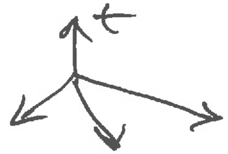
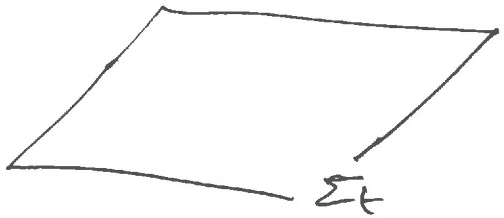
• $\mathcal{M}$ 4D manifold with metric $\underline{g}$; space and time are indistinguishable but it is convenient to split time away from space



• 

hypersurface at $t =$ const : fully spatial

• First, fix unit normal $\underline{n}$ such that $\underline{n}\cdot\underline{n} = -1$ and

$n_\mu \propto \nabla_\mu t = \Omega_\mu \Rightarrow \boxed{n_\mu = -\alpha \nabla_\mu t}$ $\quad \underline{n}\cdot\tilde{\Omega} = \alpha^{-1}$: gradient of t is a function.

• Second, build projector $\perp$ to $\underline{n}$: $\underline{\gamma} = \underline{g} + \underline{n}\,\underline{n}$

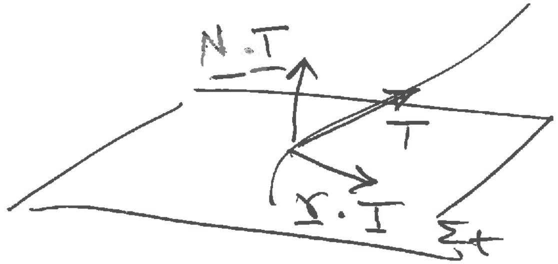
$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$: purely spatial $\quad \underline{\gamma}\cdot\underline{n} = 0$

γ can be used to raise/lower indices but only for purely spatial tensors.

- Third, build projector along $\underline{n}$: $\underline{N} = -\underline{n}\,\underline{n}$; $N^{\mu}{}_{\nu} = -n^{\mu} n_{\nu}$

$\underline{N} \cdot \underline{\gamma} = 0$

- In this way we can split any tensor in a purely spatial and in a purely time part

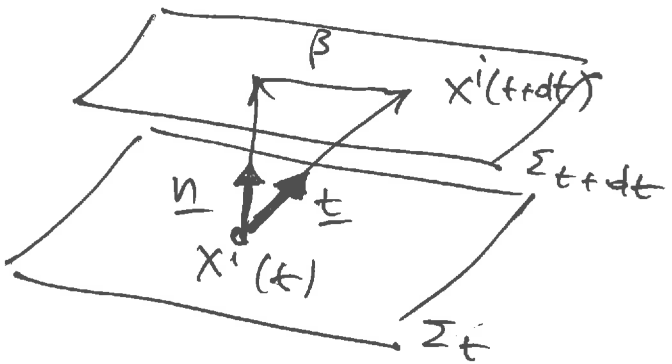


- $\underline{n}$ not parallel to $\tilde{\Omega}$; $\underline{n} \cdot \tilde{\Omega} = \alpha \neq$ const $\Rightarrow$

need 4-vector that is timelike and parallel to $\tilde{\Omega}$

$\underline{t} = \alpha \underline{n} + \underline{\beta}$ = (time part) + (space part) = $\underline{e}_t$

$\underline{t} \cdot \tilde{\Omega} = 1$



$$x^i(t+dt) = x^i(t) - \beta^i(x^i, t)\,dt$$

In this way we can build all the metric functions

Using the lapse and shift we can write the tt and ti parts of the metric as

$$g_{tt} = \underline{t}\cdot\underline{t} = -\alpha^2 + \beta^i\beta_i \qquad \left[(\alpha n^\mu + \beta^\mu)(\alpha n_\mu + \beta_\mu) = \alpha^2 n^\mu n_\mu + \cancel{2\alpha n^\mu \beta_\mu} + \beta^\mu\beta_\mu\right]$$

$$g_{ti} = \underline{t}\cdot\underline{\gamma} = t^\mu \gamma_{\mu i} = t^\mu\left(g_{\mu i} + \cancel{n_\mu n_i}\right) = t_i =$$

$$\underline{t}\cdot\underline{n} = 0$$

$$= (\alpha n^\mu + \beta^\mu) g_{\mu i} = \alpha n_i + \beta^\mu g_{\mu i}$$

$$= \alpha\cdot 0 + \beta^j \gamma_{ij}$$

$$= +\beta_i$$

so that the line element reads

$$ds^2 = -(\alpha^2 - \beta^i\beta_i)dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j$$

$$g_{\mu\nu} = \begin{pmatrix} -(\alpha^2 - \beta^i\beta_i) & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix} ; \quad g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^i/\alpha^2 \\ \beta^i/\alpha^2 & \gamma^{ij} - \beta^i\beta^j/\alpha^2 \end{pmatrix}$$

