QCD corrections to $p\bar{p} \to W^+ + X$

by Bin Yan

① Scott Willenbrock TASI 1989 Lecture

② C.-P. Yuan Lectures.

③ Li Yang's note.

A. Lagrangian

$$\mathcal{L} = \bar{\psi}_0 i \not{D} \psi_0$$

$$= \bar{\psi}_0 i [\partial_\mu - i g_{s0} T^a G^a_\mu - i g_0 \frac{\tau^a}{2} W^a_\mu] \psi_0$$

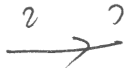
where T^a and $\frac{\tau^a}{2}$ are the generators of $SU(3)_c$ and $SU(2)_L$.

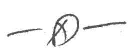
$\psi = Z_\psi^{-1/2} \psi_0$ and $g_{s0} = g_s \mu^\epsilon$

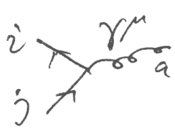
$$\Rightarrow \mathcal{L} = Z_\psi \bar{\psi} i [\partial_\mu - i g_s \mu^\epsilon T^a G^a_\mu - i g \frac{\tau^a}{2} W^a_\mu] \gamma^\mu \psi$$

define $\delta_\psi = Z_\psi - 1$ note: the electroweak coupling and field strength are not renormalized.

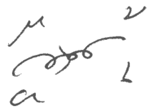
Feynman rules:

Fermion: 

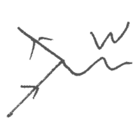
$\frac{i}{\not{p} - m} \delta_{ij}$  $i \not{p}\, \delta_\psi$



$i g_s \gamma^\mu (T^a)_{ij} \mu^\epsilon$



$\frac{-i}{k^2} \delta_{ab} [g_{\mu\nu} - \frac{(1-\xi) k_\mu k_\nu}{k^2}]$

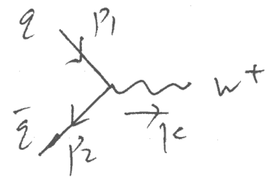


$\frac{ig}{\sqrt{2}} \gamma^\mu P_L$

$\frac{ig}{\sqrt{2}} \gamma^\mu P_L \delta_\psi$

B. Traditional method

B-1. Tree-level cross section:



$$i\mathcal{M} = \bar{v}(p_2)\, \frac{ig}{\sqrt{2}} \gamma^\mu P_L\, u(p_1)\, \epsilon_\mu(k)$$

$$= \frac{ig}{2\sqrt{2}}\, \bar{v}(p_2)\, \gamma^\mu (1-\gamma_5)\, u(p_1)\, \epsilon_\mu(k)$$

Ampitude square and average over the spin and color.

$$|\bar{\mathcal{M}}|^2 = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{3}\times\frac{1}{3}\times 3\, \frac{g^2}{8}\, Tr[\not{p}_2 \gamma^\mu (1-\gamma_5) \not{p}_1 \gamma^\nu (1-\gamma_5)]\, (-g_{\mu\nu})$$

Note: $k^\mu k^\nu$ term don't contribute to $|\bar{\mathcal{M}}|^2$ when $m_q = 0$.

Naive-γ_5 scheme: $\{\gamma^\mu, \gamma_5\} = 0$, define: $n = 4-2\epsilon$

$$|\bar{\mathcal{M}}|^2 = \frac{1}{12} g^2 [1-\epsilon]\, \hat{s} = \frac{\pi}{3} \alpha_W (1-\epsilon)\, \hat{s}$$

Parton-level cross section:

$$\hat{\sigma} = \int \frac{1}{2s} |\bar{\mathcal{M}}|^2 \frac{d^3k}{(2\pi)^3 2E} (2\pi)^4 \delta^4(p_1+p_2-k)$$

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{2E} \delta^4(p_1+p_2-k) = \int d^4q\, \delta^4(p_1+p_2-k)\, \delta^+(q^2-M^2)$$

$$= \delta(k^2-M^2)$$

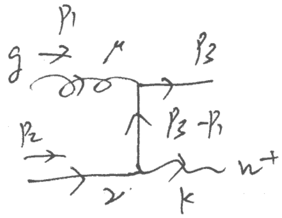
$$\hat{\sigma} = \frac{2\pi}{2\hat{s}}\,\frac{\pi}{3}\,\alpha_w\,(1-\epsilon)\,\hat{s}\;\;\delta(k^2-M^2)$$

$$= 2\times\frac{\pi^2\alpha_w}{6}\,(1-\epsilon)\;\delta(\hat{s}-M^2)$$

$$= 2\times\frac{\pi^2\alpha_w}{6\hat{s}}\,(1-\epsilon)\;\delta(1-\hat{\tau}) \qquad \text{where } \hat{\tau} = \frac{M^2}{\hat{s}}$$

$$= \frac{\pi^2\alpha_w}{3\hat{s}}\,(1-\epsilon)\;\delta(1-\hat{\tau})$$

B-2. $\quad gq \to W^+ q$



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$$\begin{cases} s = (p_1+p_2)^2 \\ t = (p_1-p_3)^2 \\ u = (p_2-p_3)^2 \end{cases}$$

$$iM_1 = \bar{u}(p_3)\left[ig_s\mu^{\epsilon}\,T^A\gamma^\mu\,\frac{i}{\not{p}_3-\not{p}_1}\,\frac{ig}{\sqrt{2}}\gamma^\nu P_L\right]u(p_2)\;\epsilon_\mu(p_1)\;\epsilon_\nu^*(k)$$

$$iM_2 = \frac{ig_s\mu^{\epsilon}\,ig}{2\sqrt{2}}\;\bar{u}(p_3)\left[\gamma^\nu(1-\gamma_5)\,\frac{i}{\not{p}_1+\not{p}_2}\,\gamma^\mu T^A\right]u(p_2)\;\epsilon_\mu(p_1)\;\epsilon_\nu^*(k)$$

$$iM = iM_1 + iM_2$$

$$= \frac{ig_s\mu^{\epsilon}\,ig}{2\sqrt{2}}\;\bar{u}(p_3)\left\{\gamma^\mu T^A\,\frac{i}{\not{p}_3-\not{p}_1}\,\gamma^\nu(1-\gamma_5) + \gamma^\nu(1-\gamma_5)\,\frac{i}{\not{p}_1+\not{p}_2}\,\gamma^\mu T^A\right\}u(p_2)$$

$$\epsilon_\mu(p_1)\;\epsilon_\nu^*(k)$$

Amplitude square and average for spin and color.

Note: There are $n-2$ transverse ~~dim~~ spatial dimensions in n dimensions.

So the gluon has $n-2 = 2(1-\epsilon)$ spin components.

[from little group $ISO(N-2)$]

$$|\bar{M}|^2 = \frac{1}{2} \times \frac{1}{2(1-\epsilon)} \times \frac{1}{3} \times \frac{1}{8} \times |M|^2$$

$$|M|^2 = \frac{g_s^2 \mu^{2\epsilon} g^2}{8} \, Tr[T^A T^A]$$

$$\{ Tr[\not{p}_3 \gamma^\mu (\not{p}_3 - \not{p}_1) \gamma^\nu (1-\gamma_5) \not{p}_2 \gamma_\nu (1-\gamma_5) (\not{p}_3 - \not{p}_1) \gamma_\mu] \frac{1}{t^2} \quad (1)$$

$$+ Tr[\not{p}_3 \gamma^\nu (1-\gamma_5)(\not{p}_1 + \not{p}_2) \gamma^\mu \not{p}_2 \gamma_\mu (\not{p}_1 + \not{p}_2) \gamma_\nu (1-\gamma_5)] \frac{1}{s^2} \quad (2)$$

$$+ 2Tr[\not{p}_3 \gamma^\mu (\not{p}_3 - \not{p}_1) \gamma^\nu (1-\gamma_5) \not{p}_2 \gamma_\mu (\not{p}_1 + \not{p}_2) \gamma_\nu (1-\gamma_5)] \frac{1}{st} \} \quad (3)$$

Note: The dirac algebra in n dimensions can be found in Peskin's book.

