 30 seconds

- Consider a field ϕ evolved with a first-order system of PDEs
- The system has a well-posed initial-value formulation

  $:\Leftrightarrow$ there exists a norm and a smooth function

  $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\forall_t \;\; ||\phi(t)|| \leq F(||\phi(0)||, t) \times ||\phi(0)||$
- Well-posed systems have unique solutions for given initial data
- There can still be rapid divergence, e.g. exponential
- A necessary condition for well-posedness is strong hyperbolicity
- The general ADM equations are only weakly hyperbolic
- Key part of PDEs: Principle part = highest derivative terms
- Details depend on gauge, constraints, discretization

Sarbach & Tiglio 1203.6443; Gundlach & Martín-García gr-qc/0604035;

Reula gr-qc/0403007

The BSSN system

- Goal: Modify ADM eqs. to get a strongly hyperbolic system

Shibata & Nakamura PRD 52 (1995), Baumgarte & Shapiro PRD gr-qc/9810065

- Use (i) conformal desomposition, (ii) trace split, (iii) aux. variables

$$\gamma := \det \gamma_{ij}\,, \quad \chi = \gamma^{-1/(D-1)}\,, \quad K = \gamma^{mn} K_{mn}\,,$$

$$\tilde{\gamma}_{ij} = \chi \gamma_{ii} \quad \Leftrightarrow \quad \tilde{\gamma}^{ij} = \frac{1}{\chi} \gamma^{ij}$$

$$\tilde{A}_{ij} = \chi \left(K_{ij} - \frac{1}{D-1} \gamma_{ij} K \right) \quad \Leftrightarrow \quad K_{ij} = \frac{1}{\chi} \left(\tilde{A}_{ij} + \frac{1}{D-1} \tilde{\gamma}_{ij} K \right)$$

$$\tilde{\Gamma}^i = \tilde{\gamma}^{mn} \tilde{\Gamma}^i_{mn}$$

- Auxiliary constraints

$$\tilde{\gamma} = 1, \qquad \tilde{\gamma}^{mn} \tilde{A}_{mn} = 0, \qquad \mathcal{G}^i \equiv \tilde{\Gamma}^i - \tilde{\gamma}^{mn} \tilde{\Gamma}^i_{mn} = 0\,.$$

The BSSN equations

$$\mathcal{H} := \mathcal{R} + \frac{D-2}{D-1}K^2 - \tilde{A}^{mn}\tilde{A}_{mn} - 16\pi\rho - 2\Lambda = 0\,,$$

$$\mathcal{M}_i := \tilde{\gamma}^{mn}\tilde{D}_m\tilde{A}_{ni} - \frac{D-2}{D-1}\partial_i K - \frac{D-1}{2}\tilde{A}^m{}_i\frac{\partial_m\chi}{\chi} - 8\pi j_i = 0\,,$$

$$\partial_t\chi = \beta^m\partial_m\chi + \frac{2}{D-1}\chi(\alpha K - \partial_m\beta^m)\,,$$

$$\partial_t\tilde{\gamma}_{ij} = \beta^m\partial_m\tilde{\gamma}_{ij} + 2\tilde{\gamma}_{m(i}\partial_{j)}\beta^m - \frac{2}{D-1}\tilde{\gamma}_{ij}\partial_m\beta^m - 2\alpha\tilde{A}_{ij}\,,$$

$$\partial_t K = \beta^m\partial_m K - \chi\tilde{\gamma}^{mn}D_mD_n\alpha + \alpha\tilde{A}^{mn}\tilde{A}_{mn} + \frac{1}{D-1}\alpha K^2 + \frac{8\pi}{D-2}\alpha[S + (D-3)\rho] - \frac{2}{D-2}\alpha\Lambda\,,$$

$$\partial_t\tilde{A}_{ij} = \beta^m\partial_m\tilde{A}_{ij} + 2\tilde{A}_{m(i}\partial_{j)}\beta^m - \frac{2}{D-1}\tilde{A}_{ij}\partial_m\beta^m + \alpha K\tilde{A}_{ij} - 2\alpha\tilde{A}_{im}\tilde{A}^m{}_j + \chi\left(\alpha\mathcal{R}_{ij} - D_iD_j\alpha - 8\pi\alpha S_{ij}\right)^{\mathrm{TF}}\,,$$

$$\partial_t\tilde{\Gamma}^i = \beta^m\partial_m\tilde{\Gamma}^i + \frac{2}{D-1}\tilde{\Gamma}^i\partial_m\beta^m - \tilde{\Gamma}^m\partial_m\beta^i + \tilde{\gamma}^{mn}\partial_m\partial_n\beta^i + \frac{D-3}{D-1}\tilde{\gamma}^{im}\partial_m\partial_n\beta^n - \tilde{A}^{im}\left[(D-1)\alpha\frac{\partial_m\chi}{\chi} + 2\partial_m\alpha\right] + 2\alpha\tilde{\Gamma}^i_{mn}\tilde{A}^{mn} - 2\frac{D-2}{D-1}\alpha\tilde{\gamma}^{im}\partial_m K - 16\pi\frac{\alpha}{\chi}j^i - \sigma\mathcal{G}^i\partial_m\beta^m\,.$$

- Note: there exist slight variations of the exact equations

The BSSN equations

- Auxiliary expressions we have used:

$$
\begin{aligned}
\Gamma^i_{jk} &= \tilde{\Gamma}^i_{jk} - \frac{1}{2\chi}\left(\delta^i{}_k\partial_j\chi + \delta^i{}_j\partial_k\chi - \tilde{\gamma}_{jk}\tilde{\gamma}^{im}\partial_m\chi\right)\,,\\
\mathcal{R}_{ij} &= \tilde{\mathcal{R}}_{ij} + \mathcal{R}^\chi_{ij}\,,\\
\mathcal{R}^\chi_{ij} &= \frac{\tilde{\gamma}_{ij}}{2\chi}\left[\tilde{\gamma}^{mn}\tilde{D}_m\tilde{D}_n\chi - \frac{D-1}{2\chi}\tilde{\gamma}^{mn}\partial_m\chi\,\partial_n\chi\right] + \frac{D-3}{2\chi}\left(\tilde{D}_i\tilde{D}_j\chi - \frac{1}{2\chi}\partial_i\chi\,\partial_j\chi\right)\,,\\
\tilde{\mathcal{R}}_{ij} &= -\frac{1}{2}\tilde{\gamma}^{mn}\partial_m\partial_n\tilde{\gamma}_{ij} + \tilde{\gamma}_{m(i}\partial_{j)}\tilde{\Gamma}^m + \tilde{\Gamma}^m\tilde{\Gamma}_{(ij)m} + \tilde{\gamma}^{mn}\left[2\tilde{\Gamma}^k_{m(i}\tilde{\Gamma}_{j)kn} + \tilde{\Gamma}^k_{im}\tilde{\Gamma}_{kjn}\right]\,,
\end{aligned}
$$

$$
D_iD_j\alpha = \tilde{D}_i\tilde{D}_j\alpha + \frac{1}{\chi}\partial_{(i}\chi\,\partial_{j)}\alpha - \frac{1}{2\chi}\tilde{\gamma}_{ij}\tilde{\gamma}^{mn}\partial_m\chi\,\partial_n\alpha\,.
$$

2.1.2 Generalized harmonic formulation

The generalized harmonic gauge (GHG)

- Harmonic gauge: Choose coordinates such that

  $$\Box x^\alpha = \nabla^\mu \nabla_\mu x^\alpha = -g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = 0$$

- D dimensional Einstein eqs. in harmonic gauge:

  $$R_{\alpha\beta} = -\frac{1}{2} g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} + \ldots$$

  principle part of wave equation $\Rightarrow$ Manifestly hyperbolic!

- Problem: Start with a hyper surface $t = \text{const}$

  Does t remain timelike?

- Goal: Generalize the harmonic gauge

  Garfinkle PRD gr-qc/0110013; Pretorius CQG gr-qc/0407110;

  Lindblom et al CQG gr-qc/0512093

  $\to$ Source function $H^\alpha = \nabla^\mu \nabla_\mu x^\alpha = -g^{\mu\nu}\Gamma^\alpha_{\mu\nu}$

The generalized harmonic equations

- Any spacetime in any coordinates can formulated in GH form!

  Problem: find the corresponding H^α

- Promote the H^α to evolution variables

- Einstein equations in GH form:

$$
\begin{aligned}
\frac{1}{2} g^{\mu\nu} \partial_\mu \partial_\nu g_{\alpha\beta} = & - \partial_\nu g_{\mu(\alpha} \partial_{\beta)} g^{\mu\nu} - \partial_{(\alpha} H_{\beta)} + H_\mu \Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\nu\alpha} \Gamma^\nu_{\mu\beta} \\
& - \frac{2}{D-2} \Lambda \, g_{\alpha\beta} - 8\pi \left(T_{\mu\nu} - \frac{1}{D-2} T \, g_{\alpha\beta} \right) .
\end{aligned}
$$

with constraints $\mathcal{C}^\alpha = H^\alpha - \Box x^\alpha = 0$

- Still has principle part of the wave equation!!! Manifestly hyperbolic

Friedrich Comm.Math.Phys. 1985; Garfinkle PRD gr-qc/0110013;

Pretorius CQG gr-qc/0407110

Constraint damping in GHG system

- One can show: GHG constraints related to ADM constraints

$$\mathcal{C}^\alpha = 0, \quad \partial_t \mathcal{C}^\alpha = 0 \quad \text{at} \quad t = 0 \quad \Rightarrow \quad \mathcal{H} = 0\,, \quad \mathcal{M}_i = 0$$

- Bianchi identies imply evolution of the $\mathcal{C}^\alpha$:

$$\Box \mathcal{C}_\alpha = -\mathcal{C}^\mu \nabla_{(\mu} \mathcal{C}_{\alpha)} - \mathcal{C}^\mu \left[8\pi \left(T_{\mu\alpha} - \frac{1}{D-2} T\, g_{\mu\alpha} \right) + \frac{2}{D-2} \Lambda\, g_{\mu\alpha} \right] .$$

- In practice: Numerical violations of $\mathcal{C}^\mu = 0 \quad \Rightarrow$ unstable modes!

- Solution: Add constraint damping terms

$$\begin{aligned} \frac{1}{2} \partial_{\mu\nu} g_{\alpha\beta} = &- \partial_\nu g_{\mu(\alpha} \partial_{\beta)} g^{\mu\nu} - \partial_{(\alpha} H_{\beta)} + H_\mu \Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\nu\alpha} \Gamma^\nu_{\mu\beta} \\ &- \frac{2}{D-2} \Lambda\, g_{\alpha\beta} - 8\pi \left(T_{\mu\nu} - \frac{1}{D-2} T\, g_{\alpha\beta} \right) - \kappa \left[2 n_{(\alpha} \mathcal{C}_{\beta)} - \lambda g_{\alpha\beta} n^\mu \mathcal{C}_\mu \right] \end{aligned}$$

Gundlach et al CQG (2005)

- E.g. Pretorius PRL gr-qc/0507014 uses $\kappa = 1.25/m\,, \quad \lambda = 1$

Summary of the GHG formulation

- Specify initial data $g_{\alpha\beta}\,,\quad \partial_t g_{\alpha\beta}$ at $t=0$

  that satisfy the constraints $\mathcal{C}_\alpha = \partial_t \mathcal{C}_\alpha = 0$

- Constraints preserved due to Bianchi identities

- Alternative first-order version of GH formulation

  Lindblom et al CQG gr-qc/0512093

  - Auxiliary variables $\rightarrow$ First-order system
  - Symmetric hyperbolic system

    $\rightarrow$ constraint preserving boundary conditions
  - Used in spectral code SXS

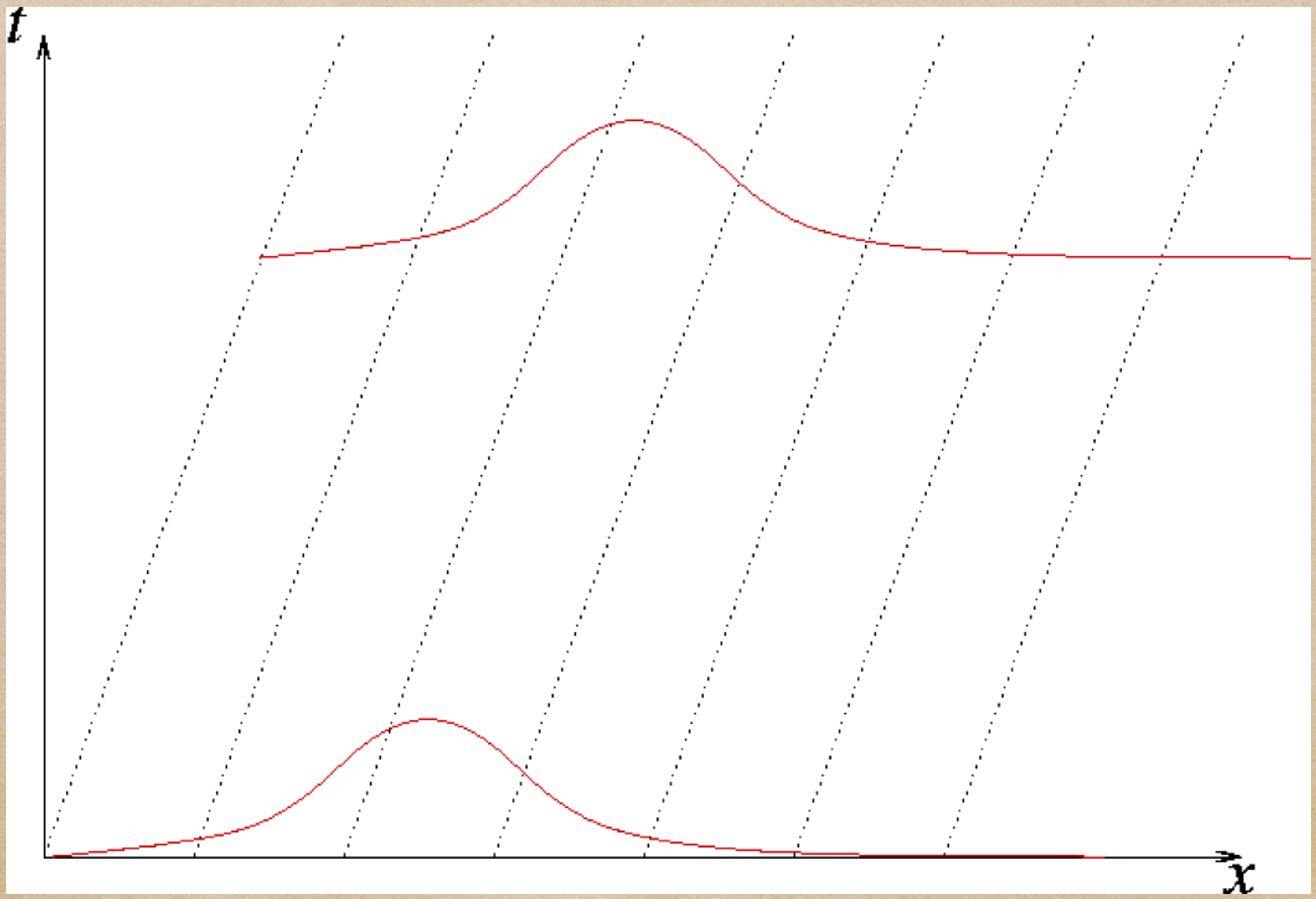
    Caltech, Cornell, CITA

2.1.3 Characteristic formulation

Characteristic coordinates

- Consider advection equation $\partial_t f + a\partial_x f = 0$
- Characteristics: Curves $\mathcal{C} : x \mapsto at + x_0 \quad \Leftrightarrow \quad \frac{dx}{dt} = a$

$$\Rightarrow \quad \frac{df}{dt}\bigg|_{\mathcal{C}} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\,\frac{dx}{dt}\bigg|_{\mathcal{C}} = \frac{\partial f}{\partial t} + a\frac{\partial f}{\partial x} = 0 \quad \Rightarrow \quad f = \text{const} \quad \text{along} \quad \mathcal{C}$$



Characteristic "Bondi-Sachs" formulation

Here: $D = 4\,, \quad \Lambda = 0\,, \quad T_{\alpha\beta} = 0$

- Write metric as

$$ds^2 = V\frac{e^{2\mathcal{B}}}{r}du^2 - 2e^{2\mathcal{B}}du\,dr + r^2 h_{\mu\nu}(dx^\mu - U^\mu du)(dx^\nu - U^\nu du)$$

$$2h_{\mu\nu}dx^\mu dx^\nu = (e^{2\mathcal{C}} + e^{2\mathcal{D}})d\theta^2 + 2\sin\theta\,\sinh(\mathcal{C}-\mathcal{D})d\theta\,d\phi + \sin^2\theta\,(e^{-2\mathcal{C}} + e^{-2\mathcal{D}})d\phi^2$$

- Introduce tetrad $\mathbf{k},\ \mathbf{l},\ \mathbf{m},\ \bar{\mathbf{m}}$ such that

  $\boldsymbol{g}(\mathbf{k},\mathbf{l}) = 1\,, \quad \boldsymbol{g}(\mathbf{m},\bar{\mathbf{m}}) = 1$ and all other products vanish

- Then the Einstein equations become
  - 4 Hypersurface equations $\quad R_{\mu\nu}\mathbf{k}^\mu\mathbf{k}^\nu = R_{\mu\nu}\mathbf{k}^\mu\mathbf{m}^\nu = R_{\mu\nu}\mathbf{m}^\mu\bar{\mathbf{m}}^\nu = 0\,,$
  - 2 evolution equations $\quad R_{\mu\nu}\mathbf{m}^\mu\mathbf{m}^\nu = 0\,,$
  - 1 trivial equation $\quad R_{\mu\nu}\mathbf{k}^\mu\mathbf{l}^\mu = 0\,,$
  - 3 supplementary equations $\quad R_{\mu\nu}\mathbf{l}^\mu\mathbf{m}^\nu = R_{\mu\nu}\mathbf{l}^\mu\mathbf{l}^\nu = 0\,.$

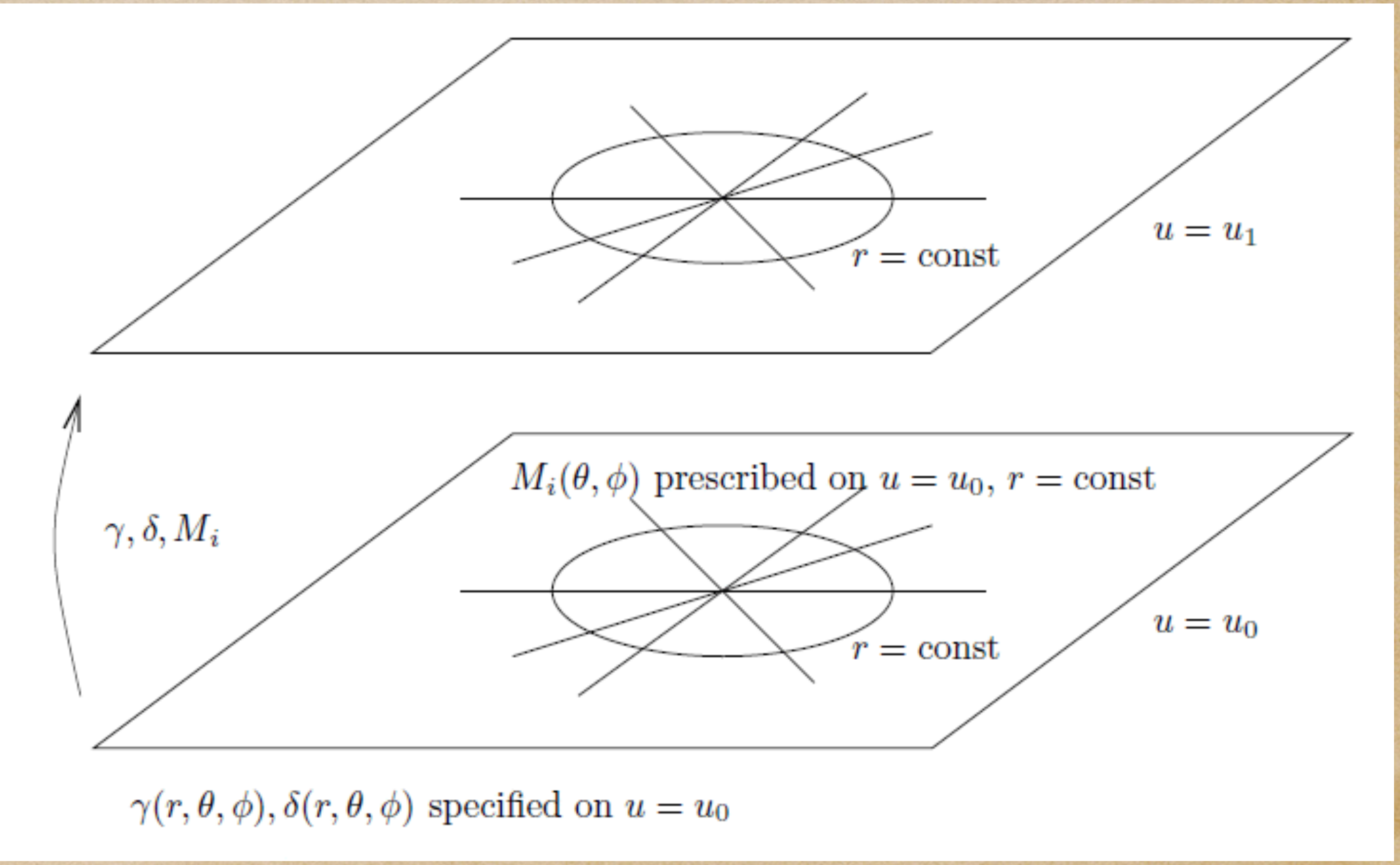
Integration of the characteristic eqs.

- Provide initial data for $\mathcal{C},\ \mathcal{D}$ on hyper surface $u = \text{const}$
- Integrate hypersurface eqs. along $r \rightarrow \mathcal{B},\ V,\ U^\alpha$ on $u = \text{const}$

  $\rightarrow$ 3 "constants" of integration $M_i(\theta,\phi)$
- Evolve $\mathcal{C},\ \mathcal{D}$ using the evolution equations

  $\rightarrow$ 2 "constants" of integration $\rightarrow$ complex news $\partial_u c(u,\theta,\phi)$
- Evolve the M_i through

  the supplementary eqs.



Features of the characteristic formulation

- Naturally adapted to the causal structure of GR
- Clear hierarchy of equations $\rightarrow$ isolated degrees of freedom
- Problem: Caustics $\rightarrow$ breakdown of coordinates
- Well suited for symmetric spacetimes, planar BHs
- Solution for binary problem?

  Work in progress; see e.g. Babiuc, Kreiss & Winicour 1305.7179
- Application to characteristic GW extraction

  Babiuc et al PRD 1011.4223; Reisswig et al CQG 0912.1285

Direct methods

- Use symmetry to write line element; e.g.

$$ds^2 = -a^2(\mu,t)dt^2 + b^2(\mu,t)d\mu^2 + R^2(\mu,t)(d\theta^2 + \sin^2\theta d\phi^2)$$

May & White PR (1966)

- Energy momentum tensor

$$T^0{}_0(1+\epsilon)\,,\quad T^1{}_1 = T^2{}_2 = T^3{}_3 = 0$$ Lagrangian coordinates

- GRTensor (Maple), Mathematica,...

$\Rightarrow$ Field equations

$$a' = \ldots$$
$$b' = \ldots$$
$$\ddot{R} = \ldots$$

Further reading

- (D-1)+1, 3+1 formalism

  Gourgoulhon gr-qc/0703035, Cardoso et al LRR-2015-1 1310.7590

- Characteristic formalism

  Winicour LRR-2012-2 gr-qc/0102085

- Numerical relativity in general

  Alcubierre: "Introduction to 3+1 Numerical Relativity" Oxford Univ. Press

  Baumgarte & Shapiro: "Numerical Relativity" Cambridge Univ. Press

  Bona, Palenzuela, Bona-Casas: "Elements of Numerical Relativity and Relativistic Hydrodynamics" Springer

- Well-posedness, Einstein eqs. as an initial-value problem

  Sarbach & Tiglio LRR-2012-15 1203.6443

2.2 Numerical Relativity beyond 4D

A list of tasks

- NR in 3+1 dimensions: $\mathcal{O}(100)$ of cores, Gb memory
- Each extra dimension can introduce factor $\mathcal{O}(100)$

  $\Rightarrow$ reduce D to $3+1$ dimensions; Symmetries
- Three approaches:
  - Dimensional reduction to 3+1 GR plus quasi-matter
  - Cartoon-type methods
  - Simplify the line element using symmetry
- Outer boundary conditions: regularization, background subtraction

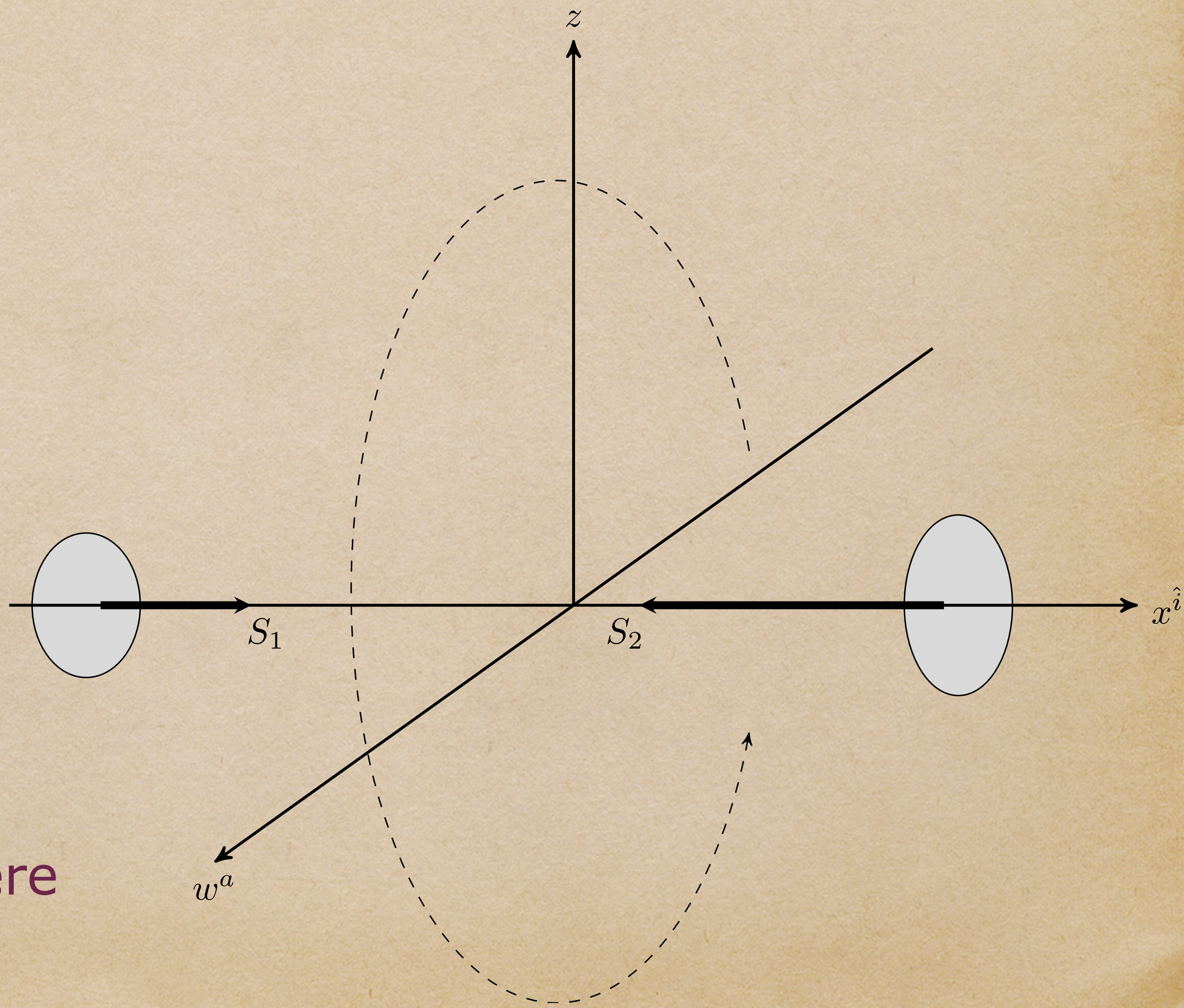
Notation

- D spacetime dimensions, $D-1$ spatial dimensions
- Computational domain: $3+1$ spacetime dims., 3 spatial dims.
- Indices:

$$\alpha,\, \beta,\, \ldots = 0,\, \ldots,\, D-1$$
$$i,\, j,\, \ldots = 1,\, \ldots,\, D-1$$
$$\bar{\alpha},\, \bar{\beta},\, \ldots = 0,\, \ldots,\, 3$$
$$\bar{i},\, \bar{j},\, \ldots = 1,\, 2,\, 3$$
$$\hat{i},\, \hat{j},\, \ldots = 1,\, 2$$
$$x^3 = z$$
$$a,\, b,\, \ldots = 4,\, \ldots,\, D-1$$

- Symmetry: $SO(D-3)$

  Rotations on S^{D-4} sphere



2.2.1 Dimensional reduction

Metric decomposition

- The general D metric can be written as

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$$
$$= (g_{\bar{\mu}\bar{\nu}} + e^2\kappa^2 g_{ab} B^a{}_{\bar{\mu}} B^b{}_{\bar{\nu}}) dx^{\bar{\mu}} dx^{\bar{\nu}} + 2e\kappa B^a{}_{\bar{\mu}} g_{ab} dx^{\bar{\mu}} dx^b + g_{ab} dx^a dx^b$$

- Comments:
  - e, κ are coupling and scale parameters; they'll drop out
  - This metric is completely general!
  - We have already used a special case of this: ADM split

Cho Phys.Lett.B (1987)

Cho & Kim J.Math.Phys. (1987)

Zilhão PhD Thesis 1301.1509

Rotational Killing vector fields

- Assumption: $g_{\alpha\beta}$ admits $D-4$ Killing vector fields $\boldsymbol{\xi}_{(a)} = \dfrac{\partial}{\partial \phi^a}$

  $\Rightarrow \mathcal{L}_{\boldsymbol{\xi}_{(a)}} g_{\alpha\beta} = 0$

- **Def.:** Dual form $\boldsymbol{\xi}^{(b)} := \xi^{(b)}_c \mathbf{d}x^c$ such that $\xi^{(b)}_c \xi^c_{(a)} = \delta^b{}_a$

- **Def.:** "Connection" $F^b{}_{cd} := -\xi^{(a)}_c \partial_d \xi^b_{(a)}$

- Then $\mathcal{L}_{\boldsymbol{\xi}_{(a)}} g_{\alpha\beta} = 0$ implies
  1) $\partial_a g_{bc} = F^d{}_{ab} g_{dc} + F^d{}_{ac} g_{db}\,,$
  2) $\partial_a B^b{}_{\bar{\mu}} = -F^b{}_{ad} B^d{}_{\bar{\mu}}\,,$
  3) $\partial_a g_{\mu\nu} = 0\,.$

- Consequences:
  - $g_{ab} = e^{2\psi(x^{\bar{\mu}})} h_{ab}\,,$ $h_{ab} =$ Metric on sphere
  - $g_{\bar{\mu}\bar{\nu}} = g_{\bar{\mu}\bar{\nu}}(x^{\bar{\alpha}})\,,$ in adapted coordinates
  - $[\boldsymbol{\xi}_{(a)}, \boldsymbol{B}_{\bar{\mu}}] = 0$ if ≥ 2 Killing fields exist

  1 KV: special case Zilhão et al PRD 1001.2302

The dimensionally reduced Einstein eqs.

- After some calculation, the D-dim. vacuum Einstein eqs. become

$$e^{2\psi}[(D-4)\partial^{\bar{\mu}}\psi\,\partial_{\bar{\mu}}\psi + \nabla^{\bar{\mu}}\partial_{\bar{\mu}}\psi] = (D-5)\,,$$

$$\mathcal{R}_{\bar{\mu}\bar{\nu}} = (D-d)(\nabla_{\bar{\nu}}\partial_{\bar{\mu}}\psi - \partial_{\bar{\mu}}\psi\,\partial_{\bar{\nu}}\psi)\,.$$

  where $\mathcal{R}_{\bar{\mu}\bar{\nu}}$ is the Ricci tensor of the base metric $g_{\bar{\mu}\bar{\nu}}$

- Note: This is merely 4-dim. GR plus a matter field

- Comments:
  - One of the $(x,\,y,\,z)$ spatial coordinates is a radius; e.g. z

    $\Rightarrow$ Computational domain: $x,\,y \in \mathbb{R}\,,\quad z \geq 0$
  - In practice: Use rescaled variable $\zeta \propto \dfrac{e^{2\psi}}{z^2}$
  - We thus can obtain BSSN with matter terms

    e.g. Zilhão et al PRD 1001.2302

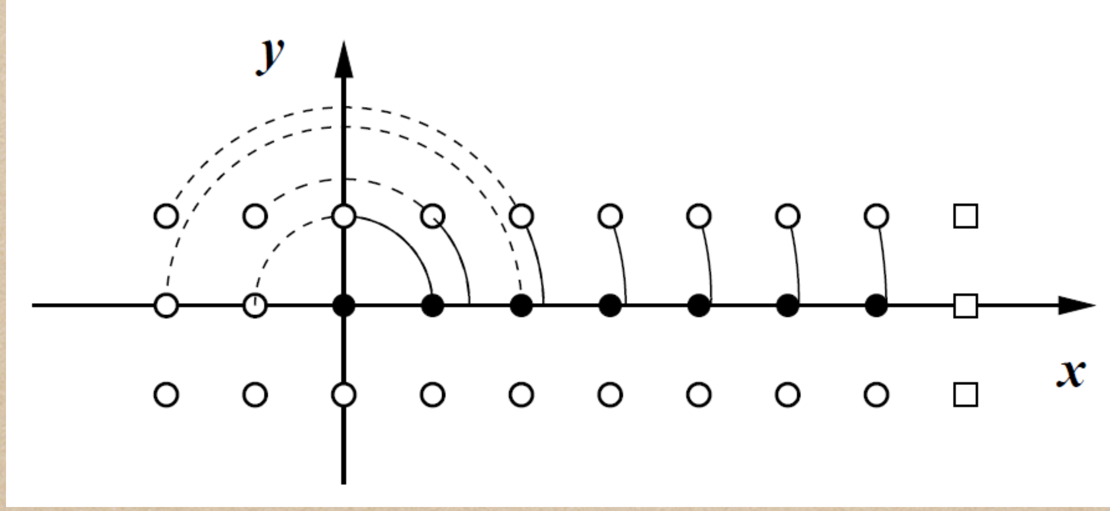
2.2.2 Cartoon methods

The original Cartoon method

- Developed for axisymmetry around z in 3+1 GR

  Alcubierre et al IJMPD gr-qc/9908012

- Coordinates $(z,\, x,\, y) \leftrightarrow (z,\, \rho, \phi)$ where $x = \rho \cos\phi\,, \quad y = r \sin\phi$
- Killing vector: $\partial_\phi = x\partial_y - y\partial_x$
- Extend 2D grid by ghostzones for derivatives; rotate, interpolate



- Problem: For large D Cartoon ghostzones require lots of memory

The modified Cartoon method

- Solution: Use symmetry to relate "off-domain" to "on-domain"

  1) Coordinates: $X^i = (x^{\hat{i}},\, z,\, w^a) \quad \leftrightarrow \quad \bar{X}^i = (x^{\hat{i}},\, \rho,\, \phi,\, w^5,\, \ldots,\, w^{D-1})$

  2) Tensor components: $\bar{T}_{\hat{i}\phi} = \frac{\partial X^\alpha}{\partial \bar{X}^{\hat{i}}} \frac{\partial X^\beta}{\partial \phi} T_{\alpha\beta} = -wT_{\hat{i}z} + zT_{\hat{i}w}\,,$ $\boxed{w := w^4}$

  3) By symmetry $\bar{T}_{\hat{i}\phi} = 0 \quad \Rightarrow \quad T_{\hat{i}w} = \frac{w}{z} T_{\hat{i}z}$

  4) Computational domain is $w = 0 \quad \Rightarrow \quad \boxed{T_{\hat{i}w} = 0}$

- Play same game for other tensor components, scalars, vectors

  and deriv's using also that Lie deriv's along $\boldsymbol{\xi} = z\boldsymbol{\partial}_w - w\boldsymbol{\partial}_z$ vanish

  $\Rightarrow$ express all w^a components and deriv's in terms of components and deriv's in the computational domain and one new func.