As a result

$$n_\mu = (-\alpha, 0, 0, 0) \; ; \qquad n^\mu = \frac{1}{\alpha}(1, -\beta^i)$$

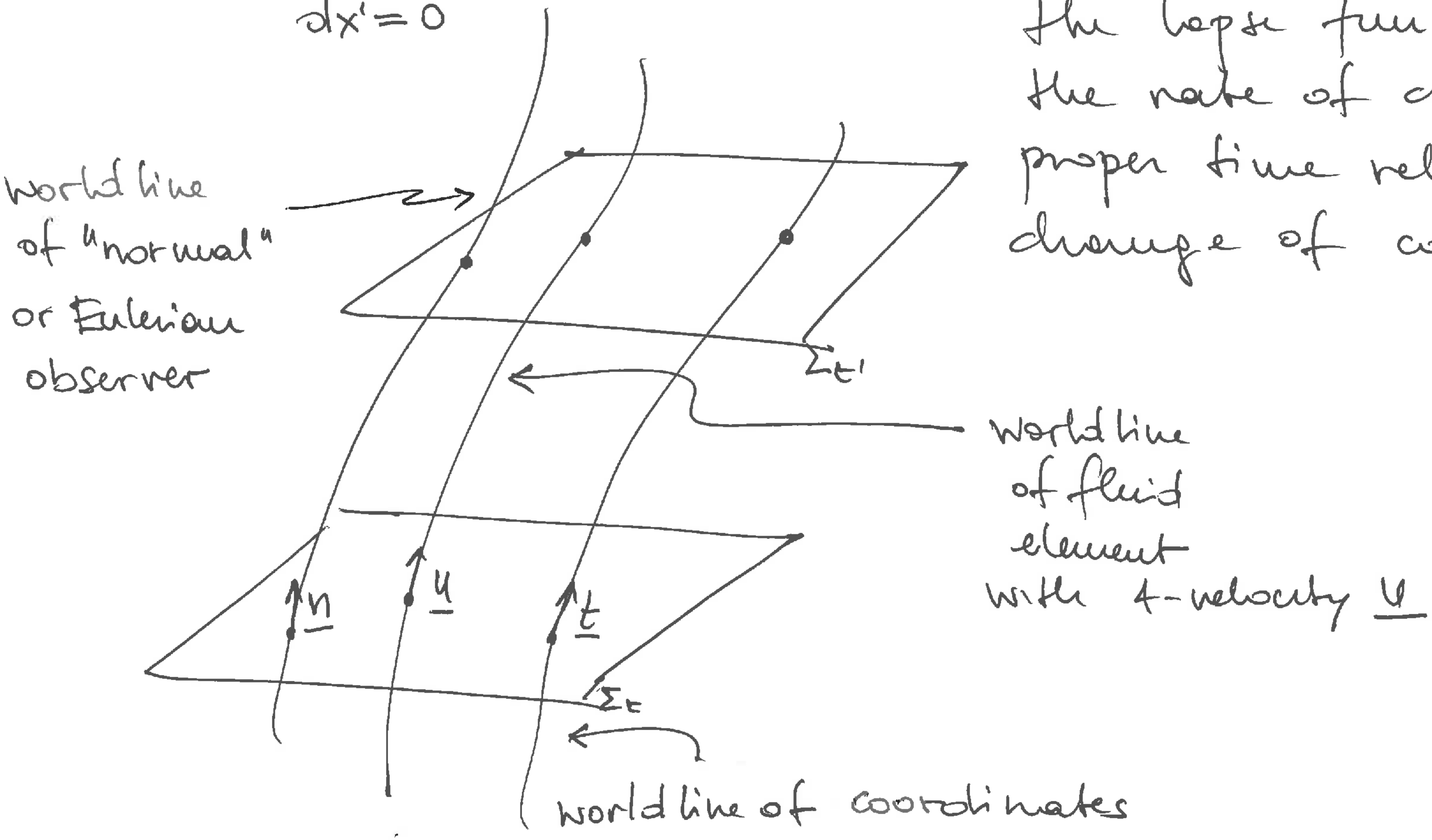
$n^i = g^{\mu i} n_\mu = g^{0i} n_0 = -\alpha\left(\frac{\beta^i}{\alpha^2}\right) = -\beta^i/\alpha$

$n^0 = g^{00} n_0 = -\frac{1}{\alpha^2}\cdot(-\alpha) = \frac{1}{\alpha}$

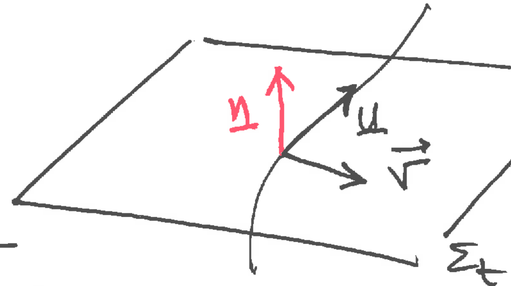
$$d\tau^2 = -ds^2 \underset{dx^i = 0}{=} +\alpha^2 dt^2$$

$$\underbrace{d\tau = \pm \alpha\, dt}$$

the lapse function expresses the rate of change of proper time relative to the change of coordinate time.