(1) only t-channel. $= -16(1-\epsilon)^2 \frac{s}{t}$

(2) only s-channel. $= -16(1-\epsilon)^2 \frac{t}{s}$. from $p_2 \to -p_3$, $t \to s$, $s \to t$

(3) The interference between s chanel and t-channel.

$$= 16[\epsilon - 1][-st\epsilon + uM^2] \frac{1}{st}$$

$$|M|^2 = \frac{g_s^2 \mu^{2\epsilon} g^2}{8} \times 16 \times 4 \, [1-\epsilon] \{ [1-\epsilon][-\frac{s}{t} - \frac{t}{s}] - \frac{2uM^2}{st} + 2\epsilon \}$$

$$|\bar{M}|^2 = \frac{g_s^2 \mu^{2\epsilon} g^2}{12} [(1-\epsilon)(-\frac{s}{t} - \frac{t}{s}) - \frac{2uM^2}{st} + 2\epsilon]$$

Two-particle phase space in n dimensions.

$$\int dPS_2 = \int \frac{d^{n-1}k}{(2\pi)^{n-1}}\frac{1}{2k_0}\frac{d^{n-1}q}{(2\pi)^{n-1}}\frac{1}{2q_0}(2\pi)^n\delta^n(p-q-k)$$

use: $$\frac{d^{n-1}\vec{q}}{2q_0} = \int d^nq\,\delta^+(q^2-Q^2)$$

$$\int dPS_2 = \frac{1}{(2\pi)^{n-2}}\int\frac{d^{n-1}\vec{k}}{2k_0}\delta^+[(p-k)^2-Q^2]$$

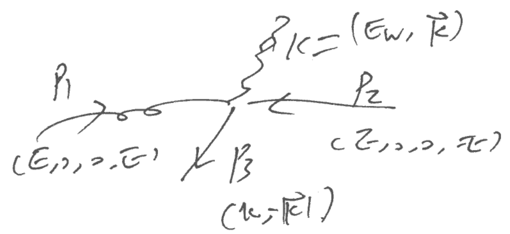
$$= \frac{1}{(2\pi)^{n-2}}\int\frac{dk\,k^{n-3}}{2}\int d\Omega_{n-2}\,\delta(\hat{s}-2k\sqrt{\hat{s}}-Q^2)$$

note: $p^2=\hat{s}$, $\vec{p}=0$, $k=|\vec{k}|$.

$$\int dPS_2 = \frac{\Omega_{n-3}}{(2\pi)^{2(1-\epsilon)}}\int\frac{dk\,k^{1-2\epsilon}}{4\sqrt{\hat{s}}}\int_0^\pi d\theta(\sin\theta)^{1-2\epsilon}\,\delta\left(k-\frac{\hat{s}-Q^2}{2\sqrt{\hat{s}}}\right)$$

define: $z=\frac{Q^2}{\hat{s}}$, $v=\frac{1}{2}(1+\cos\theta) \Rightarrow k=\frac{\sqrt{\hat{s}}}{2}(1-z)$

$$\int dPS_2 = \frac{1}{8\pi}\left(\frac{4\pi}{M^2}\right)^\epsilon\frac{z^\epsilon(1-z)^{1-2\epsilon}}{\Gamma(1-\epsilon)}\int_0^1 dv\,[v(1-v)]^{-\epsilon}$$



$$t = -s\left(1-\frac{M_W^2}{s}\right)(1-v) = -s(1-z)(1-v)$$

$$u = -s(1-z)v$$

Cross Section for $gq \to W + q$

$$\sigma = \frac{1}{2\hat{s}} \overline{|M|}^2 PS$$

$$= \frac{1}{2\hat{s}} \frac{g_s^2 \mu^{2\epsilon} g^2}{12} \frac{1}{8\pi} \left(\frac{4\pi}{\mu^2}\right)^\epsilon \frac{z^\epsilon (1-z)^{1-2\epsilon}}{\Gamma(1-\epsilon)} \times$$

$$\int_0^1 dv \, [v(1-v)]^{-\epsilon} \left[(1-\epsilon)\left(-\frac{s}{t} - \frac{t}{s}\right) - \frac{2uM^2}{st} + 2\epsilon \right]$$

$$= \frac{1}{2\hat{s}} \frac{g_s^2 \mu^{2\epsilon} g^2}{12} \frac{1}{8\pi} \left(\frac{4\pi}{\mu^2}\right)^\epsilon \frac{z^\epsilon (1-z)^{1-2\epsilon}}{\Gamma(1-\epsilon)} \times$$

$$\int_0^1 dv \, [v(1-v)]^{-\epsilon} \left[(1-\epsilon)\left(\frac{1}{(1-z)(1-v)} + (1-z)(1-v)\right) - \cancel{2v\frac{z}{1-z}} + 2\epsilon \right]$$

$$2z\frac{v}{1-v}$$

Beta function:

$$B(\alpha,\beta) = \int_0^1 dy \, y^{\alpha-1} (1-y)^{\beta-1}$$

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Thus: $$\int_0^1 dv \left\{ v^{-\epsilon}(1-v)^{-1-\epsilon} \frac{1-\epsilon}{1-z} + v^{-\epsilon}(1-v)^{1-\epsilon}(1-\epsilon)(1-z) \right.$$

$$\left. - 2v^{1-\epsilon}(1-v)^{-1-\epsilon} z + 2\epsilon v^{-\epsilon}(1-v)^{-\epsilon} \right\}$$

$$= B(-\epsilon, -1-\epsilon)\frac{1-\epsilon}{1-z} + B(-\epsilon, 1-\epsilon)(1-\epsilon)(1-z) - 2z B(1-\epsilon, -1-\epsilon)$$

$$+ 2\epsilon B(-\epsilon, -\epsilon)$$

note: $\Gamma(n+1) = n\Gamma(n)$

cross section:

$$\hat{\sigma} = \frac{1}{2\hat{s}} \frac{g_s^2 \mu^{2\epsilon} g^2}{12} \frac{1}{8\pi} \left(\frac{4\pi}{M^2}\right)^{\epsilon} \frac{z^{\epsilon}(1-z)^{1-2\epsilon}}{\Gamma(1-\epsilon)} \times$$

$$\left\{ -\frac{1}{\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[\frac{1-\epsilon}{1-z} - 2z(1+\epsilon) \right] + \frac{1}{2}(1-z) + O(\epsilon) \right\}$$

$$= \frac{1}{2\hat{s}} \frac{g_s^2 \mu^{2\epsilon} g^2}{12} \frac{1}{8\pi} \left(\frac{4\pi}{M^2}\right)^{\epsilon} z^{\epsilon}(1-z)^{1-2\epsilon} \times$$

$$\left\{ -\frac{1}{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[\frac{1-\epsilon}{1-z} - 2z(1+\epsilon) \right] + \frac{1}{2}(1-z) \right\}$$

$$= \cancel{\frac{\alpha_s}{4\pi} \cdot \frac{\pi}{12} \frac{g^2}{\hat{s}} \Big\{ z(1-z}$$

$$= \frac{\alpha_s}{4\pi} \frac{\pi}{12} \frac{g^2}{\hat{s}} \left\{ \left[z^2 + (1-z)^2\right] \left[-\frac{1}{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} + \ln\left(\frac{M^2}{4\pi\mu^2} \frac{(1-z)^2}{z} \right) \right] + \frac{3}{2} + z - \frac{3}{2} z^2 \right\}$$

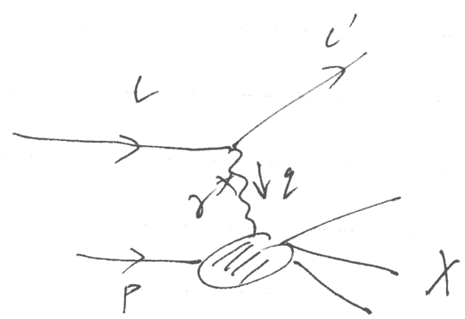
where $z = \dfrac{M^2}{\hat{s}}$

note: collinear singularity in the limit $\epsilon \to 0$, it comes from $v \to 1$

$v = \frac{1}{2}(1+\cos\theta)$, which corresponding to $\theta \to 0$ (collinear singularity)

B3. $\gamma^* g \to q\bar{q}$ corrections to the quark PDF from the presence of gluon in proton

(a) Deep inelastic scattering.