- E.g.: $\boxed{\partial_w T_{iw} = \frac{T_{iz} - \delta_{iz} T_{ww}}{z}}$; works for metric, ADM, BSSN variables

Pretorius gr-qc/0407110; Yoshino & Shibata PTPS 189/269; Cook et al 1603.00362

Further reading

- Dimensional reduction by isometry

  Cho Phys.Lett.B 186 (1987) 38

  Cho & Kim J.Math.Phys. 30 (1987) 1570

  Zilhão et al PRD, arXiv:1001.2302

  Zilhão PhD thesis, arXiv:1301.1509

- Modified Cartoon method

  Pretorius

  Yoshino & Shibata PTPS 189 (2011) 269, PTPS 190 (2011) 282

  Cook et al IJMPD, arXiv:1603.00362

2.3 Initial data, gauge

2.3.1 Initial data

Analytic initial data

- Schwarzschild, Kerr, Tangherlini, Myers-Perry,...

  e.g. Schwarzschild in isotropic coordinates

$$ds^2 = -\left(\frac{2r-M}{2r+M}\right)^2 dt^2 + \left(1+\frac{M}{2r}\right)^4 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

- Time symmetric initial data with n BHs:

  Brill & Lindquist PR 131 (1963) 471, Misner PR 118 (1960) 1110

- Problem: Find initial data for dynamic systems

- Goals: 1) Solve constraints

  2) Realistic snapshot of physical system

- This is mostly done using the ADM 3+1 split

The York-Lichnerowicz split

- We work in $D=4$; generalization to $D>4$ possible
- Conformal metric $\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}$

  Lichnerowicz J.Math.Pures Appl. 23 (1944) 37

  York PRL 26 (1971) 1656, PRL 28 (1972) 1082

- Note: In contrast to BSSN, we do not require $\det \bar{\gamma}_{ij} = 1$
- Conformal traceless split of the extrinsic curvature

$$K_{ij} = A_{ij} + \frac{1}{3} K \gamma_{ij} \,,$$

$$A^{ij} = \psi^{-10} \bar{A}^{ij} \quad \Leftrightarrow \quad A_{ij} = \psi^{-2} \bar{A}_{ij}$$

Bowen-York data

- By further splitting $\bar{A}_{ij}$ into a longitudinal and a transverse traceless part, the momentum constraints simplify substantially

  Cook LRR gr-qc/0007085

- Further assume: Vacuum, $K = 0\,,\quad \bar{\gamma}_{ij} = f_{ij}\,,\quad \lim_{r\to\infty} \psi = 0$,

  where f_{ij} is the flat metric in arbitrary coords.

  In words: Traceless E.Curv., conformal flatness, assymptotic flatness

- Then there exists an analytic solution to the momentum constraints

$$\bar{A}_{ij} = \frac{3}{2r^2}\left[P_i n_j + P_j n_i - (f_{ij} - n_i n_j) P^k n_k\right] + \frac{3}{r^3}\left(\epsilon_{kil} S^l n^k n_j + \epsilon_{kjl} S^l n^k n_i\right)\,,$$

  where r is a coordinate radius and $n^i = \frac{x^i}{r}$

  Bowen & York PRD (1980)

Properties of the Bowen-York solution

- The momentum in an asymptotically flat hyper surface associated with asymptotic translational and rotational Killing vectors $\xi^i_{(a)}$ is

$$\sum_i \Pi^i \xi^i_{(a)} = \frac{1}{8\pi} \oint_\infty (K^j{}_i - \delta^j{}_i K) \xi^i_{(a)} \, d^2 A_j$$

$\Rightarrow \ldots \Rightarrow$ P^i and S^i are the physical linear and angular momentum of the spacetime

- The momentum constraint is linear

$\Rightarrow$ we can superpose Bowen-York data. The momenta simply add up.

- Bowen-York data generalizes (analytically!) to higher D

Yoshino, Shiromizu & Shibata PRD gr-qc/0610110

Puncture data

Brandt & Bügmann PRL gr-qc/9703066

- The Hamiltonian constraint is then given by

$$\bar{\nabla}^2\psi + \frac{1}{8}\psi^{-7}\bar{A}_{mn}\bar{A}^{mn} = 0$$

- Ansatz for conformal factor $\psi = \psi_{\mathrm{BL}} + u$

  where $\psi_{\mathrm{BL}} = \sum_{a=1}^{N} \frac{m_a}{|\vec{r} - \vec{r}_a|}$ is the Brill-Lindquist conformal factor,

  i.e. the solution for $\bar{A}_{ij} = 0$.

- There then exist unique $\mathcal{C}^2$ solutions u to the Hamiltonian constr.

- The Hamiltonian constraint in this form is particularly suitable

  for numerical solution.

  E.g. Ansorg, Brügmann & Tichy PRD gr-qc/0404056

Properties of the puncture solutions

- m_a and $\vec{r}_a$ are the bare mass and position of the a^{th} BH

- In the limit of vanishing Bowen-York parameters $P^i = S^i = 0$,

  the puncture solution reduces to Brill-Lindquist data

$$\gamma_{ij} dx^i \, dx^j = \left(1 + \sum_a \frac{m_a}{2|\vec{r} - \vec{r}_a|}\right)^4 (dx^2 + dy^2 + dz^2)$$

- The numerical solution of the Hamiltonian constraint generalizes

  rather straightforwardly to higher D

  Yoshino, Shirumizu & Shibata PRD gr-qc/0610110

  Zilhão et al PRD 1109.2149

- Punctures also generalize to asymptotically de Sitter BHs

  Zilhão et al PRD 1204.2019

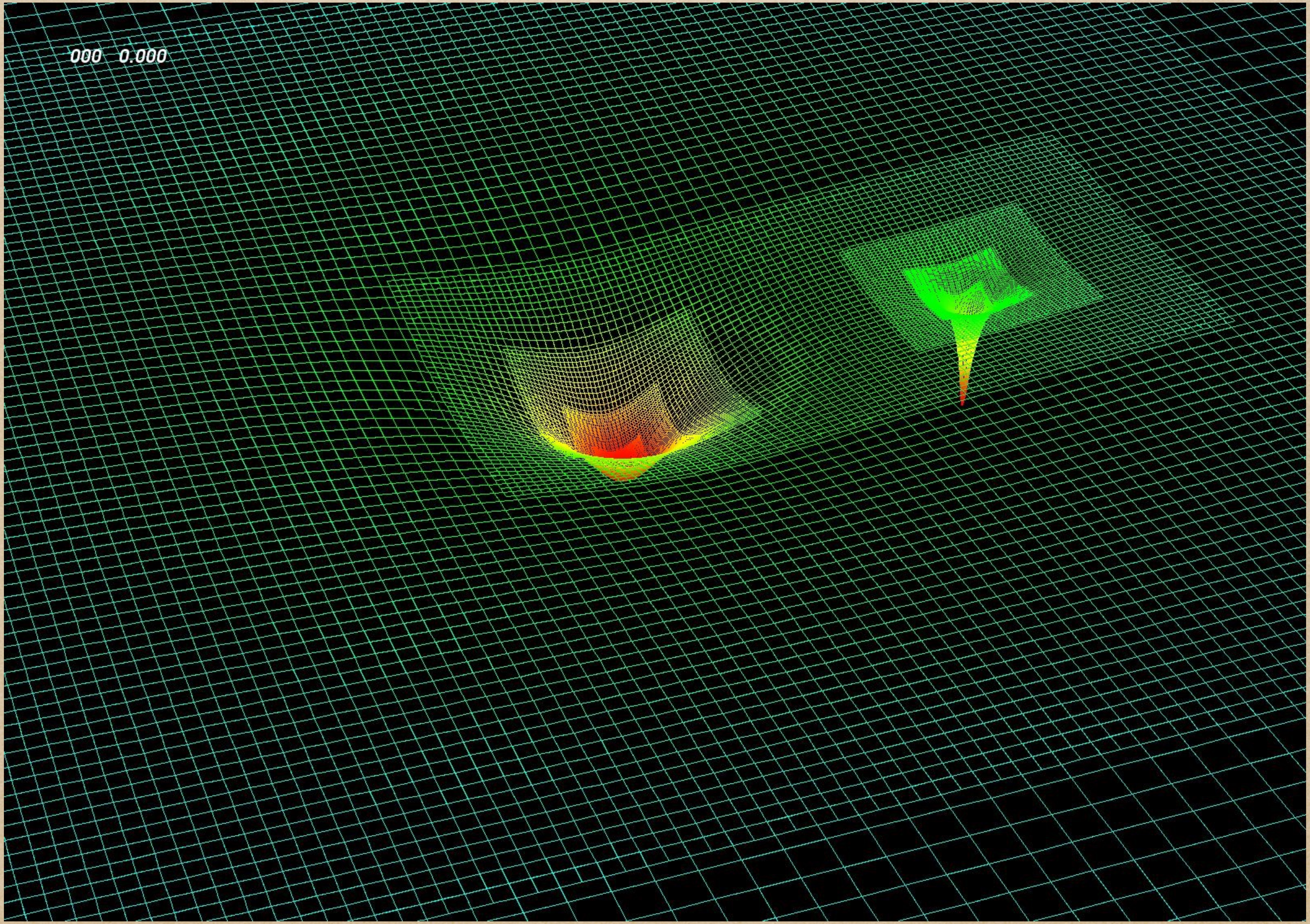
  using McVittie coordinates McVittie MNRAS (1933)

2.3.2 Gauge

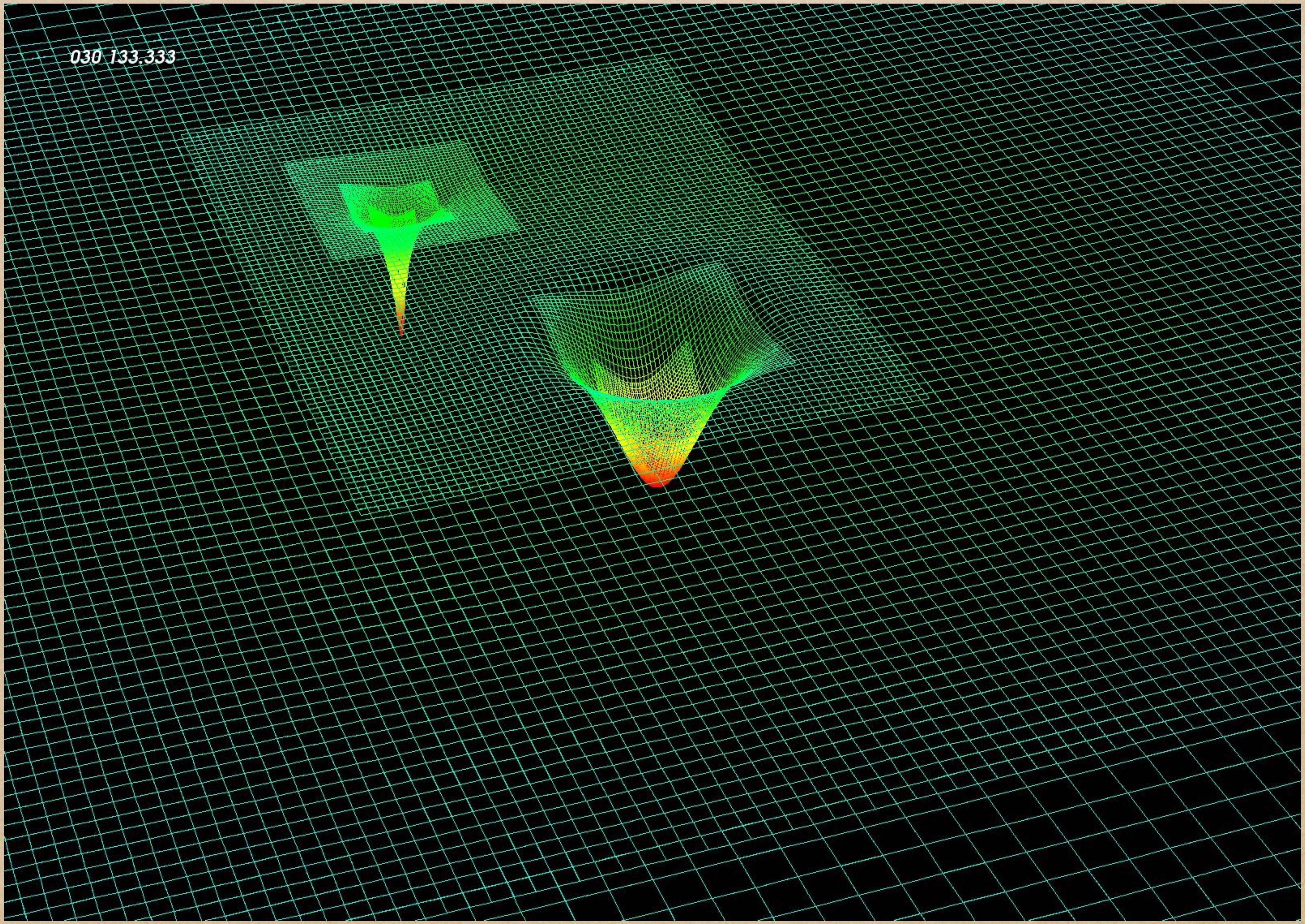
The gauge freedom

- Recall: Einstein's equations say nothing about $\alpha\,, \;\; \beta^i$
- Any choice of lapse and shift gives a solution to Einstein's eqs.
- This is the coordinate or gauge freedom of GR
- If the physics do not depend on $\alpha\,, \;\; \beta^i$, then why bother?
- Answer: The performance of the numerics DO depend very sensitively on the gauge!

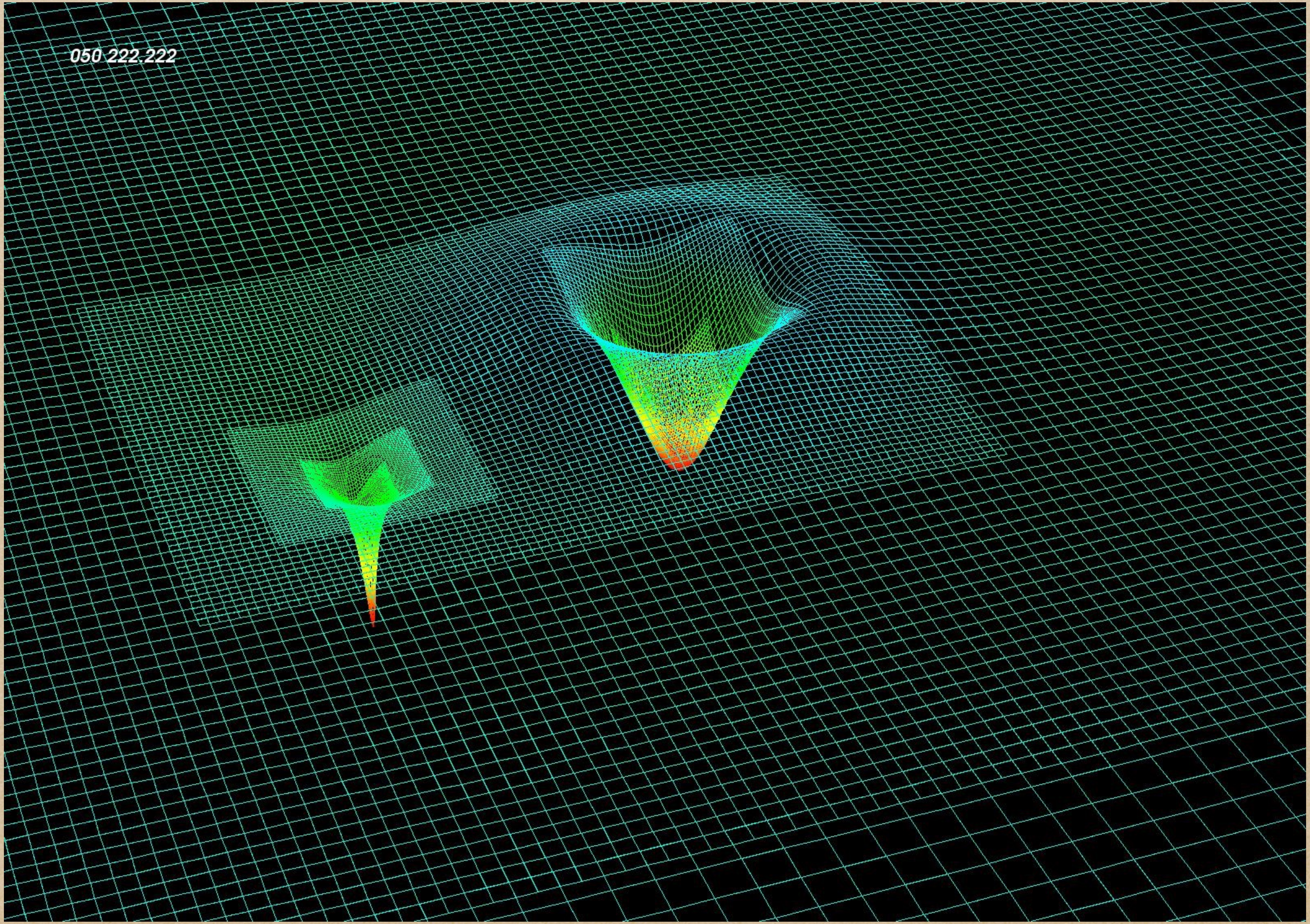
What goes wrong with bad gauge?



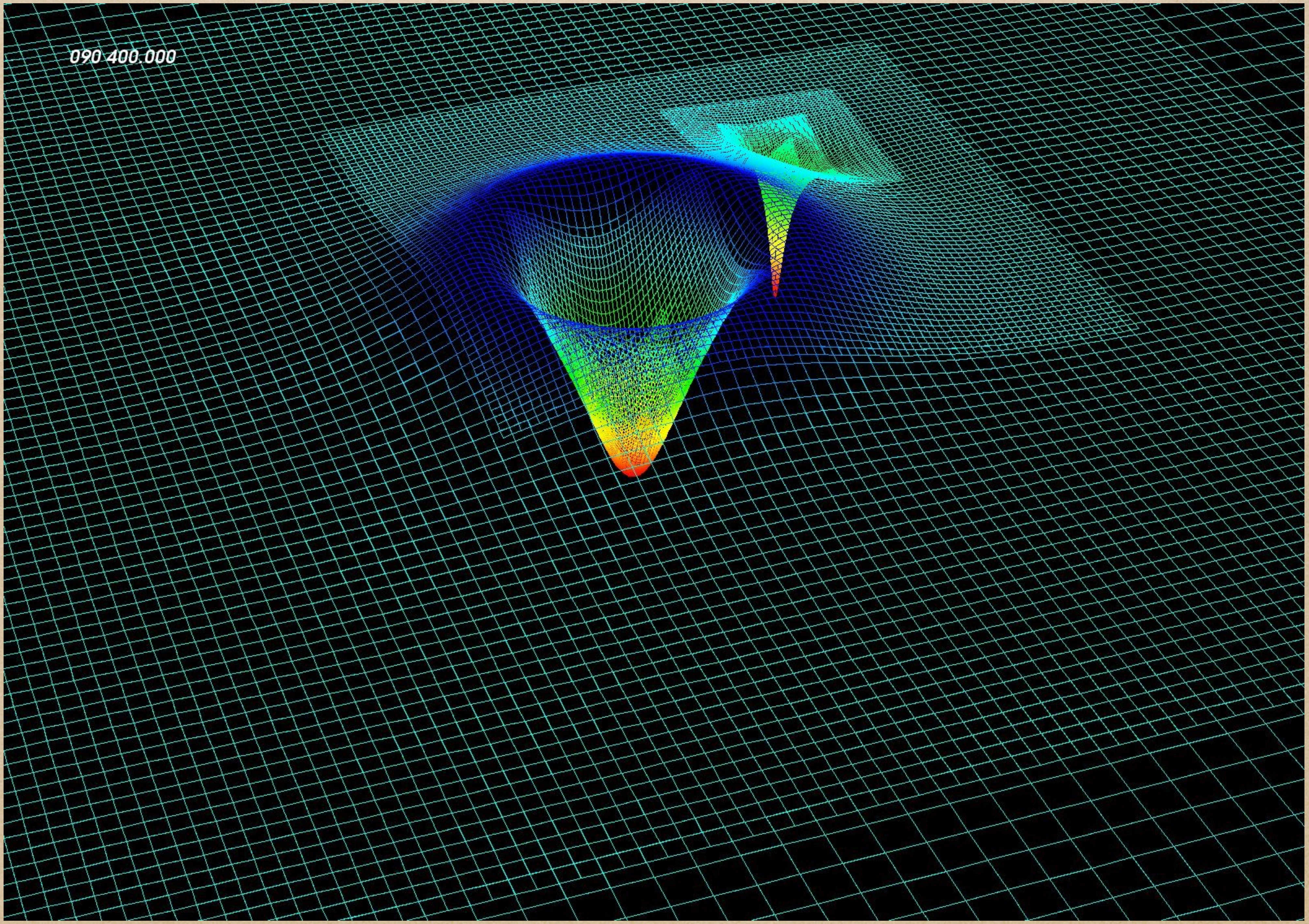
What goes wrong with bad gauge?



What goes wrong with bad gauge?



What goes wrong with bad gauge?



Ingredients for good gauge

- Singularity avoidance
- Avoid slice stretching
- Aim for stationarity in a co-moving frame
- Well-posedness of the system of PDEs
- Generalize "good" gauge, e.g. harmonic
- Lots of good luck!

Bona et al PRL (1995)

Alcubierre et al PRD gr-qc/0206072

Alcubierre CQG gr-qc/0210050

Garfinkle PRD gr-qc/0110013

Moving puncture gauge

- Moving punctures is one of the NR breakthrough methods

  Baker et al PRL gr-qc/0511103; Campanelli et al PRL gr-qc/0511048

- Gauge played a key role

- Variant of $1 + \log$ slicing and $\Gamma-$driver shift

  Alcubierre et al PRD gr-qc/0206072

- Now in use as $\partial_t \alpha = \beta^m \partial_m \alpha - 2\alpha K$

  and $\partial_t \beta^i = \beta^m \partial_m \beta^i + \frac{3}{4} B^i$

  $$\partial_t B^i = \beta^m \partial_m B^i + \partial_t \tilde{\Gamma}^i - \beta^m \partial_m \tilde{\Gamma}^i - \eta B^i$$

  or $\partial_t \beta^i = \beta^m \partial_m \beta^i + \frac{3}{4} \tilde{\Gamma}^i - \eta \beta^i$

  e.g. van Meter et al PRD gr-qc/0605030

Moving puncture gauge

Comments:

- Some people drop the advection terms $\beta^m \partial_m \ldots$
- η is a damping parameter or position-dependent function

  Alic et al CQG 1008.2212; Schnetter CQG 1003.0859;

  Müller et al PRD 1003.4681
- Modifications in higher D :
  - Change numerical values of the parameters: Trial & Error?

    Yoshino & Shibata PTPS 189 269
  - Dim. reduction by isometry: add scalar terms to Eqs.

    Zilhão et al PRD 1001.2302

Gauge conditions in the GH formulation

- How to choose the H_μ ? $\rightarrow$ Also requires some trial & error

- Pretorius' breakthrough simulations used

$$\Box H_t = -\xi_1 \frac{\alpha - 1}{\alpha^\eta} + \xi_2 n^\mu \partial_\mu H_t$$ with

$\xi_1 = 19/m\,, \quad \xi_2 = 2.5/m\,, \quad \eta = 5$, where $m =$ mass of 1 BH

- Caltech-Cornell-CITA spectral code:

Initialize H_α to minimize time derivatives of the metric,

adjust H_α to harmonic and damped harmonic gauge condition.

Lindblom & Szilágyi PRD 0904.4873; with Scheel PRD 80 (2009)

- The H_α are related to lapse and shift:

$$n^\mu H_\mu = -K - n^\mu \partial_\mu \ln \alpha\,,$$

$$\perp^{\mu i} H_\mu = -\gamma^{mn} \Gamma^i_{mn} + \gamma^{im} \partial_m \ln \alpha + \frac{1}{\alpha} n^\mu \partial_\mu \beta^i\,.$$

Further reading

- Initial data construction

  Cook LRR gr-qc/0007085

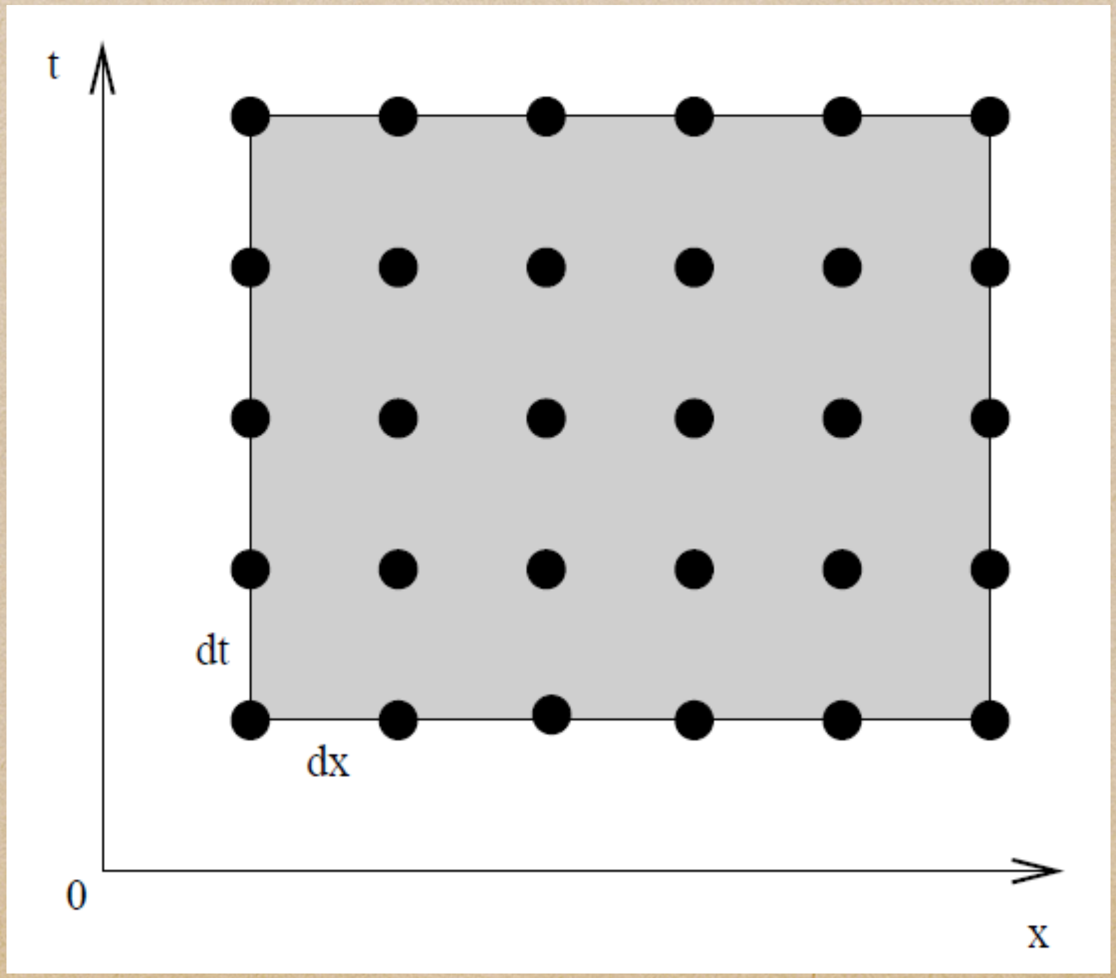
  Pfeiffer Thesis gr-qc/0510016

2.4 Discretization of the equations

Finite differencing

- Consider one spatial and one time dimension: $t, \quad x$
- Replace computational domain by discrete points

  $x_i = x_0 + i\,dx\,, \quad t_n = t_0 + n\,dt$
- Approximate function: $f(t_n, x_i) \approx f_{n,i}$



Derivatives and finite differences

- Goal: Represent $\frac{\partial^m f}{\partial x^m}$ in terms of $f_{n,i}$
- Fix index n; Taylor expand
$$f_{i-1} = f_i - f_i' dx + \frac{1}{2} f_i'' \, dx^2 + \mathcal{O}(dx^3)$$
$$f_i = f_i$$
$$f_i + 1 = f_i + f_i' \, dx + \frac{1}{2} f_i'' \, dx^2 + \mathcal{O}(dx^3)$$
- Write f_i' as linear combination: $f_i' = A f_{i-1} + B f_i + C f_{i+1}$
- Insert Taylor expressions and compare coefficients on both sides
$$\Rightarrow \; 0 = A + B + C \,, \;\; 1 = (-A + B) dx \,, \;\; 0 = \frac{1}{2} A \, dx^2 + \frac{1}{2} C \, dx^2$$
$$\Rightarrow \; A = -\frac{1}{2dx} \,, \;\; B = 0 \,, \;\; C = \frac{1}{2dx}$$
$$\Rightarrow \; f_i' = \frac{f_{i+1} - f_{i-1}}{2dx} + \mathcal{O}(dx^2)$$
- Same method in time direction; higher accuracy $\rightarrow$ more points

Mesh refinement

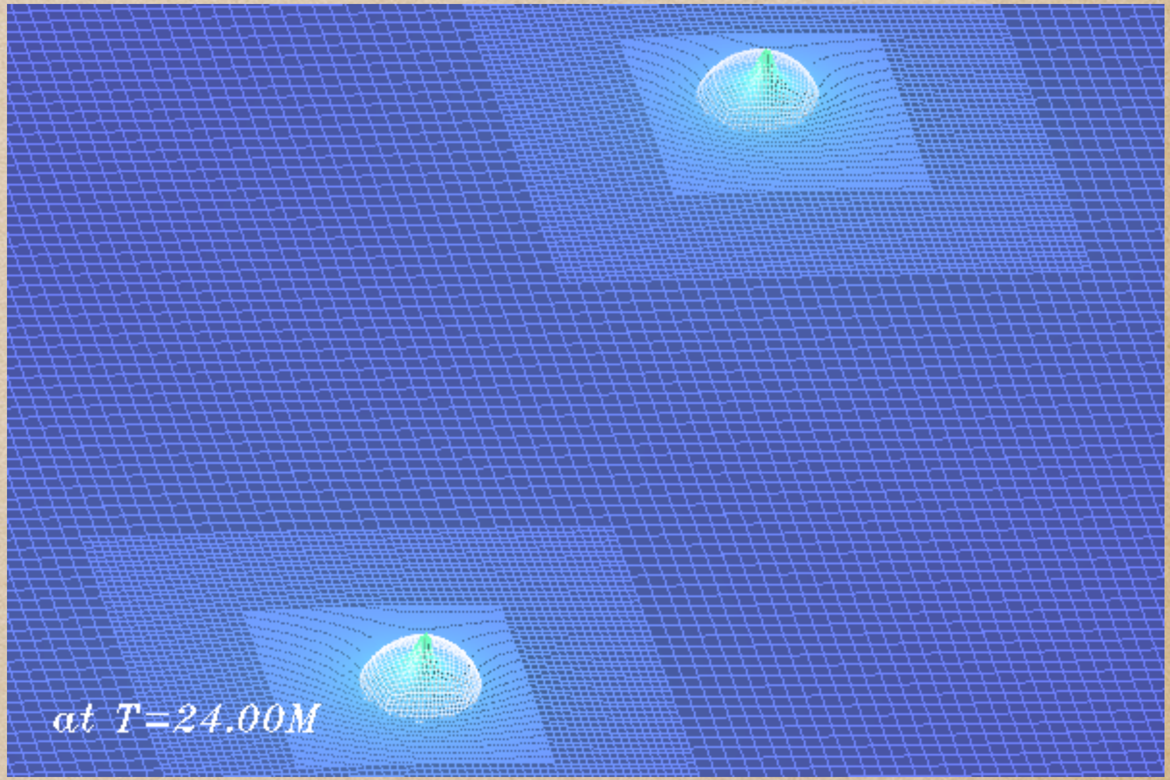
- 3 length scales: BH $\sim 1\ M$

  Wavelength $\sim 10 \ldots 100\ M$

  Wave zone $\sim 100 \ldots 1000\ M$

- First mesh refinement in GR: Critical phenomena

  Choptuik PRL **70** 9-12



- First use for BBHs

  Brügmann PRD gr-qc/9608050

- Available packages

  SAMRAI

  Paramesh MacNeice et al Comp.Phys.Comm. 136 (2000) 330

  Carpet Schnetter et al gr-qc/0310042

  Chombo Clough et al 1503.03436

Berger-Oliger mesh refinement

- Goal: Update from t to $t + dt$
- Refinement criteria: numerical error, curvature, …
- Here for $1 + 1$ dimensions

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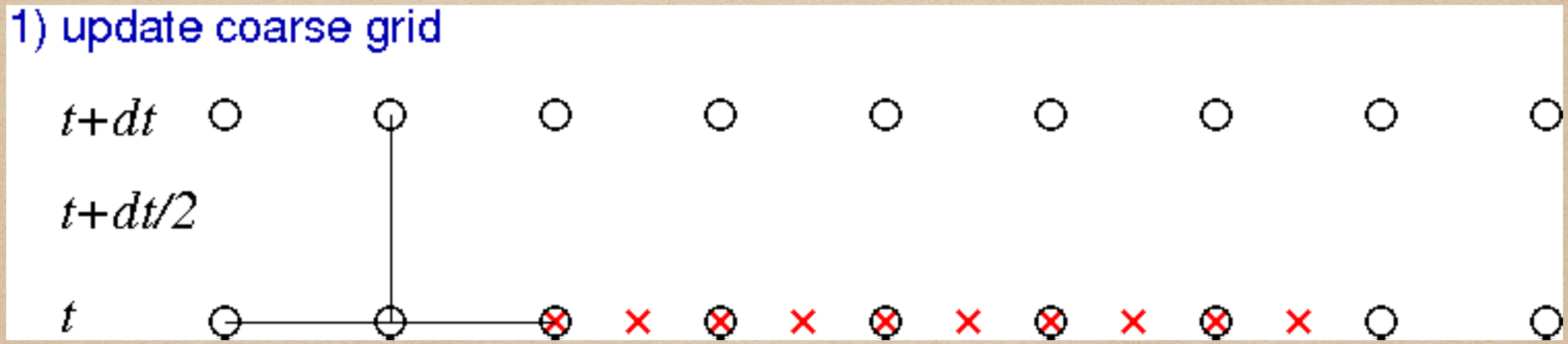