An observer with tangent vector $\underline{n}$ is said to be a "normal" observer or "Eulerian" observer. This is the standard observer in a 3+1 split and it is relative to this observer that 3+1 quantities are measured. This is the case also for the fluid four-velocity $\underline{u}$



$$\vec{v} : \begin{pmatrix} \text{spatial part} \\ \text{of } \underline{u} \end{pmatrix} = \frac{(\text{projection of } \underline{u} \text{ on } \Sigma_t)}{(\text{projection of } \underline{u} \text{ along } \underline{n})}$$

$$= \frac{(\text{space})}{(\text{time})} = \frac{\gamma^i{}_\mu u^\mu}{-u_\mu n^\mu}$$

$W := -n_\mu u^\mu = \alpha u^t$: Lorentz factor

$W = (1 - v^i v_i)^{-1/2}$ as in special relativity (Exercise)

$W \to \infty$ for $v \to 1$

Prove $W = (1 - v^i v_i)^{-1/2}$

Proof

$$v^i = \frac{1}{\alpha}\left(\frac{u^i}{u^0} + \beta^i\right) \; ; \; v_i = \gamma_{ij} v^j = \frac{\gamma_{ij}}{\alpha}\left(\frac{u^j}{u^0} + \beta^j\right)$$

$$v^i v_i = \frac{1}{\alpha^2}\left[\left(\frac{u^i}{u^0} + \beta^i\right)\gamma_{ij}\left(\frac{u^j}{u^0} + \beta^j\right)\right] = \frac{1}{\alpha^2}\left[\gamma_{ij}\frac{u^i u^j}{(u^0)^2} + 2\beta^i \frac{u_i}{u^0} + \beta^i \beta_i\right] =$$

$$-1 = u^\mu u_\mu =$$

$$= g_{00}(u^0)^2_i + 2 g_{0i} u^0 u^i + u^i u_i$$

$$= -(\alpha^2 - \beta^i \beta_i)(u^0)^2_i + 2\beta_i u^i u^0 + u^i u_i \Rightarrow$$

$$= -1 + (\alpha u^0)^2$$

$$= \frac{1}{(\alpha u^0)^2}\left[u^i u_i + 2\beta^i u_i u^0 + (u^0)^2 ...\right.$$

$$= \frac{1}{(\alpha u^0)^2}\left(-1 + (\alpha u^0)^2\right) = \frac{-1 + W^2}{W^2} \rightarrow W^2(1 - v^i v_i) = 1 \Rightarrow$$

$$W = (1 - v^i v_i)^{-1/2} \checkmark$$

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In component form:

$$v^t = 0 \quad ; \quad v^i = \frac{\gamma^i{}_\mu u^\mu}{\alpha u^t} = \frac{(\delta^i_\mu + n^i n_\mu)\, u^\mu}{\alpha u^t} = \frac{u^i}{\alpha u^t} + \left(-\frac{\beta^i}{\alpha}\right) \frac{\overbrace{n_\mu u^\mu}^{-\alpha u^t}}{\alpha u^t}$$

$$= \frac{u^i}{\alpha u^t} + \frac{\beta^i}{\alpha} = \frac{1}{\alpha}\left(\frac{u^i}{u^t} + \beta^i\right)$$

$$v_t = 0 \quad ; \quad v_i = \frac{\gamma_{i\mu} u^\mu}{\alpha u^t} = \frac{\gamma_{ij}}{\alpha}\left(\frac{u^j}{u^t} + \frac{\gamma_{i0} u^0}{u^0}\right) = \frac{\gamma_{ij}}{\alpha}\left(\frac{u^j}{u^t} + \beta^j\right) ;$$

In other words, using $W = \alpha u^t$

$$\boxed{v^i = \frac{u^i}{W} + \frac{\beta^i}{\alpha}}$$

$$\boxed{v_i = \frac{u_i}{W} = \gamma_{ij}\left(\frac{u^j}{W} + \frac{\beta^j}{\alpha}\right)}$$

Note that $\underline{\gamma}$ does not lower indices of $\underline{u}$ (4D object)

I recall that in special relativity

$$v^i = \frac{u^i}{u^t} = \frac{dx^i/dt}{dt/d\tau} = \frac{dx^i}{d\tau}$$

Recap $v^i = \frac{u^i}{W} + \frac{\beta^i}{\alpha} = \frac{u^i}{\alpha u^t} + \frac{\beta^i}{\alpha}$

compare with special-relativistic expression

$$v^i_{SR} = \frac{u^i}{u^t} = \frac{u^i}{\gamma}$$

Hence, the three-velocity gains a dependence on α and $\underline{\beta}$

$v^i = v^i_{SR}$ if $\alpha = 1$; $\underline{\beta} = 0$

It's easy to show that

$$\underline{u} = W(\underbrace{\underline{n}}_{\text{purely time part}} + \underbrace{\underline{v}}_{\text{purely spatial part}})$$

($W = -\underline{n}\cdot\underline{u} \Rightarrow \underline{n}W = \underline{u}$

$u^i = Wv^i - \frac{\beta^i}{\alpha}W = Wn^i + Wv^i$

$u^0 = Wn^0 = +\frac{1}{\alpha}W$)