$Q^2 \gg M_p^2$

define: $Q^2 = -q^2 = 2EE'(1-\cos\theta)$

$\nu = \frac{P\cdot q}{M_p} = E - E'$

where M_p is proton mass. E and E' are lepton energies

Amplitude:

$$i\mathcal{M} = ie(-ie)\frac{i}{Q^2}\bar{u}(l')\gamma^\mu u(l) \langle P| J_\mu |X\rangle$$

where $\langle P|J_\mu|X\rangle$ hadronic current

Amplitude square and summing over the spins and average over spins.

$$\overline{|\mathcal{M}|^2} = e^4 \frac{1}{Q^4} L^{\mu\nu} \frac{1}{2} \langle p|J_\mu|X\rangle\langle X|J_\nu|p\rangle$$

where $L^{\mu\nu} = \frac{1}{2}\mathrm{Tr}[\not{l}'\gamma^\mu \not{l}\gamma^\nu]$

averaging over initial spin

The differential cross section:

$$d\sigma = \frac{1}{2E_1 2E_2}\frac{1}{2}|\overline{M}|^2 \frac{d^3l'}{(2\pi)^3 2E'} dX\, (2\pi)^4 \delta^4(P+q-X)$$

$$S = (P+l)^2 = M_p^2 + 2P.l = M^2 + 2E_1E_2 \qquad \text{(Proton rest frame)}$$

$$d\sigma = \frac{1}{2(S-M_p^2)}|\overline{M}|^2 \frac{d^3l'}{(2\pi)^3 2E'} dX\, (2\pi)^4 \delta^4(P+q-X)$$

$$= \frac{1}{2(S-M_p^2)} e^4 \frac{1}{Q^4} L^{\mu\nu} \frac{1}{2} \langle P|J_\mu|X\rangle\langle X|J_\nu|P\rangle \frac{d^3l'}{(2\pi)^3 2E'} dX$$

$$(2\pi)^4\delta^4(P+q-X)$$

$$= \frac{1}{2(S-M_p^2)} e^4 \frac{1}{Q^4} L^{\mu\nu} 4\pi M_p \frac{d^3l'}{(2\pi)^3 2E'}$$

$$\frac{1}{2}\times\frac{1}{4\pi M_p}\langle P|J_\mu|X\rangle\langle X|J_\nu|P\rangle (2\pi)^4 \delta^4(P+q-X)$$

so we define:

$$W_{\mu\nu} = \frac{1}{2}\frac{1}{4\pi M_p}\int dX \langle P|J_\mu|X\rangle\langle X|J_\nu|P\rangle (2\pi)^4\delta^4(P-q-X)$$

$$d\sigma = \frac{1}{2(S-M_p^2)} e^4 \frac{1}{Q^4} L^{\mu\nu} 4\pi M_p W_{\mu\nu} \frac{d^3l'}{(2\pi)^3 2E'}$$

From the conserved currents requirements. $q^\nu W_{\mu\nu} = 0$

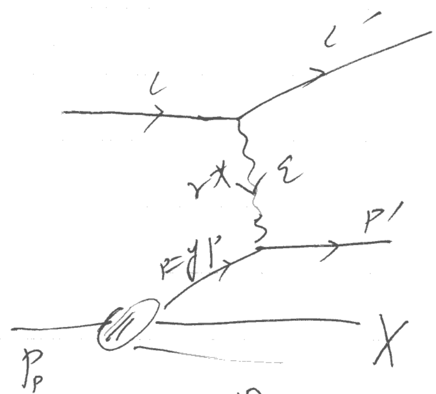
$$W_{\mu\nu} = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) W_1(\nu, Q^2)$$
$$+ \frac{1}{M_p^2}\left(P_\mu - q_\mu \frac{P.q}{q^2}\right)\left(P_\nu - q_\nu\frac{P.q}{q^2}\right) W_2(\nu, Q^2)$$

define: $x = \frac{Q^2}{2M_p\nu} = \frac{Q^2}{2P\cdot q}$

Form factors: $F_1(x, Q^2) = 2M_p W_1(\nu, Q^2)$

$F_2(x, Q^2) = \nu W_2(\nu, Q^2)$

(b) Parton model.



note: $x = \frac{Q^2}{2P_p\cdot q}$ $\qquad p = y P_p$

$\boxed{x = y}$ $\qquad (p+q)^2 = 0 \Rightarrow 2p\cdot q = Q^2$

The cross section for DIS:

$$d\sigma = \frac{1}{2s}\frac{e^4}{Q^4} L^{\mu\nu} 4\pi M_p W_{\mu\nu} \frac{d^3l'}{(2\pi)^3 2E'}$$

The subprocess:

$$d\sigma_i = \frac{1}{2\hat{s}}\frac{e^4}{Q^4} L^{\mu\nu}\hat{W}^i_{\mu\nu}\frac{d^3l'}{(2\pi)^3 2E'} \qquad \text{where } \hat{s} = (p+l)^2 \simeq y\cdot s$$

$$\hat{W}^i_{\mu\nu} = \frac{1}{2}\int \langle yP_p | J^i_\mu | X'\rangle\langle X' | J^i_\nu | yP_p\rangle \, dX' (2\pi)^4 \delta^4(P_p+q-X')$$

average from initial quark spins

The cross section in the parton model.

$$d\sigma = \sum_i \int_x^1 dy\, f_i(y)\, d\hat{\sigma}_i$$

where $f_i(y)$ is the probability density of the struck quark carries momentum fraction y. , since $(p+q)^2 \geq 0 \Rightarrow y \geq x$

Therefore the form factor:

$$W_{\mu\nu} = \frac{1}{4\pi M_p} \sum_i \int_x^1 \frac{dy}{y}\, f_i(y)\, \hat{W}^i_{\mu\nu}$$, where $\hat{s} = ys$ are used.

note: The quark current $\langle y p_p | J^i_\mu | x' \rangle = i q_i \bar{u}(p') \gamma_\mu u(p)$

※ · The coupling e is removed from the definition of the hadron current

$$W_2 = \sum_i \frac{q_i^2}{\nu} x f_i(x) \qquad\qquad F_1(x, Q^2) = \sum q_i^2 f_i(x)$$

$$\Rightarrow$$

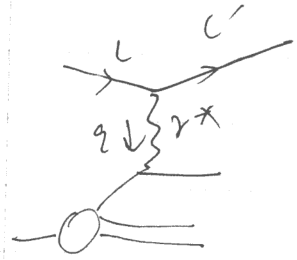
$$W_1 = \sum_i \frac{q_i^2}{2M_p} f_i(x) \qquad\qquad F_2(x, Q^2) = \sum_i q_i^2 x f_i(x)$$

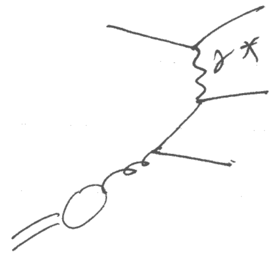
note: $\nu = \dfrac{P_p \cdot q}{M_p}$, $x = \dfrac{Q^2}{2P_p \cdot q}$

(C). <u>quark PDF</u>

※ define the quark distribution function to all orders in QCD.

$$F_2(x, Q^2) = \underbrace{\sum_i q_i^2}_{\text{quark electric charge}} x\left[q_i(x, Q^2) + \bar{q}_i(x, Q^2) \right]$$





In order to exact F_2 from $W_{\mu\nu}$, we define

$$W_T \equiv -g^{\mu\nu} W_{\mu\nu}$$
$$W_L \equiv P_p^{\mu} P_p^{\nu} W_{\mu\nu}$$

Thus
$$W_T = -g^{\mu\nu} \Big\{ -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) W_1(\nu, q^2) + \frac{1}{M_p^2}\left(P_\mu - q_\mu \frac{P\cdot q}{q^2}\right)\left(P_\nu - q_\nu \frac{P\cdot q}{q^2}\right) W_2(\nu, q^2) \Big\}$$

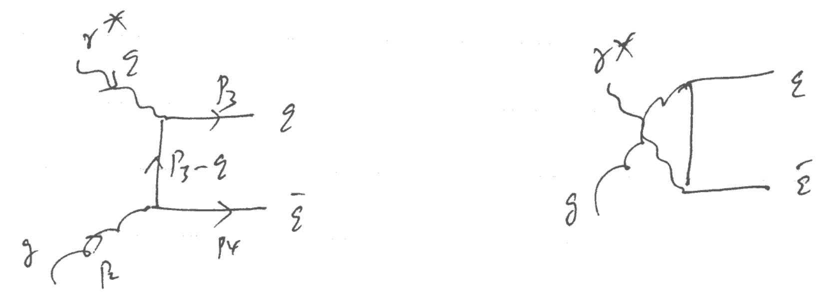
note: P_μ is the momentum of proton.