Berger-Oliger mesh refinement

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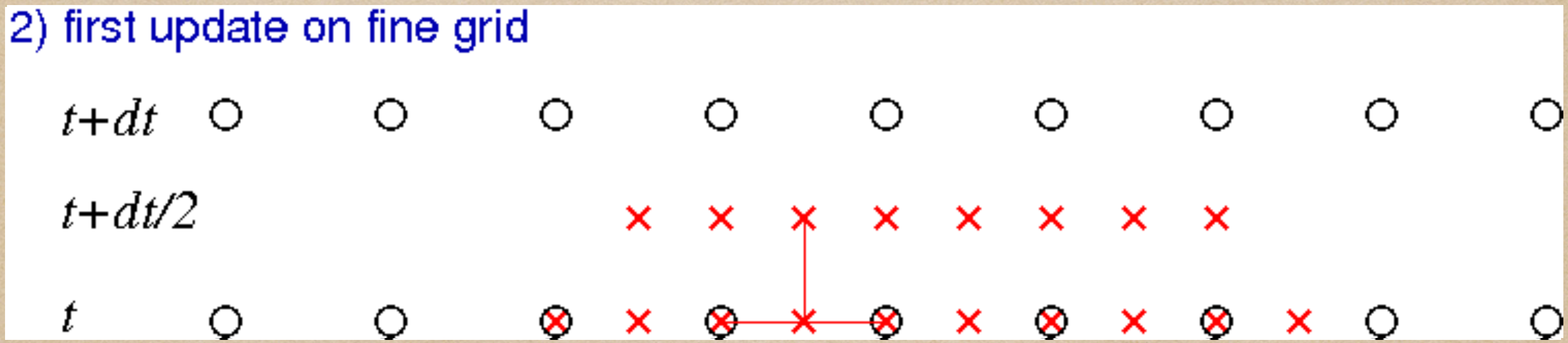

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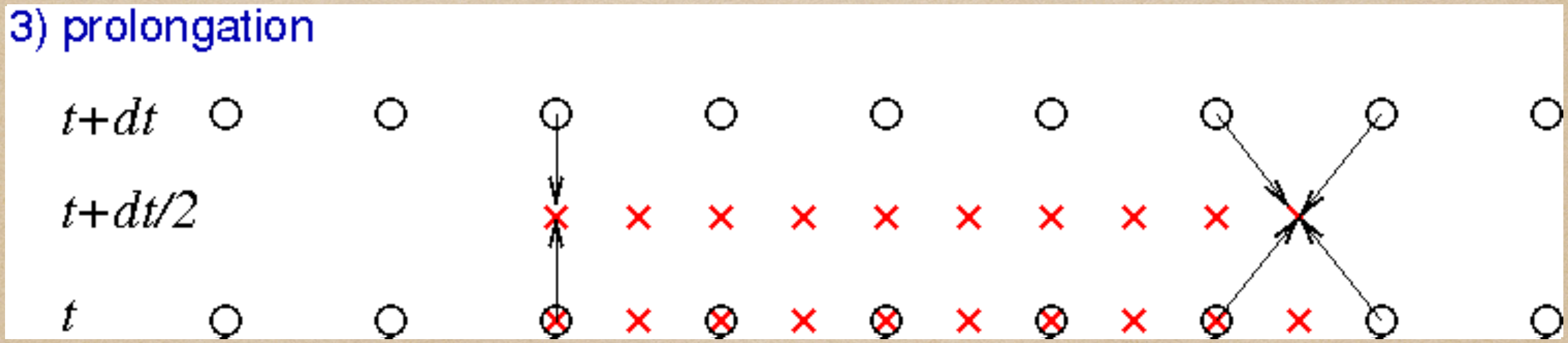

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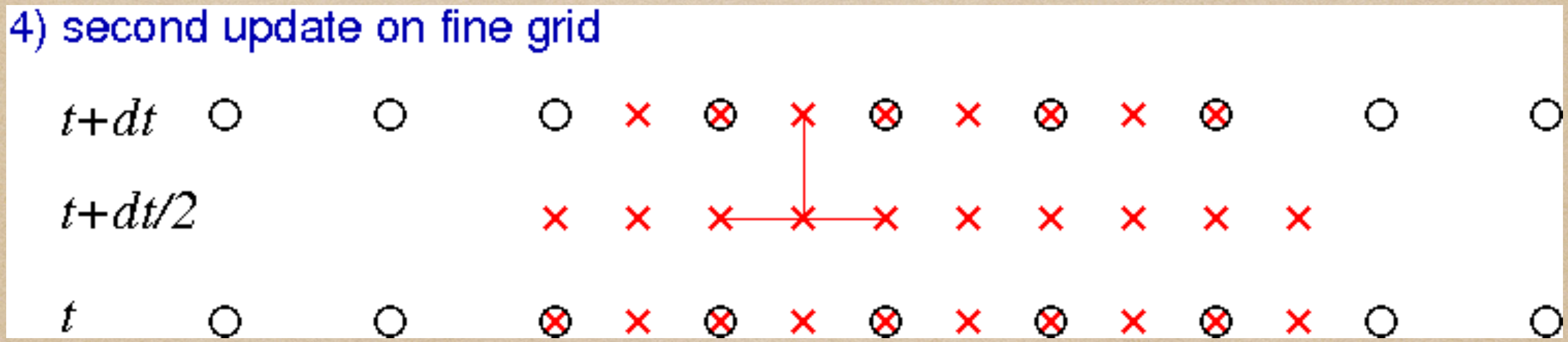

Berger-Oliger mesh refinement

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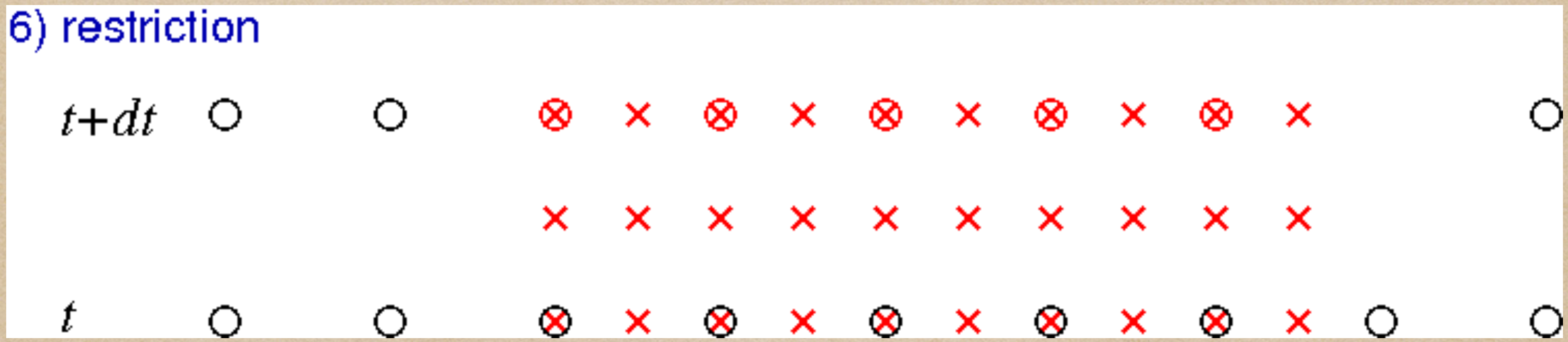


Alternative discretization schemes

- Spectral methods: high accuracy, efficiency, complexity

  e.g. Caltech-Cornell-CITA code; `http://www.black-holes.org/SpEC.html`

  Application to moving punctures hard

  e.g. Tichy PRD 0911.0973

  Also used in symmetric asymptotically AdS spacetimes

  e.g. Chesler & Yaffe PRL 1011.3562; Santos & Sopuerta PRL 1511.04344

- Finite volume methods

- Finite element methods

  e.g. Arnold, Mukherjee & Pouly gr-qc/9709038
  Sopuerta et al CQG gr-qc/0507112
  Sopuerta & Laguna PRD gr-qc/0512028

Further reading

- Numerical Methods

  Press et al "Numerical Recipes" Cambridge University Press

2.5 Boundaries

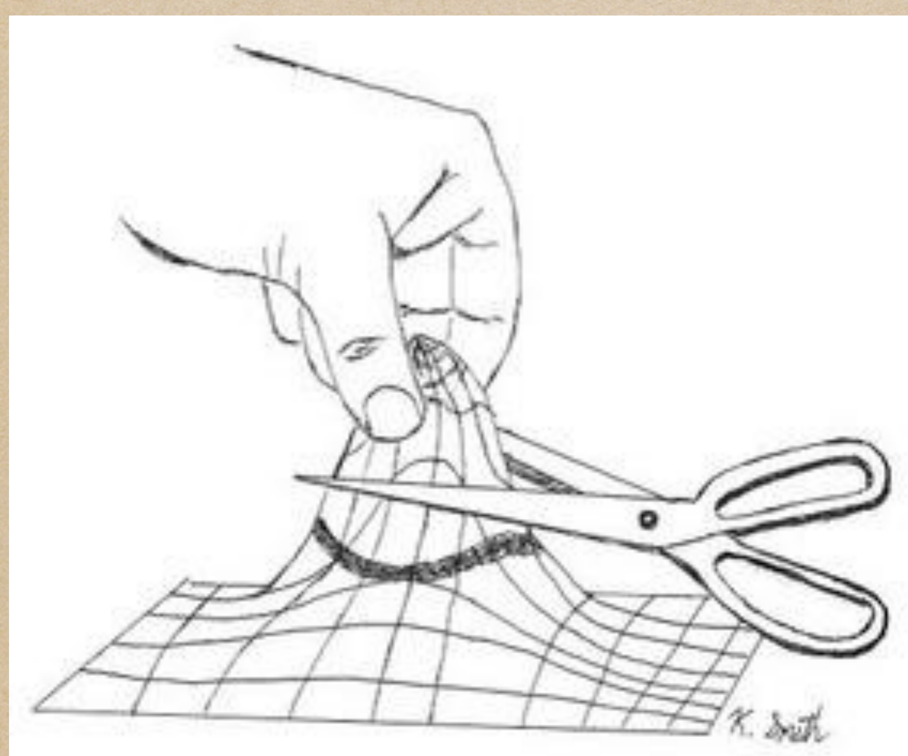
Inner boundary: Singularity treatment

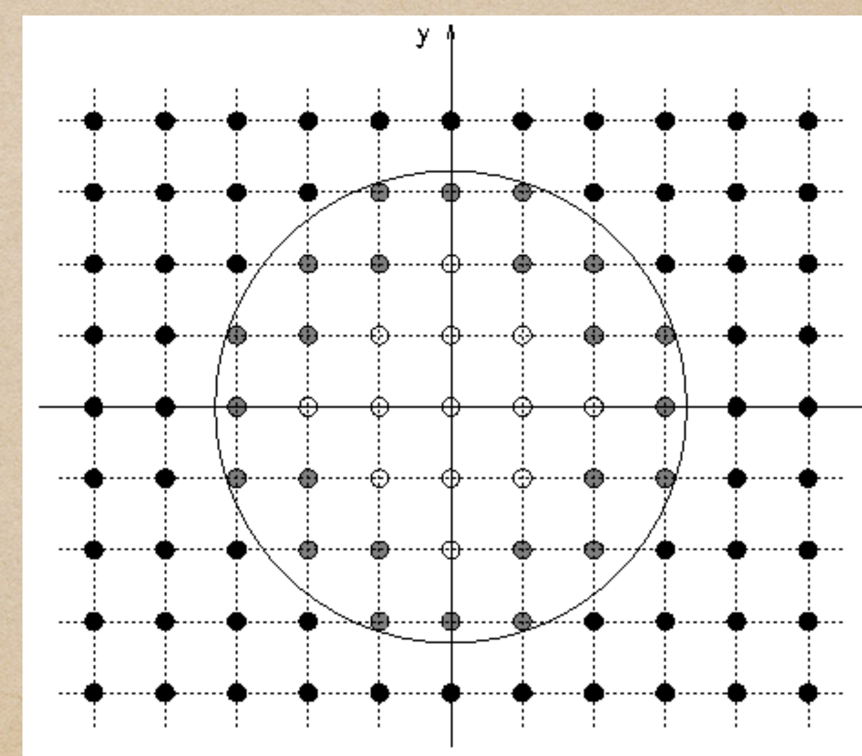
- Cosmic censorship $\Rightarrow$ horizon protects outside from singularity
- Moving puncture method: "we get away with it…"

  Baker et al PRL gr-qc/0511103; Campanelli et al PRL gr-qc/0511048

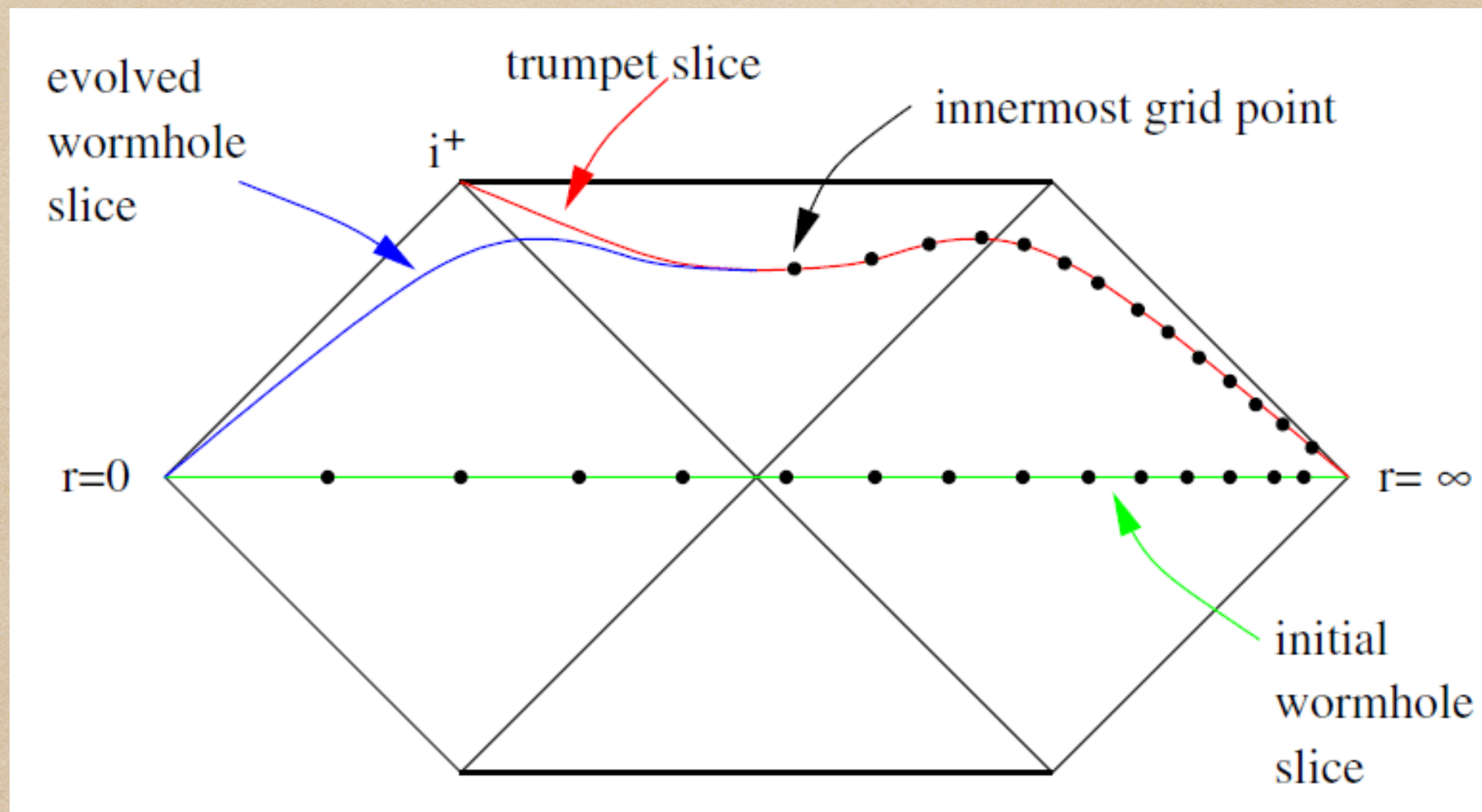
- Excision: Cut out region around the singularity

  Caltech-Cornell-CITA code, Pretorius' code





Moving puncture slices: Schwarzschild



- Wormhole evolves into "Trumpet slice" $=$ stationary $1+\log$ slice.

  Hannam et al PRL gr-qc/0606099, PRD 0804.0628

  Brown PRD 0705.1359, CQG 0705.3845

- Note: Gauge might propagate at $> c$, but: no pathologies apparent

  Moving puncture $=$ "Natural excision" Brown PRD 0908.3814

Outer boundary: Outgoing radiation

- Computational domains typically don't extend to ∞

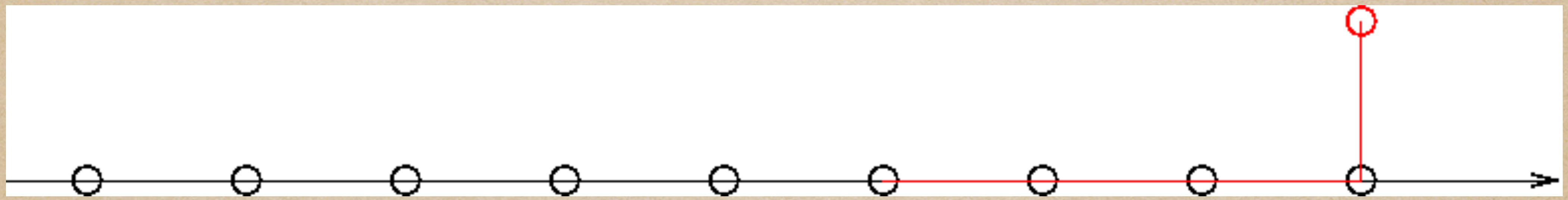
- Outgoing Sommerfeld condition:

Assume: $f = f_0 + \frac{u(t-r)}{r^n}$ where f_0 is the asymptotic value

$$\Rightarrow \partial_t u + \partial_r u = 0$$

$$\Rightarrow \partial_t f + n\frac{f - f_0}{r} + \frac{x^m}{r}\partial_m f = 0$$

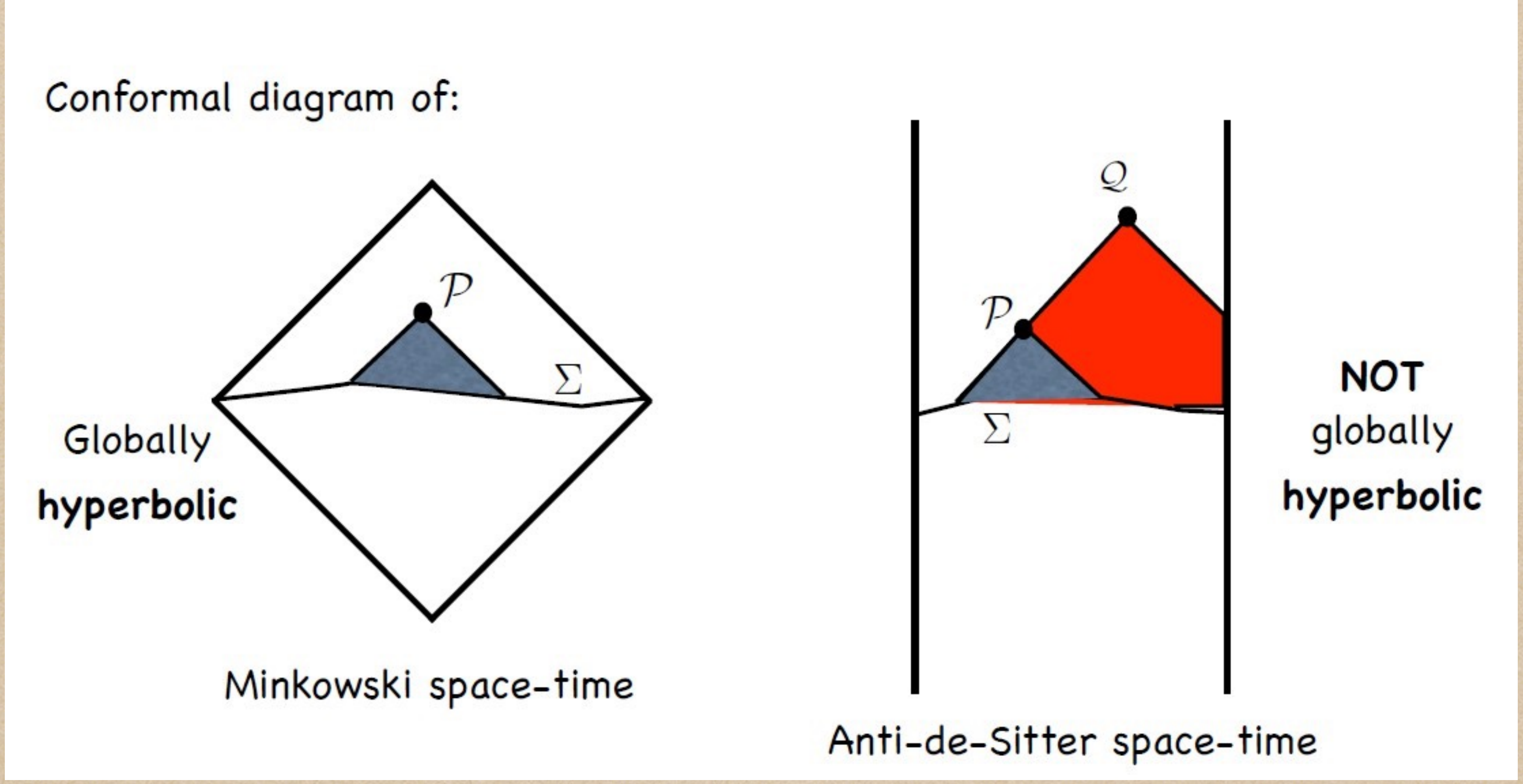
- Implemented through upwinding, i.e. one sided derivatives



- This method is straightforwardly generalized to asymptotically de Sitter spacetimes Zilhão et al PRD 1204.2019

Outer boundary: Anti-de Sitter

- Much more complicated! Penrose diagram of Minkowski and AdS:



- AdS: Timelike outer boundary affects interior
- AdS metric diverges at outer boundary

The Anti-de Sitter metric

- Maximally symmetric solution to Einstein's eqs. with $\Lambda < 0$
- Hyperboloid embedded in $D+1$ dimensional flat spacetime of signature $--++\ldots+$: $X_0^2 + X_D^2 - \sum_{i=1}^{D-1} X_i^2 = L^2$
- 1) Global AdS: $ds^2 = \dfrac{L^2}{\cos^2\rho}\left(-d\tau^2 + d\rho^2 + \sin^2\rho\, d\Omega_{D-2}^2\right)$

  where $0 \leq \rho < \pi/2\,, \quad -\pi < \tau < \pi$ and outer boundary at $\rho = \pi/2$

  2) Poincaré patch: $ds^2 = \dfrac{L^2}{z^2}\left[-dt^2 + dz^2 + \sum_{i=1}^{D-2}(dx^i)^2\right]$

  where $z > 0\,,\ \ t \in \mathbb{R}$ and outer boundary at $z = 0$

see e.g. Ballón Bayona & Braga hep-th/0512182

The outer boundary of AdS

- AdS boundary: $\rho \to \pi/2$ (global)

  $z \to 0$ (Poincaré)

- AdS metric becomes singular

  $\Rightarrow$ induced metric determined up to conformal rescaling only

- Global: $ds^2_{\rm gl} \sim -d\tau^2 + d\Omega^2_{D-2}$

  Poincaré: $ds^2_{\rm P} \sim -dt^2 + \sum_{i=1}^{D-2} d(x^i)^2$

  $\Rightarrow$ Different topology: $\mathbb{R} \times S_{D-2}$ and $\mathbb{R}^{D-1}$

- The dual theories live on spacetimes of different topology

Regularization methods

- Decompose metric into AdS part plus deviation

  e.g. Bantilan, Pretorius & Gubser PRD 1201.2132

- Factor out appropriate factors of the bulk coordinate

  e.g. Chesler & Yaffe PRL 1011.3562;

  Heller, Janik & Witaszczyk PRL 1103.3452

- Factor out singular term of the metric

  e.g. Bizón & Rostworowski PRL 1104.3702

- Regularity of the outer boundary may constrain the gauge freedom

  e.g. Bantilan, Pretorius & Gubser PRD 1201.2132

2.6 Diagnostics

The subtleties of diagnostics in GR

- Successful NR simulation $\rightarrow$ Tons of numbers for grid functions
- Typically: Spacetime metric $g_{\alpha\beta}$ and time derivative $\partial_t g_{\alpha\beta}$, or

  ADM variables $\gamma_{ij}\,, \quad K_{ij}\,, \quad \alpha\,, \quad \beta^i$
- Challenges
  - Coordinate dependence of numbers $\Rightarrow$ Gauge invariants
  - Global quantities at ∞, domain finite $\Rightarrow$ Extrapolation
  - Complexity of variables, e.g. GWs $\Rightarrow$ Spherical harmonics
  - Local quantities meaningful? $\Rightarrow$ Horizons
- AdS/CFT correspondence: Dictionary

Conventions: Newton's constant

- Einstein eqs. without Λ : $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G T_{\alpha\beta}$
- The (areal) horizon radius of a static BH in D dimensions then is

$$r_{\rm S}^{D-3} = \frac{16\pi G M}{(D-2)\Omega_{D-2}} \,,$$

where $\Omega_{D-2} = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)}$ is the area of the $D-2$ hypersphere

- The Hawking entropy formula is $S = \frac{\mathcal{A}_{\rm AH}}{4G}$
- But Newton's force law picks up geometrical factors:

$$\boldsymbol{F} = \frac{(D-3) 8\pi G}{(D-2)\Omega_{D-2}} \frac{Mm}{r^{D-2}} \hat{\boldsymbol{r}}$$

e.g. Emparan & Reall LRR 0801.3471

Global quantities

- Assumptions:
  - Asymptotically, the metric is flat and time independent
  - Our expressions refer to Cartesian coordinates
- ADM mass = Total mass-energy of the spacetime

$$M_{\rm ADM} = \frac{1}{4\Omega_{D-2}G} \lim_{r\to\infty} \int_{S_r} \sqrt{\gamma} \left[\gamma^{mn}\gamma^{kl}(\partial_n\gamma_{mk} - \partial_k\gamma_{mn})\right] dS_l$$

- Linear momentum of spacetime

$$P_i = \frac{1}{2\Omega_{D-2}G} \lim_{r\to\infty} \int_{S_r} \sqrt{\gamma}\,(K^m{}_i - \delta^m{}_i\,K)\,dS_m$$

- Angular momentum in $D = 4$

$$J_i = \frac{1}{8\pi}\epsilon_{il}{}^m \lim_{r\to\infty} \int_{S_r} \sqrt{\gamma}\,x^l\,(K^n{}_m - \delta^n{}_m\,K)\,dS_n$$

- By construction, these are time independent!

Apparent horizons

- By cosmic censorship, existence of an apparent horizon implies an event horizon
- Consider outgoing null geodesics with tangent vector k^μ
- **<u>Def.:</u>** Expansion $\Theta := \nabla_\mu k^\mu$
- **<u>Def.:</u>** Apparent horizon $:=$ Outermost surface on Σ_t where $\Theta = 0$
- On a hypersurface Σ_t, the condition for $\Theta = 0$ becomes
  $\hat{D}_m s^m - K + K_{mn} s^m s^n = 0\,,$
  where $s^i =$ unit normal to the $D-2$ dimensional AH surface
  and $\hat{D}_i =$ the cov.deriv. of the metric induced on this surface;
  e.g. Thornburg PRD gr-qc/9508014

Apparent horizons in D=4

- Parametrize the horizon by $r = f(\varphi^i)$,
  where r is the radial and φ^i are angular coordinates
- Rewrite the condition $\Theta = 0$ in terms of $f(\varphi)$
  $\Rightarrow$ Elliptic equation for $f(\varphi)$
- This can be solved e.g. with Flow, Newton methods

  Thornburg PRD gr-qc/9508014; Gundlach PRD gr-qc/9707050

  Alcubierre CQG gr-qc/9809004; Schnetter CQG gr-qc/0306006
- Irreducible mass: $M_{\rm irr} = \sqrt{\frac{A_{\rm AH}}{16\pi G^2}}$
- Total BH mass: $M^2 = M_{\rm irr}^2 + \frac{S^2}{4M_{\rm irr}^2} \quad (+\,P^2)$
  where S is the spin of the BH

  Christodoulou PRL 25 1596

GW extraction in D=4: Newman Penrose

- Construct a tetrad
  - $n^\alpha =$ Timelike unit normal field
  - Spatial triad $\boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w}$ Gram-Schmidt orthonormalization

    E.g. starting with $u^i = [x,\, y,\, z]\,, \quad v^i = [xz,\, yz,\, -x^2 - y^2]\,,$

    $$w^i = \epsilon^i{}_{mn} v^m w^n \,.$$
  - $l^\alpha = \frac{1}{\sqrt{2}}(n^\alpha + u^\alpha)\,, \quad k^\alpha = \frac{1}{\sqrt{2}}(n^\alpha - u^\alpha)\,, \quad m^\alpha = \frac{1}{\sqrt{2}}(v^\alpha + i\, w^\alpha)$

    $\Rightarrow \quad -\boldsymbol{l}\cdot\boldsymbol{k} = 1 = \boldsymbol{m}\cdot\bar{\boldsymbol{m}}$, all other products vanish
- Newman-Penrose scalar $\Psi_4 = -C_{\alpha\beta\gamma\delta} k^\alpha \bar{m}^\beta k^\gamma \bar{m}^\delta$
- In vacuum: $R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta}$
- For more details, see e.g.

Nerozzi PRD gr-qc/0407013; Brügmann et al PRD gr-qc/0610128

Analysis of Ψ_4

- Multipolar decomposition: $\Psi_4 = \sum_{\ell,m} \psi_{\ell m}(t,r) Y^{-2}_{\ell m}(\theta,\phi)$

  where $\psi_{\ell m} = \int_0^{2\pi}\int_0^{\pi} \Psi_4 \overline{Y^{-2}_{\ell m}} \sin^2\theta \, d\theta \, d\phi$

- Radiated energy $\dfrac{dE}{dt} = \lim_{r\to\infty}\left[\dfrac{r^2}{16\pi}\int_\Omega \left|\int_{-\infty}^{t} \Psi_4 \, d\tilde{t}\right|^2 d\Omega\right]$

- Momentum: $\dfrac{dP_i}{dt} = -\lim_{r\to\infty}\left[\dfrac{r^2}{16\pi}\int_\Omega \ell_i \left|\int_{-\infty}^{t} \Psi_4 \, d\tilde{t}\right|^2 d\Omega\right],$

  where $\ell_i = [-\sin\theta\,\cos\phi, \quad -\sin\theta\,\sin\phi, \quad -\cos\theta]$

- Angular momentum:

$$\frac{dJ_z}{dt} = -\lim_{r\to\infty}\left\{\frac{r^2}{16\pi}\mathrm{Re}\left[\int_\Omega \left(\partial_\phi \int_{-\infty}^{t} \Psi_4 d\tilde{t}\right)\left(\int_{-\infty}^{t}\int_{-\infty}^{\hat{t}} \bar{\Psi}_4 \, d\tilde{t}\, d\hat{t}\right) d\Omega\right]\right\}$$

see e.g. Ruiz et al GRG 0707.4654

Alternative extraction methods

- Landau-Lifshitz pseudo tensor: simple but gauge dependent

  see e.g. Lovelace et al PRD 0907.0869

- Regge-Wheeler-Zerilli-Moncrief perturbation formalism:

  perturbations on Schwarzschild $\to$ gauge invariant master function

  Regge & Wheeler PR (1957); Zerilli PRL (1970);

  Moncrief Ann.Phys. (1974);

  For applications in NR see e.g. Reisswig et al PRD 1012.0595

  Sperhake et al PRD gr-qc/0503071; Rezzolla gr-qc/0302025

- Cauchy-characteristic extraction at $\mathcal{I}^+$ using a compactified

  exterior vacuum patch with characteristic coordinates: very accurate

  Reisswig et al PRL 0907.2637, CQG 0912.1285;

  Babiuc et al PRD 1011.4223

GW extraction in D>4

- Generalization of Regge-Wheeler-Zerilli-Moncrief to higher D

  Kodama-Ishibashi formalism

  Kodama & Ishibashi PTP hep-th/0305147, PTP hep-th/0308128

  Applications in NR:

  Witek et al PRD 1006.3081, PRD 1011.0742, PRD 1406.2703

- Landau-Lifshitz pseudo tensor:

  Yoshino & Shibata PRD 0907.2760, PTPS 189 269-310

- Generalization of the Newman-Penrose scalars:

  Peeling properties of Weyl tensor: Godazgar & Reall 2012 1201.4373

  Numerical implementation: Cook & Sperhake in preparation

The AdS/CFT dictionary: Fefferman-Graham coords.

- Note: D dimensional bulk, $d = D - 1$ dimensional boundary

- AdS/CFT correspondence

  $\Rightarrow$ Vacuum expectation values $\langle T_{ij} \rangle$ of the field theory given by quasi-local Brown-York stress-energy tensor

  Brown & York PRD gr-qc/9209012

- Consider asymptotically AdS metric in Fefferman-Graham coords.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{L^2}{z^2}(-dt^2 + dz^2 + \gamma_{ij} dx^i dx^j)$$

one can show that at $t = \mathrm{const}$

$$\gamma_{ij}(z, x^i) = \gamma_{ij}^{(0)} + z^2 \gamma_{ij}^{(2)} + \ldots + z^d \gamma_{ij}^{(d)} + h_{ij}^{(d)} z^d \log z^2 + \mathcal{O}(z^{d+1})$$

- Note: This asymptotes to Poincaré coordinates as $z \to 0$

The AdS/CFT dictionary: Fefferman-Graham coords.

- Here, the $\gamma^{(s)}_{ij}$, $h^{(d)}_{ij}$ are functions of x^i
  - Logarithmic terms only appear for even d
  - Powers of z are exclusively even up to order $d-1$

- Vacuum expectation values of CFT momentum tensor for $d=4$ is

$$\langle T_{ij}\rangle = \frac{4L^3}{16\pi G}\Big\{\, \gamma^{(4)}_{ij} - \frac{1}{8}\gamma^{(0)}_{ij}\left[\gamma^2_{(2)} - \gamma^{km}_{(0)}\gamma^{ln}_{(0)}\gamma^{(2)}_{kl}\gamma^{(2)}_{mn}\right] - \frac{1}{2}\gamma_{(2)i}{}^{m}\gamma^{(2)}_{jm} - \frac{1}{4}\gamma^{(2)}_{ij}\gamma_{(2)}\Big\}\,,$$

where $\gamma_{(n)} := \mathrm{Tr}\big(\gamma^{(n)}_{ij}\big) = \gamma^{ij}_{(0)}\gamma^{(n)}_{ij}$

de Haro et al Comm.Math.Phys. hep-th/0002230 ; also for other d

- Note: $\gamma^{(2)}_{ij}$ is determined by $\gamma^{(0)}_{ij}$ $\Rightarrow$ CFT freedom given by $\gamma^{(4)}_{ij}$

Further reading

- Isolated and dynamic horizon

  Ashtekar & Krishnan LRR gr-qc/0407042

- AdS/CFT dictionary

  Balasubramanian & Kraus Comm.Math.Phys. hep-th/9902121

  Skenderis CQG hep-th/0209067

  Bantilan, Pretorius & Gubser PRD 1201.2132

3. Results from BH simulations

3.1 BHs in GW physics

Gravitational waves: weak-field solutions

- Consider small deviations from Minkowski in Cartesian coordinates

  "Background": Manifold $\mathcal{M} = \mathbb{R}^4, \quad \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$

  "Perturbation": $h_{\mu\nu} = \mathcal{O}(\epsilon) \ll 1 \quad \Rightarrow \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

- Coordinate freedom: "Transverse-traceless (TT)" gauge

$$h^\mu{}_\mu = 0, \quad \partial^\nu h_{\mu\nu} = 0$$

- Vacuum, no cosmological constant: $T_{\mu\nu} = 0, \quad \Lambda = 0$
- Einstein's eqs.: $\Box h_{\mu\nu} = 0$
- Plane wave solution in z direction: $h_{\mu\nu} = H_{\mu\nu} e^{ik_\sigma x^\sigma}$

$$k^\mu = \omega(1, 0, 0, 1) \qquad H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Effect on particles

- Geodesic eq.

  Particle at rest at x^μ stays at $x^\mu = \mathrm{const}$ in TT gauge

- Proper separation:

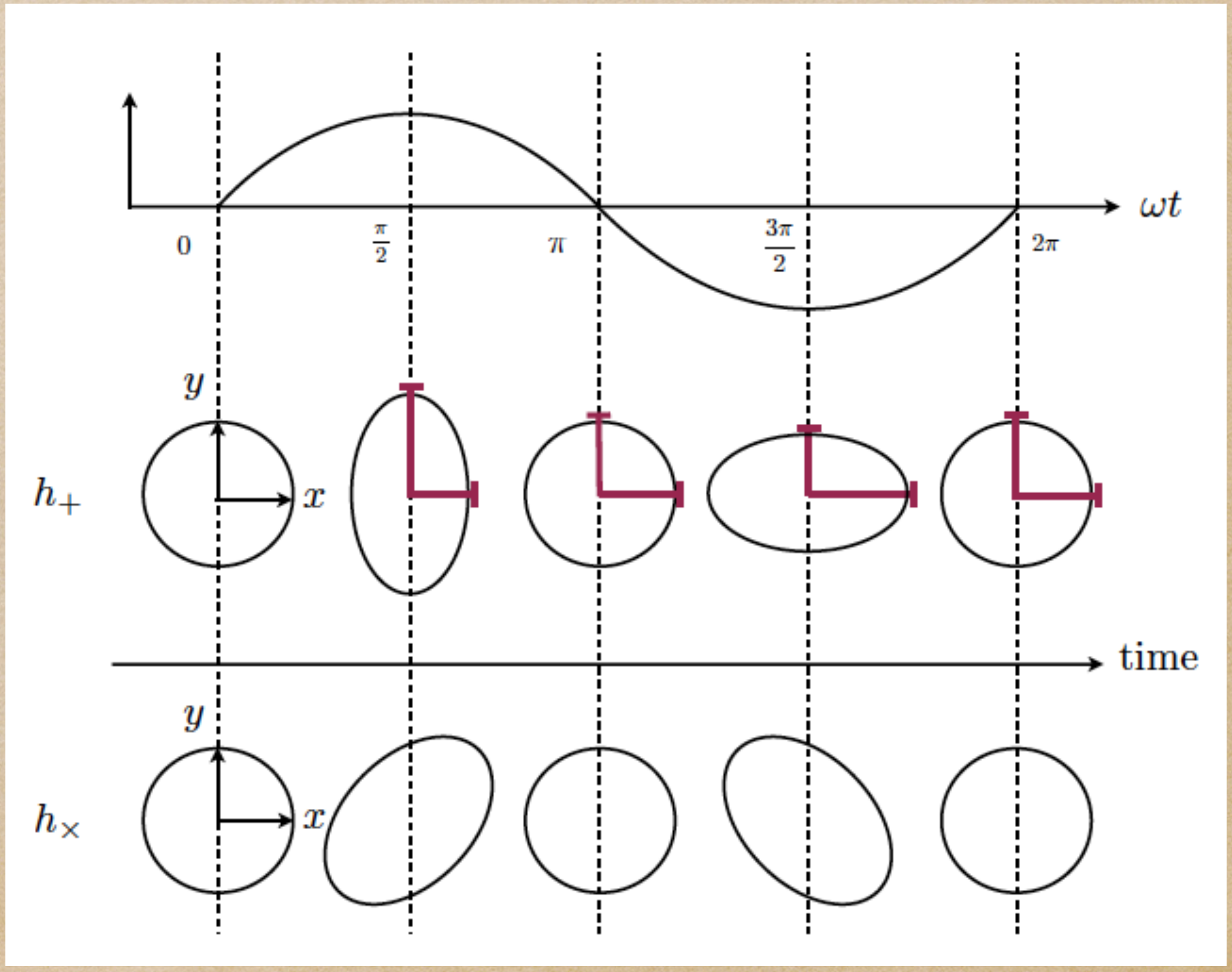
$$ds^2 = -dt^2 + (1 + h_+)\, dx^2 + (1 - h_+)\, dy^2 + 2h_\times\, dx\, dy + dz^2$$

- Effect on test particles:

  Mirshekari 1308.5240

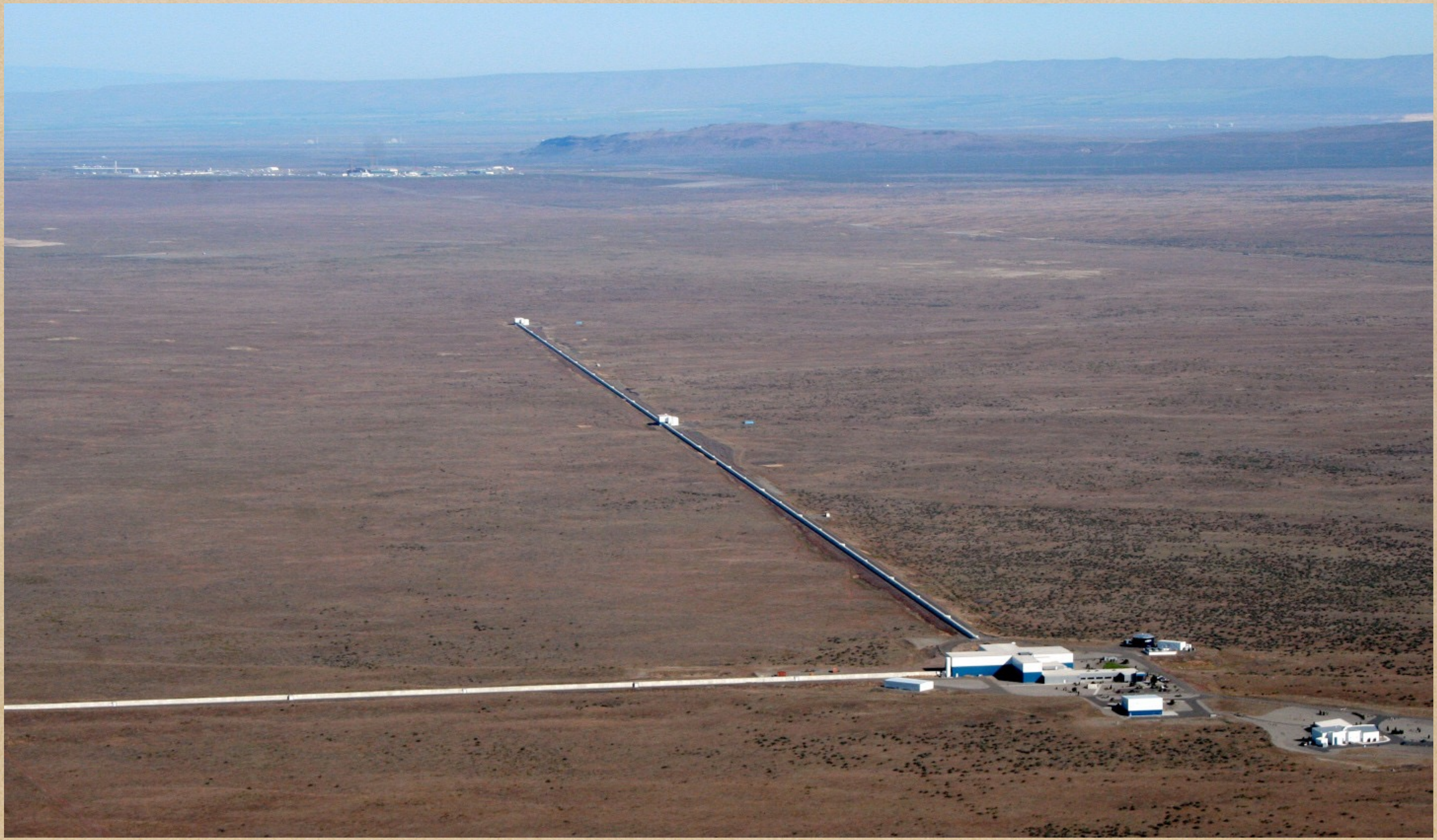
- Debate on physical reality until late 1950s

  e.g.Saulson GRG (2011)



Effect on particles

- Measure this effect; Michelson-Morley type interferometer



The gravitational wave spectrum

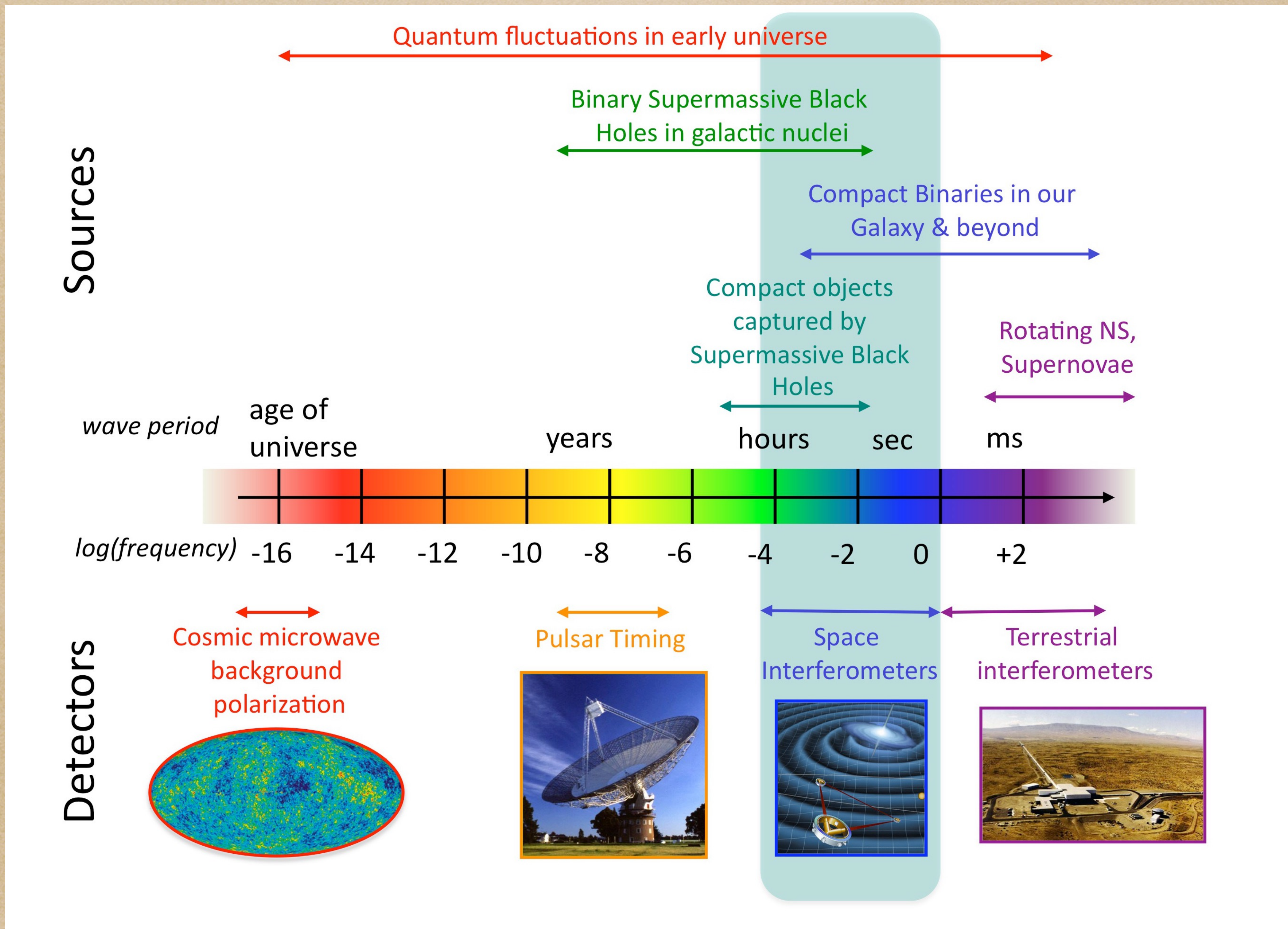
- Source types and detection strategies $\Rightarrow$ 4 regimes

Ultra low	$f \sim 10^{-18} \dots 10^{-15}$ Hz
Very low	$f \sim 10^{-9} \dots 10^{-6}$ Hz
Low	$f \sim 10^{-4} \dots 10^{-1}$ Hz
High	$f \sim 10^{1} \dots 10^{3}$ Hz

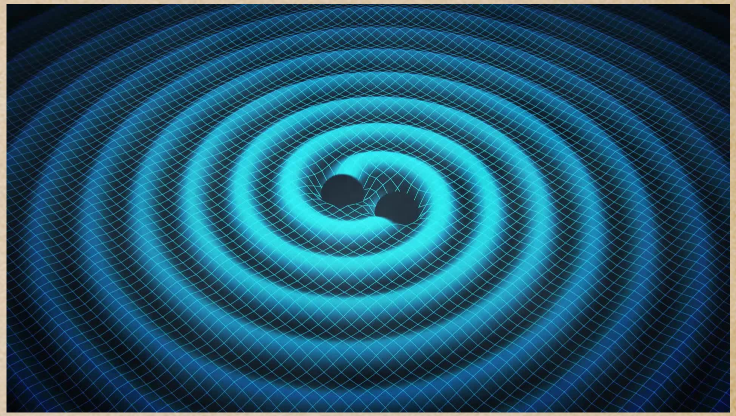
- Major sources

Ultra low:	Fluctuations in the early universe
Very low:	Supermassive BH binaries (high M, z)
Low:	SMBHs, EMRIs, Compact binaries,...
High:	Neutron star / BH binaries, supernovae,...

The gravitational wave spectrum

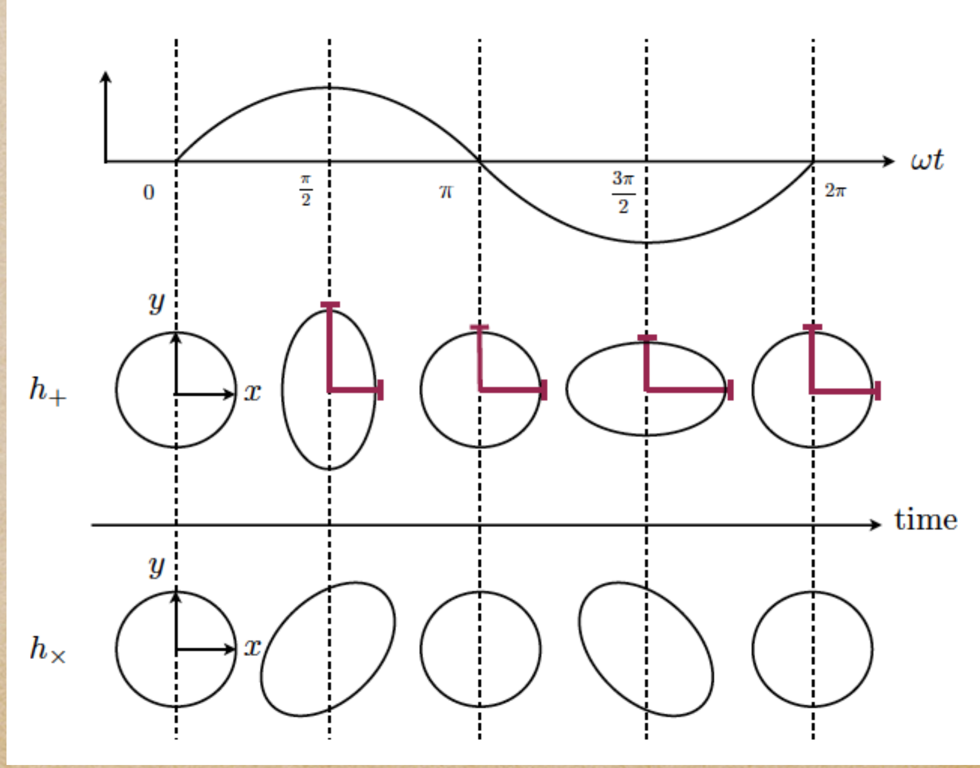


The search for GWs in the data stream



$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}\,; \qquad \frac{8\pi G}{c^4} = 2.07 \times 10^{-43} \frac{\mathrm{s}^2}{\mathrm{m\,kg}}$$

- Weak effect of matter on geometry
- GWs carry huge energy but barely interact with anything
- Induced changes in length: < atomic nucleus / km



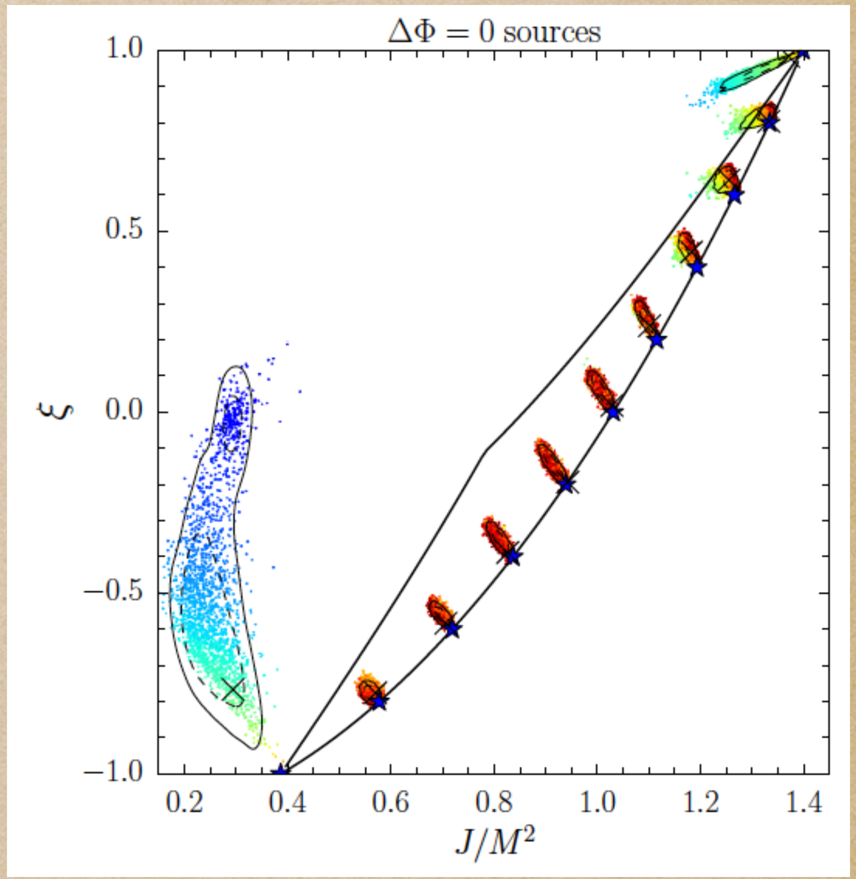
Detection and parameter estimation

Generic transient search

- No specific waveform model
- Identify excess power in detector strain data
- Use multi detector maximum likelihood   Klimenko et al. 1511.05999

Binary coalescence search

- "Matched Filtering" e.g. Allen et al. PRD 2012
- Compare data stream with GW templates ("Finger print search")
- Bayesian analysis: Prior $\longrightarrow$ Posterior



Trifiró et al 1507.05587

Black-hole binaries: parameters

- 8+2 Intrinsic parameters

  Masses $m_1\,,\ m_2$

  Spins $\boldsymbol{S}_1\,,\ \boldsymbol{S}_2$

  Eccentricity (often ignored; GW emission circularizes orbit)

- 7 Extrinsic parameters

  Location: Luminosity distance D_L, Right ascension α, Declination δ

  Orientation: Inclination ι, Polarization ψ

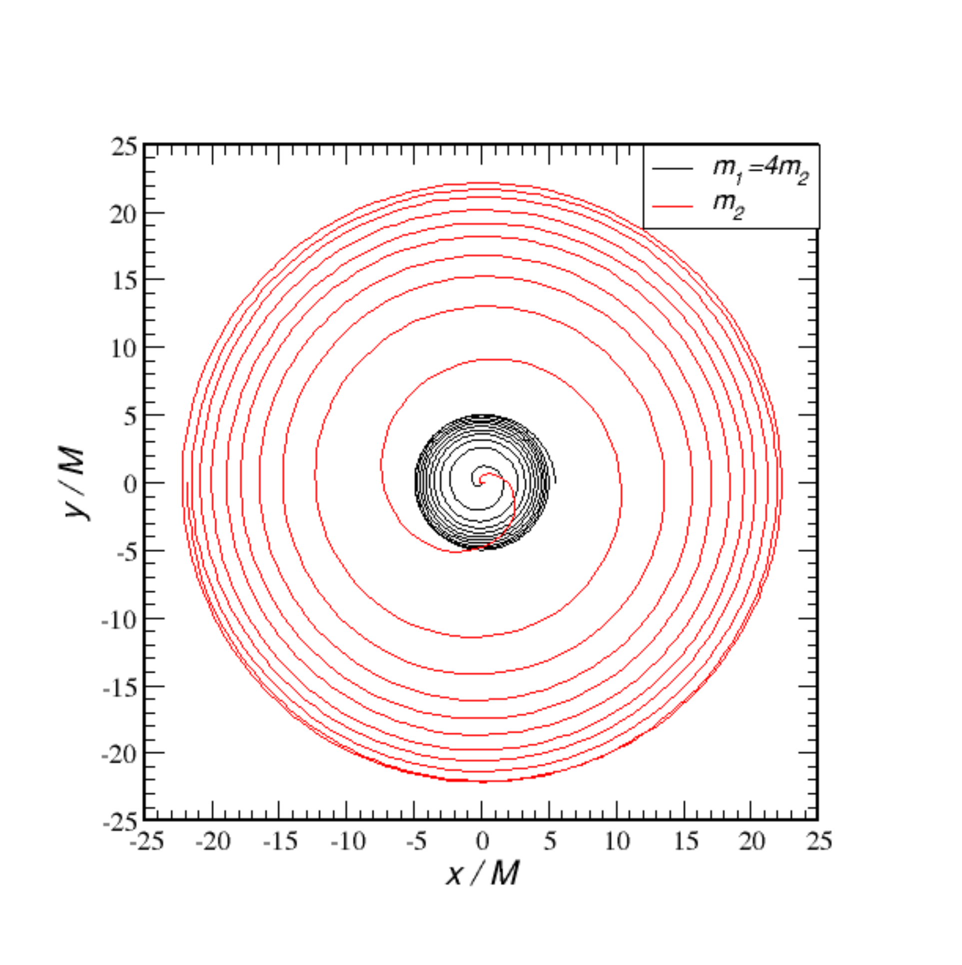
  Time t_c and Phase ϕ_c of coalescence

Binary BH trajectory and waveform

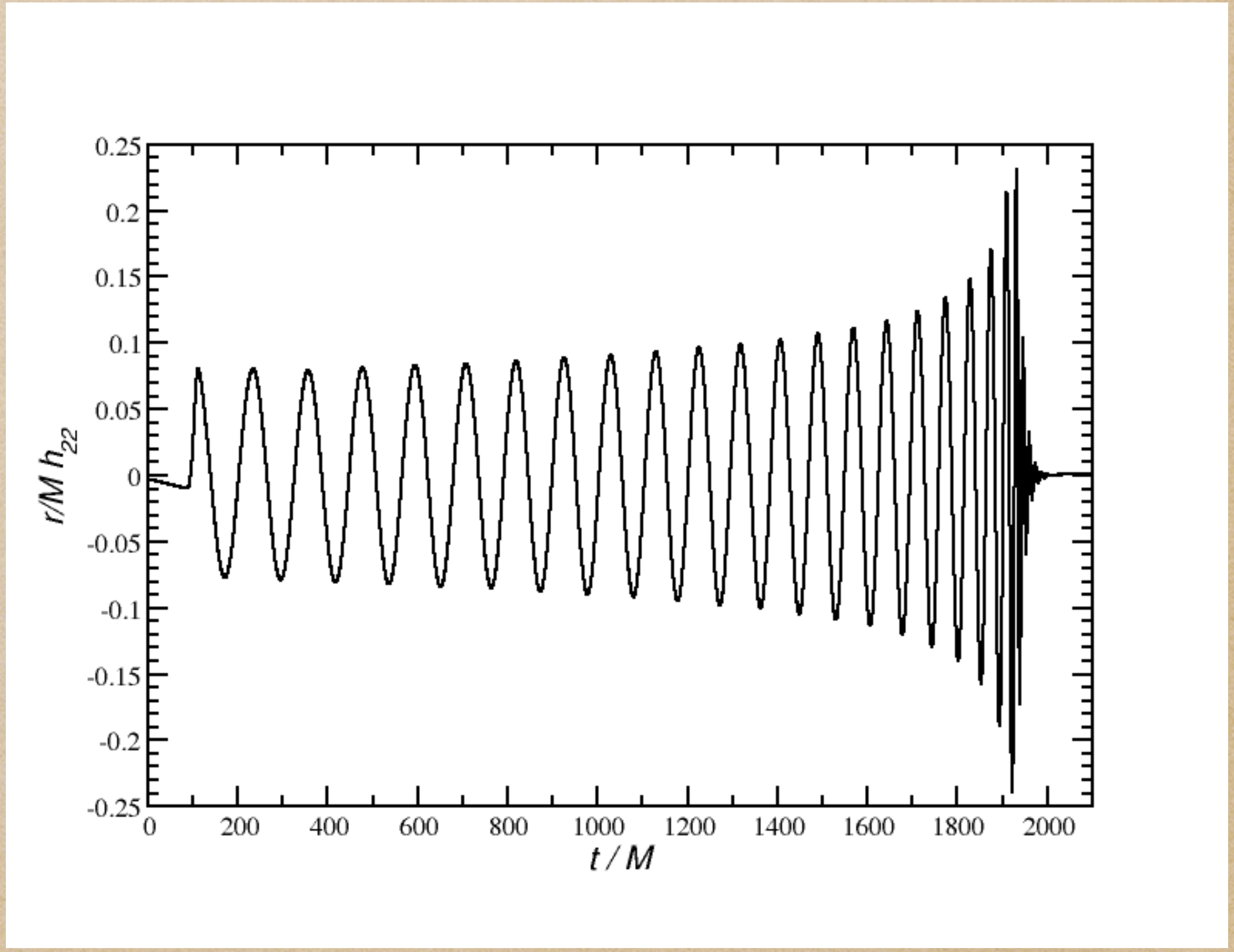
- $\frac{m_1}{m_2} = 4$ non-spinning binary; ≈ 11 orbits

Sperhake et al CQG 1012.3173

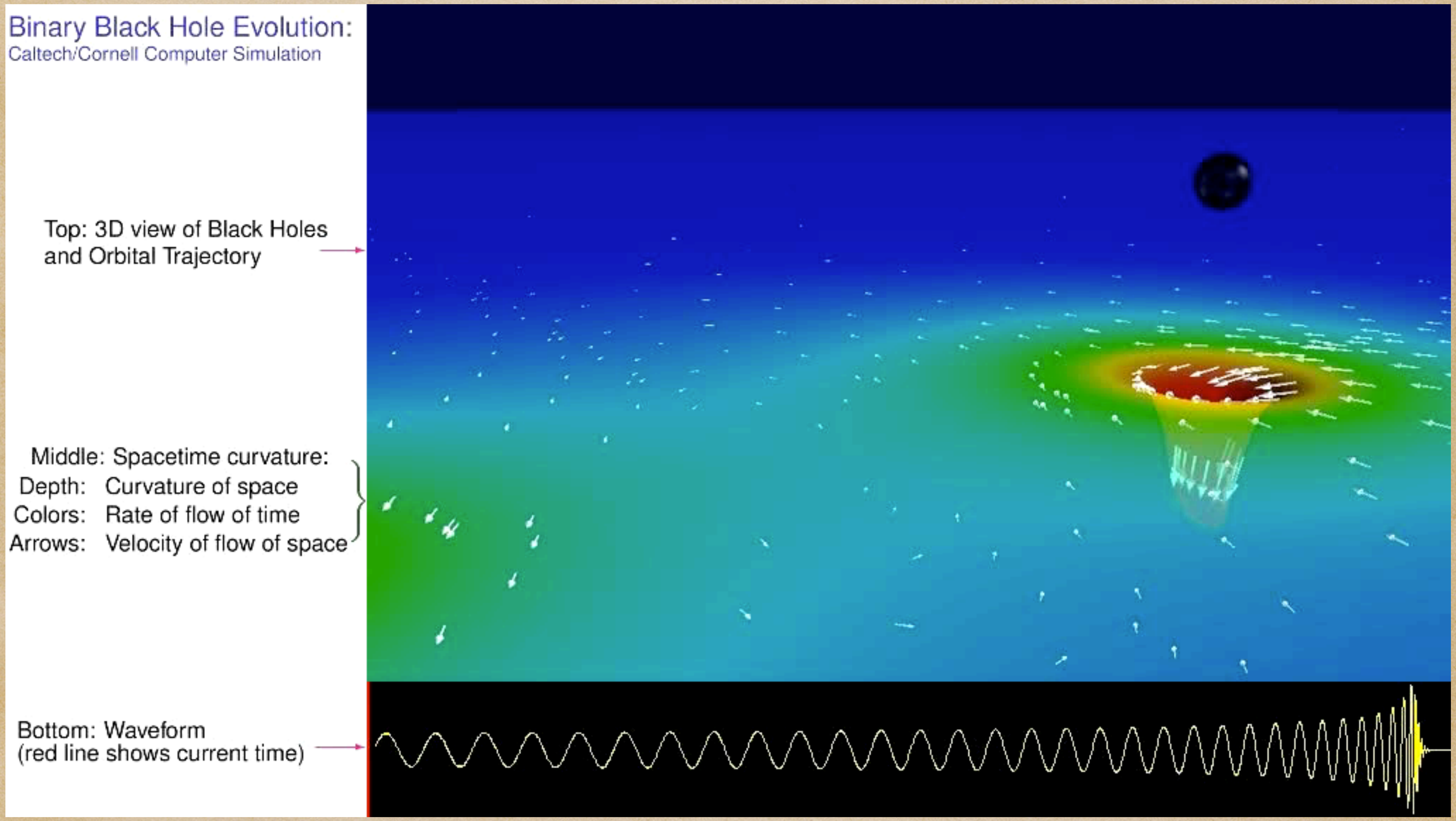
Trajectory



Quadrupole mode



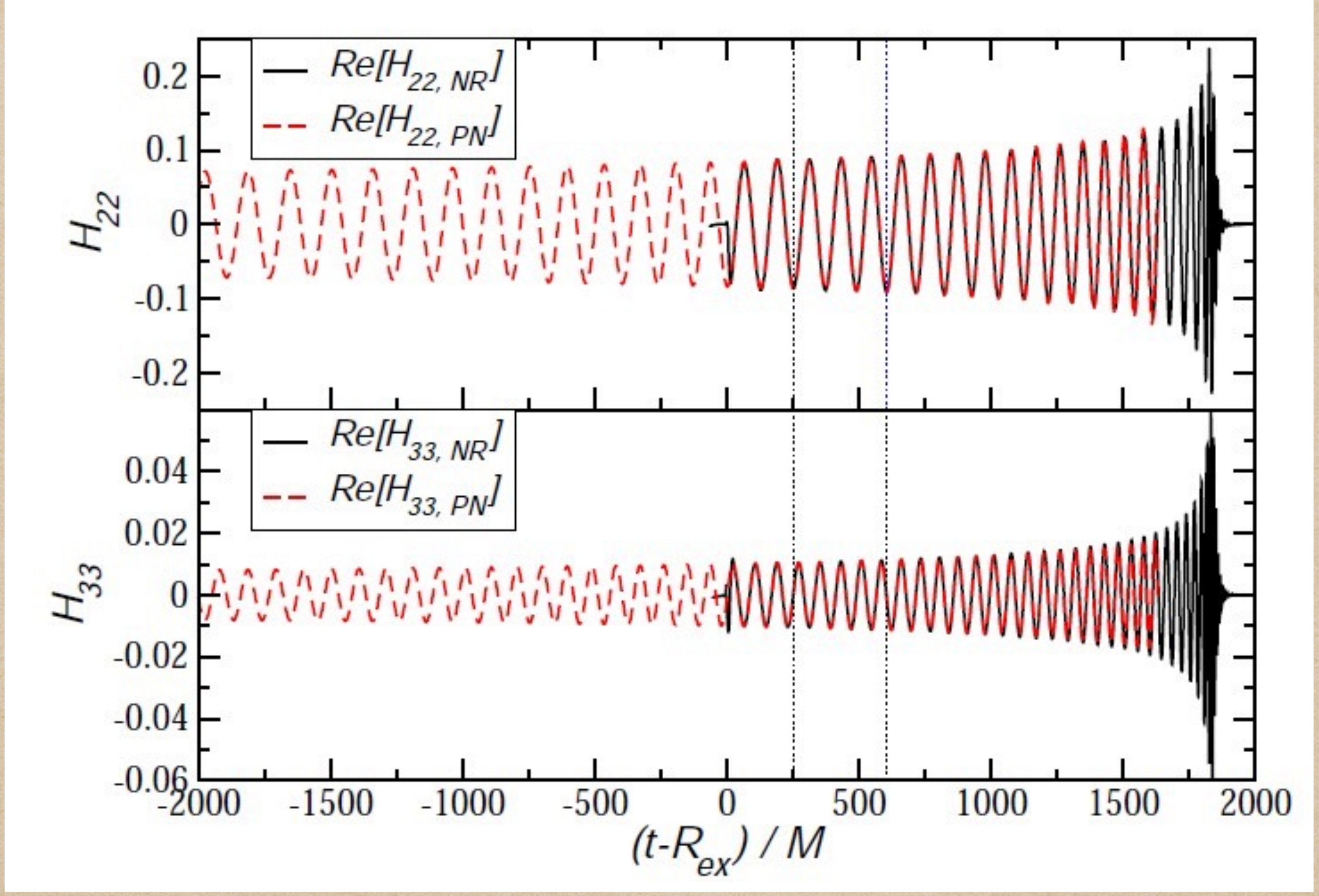
Anatomy of a BHB coalescence



Thanks to Caltech-Cornell groups

Hybrid waveforms and catalogs

- Stitch together PN and NR waveforms



Sperhake et al CQG 2011

- Mass produce waveforms; Hinder et al CQG 1307.5307;

  Mroué et al PRL 1004.4697

GW source modeling

- Key requirement for matched filtering: GW template catalog
- Model black holes in general relativity
  - Post Newtonian theory $\rightarrow$ Inspiral   Blanchet LRR-2006-4
  - Numerical relativity $\rightarrow$ final orbits, merger