$$\begin{cases} W_T = (3-2\epsilon) W_1 - \dfrac{\nu^2}{Q^2} W_2 \\[2ex] W_L = -\dfrac{M_p^2 \nu^2}{Q^2} W_1 + \dfrac{M^2 \nu^4}{Q^4} W_2 \end{cases}$$
where $Q^2 \equiv -q^2$, $\nu = \dfrac{P\cdot q}{M_p}$

$$\Rightarrow (1-\epsilon)\frac{1}{M_p} F_2 = x W_T + 4\frac{x^3}{Q^2}(3-2\epsilon) W_L$$
with $\begin{cases} x = \dfrac{Q^2}{2M\nu} \\ F_2 = \nu W_2 \end{cases}$

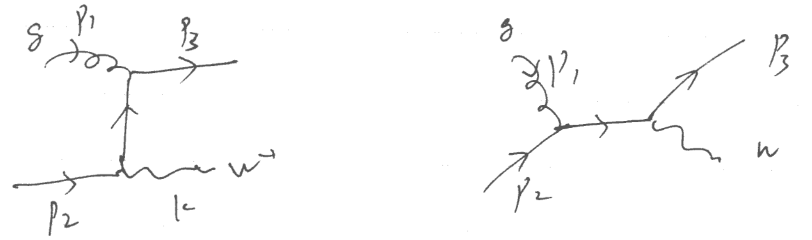
thus: $$W_T = \frac{1}{4\pi M_p} \sum_i \int_x^1 \frac{dy}{y}\, g(y)\, \hat{W}_T^i$$

Feynman diagrams:



v.s

gq → W+q



Crossing symmetry:

The S-matrix for any process involving a particle with momentum p in the initial state is equal to the S-matrix for an otherwise identical process but with an antiparticle of momentum $k = -p$ in the final state.

$gq \to W^+q$ $\quad$ $\gamma^* g \to q\bar{q}$

$p_1 \to p_2$ $\qquad$ $s = (p_1+p_2)^2 \to (p_2-p_4)^2 = t$

$p_2 \to -p_4$ $\quad \Rightarrow \quad$ $t = (p_1-p_3)^2 \to (p_2-p_3)^2 = u$

$p_3 \to p_3$ $\qquad$ $u = (p_2-p_3)^2 \to (p_3+p_4)^2 = s$

$k = -q$

Note the coupling is different.

$gq \to W^+q$ $\qquad \frac{g}{\sqrt{2}} P_L = \frac{g}{2\sqrt{2}}(1-\gamma_5)$, while V and A coupling gives a same result

$\Rightarrow \left(\frac{g}{2\sqrt{2}}\right)^2 \times 2 = \frac{g^2}{4}$, thus $g^2 \to 4 q_i^2 e^2$

There is a relative minus sign due to the fermion is involved.

$$|M|^2 = -8\, q_i^2\, e^2 g_s^2 \mu^{2\epsilon}\, \mathrm{Tr}[T^A T^A]\, (1-\epsilon) \left\{ (1-\epsilon)\left[-\frac{t}{u} - \frac{u}{t}\right] - \frac{2sQ^2}{tu} + 2\epsilon \right\}$$

only for hadron current, (note) not average over the photon spin

$$|\overline{M}|^2 = \underbrace{\frac{1}{2(1-\epsilon)}}_{\text{spin for gluon}} \times \frac{1}{8} \times 32\, q_i^2 e^2 \underset{\mu^{2\epsilon}}{g_s^2}\, (1-\epsilon) \left\{ (1-\epsilon)\left(\frac{t}{u} + \frac{u}{t}\right) - \frac{2sQ^2}{tu} - 2\epsilon \right\}$$

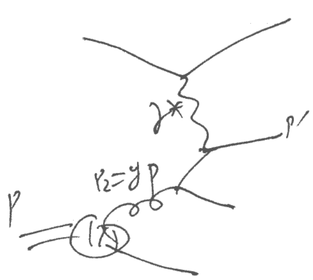
$$= 2\, q_i^2 e^2 g_s^2 \mu^{2\epsilon} \left\{ (1-\epsilon)\left(\frac{t}{u} + \frac{u}{t}\right) - \frac{2sQ^2}{tu} - 2\epsilon \right\}$$

phase space:

$$\int PS_2 = \frac{1}{8\pi}\left(\frac{4\pi}{s}\right)^{\epsilon} \frac{1}{\Gamma(1-\epsilon)} \int_0^1 dv\, v^{-\epsilon}(1-v)^{-\epsilon}$$

where $v = \frac{1}{2}(1+\cos\theta)$

$$t = -s\left(1+\frac{Q^2}{s}\right)(1-v)$$
$$u = -s\left(1+\frac{Q^2}{s}\right)v$$



note: $x = \frac{Q^2}{2P\cdot q} = y\,\frac{Q^2}{2p_2\cdot q}$

$$s = (q+p_2)^2 = 2p_2\cdot q - Q^2 = \frac{y}{x}Q^2\left(1-\frac{x}{y}\right)$$

define: $z = \frac{x}{y}$

$$\Rightarrow \begin{cases} s = \frac{Q^2}{z}(1-z) \\ t = -\frac{Q^2}{z}(1-v) \\ u = -\frac{Q^2}{z}\,v \end{cases}$$

The transverse structure function

$$\hat{W}_T^i = \int |\bar{M}|^2 \, P.S$$

$$= 2 e_i^2 g_s^2 \mu^{2\epsilon} \frac{1}{8\pi} \left(\frac{4\pi}{s}\right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \int_0^1 dv \, [v(1-v)]^{-\epsilon}$$

$$\left\{ (1-\epsilon)\left(\frac{t}{u} + \frac{u}{t}\right) - \frac{2sQ^2}{tu} - 2\epsilon \right\}$$

note the electromagnetic couple e^2 is not included in the definition of $\hat{W}_T^i$.

Beta functions $\quad B(\alpha, \beta) = \int_0^1 dy \; y^{\alpha-1} (1-y)^{\beta-1}$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\hat{W}_T^i = 2 e_i^2 g_s^2 \mu^{2\epsilon} \frac{1}{8\pi} \left[\frac{4\pi}{Q^2} \frac{z}{1-z}\right]^\epsilon \frac{1}{\Gamma(1-\epsilon)}$$

$$\int_0^1 dv \, [v(1-v)]^{-\epsilon} \left\{ (1-\epsilon)\left[\frac{1-v}{v} + \frac{v}{1-v}\right] - 2z(1-z)\frac{1}{v(1-v)} - 2\epsilon \right\}$$

$$\int v^{-\epsilon}(1-v)^{-\epsilon}(1-\epsilon)\frac{1-v}{v} = \int v^{-1-\epsilon}(1-v)^{1-\epsilon}(1-\epsilon) = (1-\epsilon) B(-\epsilon, 2-\epsilon)$$

$$\int v^{-\epsilon}(1-v)^{-\epsilon}(1-\epsilon)\frac{v}{1-v} = (1-\epsilon) B(2-\epsilon, -\epsilon)$$

$$\int v^{-\epsilon}(1-v)^{-\epsilon}\frac{1}{v(1-v)} = \int v^{-\epsilon-1}(1-v)^{-1-\epsilon} = B(-\epsilon, -\epsilon)$$

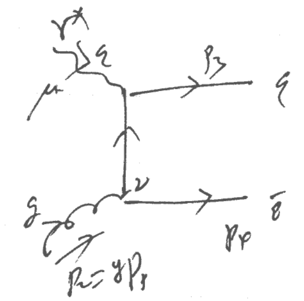
$$\hat{W}_T^i = 2 e_i^2 \alpha_s [z^2 + (1-z)^2]\left[-\frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} + \ln\left(\frac{Q^2}{4\pi\mu^2}\frac{1-z}{z}\right)\right]$$

The longitudinal structure function

$$W_L = p_p^\mu p_p^\nu W_{\mu\nu} = \frac{1}{4\pi M_p} \sum_i \int_x^1 \frac{dy}{y^3} f_i(y)\, \hat{W}_L^i$$

where $\hat{W}_L^i = p_2^\mu p_2^\nu \hat{W}_{\mu\nu}^i$, $p_2 = y\, p_p$

Feynman diagrams:



$s = (p_2+q)^2$

$t = (p_2-p_4)^2$

$u = (p_2-p_3)^2$

$$p_2^\mu M_\mu = i q_i e\, i g_s \mu^\epsilon\, \bar{u}(p_3)\left[\not{p}_2 \frac{i}{\not{p}_2-\not{p}_4}\gamma^\nu T^A + \gamma^\nu T^A \frac{i}{\not{p}_3-\not{p}_2}\not{p}_2\right] v(p_4)\, \epsilon_\nu(p_2)$$

$$= -q_i e g_s \mu^\epsilon\, \bar{u}(p_3)\left[\frac{\not{p}_2(\not{p}_2-\not{p}_4)}{t}\gamma^\nu T^A + \gamma^\nu T^A \frac{i(\not{p}_3-\not{p}_2)}{u}\not{p}_2\right] v(p_4)\, \epsilon_\nu(p_2)$$

note: $\not{p}_2^2 = p_2^2 = 0$, $\bar{u}(p_3)\not{p}_3 = \not{p}_4 v(p_4) = 0$ for on-shell conditions.

$$p_2^\mu M_\mu = -q_i e g_s \mu^\epsilon\, \bar{u}(p_3)\left[-2p_4^\nu \not{p}_2 \frac{i}{t} T^A + T^A\, 2p_3^\nu \not{p}_2 \frac{i}{u}\right] v(p_4)\, \epsilon_\nu(p_2)$$

square the amplitude and summing over gluon spin,
only the cross term survives due to $p_3^2 = p_4^2 = 0$

$$p_2^\mu p_2^\nu M_\mu M_\nu^* = -q_i^2 e^2 g_s^2 \mu^{2\epsilon}\, \mathrm{Tr}[T^A T^A] \times 2 \times (-4)\, p_3\cdot p_4 \frac{1}{tu}\, \mathrm{Tr}[\not{p}_3 \not{p}_2 \not{p}_4 \not{p}_2]$$

$-g^{\mu\nu}$

$$p_2^\mu p_2^\nu M_\mu M_\nu^* = q_i^2 e^2 g_s^2 \mu^{2\epsilon} \cdot \times 4 \cdot 8 \hat{s}$$

note: the factor $\frac{1}{\hat{t}\hat{u}}$ has been cancelled by the numerator, there are no collinear divergences.

spin and color for gluon

$$\overline{p_2^\mu p_2^\nu M_\mu M_\nu^*} = \frac{1}{2} \times \frac{1}{8} \times q_i^2 e^2 g_s^2 \mu^{2\epsilon} \times 4 \times 8 \hat{s}$$

$$= 2 q_i^2 e^2 g_s^2 \mu^{2\epsilon} \hat{s}$$

$$\hat{W}_L^i = |\overline{M}|^2 \cdot P_S \cdot = 2 q_i^2 \cancel{e^2} g_s^2 \mu^{2\epsilon} \hat{s} \cdot \frac{1}{8\pi}$$

$$= q_i^2 \alpha_s \hat{s} = q_i^2 \alpha_s \cdot \frac{Q^2}{z}(1-z)$$

note: the factor e^2 is removed.

The form factor F_2

$$(1-\epsilon) \frac{1}{M_p} F_2 = x W_T + \frac{4x^3}{Q^2}(3-2\epsilon) W_L$$

$$= \frac{x}{4\pi M_p} \sum_i \int_x^1 \frac{dy}{y} g(y) \hat{W}_T^i + \frac{4x^3}{Q^2} \frac{(3-2\epsilon)}{4\pi M_p} \sum_i \int_x^1 \frac{dy}{y^3} g(y) \hat{W}_L^i$$

$$= \frac{x}{4\pi M_p} \sum_i \int_x^1 \frac{dy}{y} g(y)\, 2 q_i^2 \alpha_s [z^2 + (1-z)^2]$$

$$\left[-\frac{1}{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} + \ln\left(\frac{Q^2}{4\pi\mu^2} \frac{1-z}{z}\right)\right]$$

$$+ \frac{4x^3}{Q^2} \frac{(3-2\epsilon)}{4\pi M_p} \sum_i \int_x^1 \frac{dy}{y^3} g(y)\, q_i^2 \alpha_s Q^2 \frac{1-z}{z}$$

$$(1-\epsilon)F_2 = \frac{1}{4\pi}\sum_i \int_x^1 \frac{x}{y}\, dy\, g(y)\; 2 e_i^2 \alpha_s\, [z^2+(1-z^2)]$$

$$\left[-\frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} + \ln\left(\frac{Q^2}{4\pi\mu^2}\,\frac{1-z}{z}\right)\right]$$

$$+\frac{1}{4\pi}\, 4(3-2\epsilon)\sum_i \int_x^1 \frac{x^3}{y^3}\, dy\, g(y)\, e_i^2 \alpha_s \frac{1-z}{z}$$

$$\overset{\epsilon\to 0}{=} \sum_i e_i^2 \frac{\alpha_s}{2\pi}\int_x^1 dy\, g(y)\, z$$

$$\Big\{\underbrace{[z^2+(1-z)^2]\left[-\frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} + \ln\left(\frac{Q^2}{4\pi\mu^2}\,\frac{1-z}{z}\right)\right]}_{\text{transverse structure function}} + \underbrace{6z(1-z)}_{\text{longitudinal structure functions.}}\Big\}$$

note: $F_2(x,Q^2) = \sum_i e_i^2\, x\,[q_i(x,Q^2) + \bar{q}_i(x,Q^2)]$, $z = \frac{x}{y}$

"bare" distribution function

$$q(x,Q^2) = q_0(x) + \frac{\alpha_s}{4\pi}\frac{1}{1-\epsilon}\int_x^1 \frac{dy}{y}\, g(y)$$

observes experimentally

$$\Big\{[z^2+(1-z)^2]\left[-\frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} + \ln\left(\frac{Q^2}{4\pi\mu^2}\,\frac{1-z}{z}\right)\right] + 6z(1-z)\Big\}$$

~~Here~~ note:  contributes equally to the quark and antiquark distribution functions.

$$\bar{q}(x,Q^2) = q(x,Q^2)$$

B 4. $pp \to w^+ X$: initial gluons.

a. leading order cross section

$$\sigma_0 = \frac{\pi^2}{3} \alpha_w (1-\epsilon) \sum_{i,j} \int_{\tau_0}^1 dx_1 \int_{\tau_0/x_1}^1 dx_2 \, [q_{0i}(x_1) \bar{q}_j(x_2) + \bar{q}_i(x_1) q_j(x_2)] \, \delta(x_1 x_2 S - M_w^2)$$

where $\hat{s} = x_1 x_2 S$, $\tau_0 = \frac{M_w^2}{S}$

$$= \frac{\pi^2}{3} \frac{\alpha_w}{S} (1-\epsilon) \sum_{i,j} \int_{\tau_0}^1 \frac{dx_1}{x_1} [q_{0i}(x_1) \bar{q}_{0j}(\tau_0/x_1) + \bar{q}_{0i}(x_1) q_{0j}(\tau_0/x_1)]$$

where $q_0, \bar{q}_0$ labels the bare quark distribution function.

b. QCD correction ~~due~~ to the quark distribution function.

$$q_0(x) = q(x, Q^2) - \frac{\alpha_s}{4\pi} \frac{1}{1-\epsilon} \int_x^1 \frac{dy}{y} g(y) \times$$

$$\left\{ [z^2 + (1-z)^2] \left[-\frac{1}{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} + \ln\left(\frac{Q^2}{4\pi\mu^2} \frac{1-z}{z}\right) \right] + 6z(1-z) \right\}$$

only focus on a particular distribution function, say $\bar{q}_{0j}(\tau_0/x_1)$

$$\sigma_1 = -\frac{\pi^2}{3} \frac{\alpha_w}{S} (1-\epsilon) \frac{\alpha_s}{4\pi} \frac{1}{1-\epsilon} \sum_{i} \int_{\tau_0}^1 \frac{dx_1}{x_1} \int_{\tau_0/x_1}^1 \frac{dx_2}{x_2} q_{0i}(x_1) g(x_2) \times$$

$$\left\{ [z^2 + (1-z)^2] \left[-\frac{1}{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} + \ln\left(\frac{Q^2}{4\pi\mu^2} \frac{1-z}{z}\right) \right] + 6z(1-z) \right\}$$

where $z = \frac{\tau_0/x_1}{x_2} = \frac{\tau_0}{x_1 x_2}$

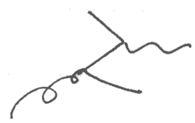
$$\sigma_1 = -\frac{\pi}{12}\alpha_s \alpha_W \sum_i \int_{\tau_0}^{1} dx_1 \int_{\tau_0/x_1}^{1} dx_2\, q_{0i}(x_1)\, g(x_2)\, \frac{1}{\hat{s}}$$

$$\left\{ (z^2+(1-z)^2)\left[-\frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} + \ln\left(\frac{Q^2}{4\pi\mu^2}\frac{1-z}{z}\right)\right] + 6z(1-z) \right\}$$

where $\hat{s} = x_1 x_2 s$

other term is similars to this.