    Pretorius PRL 2005, Baker et al PRL 2006, Campanelli et al PRL 2006
  - Perturbation theory $\rightarrow$ Ringdown
- Combine "NR" with "Post-Newtonian", "Effective one body" methods
- 2 families in use: Phenomenological, Effective one body
- Use reduced bases or similar to cover parameter space
- Multipolar decomposition

$$h_+ - ih_\times = \sum_{\ell m} {}_{-2}Y_{\ell m}(\theta, \phi) h_{\ell m}(t)$$

Template construction

- Phenomenological waveform models
  - Model phase, amplitude with simple funcs. $\rightarrow$ Model parameters
  - Create map between physical and model parameters
  - Time or frequency domain; see e.g.:

    Ajith et al CQG 0704.3764, PRD 0710.2335, PRL 0909.2867;

    Santamaria et al PRD 1005.3306; Khan et al PRD 1508.07253
- Effective-one-body (EOB) models
  - Particle in effective metric, PN, bringdown model

    Buonanno & Damour PRD gr-qc/9811091, PRD gr-qc/0001013
  - Resum PN, calibrate pseudo-PN parameters using NR; see e.g.:

    Buonanno+ PRD 0709.3839; Pan+ PRD 1106.1021, PRD 1307.6232;

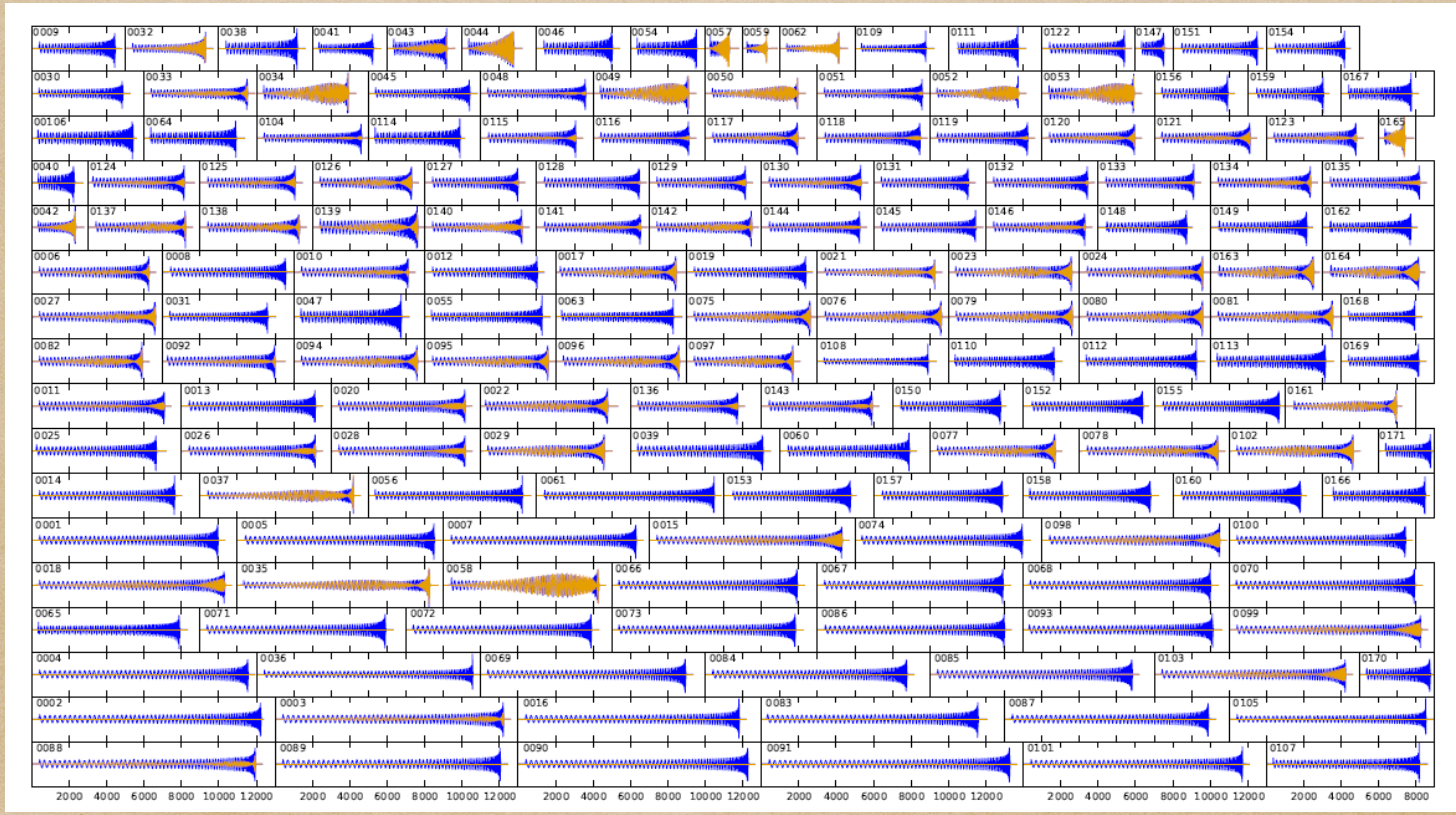
    Tarachini+ PRD 1311.2544; Damour & Nagar PRD 1406.6913

Tools of mass production

- Explore seven-dim. parameter space. E.g. SpEC catalogue:

  171 waveforms: $m_1/m_2 \leq 8$ up to 34 orbits

  Mroué et al PRL 1304.6077



Limits in parameter space

- Mass ratio: $m_1/m_2 = 100$; better waveforms needed

  Lousto & Zlochower PRL 1009.0292

- Spins: $S/M^2 = 0.994$

  Superposed Kerr-Schild data better than punctures here

  Lovelace et al CQG 1411.7297

- Length ≈ 175 orbits

  Szilágyi et al PRL 1502.04953

- Spin precession remains a considerable challenge

  e.g. Ossokine PRD 1502.01747; Hannam et al PRL 1308.3271;

  Gerosa et al PRD 1506.03492

3.2 BHs in Astrophysics

Evidence for astrophysical BHs

- X-ray binaries

  e.g. Cygnus X-1 (1964)

  MS star + compact star

  stellar mass BHs $M \sim 5 \ldots 50 \ M_\odot$

- LIGO GW observations

  GW 150914, GW 151226

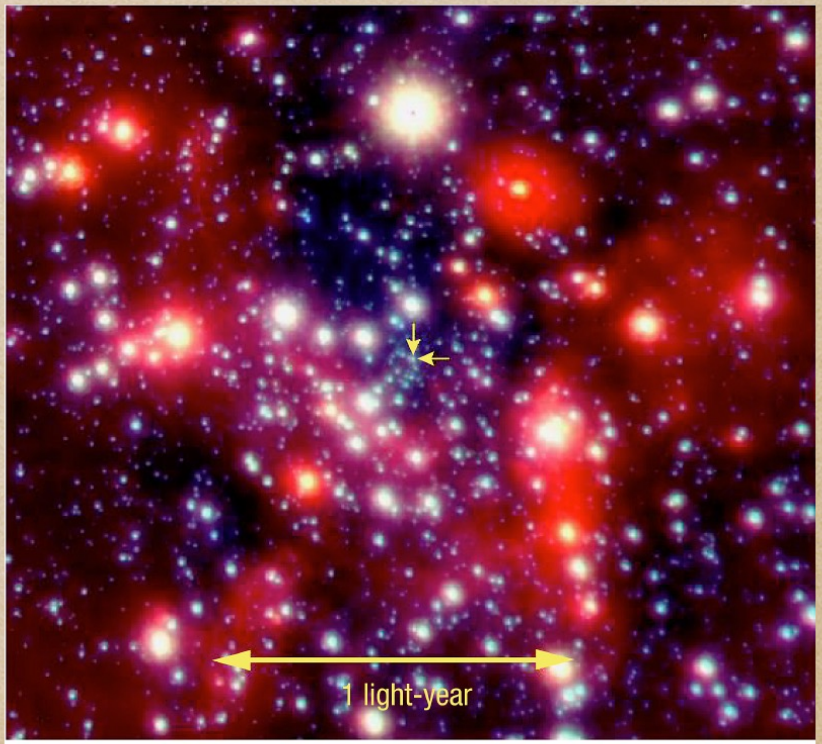
  $M \sim 7.5 \ldots 36 \ M_\odot$

- Dynamics near galactic centers

  and iron emission line profiles

  $\Rightarrow$ supermassive BHs

  AGN engines; $M \sim 10^5 \ldots 10^{10} \ M_\odot$



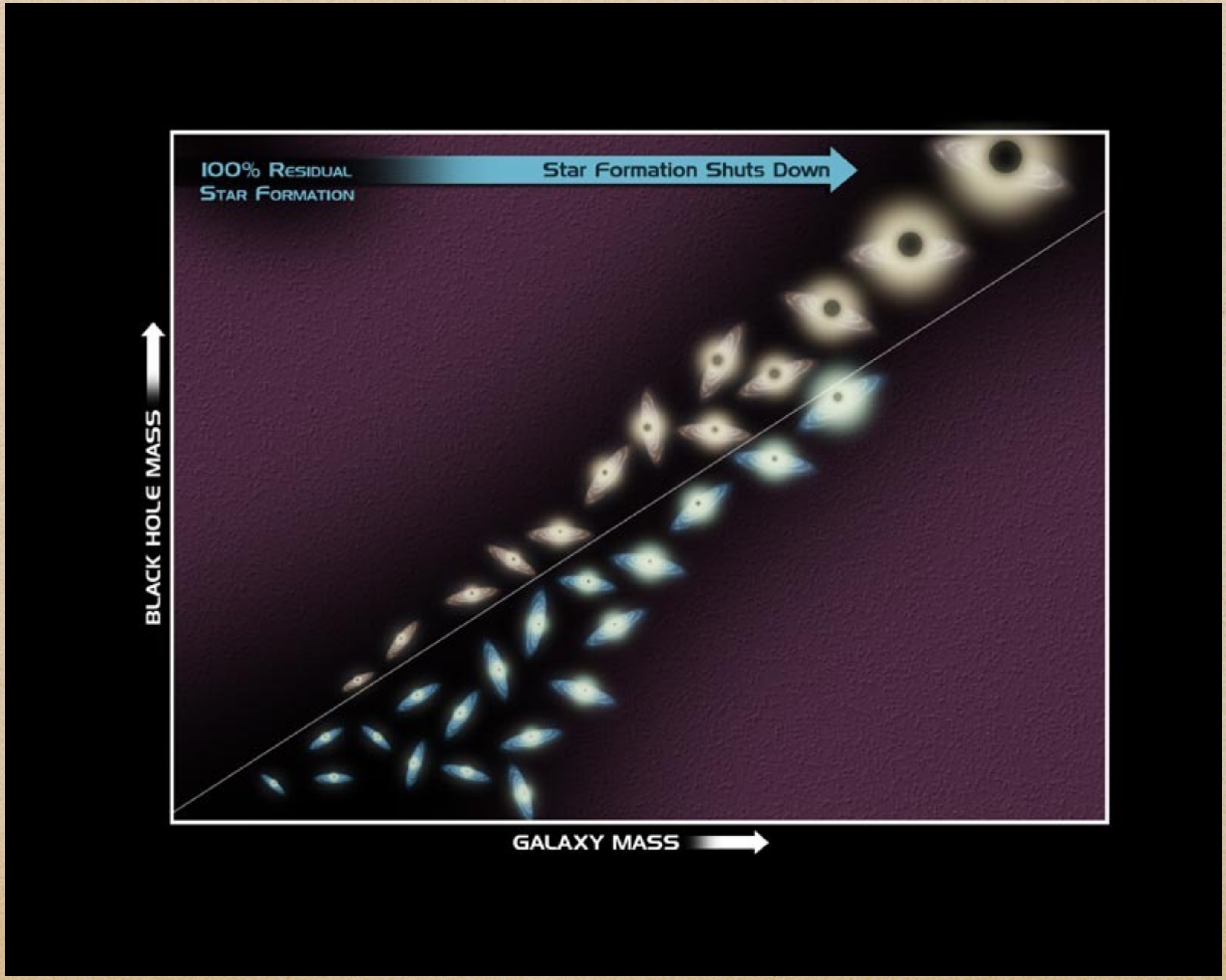
The Centre of the Milky Way
(VLT YEPUN + NACO)

ESO PR Photo 23a/02 (9 October 2002)

Correlation SMBH and host galaxy properties

- Galaxies ubiquitously harbor BHs
- BH properties correlated with bulge properties

e.g. Magorrian et al Astron.J. astro-ph/9708072

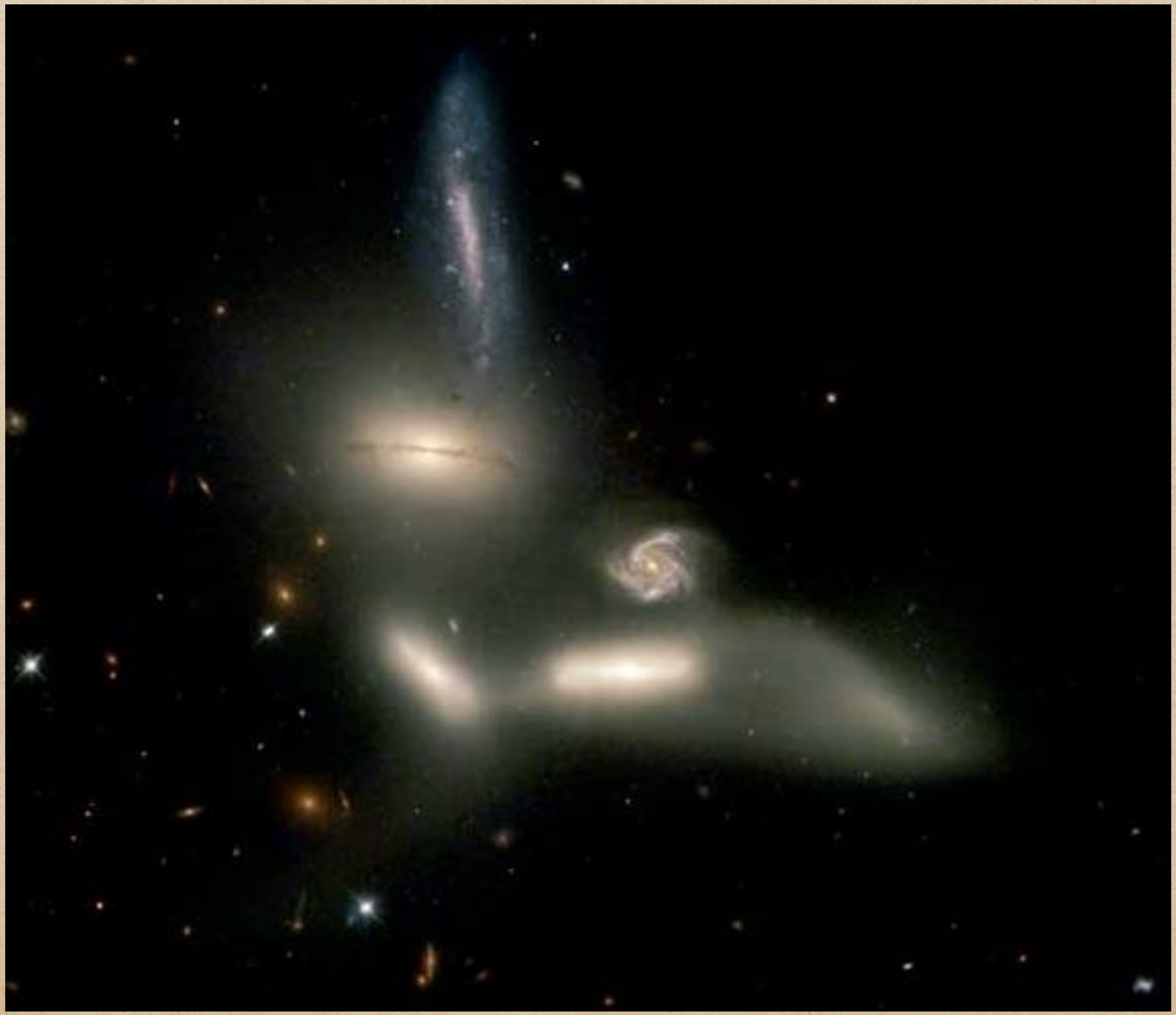


SMBH formation

- Most widely accepted scenario for galaxy formation:

  hierarchical growth “bottom-up”

- Galaxies undergo frequent mergers $\Rightarrow$ BH merger



Gravitational recoil

- Anisotropic GW emission $\Rightarrow$ recoil of remnant BH

  Bonnor & Rotenburg Proc.R.Soc.Lond.A. (1961);

  Peres PR (1962); Bekenstein ApJ (1973)

- Escape velocities:

Globular clusters	~ 30 km/s
dSph	$20 \ldots 100$ km/s
dE	$100 \ldots 300$ km/s
Giant galaxies	$\sim 1\,000$ km/s

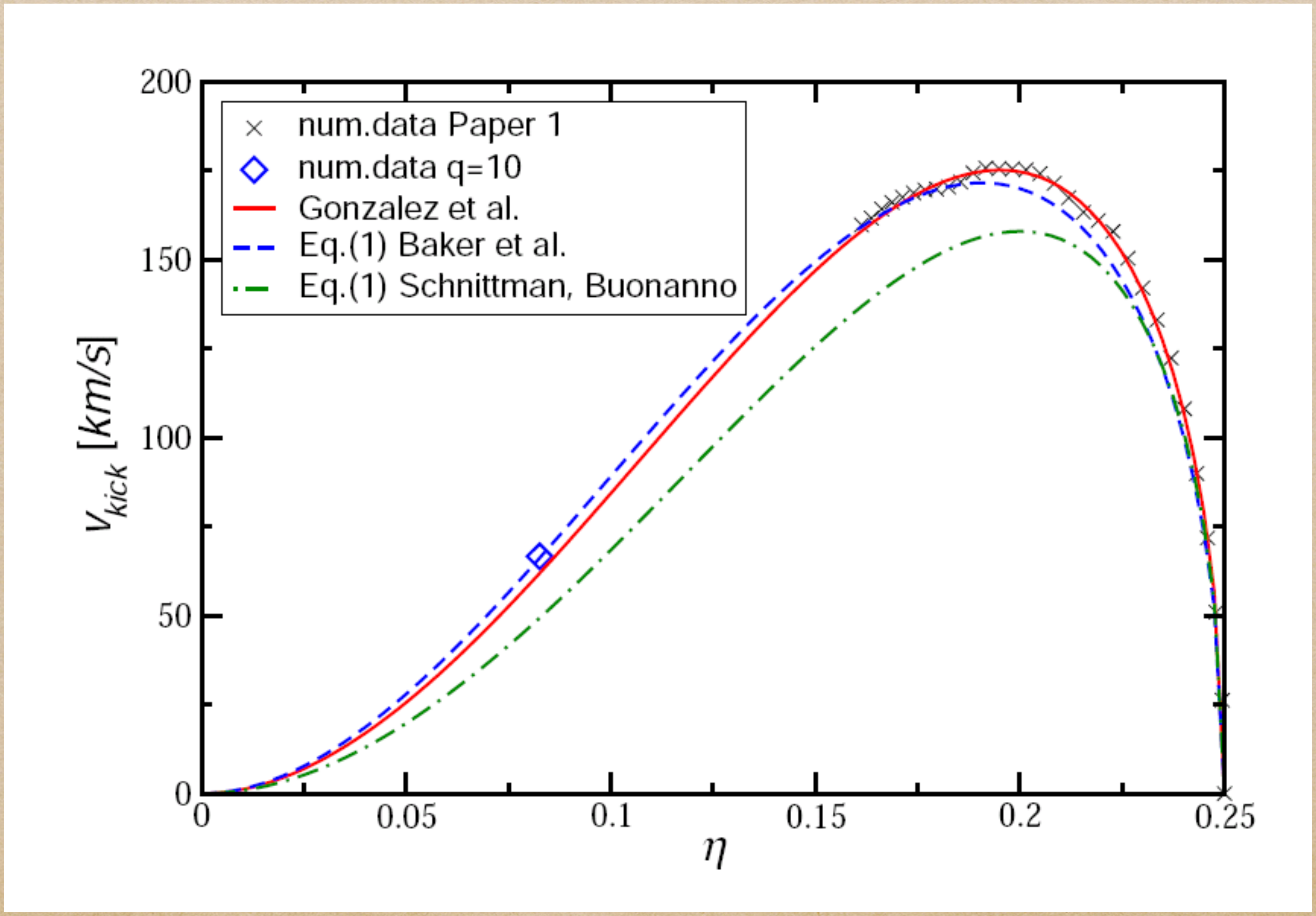
- Ejection/displacement of BHs affects
  - Growth history of SMBHs
  - BH populations, IMBHs
  - galaxy structure
  - observational "footprints"

  Komossa Adv.Astron. 1202.1977

Kicks from non-spinning BH binaries

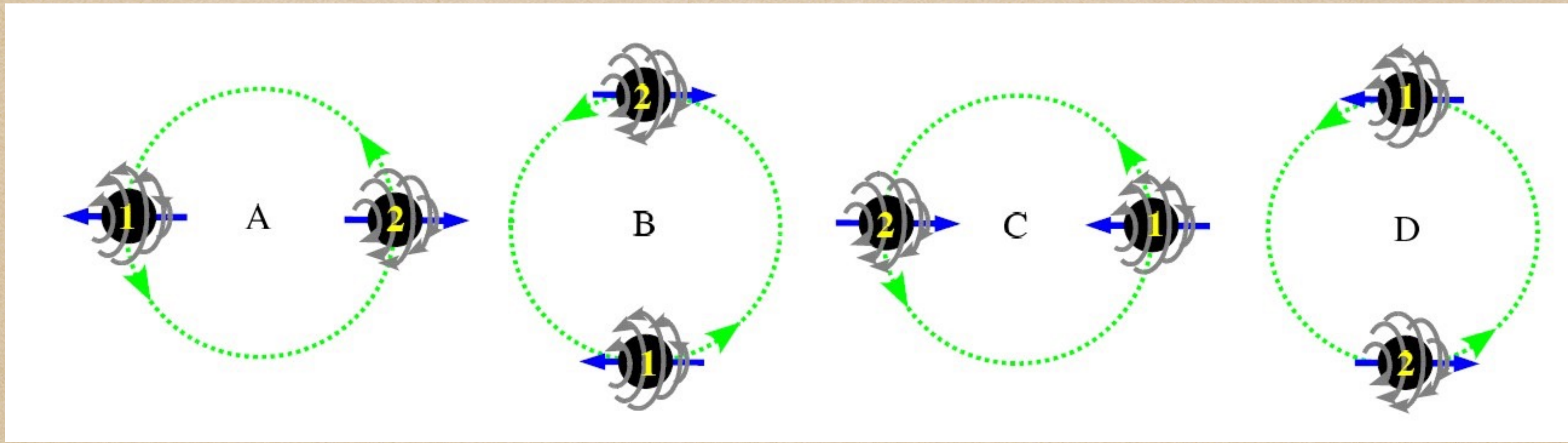
- Maximal kick: $\sim 180\ \mathrm{km/s}$ pretty harmless!

González et al PRL gr-qc/0610154



Spinning BHs: Superkicks

- Superkick configurations; Kidder gr-qc/9506022; Pretorius 0710.1338



- Kicks up to $v_{\mathrm{max}} \approx 4\,000 \mathrm{~km/s}$

  González et al PRL gr-qc/0702052

  Campanelli et al PRL gr-qc/0702133

- Suppression via spin alignment and resonance effects in inspired

  Schnittman PRD astro-ph/0409174

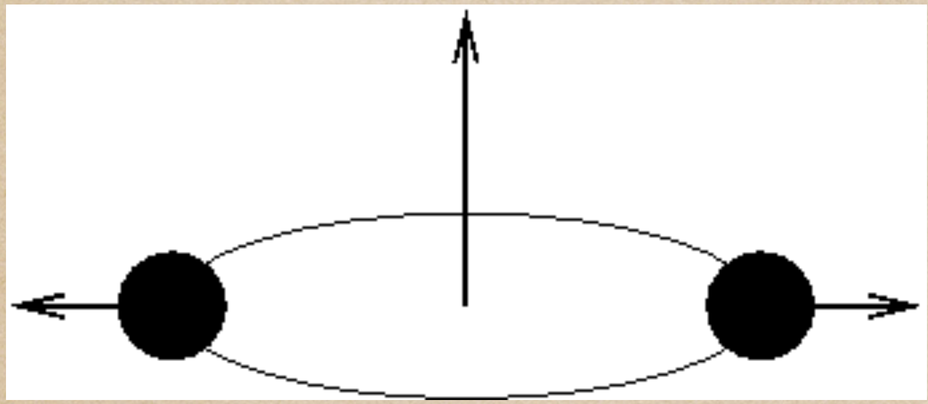
  Bogdanovicz et al ApJ astro-ph/0703054

  Kesden et al PRD 1002.2643; ApJ 1003.4993

Yet larger kicks: superkick + hang-up

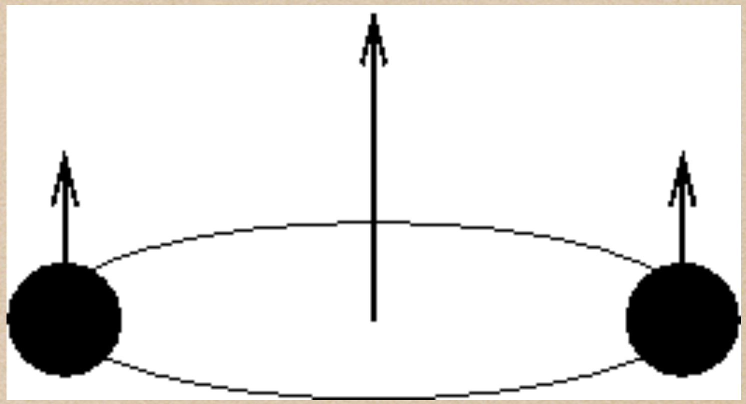
Lousto & Zlochower PRL 1108.2009

Superkick

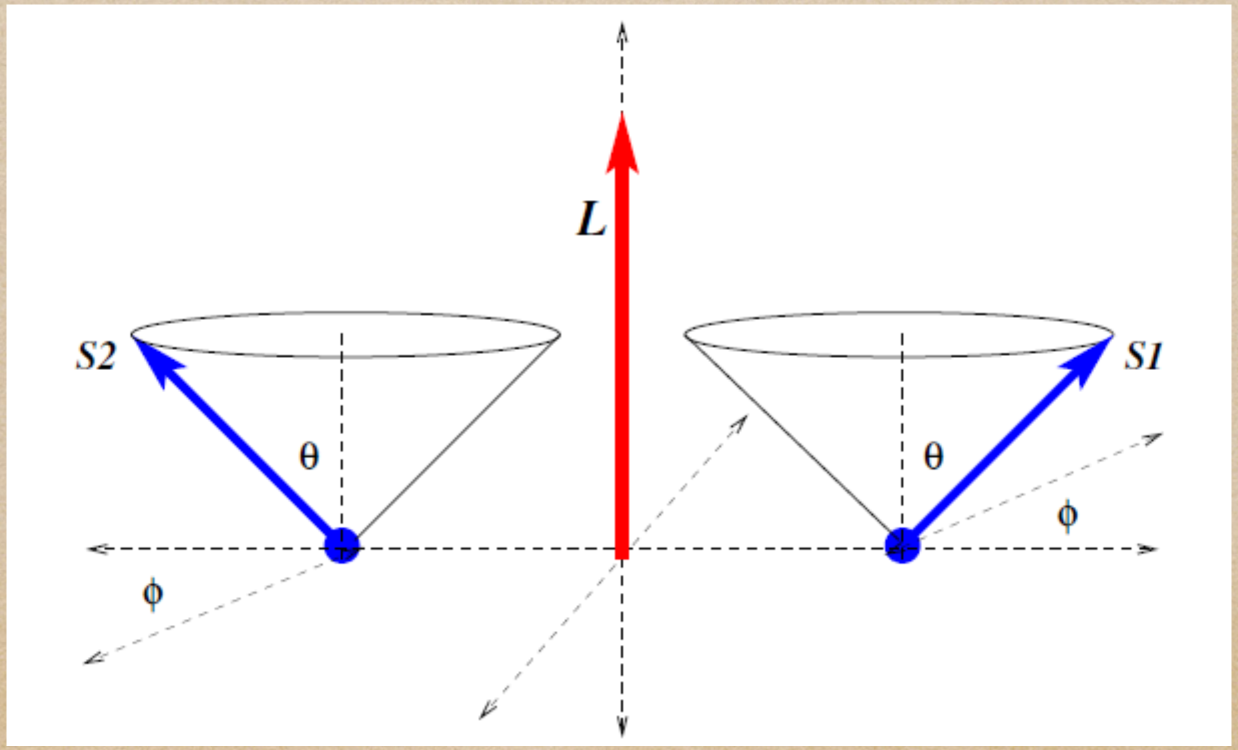


- Moderate GW generation
- Large kicks

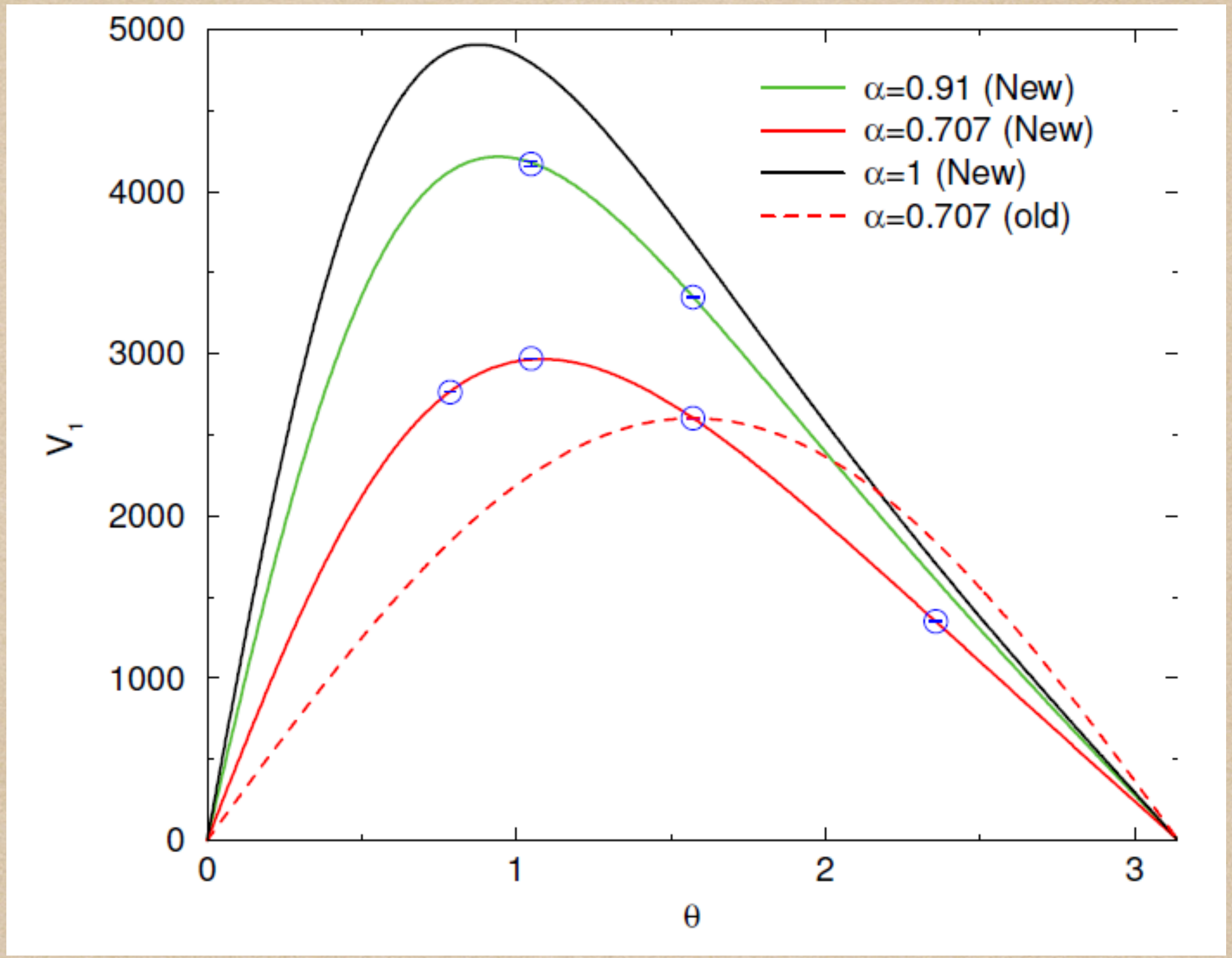
hang-up



- Strong GW generation
- No kicks



Yet larger kicks: superkick + hang-up



- Maximum kick $\sim 25\ \%$ larger: $v_{\rm max} \approx 5\,000\ {\rm km/s}$
- Distribution asymmetric in θ ; $v_{\rm kick}$ maximal for partial alignment
- Suppression through resonances still works

Berti et al PRD 1203.2920

Spin precession and flips

- X-shaped radio sources

  Merritt & Ekers Science (2002)

- Jet along spin axis

- Spin re-alignment

  new + old jet

- Example simulation:

  Spin precession

  Spin flip

  Campanelli et al PRD
  gr-qc/0612076

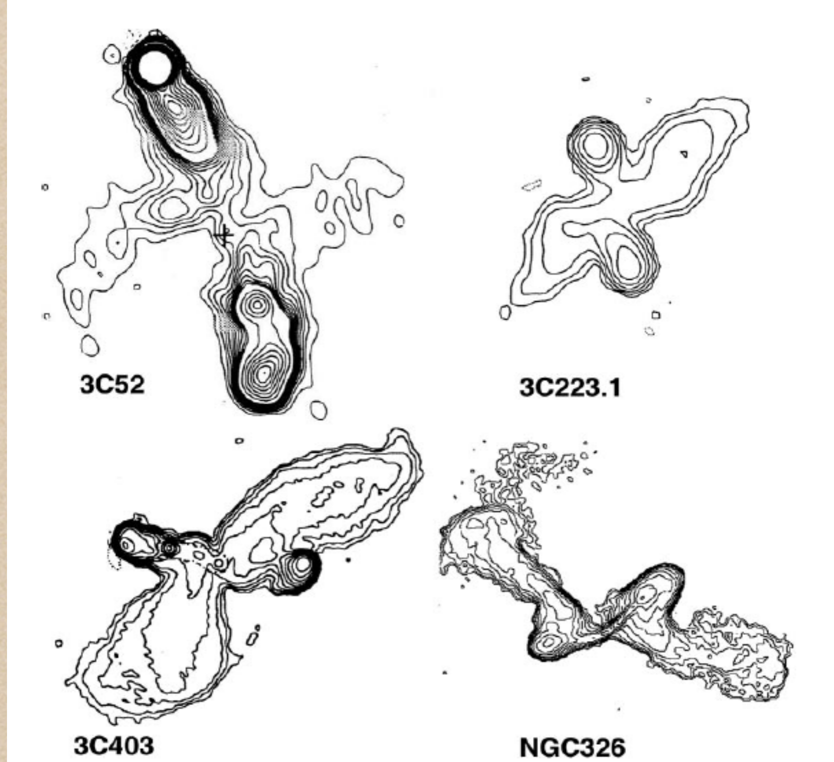
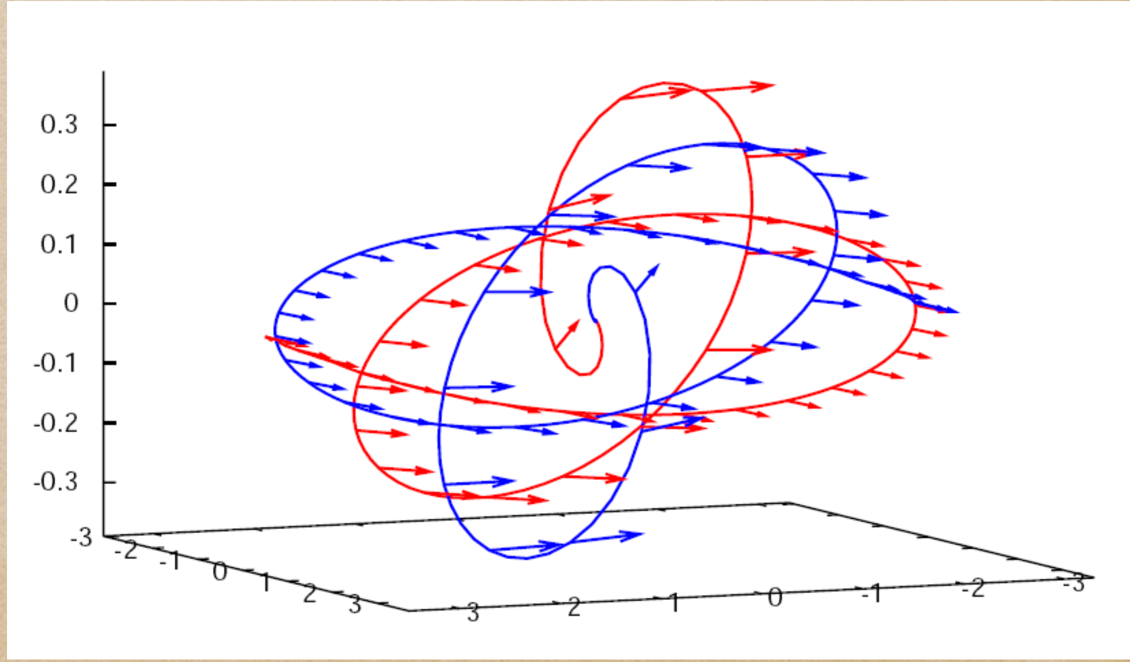


Fig. 2. A composite of four clear examples of the X-shape morphology. VLA radio observations of 3C52 (*28*), 3C223.1 (*29*), 3C403 (*29*), and NGC 326 (*12*).



3.3 High-energy collisions of BHs

The hierarchy problem of physics

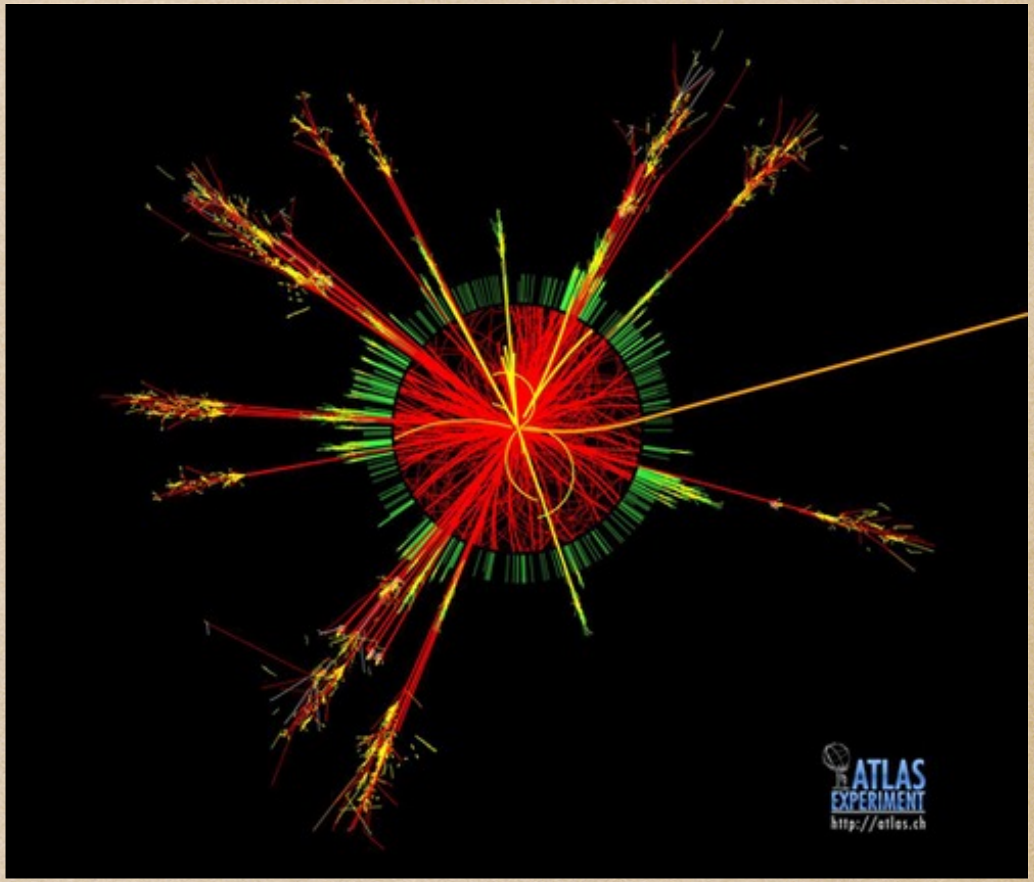
- Gravity $\sim 10^{-39}\times$ other forces
- Gravity not measured below $\sim 0.1\ \mathrm{mm}$! May be diluted due to
  - Large extra dimensions Arkani-Hamed, Dimopoulos & Dvali Phys.Lett.B (1998)
  - Extra dimensions with warp Randall & Sundrum PRL (1999)
- Non-grav. interactions confined to 3+1 brane
- Planck scale may be as low as $\mathcal{O}(\mathrm{TeV})$ instead of $10^{19}\ \mathrm{GeV}$
- BHs may be produced in collision experiments

Eardley & Giddings PRD gr-qc/0201034;

Dimopoulos & Landsberg PRL hep-th/0106295

Stages of BH formation

- Experimental signature:
  - Number of jets, leptons
  - Large transverse energy
- TODO:
  - Cross section for BH formation
  - Energy loss in GWs
  - Spin of formed BHs



- Matter does not matter at energies $\gg E_{\text{Planck}}$

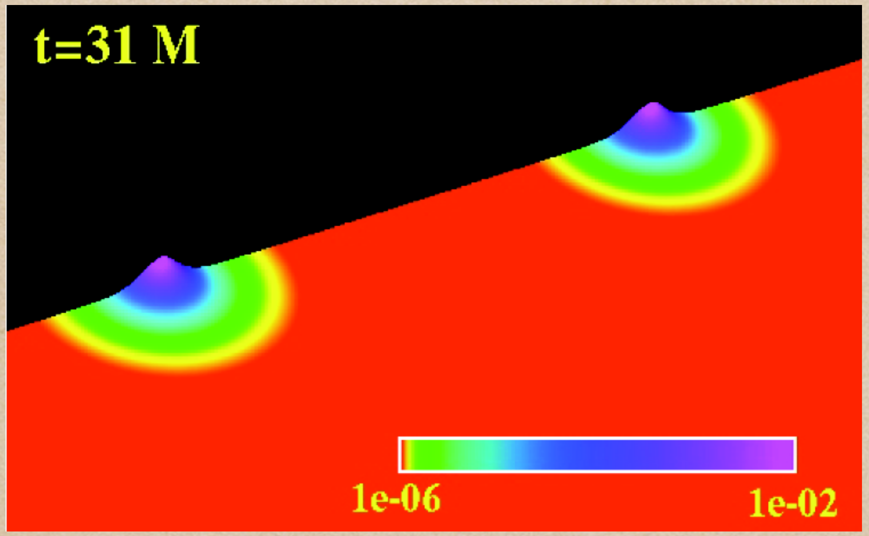
  $\Rightarrow$ model particles by BHs

Banks & Fischler hep-th/9906038; Giddings & Thomas PRD hep-ph/0106219

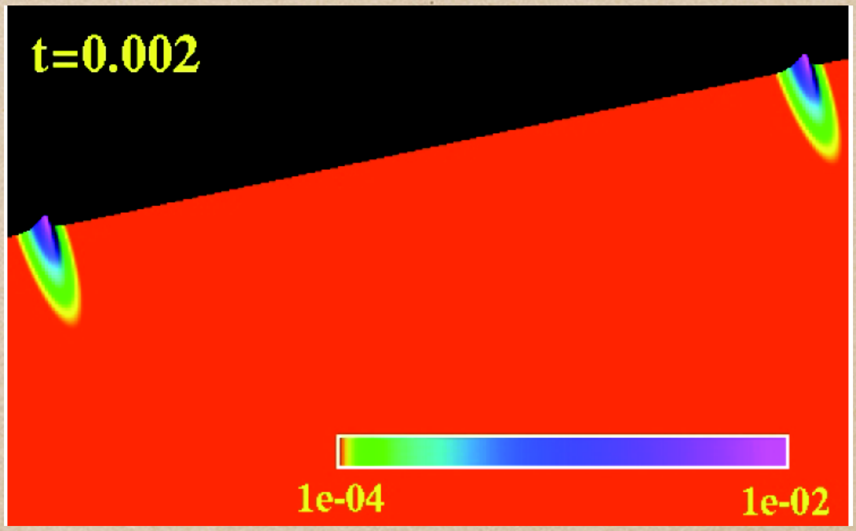
Does matter matter? Collisions of matter balls

- Einstein + minimally coupled, complex scalar field

Choptuik & Pretorius PRL 0908.1780



$\gamma = 1$



$\gamma = 4$

BH formation threshold: $\gamma_{\rm thr} = 2.9 \pm 10\,\% \sim \gamma_{\rm Hoop}/3$

- Einstein + perfect fluid balls

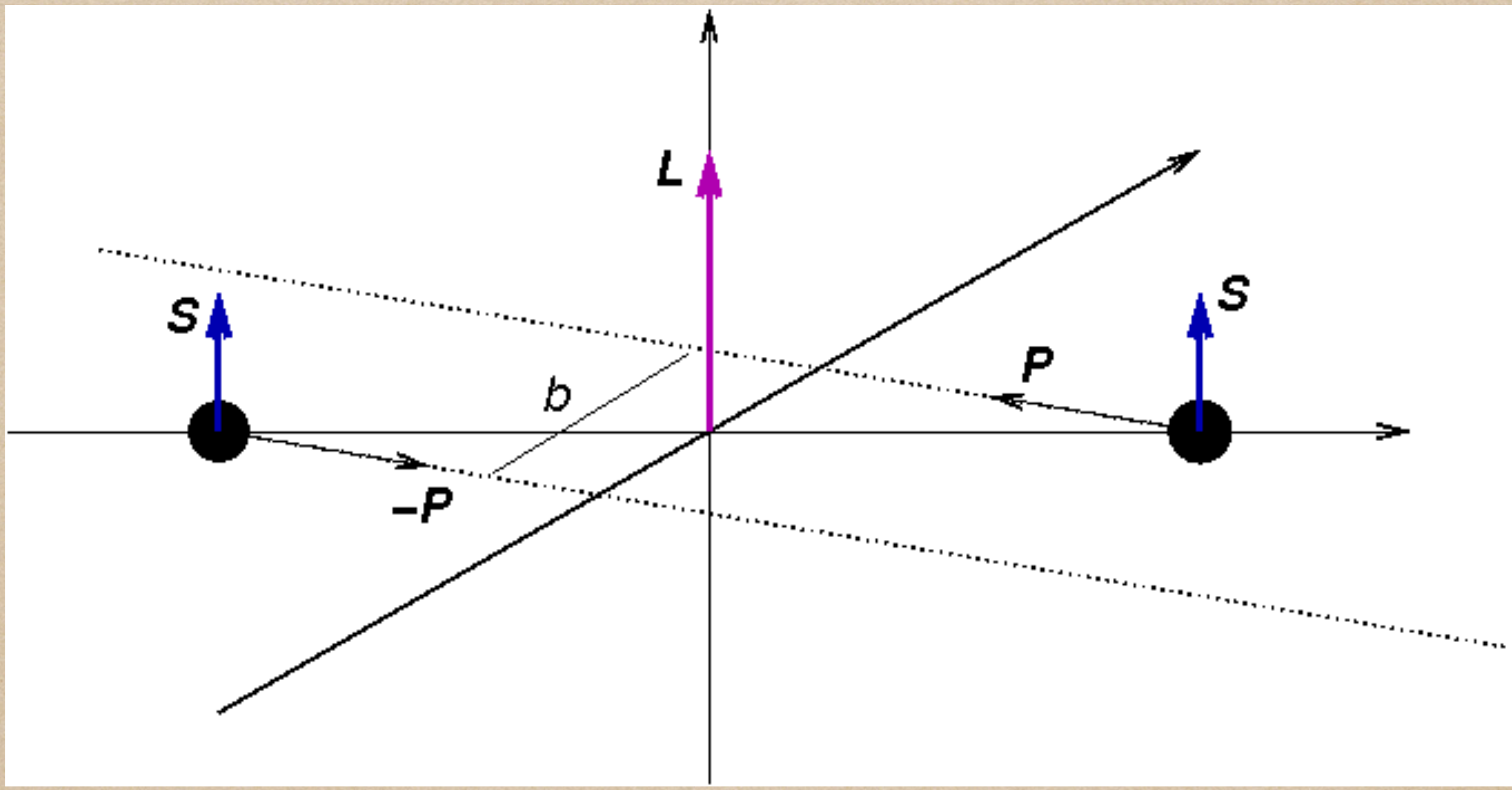
East & Pretorius PRL 1210.0443, Rezzolla & Tanaki CQG 1209.6138

  - BH formation also compatible with Hoop predictions
  - Signature of Type I critical behavior

Collisions of spinning BHs in D=4

- Orbital hang-up: Campanelli et al. PRD (2006)
- Equal-mass BHs, Boost $\gamma = 1/\sqrt{1-v^2}$

  Impact parameter $b = L/P$



- How are scattering threshold and radiated GW energy affected?

Collisions of spinning BHs in D=4

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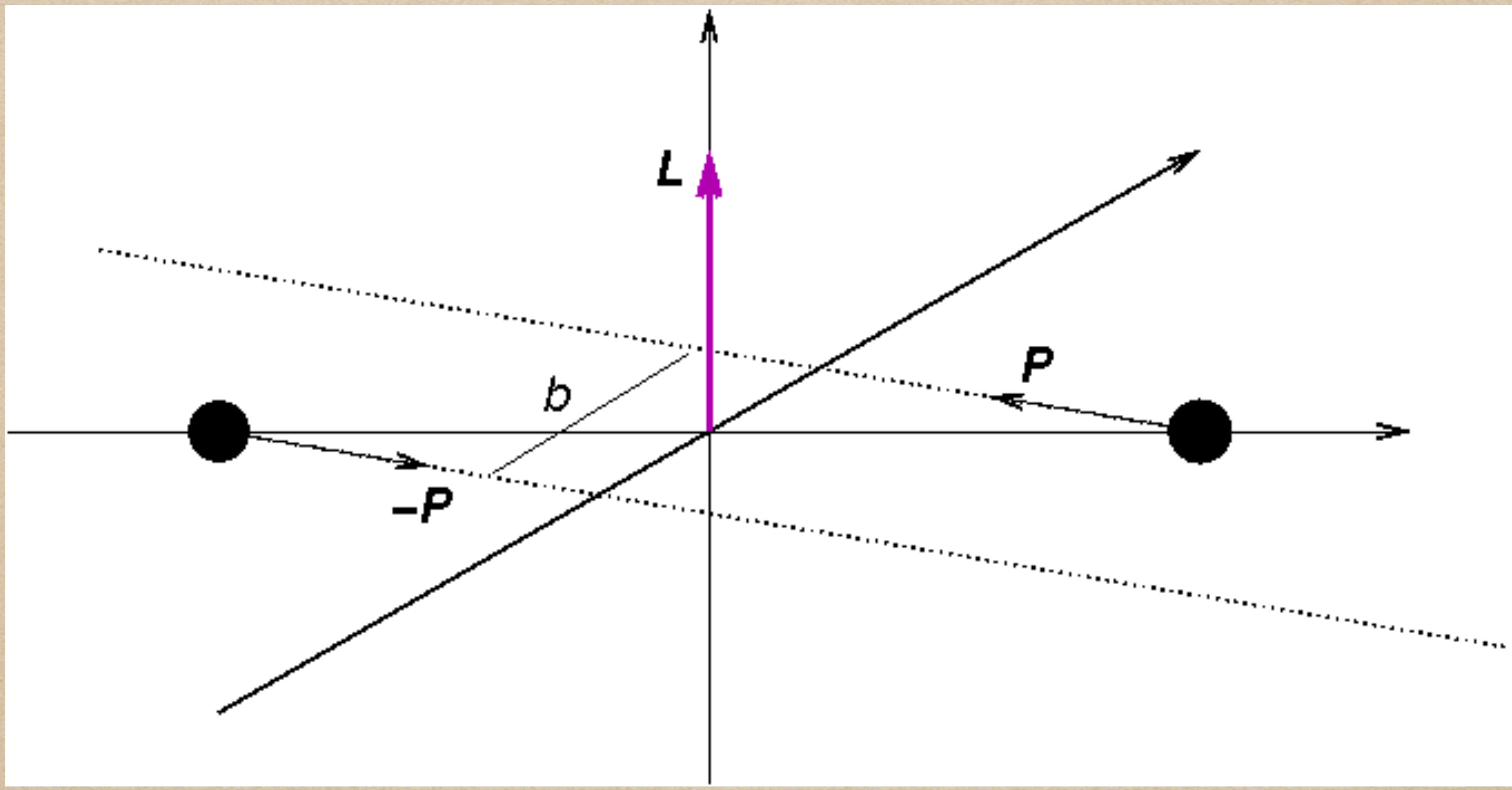

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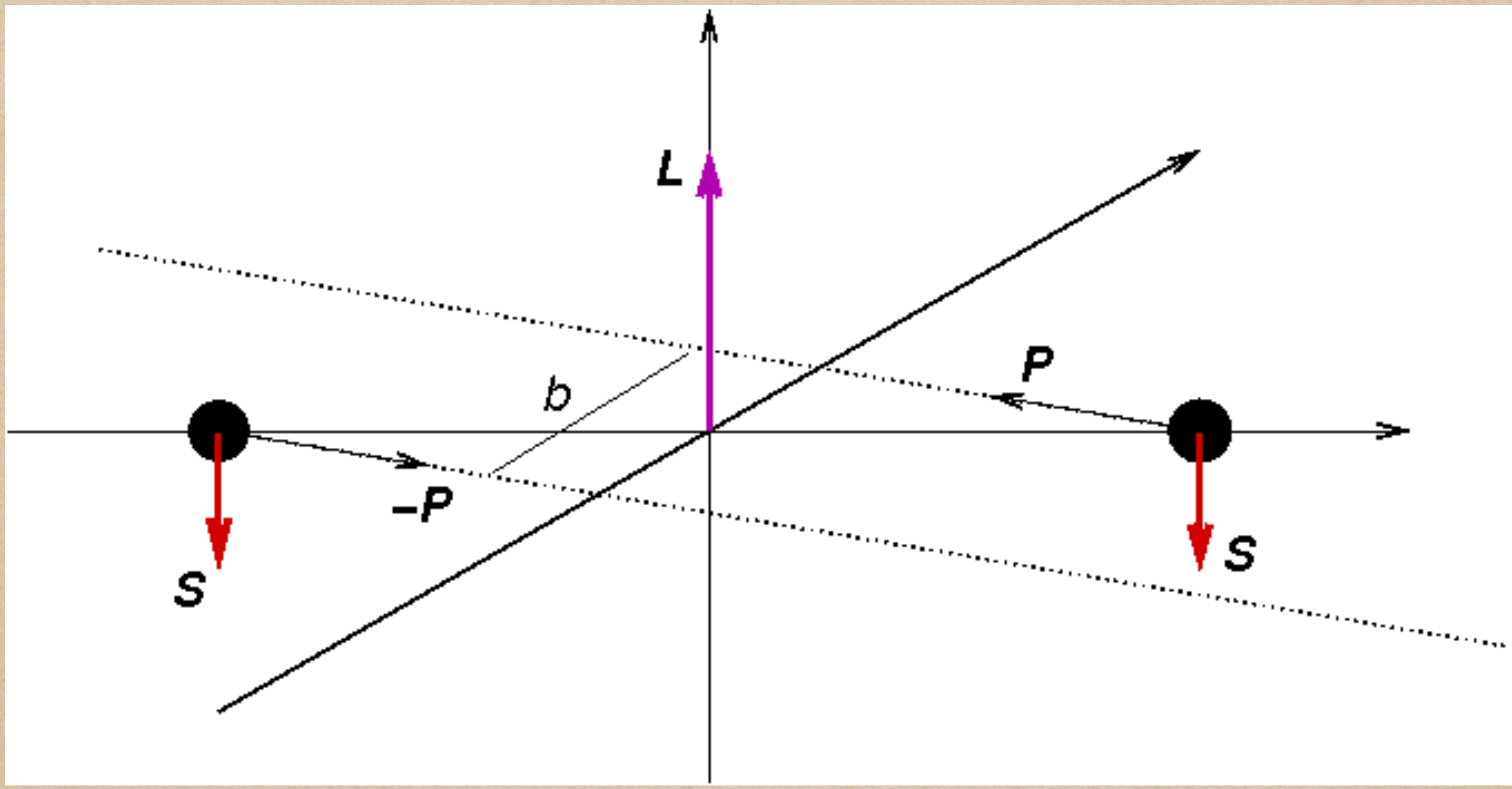
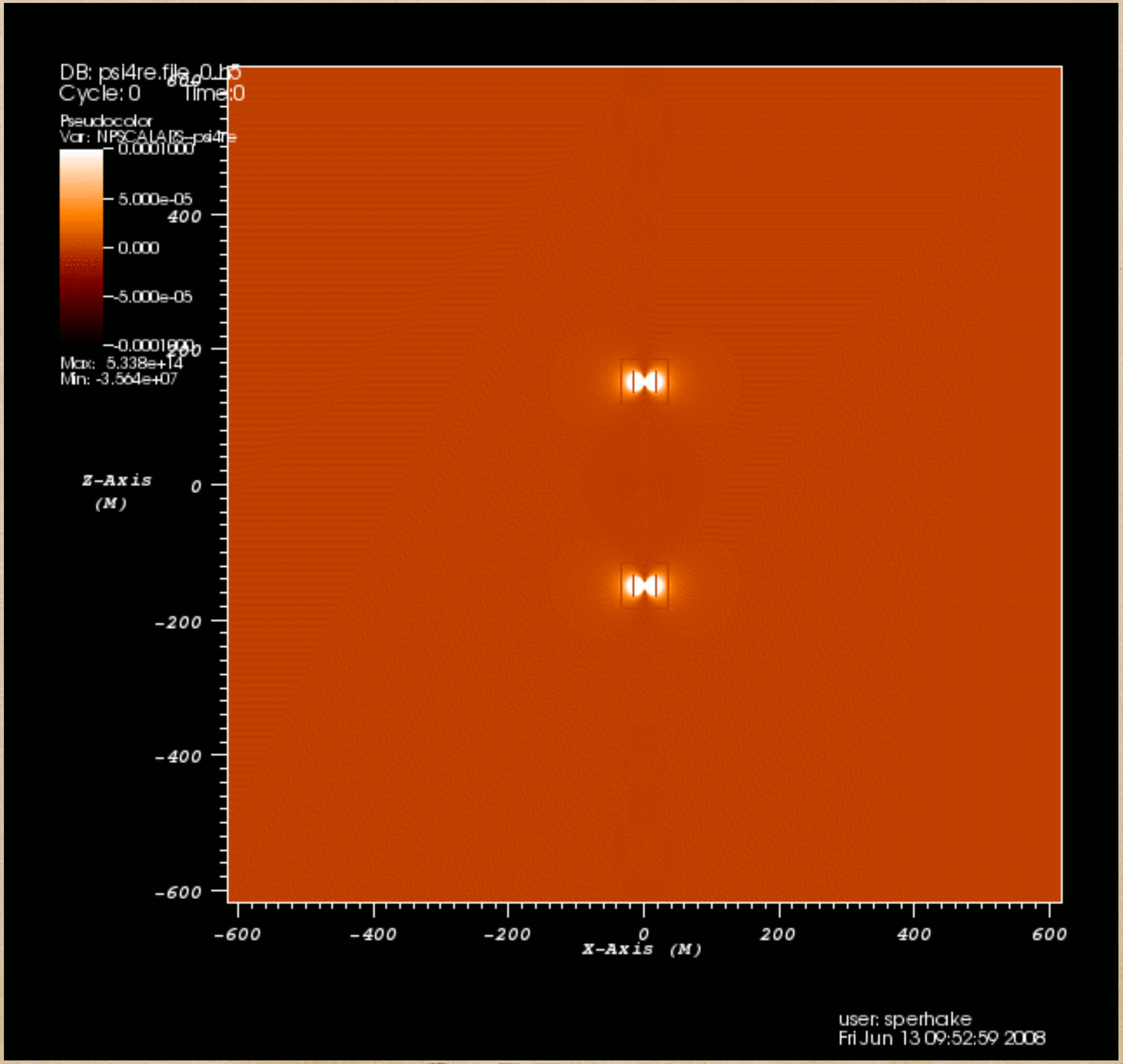
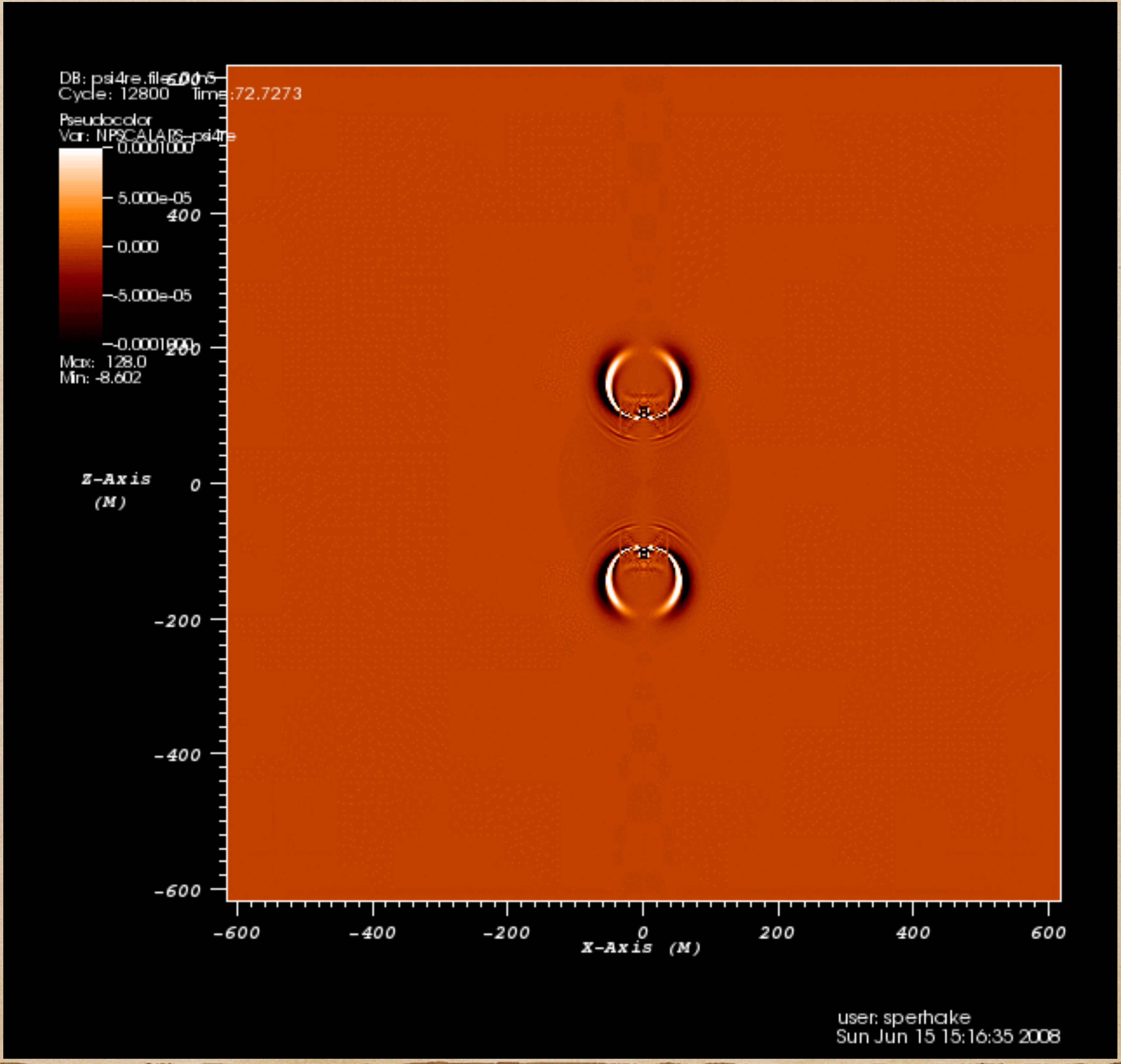


- How are scattering threshold and radiated GW energy affected?

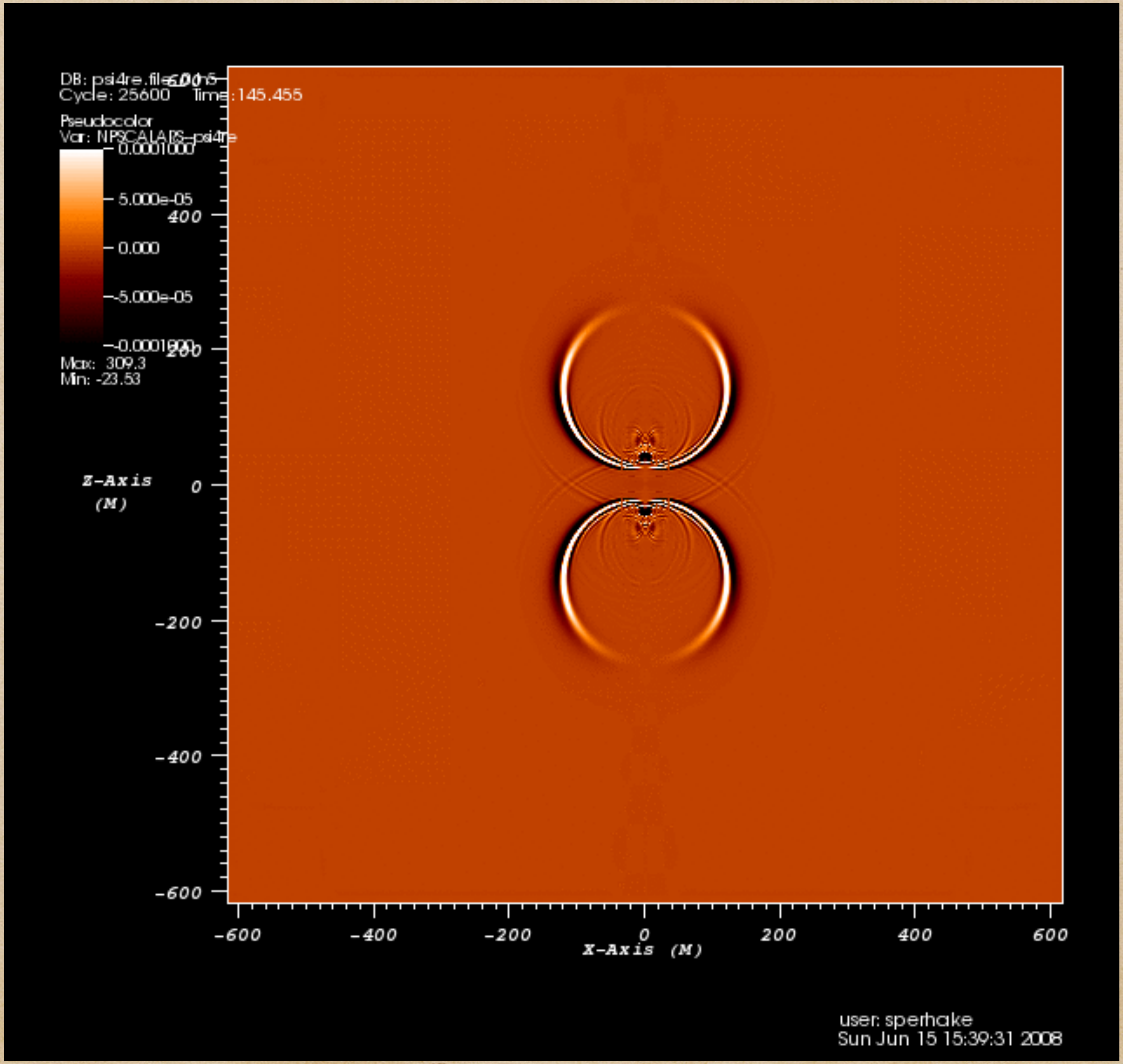
D=4 head-on collisions: $b=0\,,\quad \vec{S}=0\,,\quad \gamma=1.52$



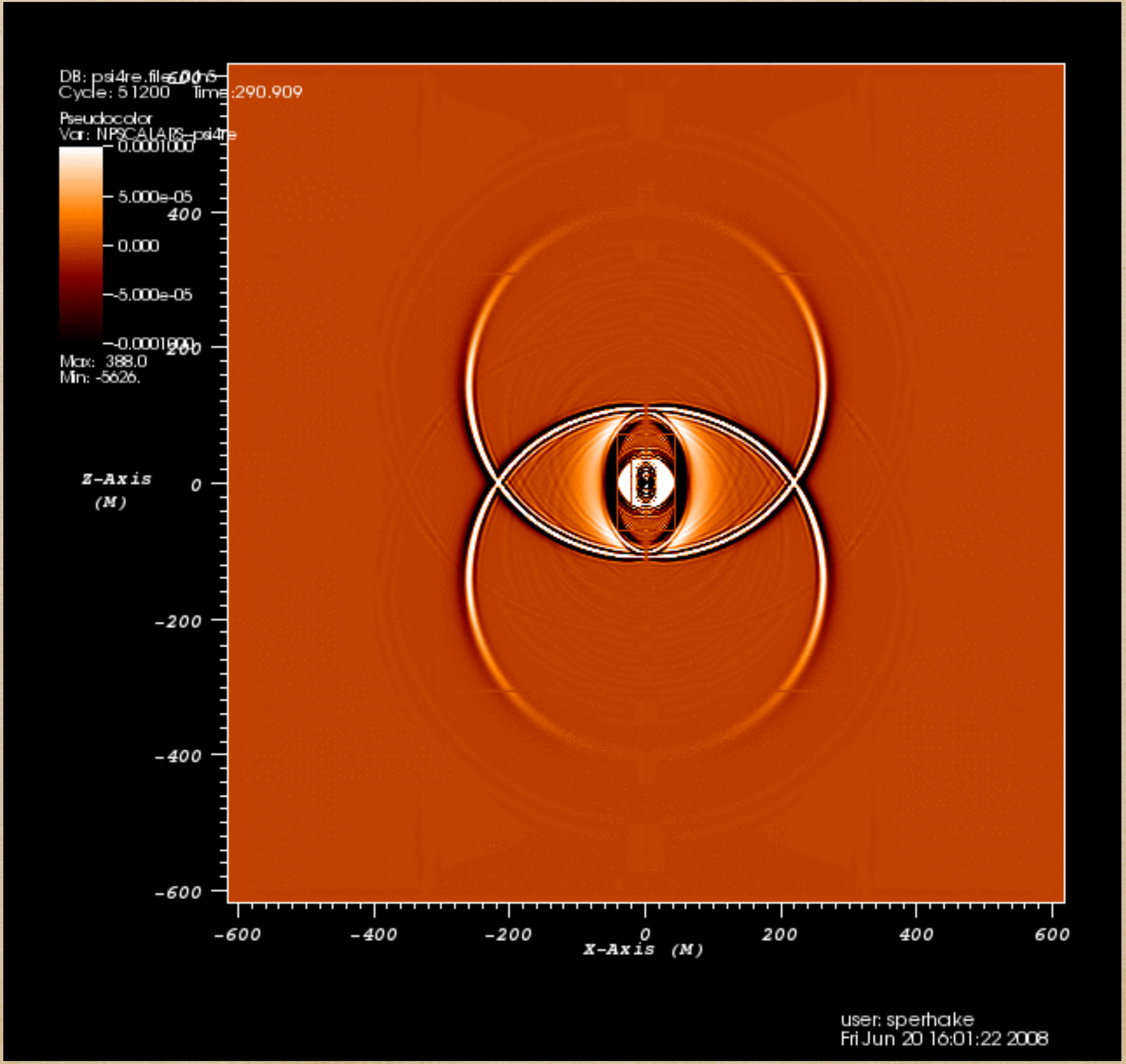
D=4 head-on collisions: $b = 0\,,\quad \vec{S} = 0\,,\quad \gamma = 1.52$