C. QCD correction to the subprocess $q\bar{q} \to W^+$ due to initial gluon



$$\sigma_1 = \sum_i \int_{\tau_0}^{1} dx_1 \int_{\tau_0/x_1}^{1} dx_2 \left[q_{0i}(x_1)\, g(x_2) + \bar{q}_{0i}(x_1)\, g(x_2) + (x_1 \leftrightarrow x_2) \right] \hat{\sigma}(gq \to W + q)$$

For the first term.

$$\hat{\sigma}(gq \to W+q) = \frac{\alpha_s}{4\pi}\frac{\pi}{12}\frac{g^2}{\hat{s}} \left\{ [z'^2 + (1-z')^2]\left[-\frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} + \ln\left(\frac{M_W^2}{4\pi\mu^2}\frac{(1-z')^2}{z'}\right)\right] + \frac{3}{2} + z' - \frac{3}{2}z'^2 \right\}$$

where $z' = \frac{M_W^2}{\hat{s}} = \frac{M_W^2}{x_1 x_2 s} = \frac{\tau_0}{x_1 x_2} = z$

$$\hat{\sigma}(gq \to W+q) = \frac{\alpha_s \pi \alpha_W}{12}\frac{1}{\hat{s}} \left\{ [z^2 + (1-z)^2]\left[-\frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} + \ln\left(\frac{M_W^2}{4\pi\mu^2}\frac{(1-z)^2}{z}\right)\right] + \frac{3}{2} + z - \frac{3}{2}z^2 \right\}$$

$$\sigma_1 = \frac{\pi}{12}\alpha_s \alpha_W \sum_i \int_{\tau_0}^{1} dx_1 \int_{\tau_0/x_1}^{1} dx_2\, q_{0i}(x_1)\, g(x_2) \cdot \frac{1}{\hat{s}}$$

$$\left\{ [z^2+(1-z)^2] \ln\left(\frac{M_W^2}{4\pi\mu^2}\frac{(1-z)^2}{z}\right) + \frac{3}{2} - z - \frac{3}{2}z^2 \right\}$$

(b) + (c) collinear singularities cancel.

$$\sigma = \frac{\pi}{12}\alpha_s\alpha_w \sum_i \int_{\tau_0}^1 dx_1 \int_{\tau_0/x_1}^1 dx_2\, q_{0i}(x_1)\, g(x_2)\, \frac{1}{\hat{s}}$$

$$\left\{ \left(z^2+(1-z)^2\right) \ln\left(\frac{M_w^2}{4\pi\mu^2}\frac{(1-z)^2}{z} \times \frac{4\pi\mu^2}{Q^2}\frac{z}{1-z}\right) + \frac{3}{2} - 5z + \frac{9}{2}z^2 \right\}$$

$$= \frac{\pi}{12}\alpha_s\alpha_w \sum_i \int_{\tau_0}^1 dx_1 \int_{\tau_0/x_1}^1 dx_2\, q_{0i}(x_1)\, g(x_2)\, \frac{1}{\hat{s}}$$

$$\left\{ \left[z^2+(1-z)^2\right] \ln\left[\frac{M_w^2}{Q^2}(1-z)\right] + \frac{3}{2} - 5z + \frac{9}{2}z^2 \right\}$$

where $\hat{s} = x_1 x_2 s$, $\tau_0 = \frac{M_w^2}{s}$, $z = \frac{\tau_0}{x_1 x_2}$

Total cross section.

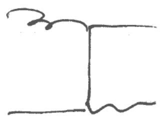
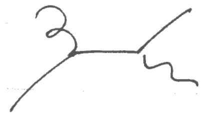
$$\sigma = \frac{\pi^2}{3}\frac{\alpha_w}{s} \sum_{i,j} \int_{\tau_0}^1 dx_1 \int_{\tau_0/x_1}^1 dx_2\, \delta(x_1 x_2 - \tau_0)$$

$$\times \left[q_i(x_1,Q^2)\,\bar{q}_j(x_2,Q^2) + \bar{q}_i(x_1,Q^2)\, q_j(x_2,Q^2) \right]$$

$$+ \frac{\pi}{12}\alpha_s\alpha_w \sum_i \int_{\tau_0}^1 dx_1 \int_{\tau_0/x_1}^1 dx_2 \left[q_{0i}(x_1)\, g(x_2) + \bar{q}_{0i}(x_1)\, g(x_2) + (x_1 \leftrightarrow x_2) \right]$$

$$\frac{1}{\hat{s}} \left\{ \left[z^2+(1-z)^2\right] \ln\left[\frac{M_w^2}{Q^2}(1-z)\right] + \frac{3}{2} - 5z + \frac{9}{2}z^2 \right\}$$

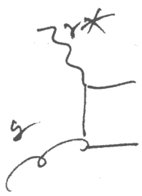
Summary 1:

①  

$$\hat{\sigma} = \frac{\pi}{12}\alpha_s \alpha_w \frac{1}{\hat{s}} \Big\{ [z^2 + (1-z)^2]\Big[-\frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} + \ln\Big(\frac{M_W^2}{4\pi\mu^2}\frac{(1-z)^2}{z}\Big)\Big] + \frac{3}{2} + z - \frac{3}{2}z^2 \Big\}$$

where $z = \frac{M_W^2}{\hat{s}}$

② quark PDF



$+$

$$q(x,Q^2) = q_0(x) + \frac{\alpha_s}{4\pi}\frac{1}{1-\epsilon}\int_x^1 \frac{dy}{y} g(y)$$

$$\Big\{ [z^2 + (1-z)^2]\Big[-\frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} + \ln\Big(\frac{Q^2}{4\pi\mu^2}\frac{1-z}{z}\Big)\Big] + 6z(1-z) \Big\}$$

The collinear singularities cancel $\underline{\underline{① + ②}}$

B5. ALTARELLI - PARISI equation

The collinear divergences leads to $\ln \frac{M_W^2}{Q^2}$, $Q^2 \simeq (1-10\ \text{GeV})^2$

$\Rightarrow$ $q(x, Q^2) \Longrightarrow q(x, M_W^2)$ we can eliminate the $\ln \frac{M_W^2}{Q^2}$ term

$$\frac{d}{d\ln Q^2} q(x, Q^2) = \frac{\alpha_s}{4\pi} \frac{1}{1-\epsilon} \int_x^1 \frac{dy}{y} g(y) \left[z^2 + (1-z)^2 \right]$$

$$= 2 \frac{\alpha_s}{4\pi} \int_x^1 \frac{dy}{y} g(y) P_{qg}\left(\frac{x}{y}\right)$$

where $P_{qg}(z) = \frac{1}{2}\left[z^2 + (1-z)^2\right]$ Splitting function $z = \frac{x}{y}$