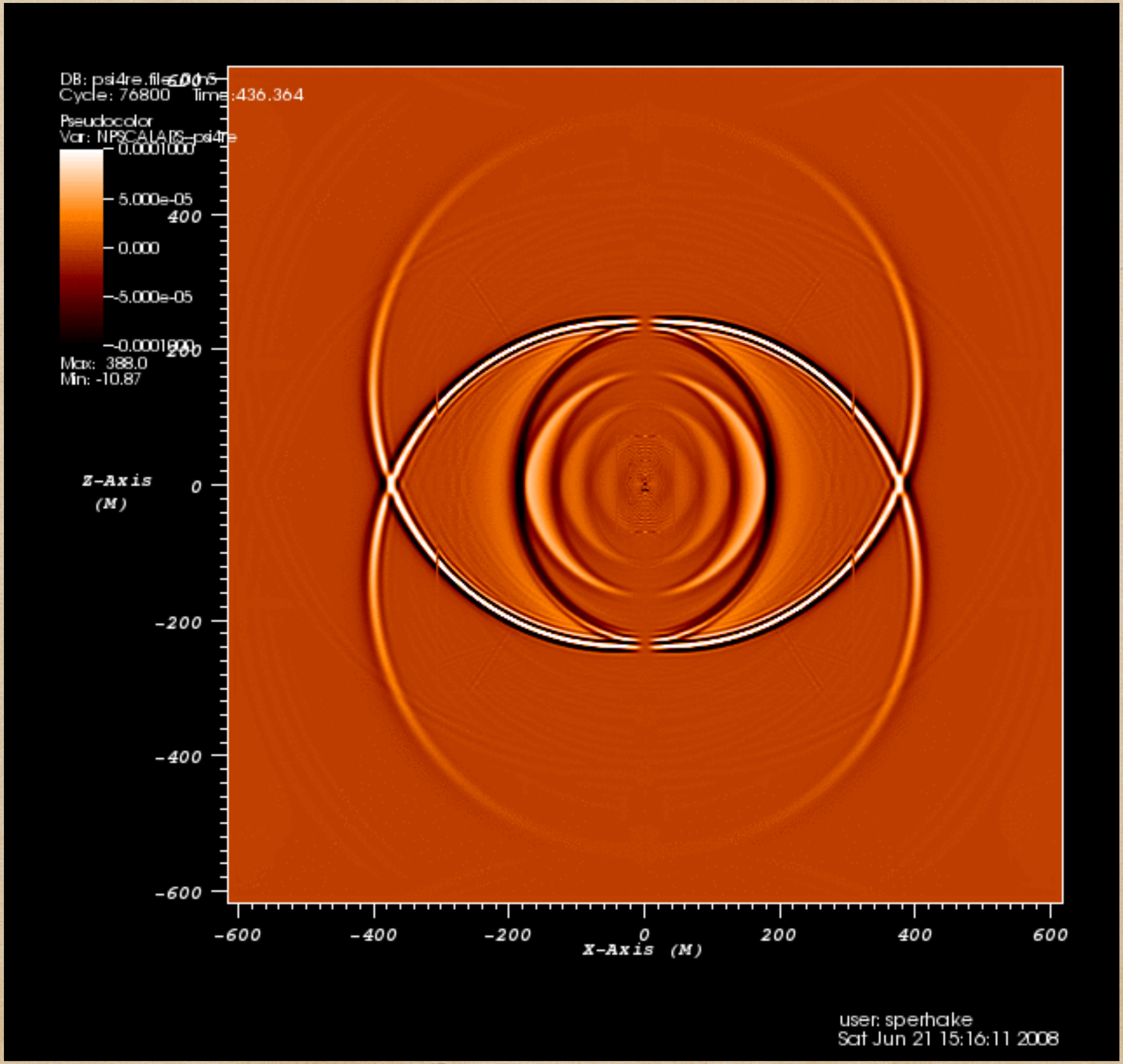
D=4 head-on collisions: $b = 0\,,\quad \vec{S} = 0\,,\quad \gamma = 1.52$



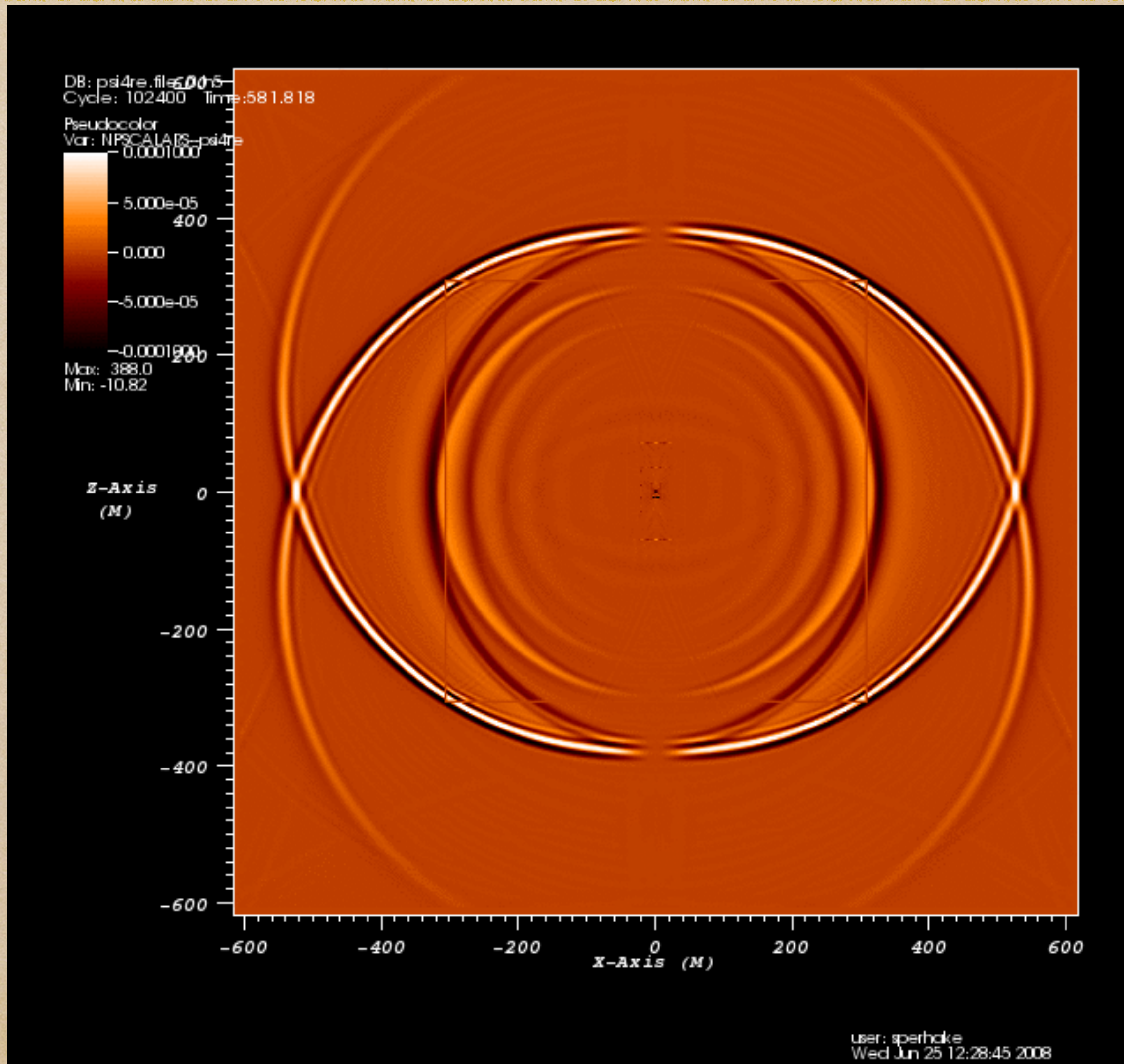
D=4 head-on collisions: $b = 0\,, \quad \vec{S} = 0\,, \quad \gamma = 1.52$



D=4 head-on collisions: $b = 0\,,\quad \vec{S} = 0\,,\quad \gamma = 1.52$

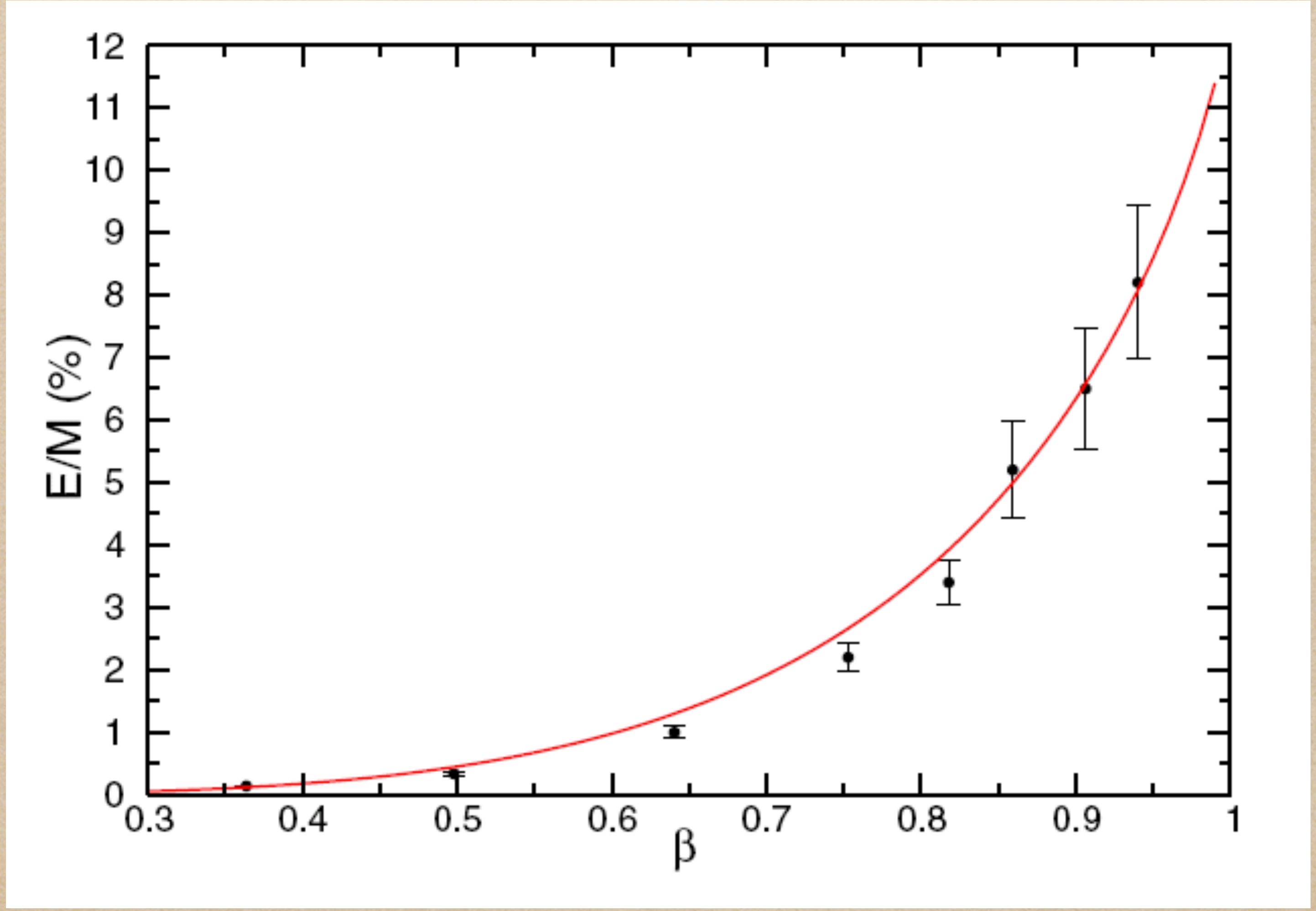


D=4 head-on collisions: $b = 0\,,\quad \vec{S} = 0\,,\quad \gamma = 1.52$



Boosted BH head-on collisions in D=4

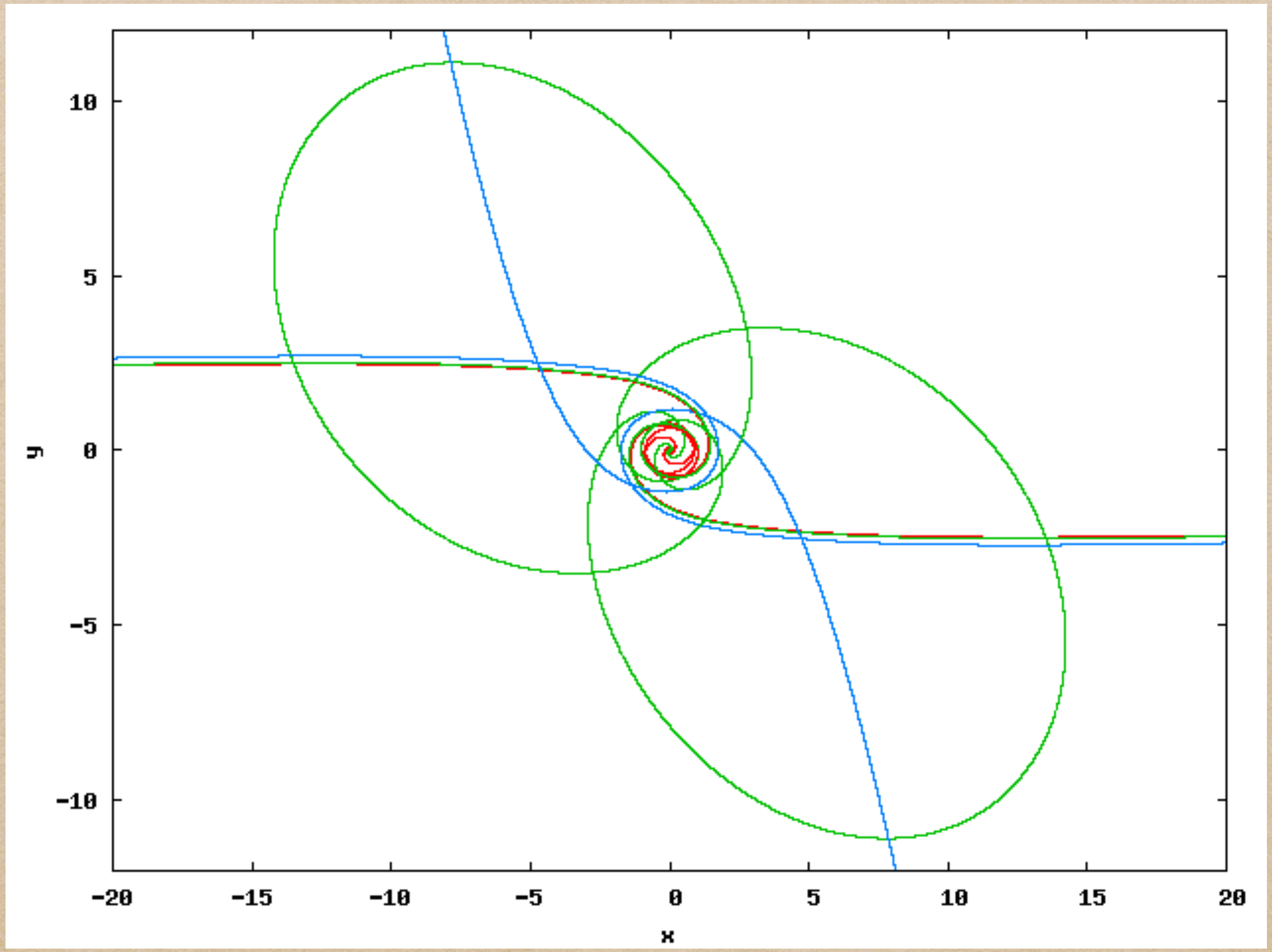
- BSSN, Cactus, Carpet, Moving Puncture, TwoPunctures, AHFinderDirect
- Equal-mass BHs, no spin $\lim_{\beta \to 1} E_{\rm rad} = 14 \pm 3\ \%$
- Agrees well with perturbative studies Berti et al PRD 1003.0812



Sperhake et al PRL 0806.1738; Healy et al 1506.06153

D=4 grazing collisions: $b = 0\,, \quad \vec{S} = 0\,, \quad \gamma = 1.52$

- Radiated energy up to at least $\approx 35\ \%\ M$
- Immediate vs. Delayed vs. No merger



Sperhake et al PRL 0907.1252

Scattering threshold

- $b < b_{\text{scat}} \quad \Rightarrow \quad$ Merger

  $b > b_{\text{scat}} \quad \Rightarrow \quad$ Scattering

- Numerical study: $b_{\text{scat}} = \dfrac{2.5 \pm 0.05}{v} M$

  Shibata et al PRD 0810.4735

- Limit from Penrose construction: $b_{\text{scat}} = 1.685\ M$

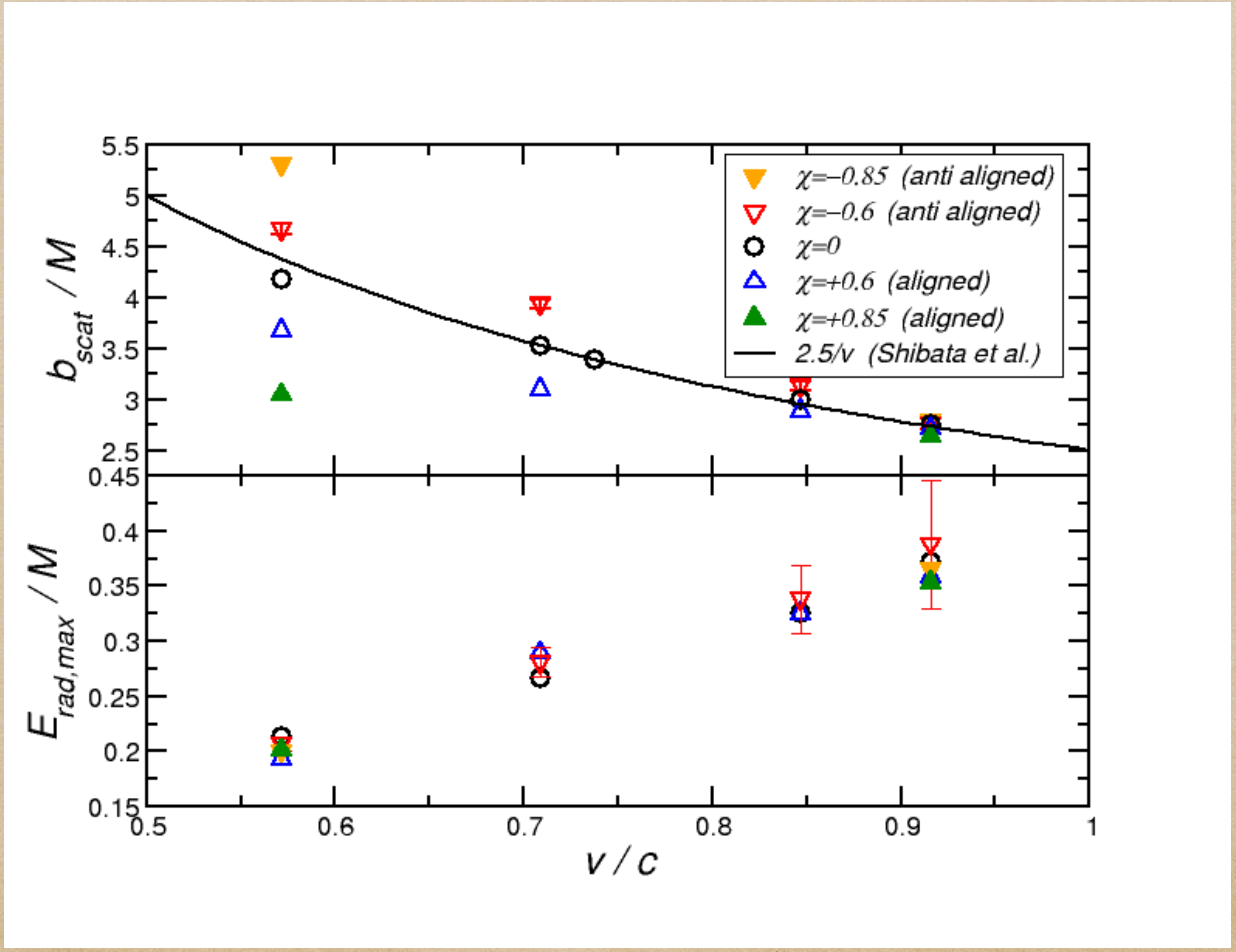
  Yoshino & Rychkov PRD hep-th/0503171

- Impact of structure of the colliding BHs?

  $\rightarrow$ Collide spinning BHs

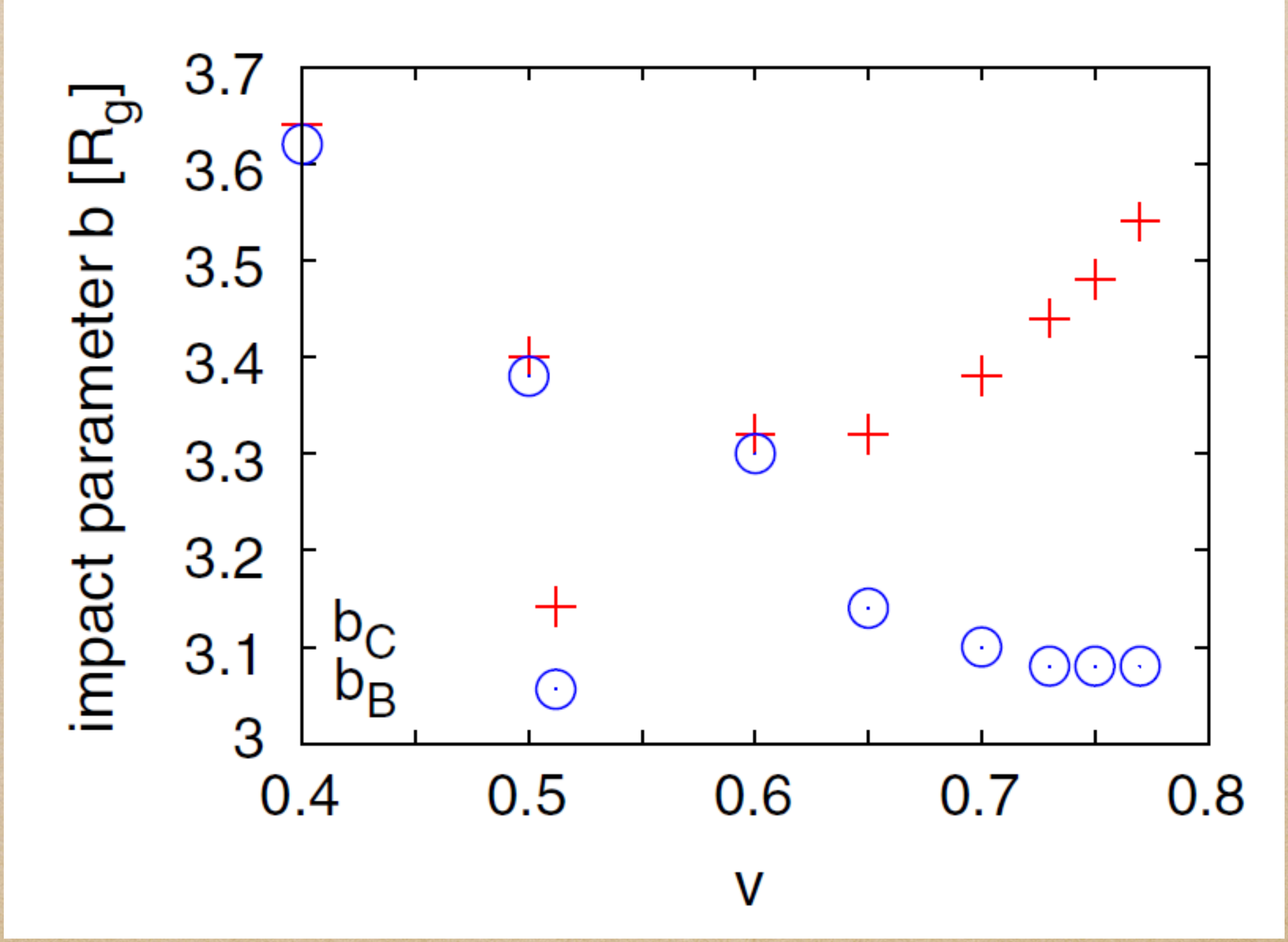
Grazing collisions in D=4

- Spins: aligned, zero, anti aligned Sperhake et al PRL 1211.6114
- $b_{\rm scat}$, $E_{\rm rad}$: spin effects washed out as $v \to c$



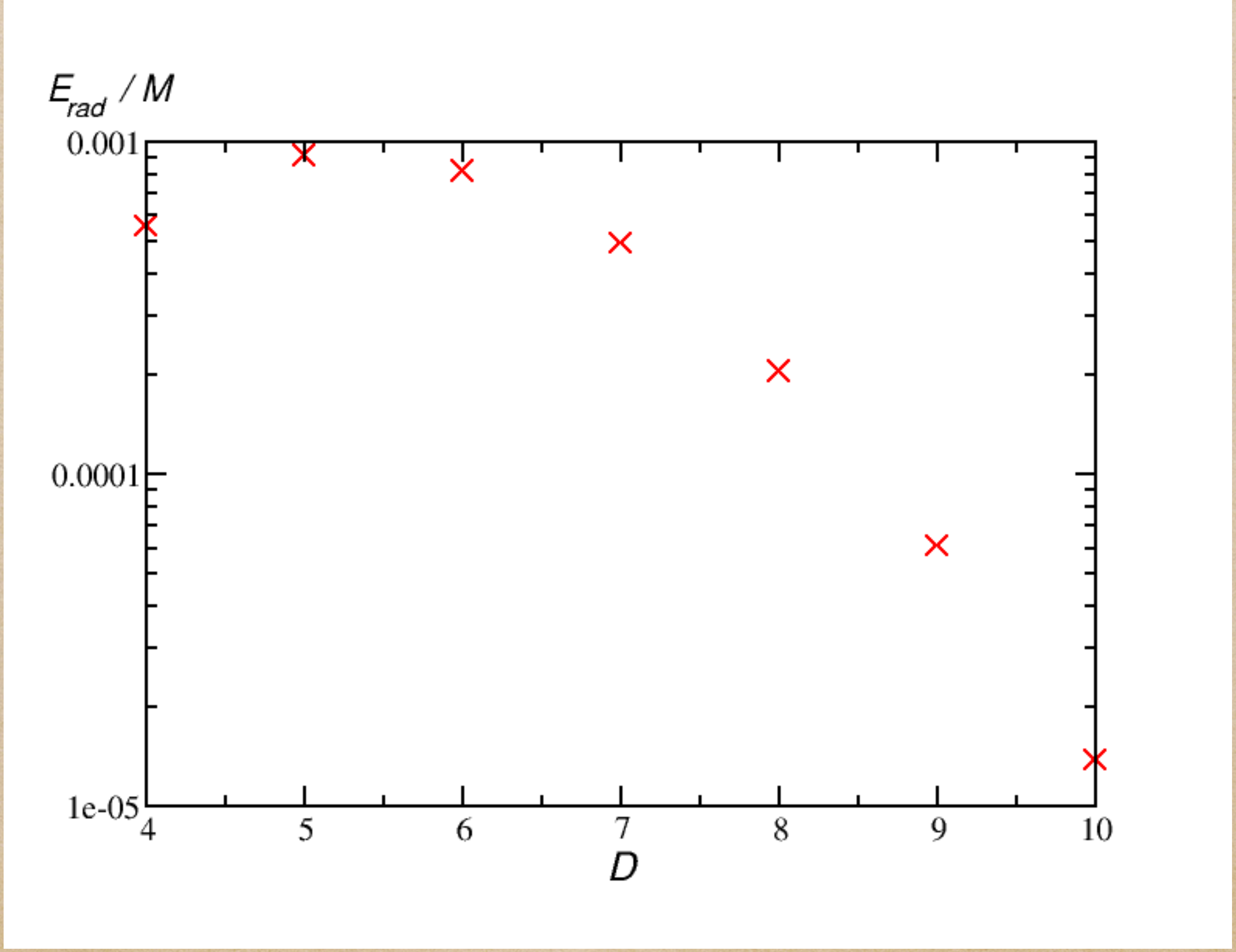
D=5: Scattering threshold

- Cartoon method   Okawa et al PRD 1105.3331
- Numerical stability still an issue



Radiated energy in D>4

- Head-on, non-spinning, from rest: $b=0\,,\quad \vec{S}=0\,,\quad \gamma=1$
- Modified Cartoon Cook et al (in preparation)

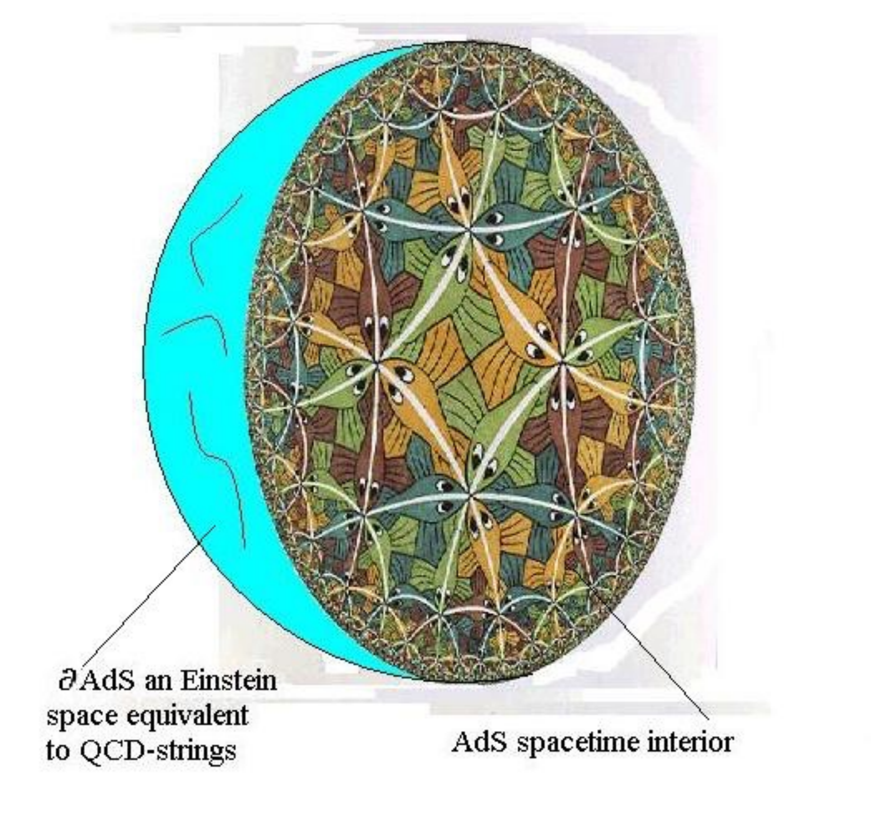


3.4 BH Holography

Large N and holography

- Holography
  - BH entropy $\propto A_{\rm hor}$
  - For a local Field theory: entropy $\propto V$
  - Gravity in D dims.

    $\Rightarrow$ local FT in $D-1$ dims.



- Best understood for:

  $\mathcal{N}=4$ super Young Mills theory (cousin of QCD)

  equivalent to $D=5$ Anti-de Sitter.

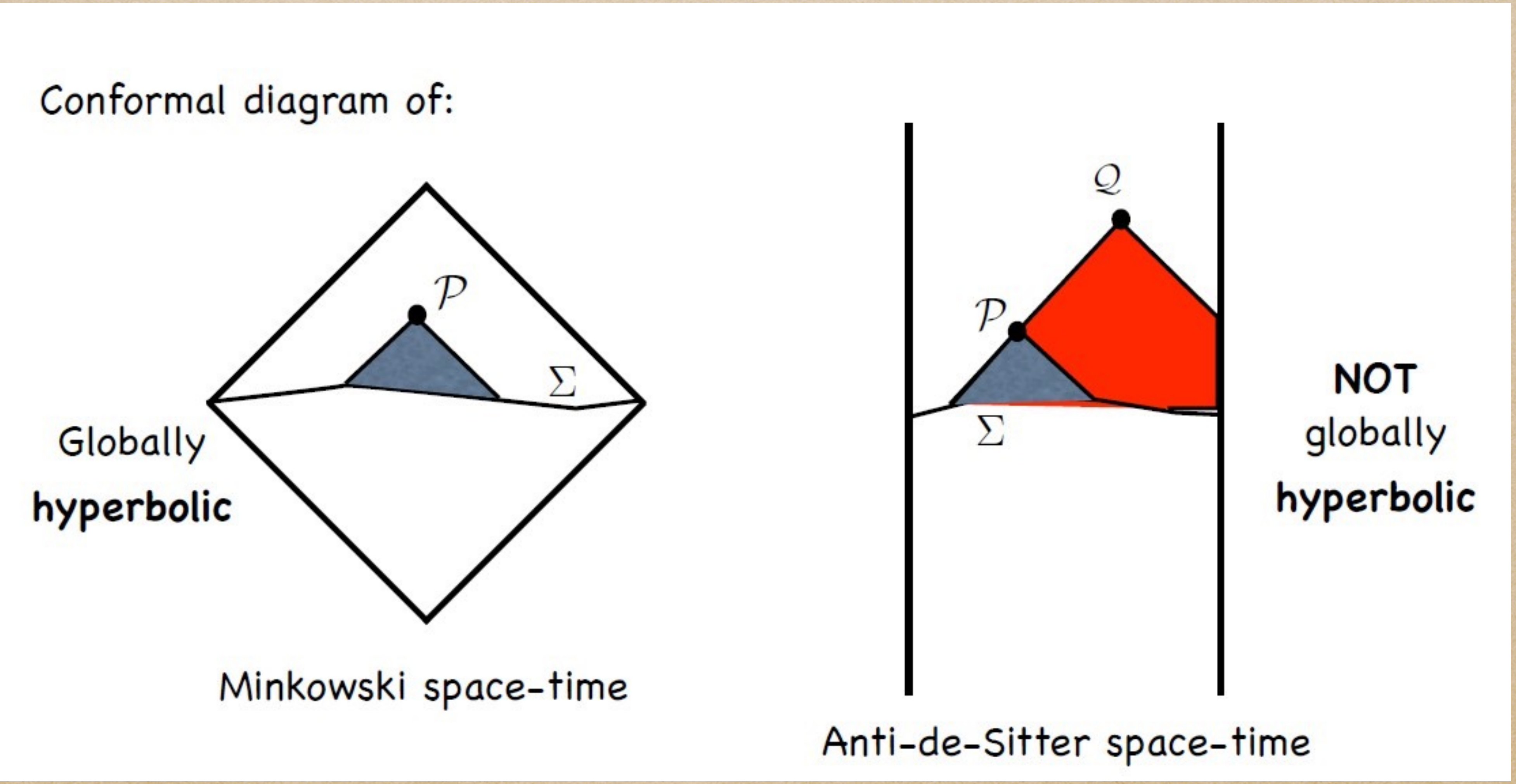
  "AdS/CFT" correspondence Maldacena Adv.Th.Math.Ph. hep-th/9711200

- E.g. Stationary AdS BH $\Leftrightarrow$ Thermal equilibrium with $T_{\rm Haw}$ in FT

  Witten Adv.Th.Math.Ph. hep-th/9803131

The boundary in AdS

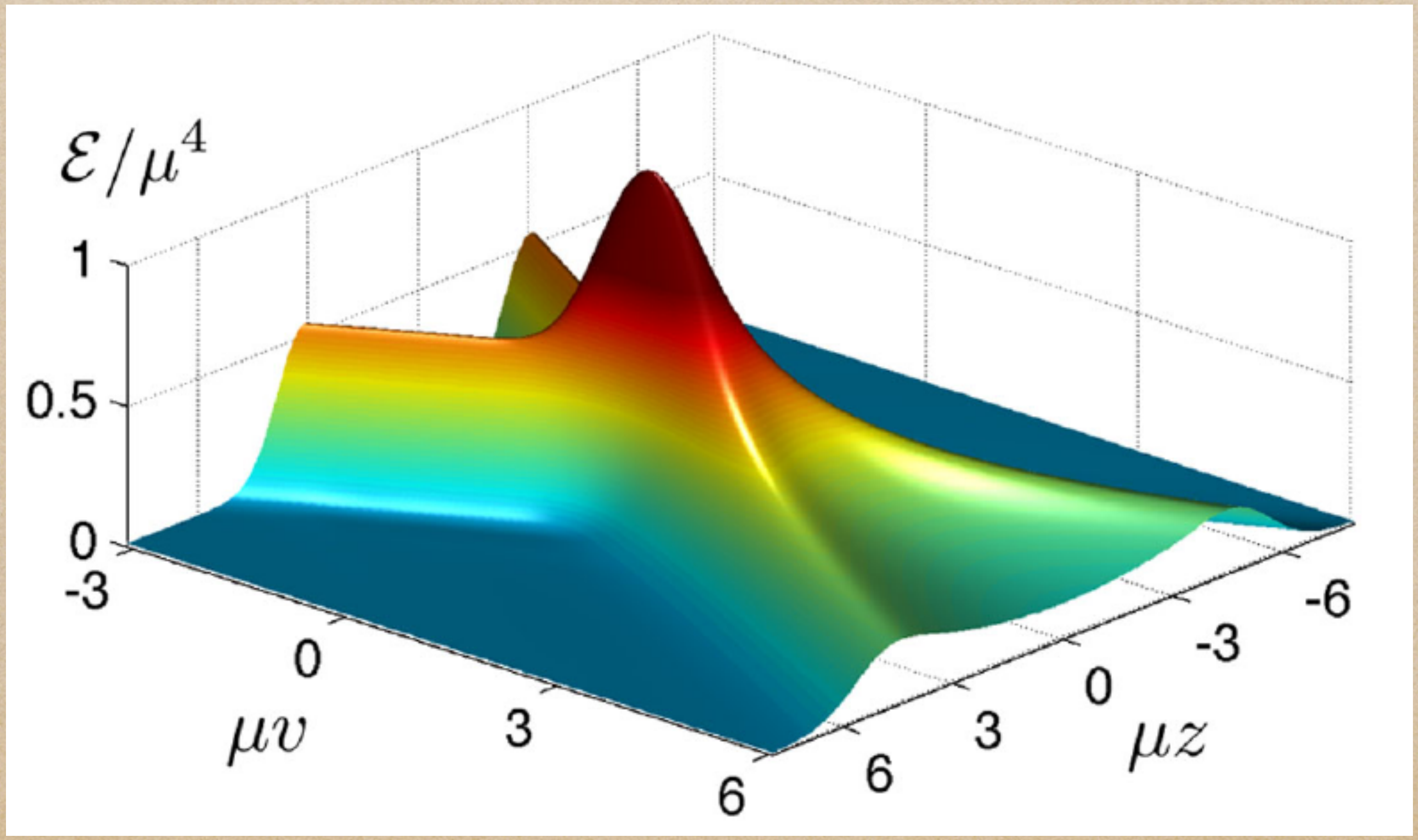
- $\exists$ Dictionary between metric properties and vacuum expectation values of CFT operators.
  E.g. $T_{\alpha\beta}$ operator of CFT $\leftrightarrow$ transverse metric on AdS boundary
- The boundary plays an active role in AdS! Metric singular!



Collision of planar shock waves in $\mathcal{N} = 4$ SYM

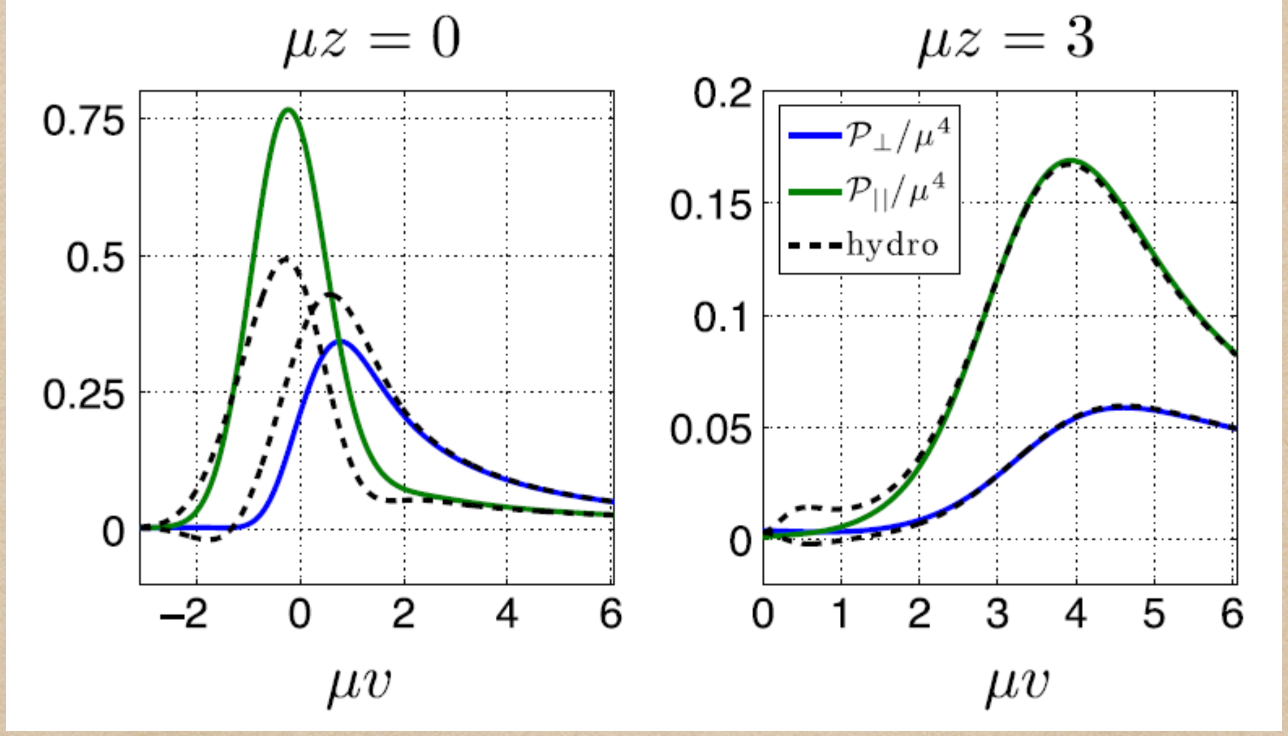
- Dual to colliding shock waves in asymptotically AdS
- Characteristic formalism with translational invariance

  Chesler & Yaffe 0812.2053 0906.4426 1011.3562 1506.02209
- Initial data: 2 superposed shockwaves



Collision of planar shock waves in $\mathcal{N}=4$ SYM

- Initially: System far from equilibrium
- Isotropization after $\Delta v \sim 4/\mu \sim 0.35 \text{ fm}/c$
- Hydro sims. of Quark-Gluon Plasma: $\sim 1 \text{ fm}/c$ Heinz nucl-th/0407067



- Non-linear vs. linear Einstein eqs. agree within ~ 20 %

  Heller PRL 1202.0981
- Thermalization in ADM formalism: Heller PRL 1203.0755

Cauchy evolutions in 4+1 asympt. AdS

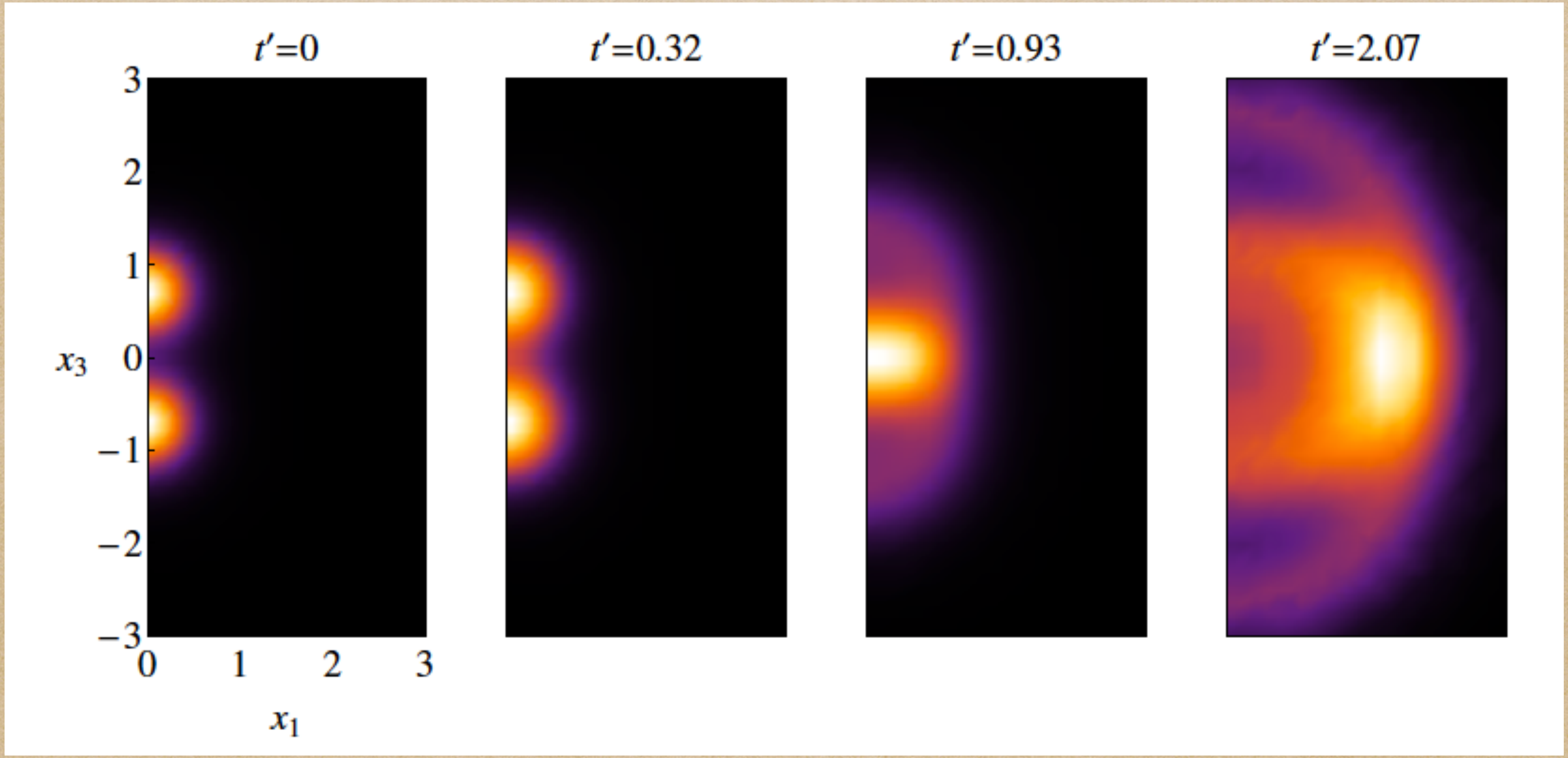
- Characteristic coordinates successful numerical tool in AdS/CFT
- But: Restricted to symmetries, caustic problem ...
- Cauchy evolution needed for general configs.? Cf. BBH inspiral!
- Cauchy scheme based on generalized harmonic formulation

  Bantilan et al PRD 1201.2132, PRL 1410.4799

  - $SO(3)$ Symmetry
  - Compactify bulk radius
  - Decompose metric into AdS_5 piece + deviations
  - Gauge must preserve asymptotic fall-off!

Cauchy evolutions in 4+1 asympt. AdS

- First BBH collision in asymptotically AdS
- Qualitative picture: similar to shock wave collisions
- Future goals: Relax symmetry, use BBHs with boost



3.5 Fundamental properties of BHs

Critical collapse & stability of AdS

- $m = 0$ scalar field, spherical symmetry, asymptotically flat

  $p > p^* \Rightarrow$ BH ; $p < p^* \Rightarrow$ flat Choptuik PRL (1993)

- $m = 0$ scalar field, spherical symmetry, asympt. AdS

  Bizon & Rostworowski PRL 1104.3702

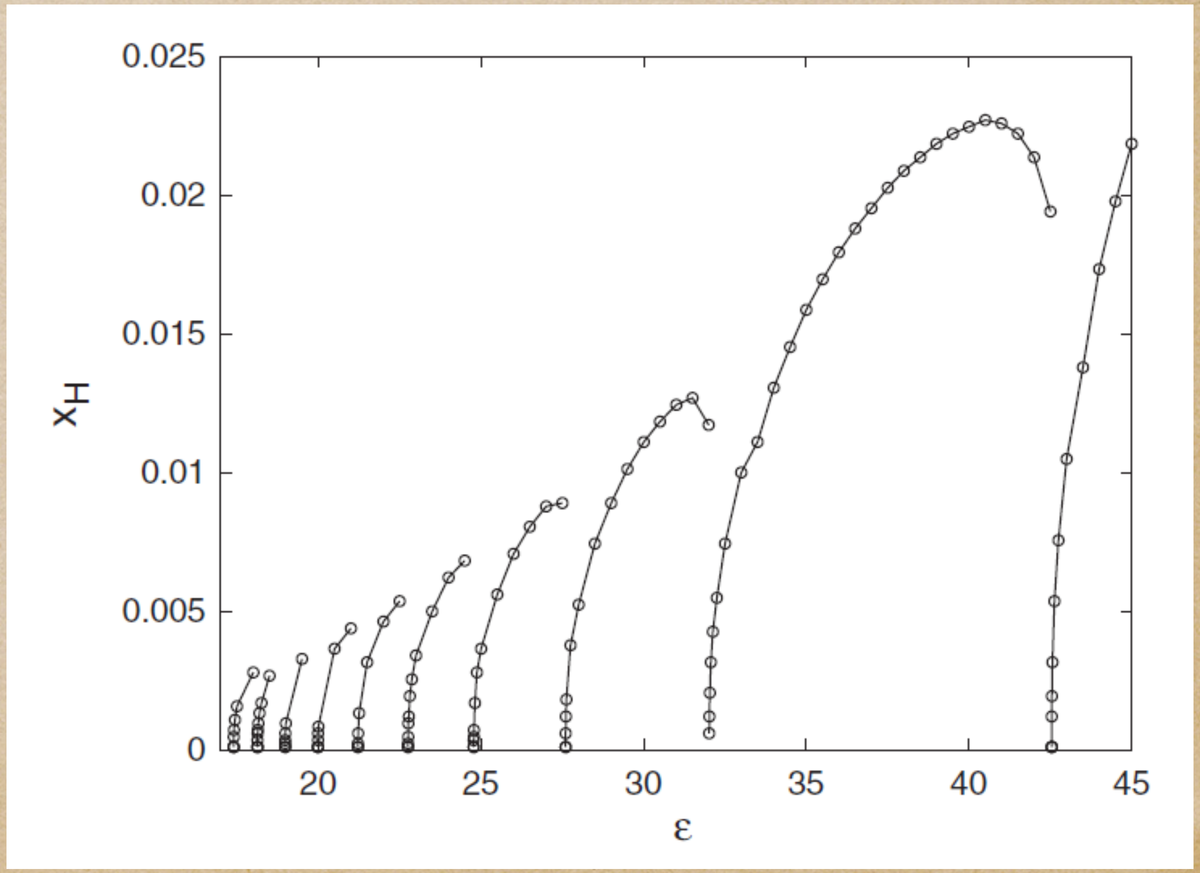
- BH always forms; pulse reflected off outer boundary

- Similar behaviour for eons

  Dias et al CQG 1109.1825

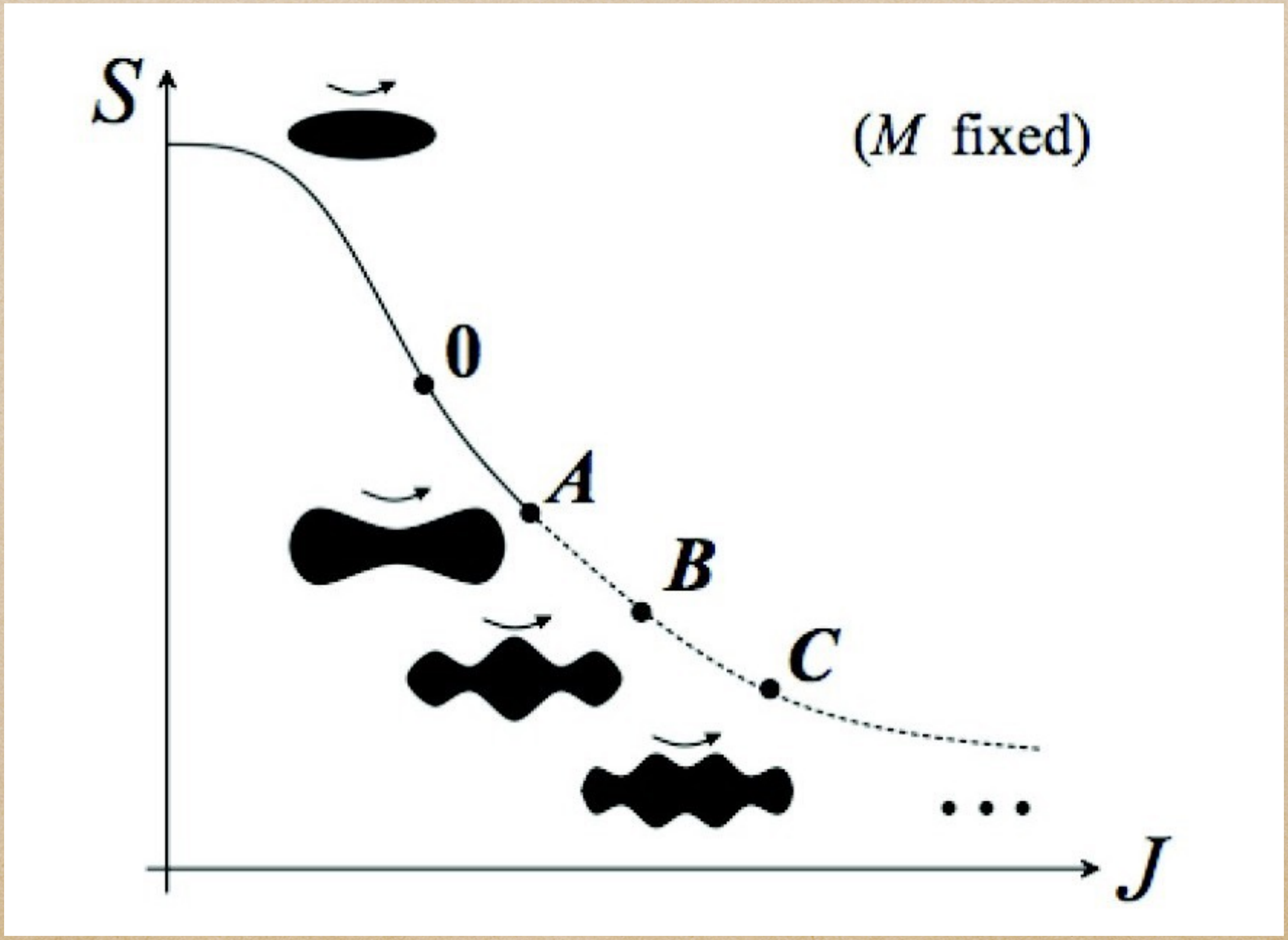
- Same in $D > 4$ dimensions

  Jalmuszna et al PRD 1108.4539



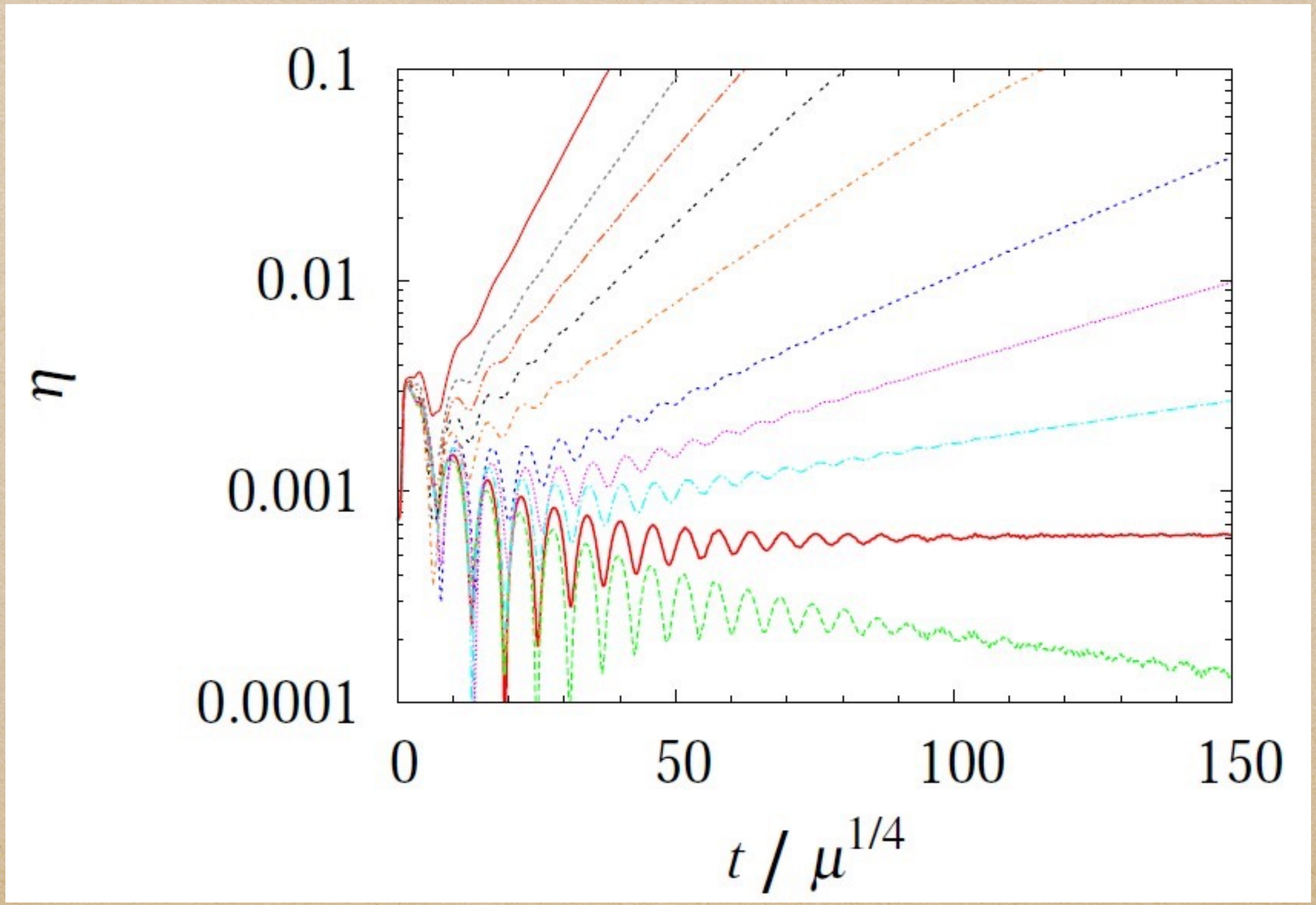
Bar mode instability of Myers Perry BHs

- Rotating BHs in $D > 5$ should be unstable if ang.mom. large
- Linearized study   Dias et al PRD 0907.2248



Bar mode instability of Myers Perry BHs

- Myers Perry metric; transformed to puncture coords.
- Add small bar-mode perturbation
- Monitor deformation η



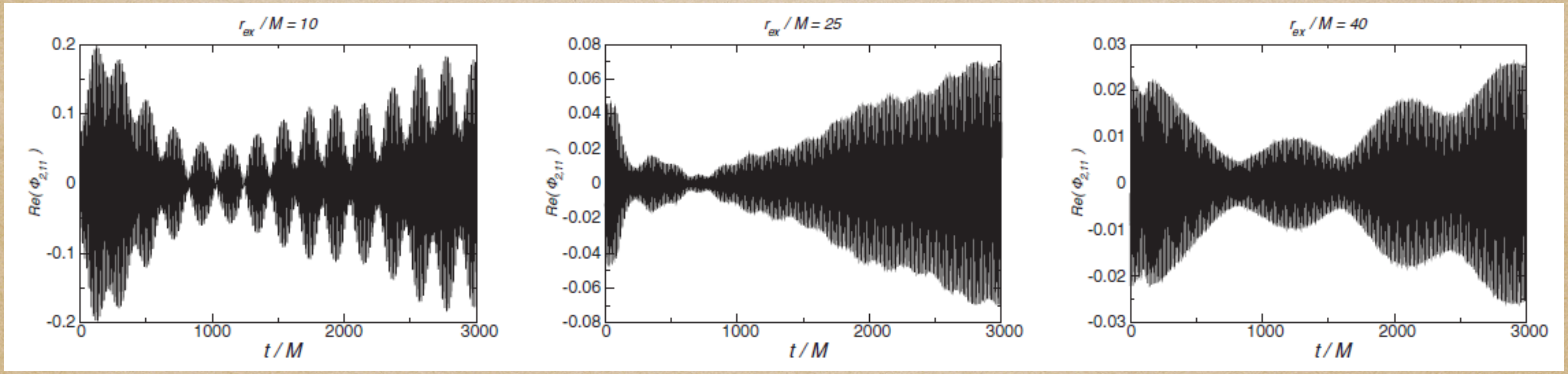
Superradiant instability

- Scattering of waves with $\mathrm{Re}[\omega]$ off BH with ang. horizon velocity Ω_h

  $\Rightarrow$ amplification $\Leftrightarrow \mathrm{Re}[\omega] < m\Omega_h$

- Measure photon mass? Pani et al PRL 1209.0465
- Numerical simulations: Dolan PRD 1212.1477;
  Witek et al PRD 1212.0551; Zilhão et al CQG 1505.00797

- Instability of spinning hairy BHs, beating effects



Witek et al PRD 1212.0551

Cosmic censorship in D=5

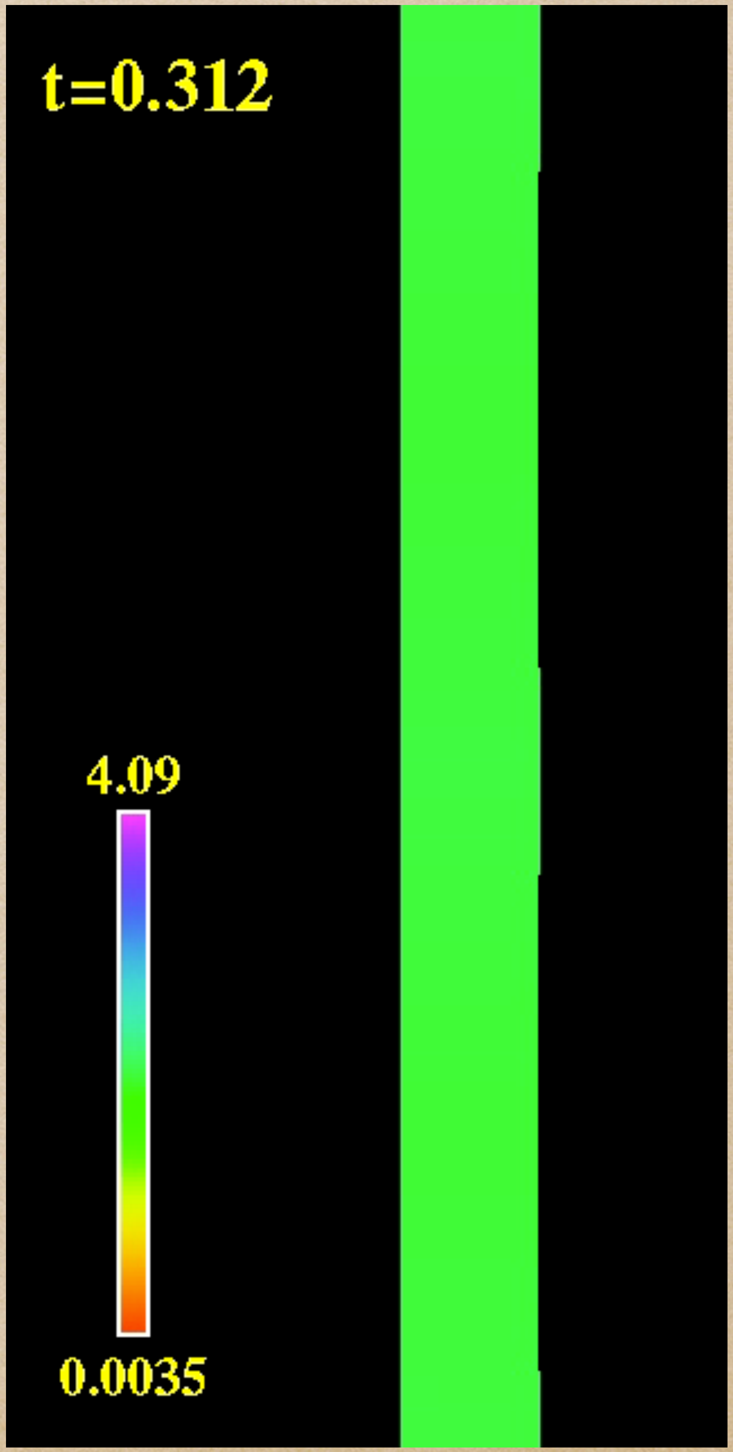
Lehner & Pretorius PRL 1006.5960

- Axisymmetric code
- Evolution of black string...
- Gregory-Laflamme instability;

  cascades down in finite time

  until string has zero width

  $\Rightarrow$ Naked singularity
- Note: spacetime not asympt.flat!



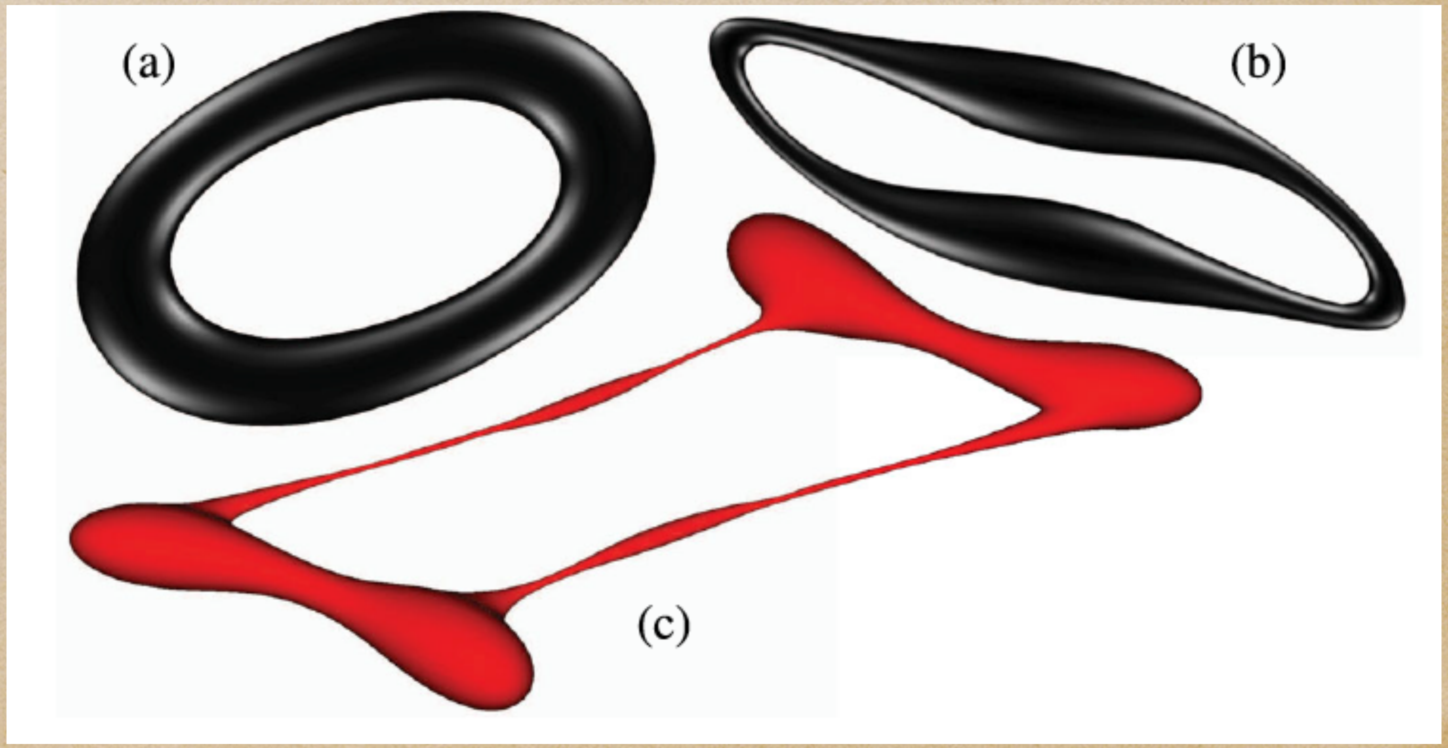
Cosmic censorship in D=5

Figueras, Kunesch & Tunyasuvunakool PRL 1512.04532

- 3+1 code with modified cartoon for 5th dimension
- Conformal Z4 system
- Black ring: assympt.flat!
- Gregory-Laflamme instability

  develops for thin ring

  $\Rightarrow$ Violation of CC!



Further reading

- Reviews of numerical relativity

Centrella et al Rev.Mod.Phys. 1010.5260

Pretorius 0710.1338

Sperhake et al Comptes Rend. phys. 1107.2819

Pfeiffer CQG 1203.5166

Hannam CQG 0901.2931

Cardoso et al LRR 1409.0014