$\downarrow$

quarks coming from gluon splittings

$$q(x, M_W^2) = q(x, Q^2) + \frac{\alpha_s}{4\pi} \ln \frac{M_W^2}{Q^2} \int_x^1 \frac{dy}{y} g(y) \left[z^2 + (1-z)^2\right]$$

a. zeroth-order cross section

$$\sigma_0 = \frac{\pi^2}{3} \frac{\alpha_W}{s} \sum_{i,j} \int_{\tau_0}^1 dx_1 \int_{\tau_0/x_1}^{1} dx_2 \, \delta(x_1 x_2 - \tau_0)$$

$$\left[q_i(x_1, Q^2) \bar{q}_j(x_2, Q^2) + \bar{q}_i(x_1, Q^2) q_j(x_2, Q^2) \right]$$

$$= \frac{\pi^2}{3} \frac{\alpha_W}{s} \sum_{i,j} \int_{\tau_0}^1 dx_1 \int_{\tau_0/x_1}^1 dx_2 \, \delta(x_1 x_2 - \tau_0)$$

$$\left\{ \left[q_i(x_1, M_W^2) - \frac{\alpha_s}{4\pi} \ln \frac{M_W^2}{Q^2} \int_{x_1}^1 \frac{dx_1'}{x_1'} g(x_1') \left[z^2 + (1-z)^2\right] \right] \cancel{\bar{q}_j(x_2, Q^2)} \right.$$

$$\left[\bar{q}_j(x_2, M_W^2) - \frac{\alpha_s}{4\pi} \ln \frac{M_W^2}{Q^2} \int_{x_2}^1 \frac{dx_2'}{x_2'} g(x_2') \left[z^2 + (1-z)^2\right] \right]$$

$$\left. + (q_i \to \bar{q}_i)(\bar{q}_j \to q_j) \right\}$$

$$\sigma_0 = \frac{\pi^2}{3}\frac{\alpha_w}{s} \sum_{i,j} \int_{\tau_0}^1 dx_1 \int_{\tau_0/x_1}^1 dx_2\, \delta(x_1 x_2 - \tau_0)$$

$$[q_i(x_1, M_W^2)\, \bar{q}_j(x_2, M_W^2) + \bar{q}_i(x_1, M_W^2)\, q_j(x_2, M_W^2)]$$

$$\left.\begin{aligned} &- \frac{\pi}{12 s}\alpha_s \alpha_w \sum_i \int_{\tau_0}^1 dx_1 \int_{\tau_0/x_1}^1 dx_2\, \delta(x_1 x_2 - \tau_0) \ln\frac{M_W^2}{Q^2} \\ &\Big\{ \int_{x_1}^1 \frac{dx_2'}{x_2'}\, g(x_2')\, [z^2 + (1-z)^2]\, \bar{q}_j(x_2, M_W^2) \\ &+ \int_{x_2}^1 \frac{dx_2'}{x_2'}\, g(x_2')\, [z'^2 + (1-z')^2]\, q_i(x_1, M_W^2) \Big\} \end{aligned}\right\} \Delta\sigma$$

$+ \cdots$

$$\Delta\sigma = -\frac{\pi}{12s}\alpha_s\alpha_w \Big[\sum \int_{\tau_0}^1 \frac{dx_1}{x_1} \int_{x_1}^1 \frac{dx_2'}{x_2'}\, g(x_2')\, [z^2 + (1-z)^2]\, \bar{q}_j\left(\frac{\tau_0}{x_1}, M_W^2\right) \ln\frac{M_W^2}{Q^2}$$

$$+ \sum \int_{\tau_0}^1 \frac{dx_1}{x_1} \int_{\tau_0/x_1}^1 \frac{dx_2'}{x_2'}\, g(x_2')\, [z'^2 + (1-z')^2]\, q_i(x_1, M_W^2) \ln\frac{M_W^2}{Q^2} \Big]$$

where $z = \frac{x_1}{x_2'}$, $z' = \frac{x_2}{x_2'} = \frac{\tau_0}{x_1 x_2'}$

$$\cancel{-\frac{\pi}{12s}\alpha_s\alpha_w \Big[\sum_i}$$

define: $x_1' = \frac{\tau_0}{x_1}$, then $z = \frac{x_1}{x_2'} = \frac{\tau_0}{x_1' x_2'}$

$$\Delta\sigma = -\frac{\pi}{12s}\alpha_s\alpha_w \Big[\sum \int_{\tau_0}^1 \frac{dx_1'}{x_1'} \int_{\tau_0/x_1'}^1 \frac{dx_2'}{x_2'}\, [z^2 + (1-z)^2]\, g(x_2')\, \bar{q}_j(x_1', M_W^2) \ln\frac{M_W^2}{Q^2}$$

$$+ \sum \int_{\tau_0}^1 \frac{dx_1}{x_1} \int_{\tau_0/x_1}^1 \frac{dx_2'}{x_2'}\, [z'^2 + (1-z')^2]\, g(x_2')\, q_i(x_1, M_W^2) \ln\frac{M_W^2}{Q^2} \Big]$$

The $\ln\frac{M_W^2}{Q^2}$ is canceled with $\sigma^{(1)}$

Total cross section:

$$\sigma = \frac{\pi^2}{3}\frac{\alpha_w}{s}\sum_{ij}\int_{\tau_0}^{1} dx_1 \int_{\tau_0/x_1}^{1} dx_2\, \delta(x_1 x_2 - \tau_0) \times$$

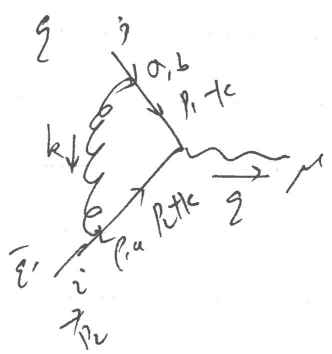
$$\left[q_i(x_1, M_W^2)\, \bar{q}_j(x_2, M_W^2) + \bar{q}_i(x_1, M_W^2)\, q_j(x_2, M_W^2) \right]$$

$$+ \frac{\pi}{12}\alpha_s \alpha_w \sum_i \int_{\tau_0}^{1} dx_1 \int_{\tau_0/x_1}^{1} dx_2 \left[q_i(x_1, M_W^2)\, g(x_2, \cancel{M_W^2}) + \bar{q}_i(x_1, M_W^2)\, g(x_2 \cancel{M_W^2}) + (x_1 \leftrightarrow x_2) \right]$$

$$\times \frac{1}{\hat{s}}\left\{ [z^2 + (1-z)^2] \ln(1-z) + \frac{3}{2} - 5z + \frac{7}{2} z^2 \right\}$$

where $\hat{s} = x_1 x_2 s$, $\tau_0 = \frac{M_W^2}{s}$, $z = \frac{\tau_0}{x_1 x_2}$

B-6. Virtual correction



$$G^\mu = \bar v(p_2)\,(i g_s \mu^\epsilon)^2\,(T^a T^b)_{ij} \int \frac{d^n k}{(2\pi)^n}\, \frac{-i g^{\sigma\rho}}{k^2+i\epsilon}\,\delta_{ab}\, \gamma_\rho\, \frac{i(-\not p_2 - \not k)}{(p_2+k)^2}\, \frac{i g_W}{\sqrt{2}}\gamma^\mu P_L$$

$$\frac{i(\not p_1 - \not k)}{(p_1-k)^2+i\epsilon}\,\gamma_\sigma\, u(p_1)$$

$$= -\frac{g_W}{\sqrt{2}}\, g_s^2 \mu^{2\epsilon}\,(T^a T^b)_{ij}\, \bar v(p_2) \int \frac{d^n k}{(2\pi)^n}\, \frac{\gamma_\sigma (\not p_2 + \not k)\gamma^\mu P_L (\not p_1 - \not k)\gamma_\sigma\, \delta^{ab}}{(p_2+k)^2 (p_1-k)^2 k^2}\, u(p_1)$$

$$= -\frac{g_W}{\sqrt{2}}\, C_F\, g_s^2 \mu^{2\epsilon}\, \bar v(p_2)\,\gamma^\sigma \gamma_\nu \gamma^\mu P_L \gamma_\rho \gamma_\sigma\, u(p_1)$$

$$\times \int \frac{d^n k}{(2\pi)^n}\, \frac{(p_2+k)^\nu (p_1-k)^\rho}{(p_2+k)^2 (p_1-k)^2 k^2}$$

Naive-γ^5 scheme: $\{\gamma^5, \gamma^\mu\} = 0$

$$= \frac{i g_W}{\sqrt{2}}\, C_F\, g_s^2\, \bar v(p_2)\,\gamma^\sigma \gamma_\nu \gamma^\mu \gamma_\rho \gamma_\sigma P_L\, u(p_1) \cdot M_1$$

$$M_1 = i\mu^{2\epsilon} \int \frac{d^n k}{(2\pi)^n}\, \frac{(p_2+k)^\nu (p_1-k)^\rho}{(p_2+k)^2 (p_1-k)^2 k^2}$$

Note. $\displaystyle \frac{1}{ab} = \int_0^1 dx\, \frac{1}{[xa + (1-x)b]^2}$, $\displaystyle \frac{1}{a^2 b} = \int_0^1 dx\, \frac{2x}{[xa+(1-x)b]^3}$

$$\frac{1}{(P_2+k)^2(P_1-k)^2} = \int_0^1 dx \frac{1}{[x(P_2+k)^2 + (1-x)(P_1-k)^2]^2}$$

$$= \int_0^1 dx \frac{1}{\{k^2 + 2k[P_2 x - (1-x)P_1]\}^2}$$

define: $p = x P_2 - (1-x) P_1$

$$\frac{1}{(P_2+k)^2(P_1-k)^2} = \int_0^1 dx \frac{1}{[k^2+2k\cdot p]^2}$$

$$\frac{1}{(P_2+k)^2(P_1-k)^2 k^2} = \int_0^1 dx \frac{1}{(k^2+2k\cdot p)^2 k^2} = \int_0^1 dx \int_0^1 dy \frac{2y}{[y(k^2+2k\cdot p) + (1-y)k^2]^3}$$

$$y(k^2+2k\cdot p) + (1-y)k^2 = k^2 + 2k\cdot y p$$

define: $l = k + y p \qquad \Delta = y p$

then $$\int \frac{d^n k}{k^2(P_2+k)^2(P_1-k)^2} = \int_0^1 dx \int_0^1 dy \, 2y \frac{d^n l}{(l^2-\Delta^2)^3}$$

where: $$\Delta^2 = y^2 p^2 = y^2[xP_2 - (1-x)P_1]^2 = -2y^2 x(1-x) P_1\cdot P_2$$
$$= -y^2 x(1-x) q^2$$

where $P_2^2 = P_1^2 = 0$, $\quad q = P_1 + P_2$

The numerator: $\quad P_2 + k = l + P_2 - yp = l + P_2 - \Delta$

$$P_1 - k = P_1 - l + yp = P_1 - l + \Delta$$

$$M_1 = i\mu^{2\epsilon}\int\frac{d^n k}{(2\pi)^n}\,\frac{(P_2+k)^\nu (P_1-k)^\rho}{(P_2+k)^2 (P_1-k)^2 k^2}$$

$$= i\mu^{2\epsilon}\int_0^1 dx\int_0^1 dy\, 2y\int\frac{d^n l}{(2\pi)^n}\,\frac{(l+P_2-a)^\nu (P_1-l+a)^\rho}{[l^2-\Delta^2]^3}$$

$$= i\mu^{2\epsilon}\int_0^1 dx\int_0^1 dy\, 2y\int\frac{d^n l}{(2\pi)^n}\left(\underbrace{\frac{-l^\nu l^\rho}{[l^2-\Delta^2]^3}}_{(\mathrm{I})} + \underbrace{\frac{(P_2-a)^\nu (P_1+a)^\rho}{[l^2-\Delta^2]^3}}_{(\mathrm{II})}\right)$$

note the odd power of l is vanished after integrated with l.

where $\Delta^2 = -y^2 x(1-x) q^2$

① (I): contains both UV and IR.

② (II): only IR $\quad l\to 0$

The IR divergence comes from the posibility of Δ^2 ~~being~~ zero.

To seperate the UV and IR in (I), we introduce a fixed scale Λ^2 and manipulate in (I) like this.

$$\frac{1}{[l^2-\Delta^2]^3} = \underbrace{\frac{1}{[l^2-\Delta^2]^3} - \frac{1}{[l^2-\Lambda^2]^3}}_{IR} + \underbrace{\frac{1}{[l^2-\Lambda^2]^3}}_{UV}$$

The uv part of M_1

$$M_1^{uv} = i\mu^{2\epsilon} \int_0^1 dx \int_0^1 dy\, 2y \int \frac{d^n l}{(2\pi)^n} \frac{-l^\nu l^\rho}{[l^2-\Delta^2]^3}$$

$$= -i\mu^{2\epsilon} \int_0^1 dx \int_0^1 dy\, 2y\, \frac{(-1)^2}{(4\pi)^{n/2}} \frac{i g^{\nu\rho}}{2} \frac{\Gamma(2-\frac{n}{2})}{\Gamma(3)} \left(\frac{1}{\Delta^2}\right)^{2-\frac{n}{2}}$$

with $n = 4 - 2\epsilon_{uv}$, $\epsilon_{uv} > 0$

$$= -i\mu^{2\epsilon} \int_0^1 dx \int_0^1 dy\, 2y\, \frac{i g^{\nu\rho}}{(4\pi)^{\frac{n}{2}} \times 2} \frac{\Gamma(\epsilon_{uv})}{2} \left(\frac{1}{\Delta^2}\right)^{\epsilon_{uv}}$$

$$= \frac{\mu^{2\epsilon} g^{\nu\rho}}{16\pi^2} \left(\frac{1}{4\pi}\right)^{-\epsilon} \frac{\Gamma(\epsilon_{uv})}{4} \int_0^1 dx \int_0^1 dy \cdot 2y \cancel{\left[\frac{1}{y^2 x(1-x) q^2}\right]^{\epsilon_{uv}}} \left[\frac{1}{\Lambda^2}\right]^{\epsilon_{uv}}$$

$$= \frac{1}{16\pi^2} \left[\frac{\Lambda^2}{4\pi\mu^2}\right]^{-\epsilon_{uv}} \frac{1}{4} g^{\nu\rho}$$

γ matrix term:

$$\gamma^\sigma \gamma_\nu \gamma^\mu \gamma_\rho \gamma_\sigma \times \frac{1}{4} g^{\nu\rho}$$

$$= \frac{1}{4} \gamma^\sigma \gamma^\rho \gamma^\mu \gamma_\rho \gamma_\sigma \qquad \gamma^\rho \gamma^\mu \gamma_\rho = -(n-2)\gamma^\mu$$

$$= \frac{1}{4} \gamma^\sigma [-2(1-\epsilon)\gamma^\mu] \gamma_\sigma$$

$$= \frac{1}{4} [-2(1-\epsilon)]^2 \gamma^\mu = (1-\epsilon)^2 \gamma^\mu$$

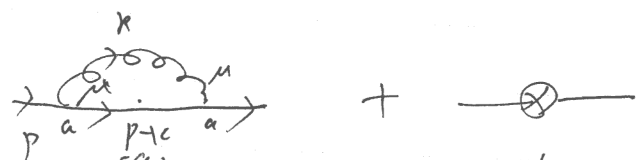
thus: $G^\mu_{UV} = \frac{ig_w}{\sqrt{2}} C_F g_s^2 (1-\epsilon)^2 \gamma^\mu P_L \times \frac{1}{16\pi^2} \left[\frac{\Lambda^2}{4\pi\mu^2}\right]^{-\epsilon} \Gamma(\epsilon)$

$$= \frac{ig_w}{\sqrt{2}} C_F \frac{\alpha_s}{4\pi} \gamma^\mu P_L \left\{\left(1-\epsilon \ln\frac{\Lambda^2}{4\pi\mu^2}\right)\left(\frac{1}{\epsilon}-\gamma_E\right)\right\}(1-\epsilon)^2$$

$$= \frac{ig_w}{\sqrt{2}} \gamma^\mu P_L \frac{\alpha_s}{4\pi} C_F \left\{\frac{1}{\epsilon} - \ln\frac{\Lambda^2}{\bar{\mu}^2} - 2\right\}$$

where $\frac{1}{\bar{\mu}^2} = \frac{e^{\gamma_E}}{4\pi\mu^2}$, γ_E is Euler constant.

✡ wave-function of fermion.



(a) (b)

$$(a) = \mu^{2\epsilon}\int\frac{d^nk}{(2\pi)^n}(ig_s)^2(-i)(i)(T^aT^a)\frac{\gamma^\mu(\not{p}-\not{k})\gamma_\mu}{k^2(p-k)^2}$$

$$= -g_s^2\mu^{2\epsilon}C_F\int\frac{d^nk}{(2\pi)^n}\frac{[-2(1-\epsilon)](\not{p}-\not{k})}{k^2(p-k)^2}$$

$$= -g_s^2\mu^{2\epsilon}C_F\int_0^1 dx\int\frac{d^nk}{(2\pi)^n}\frac{[-2(1-\epsilon)](\not{p}-\not{k})}{[k^2-2p\cdot k x+xp^2]^2}$$

on-shell $p^2=0$

define $l = k - xp$

$$= -g_s^2\mu^{2\epsilon}C_F[-2(1-\epsilon)]\int_0^1 dx\int\frac{d^nl}{(2\pi)^n}\frac{(1-x)\not{p}-\not{l}}{(l^2)^2}$$

$$= 2g_s^2\mu^{2\epsilon}C_F(1-\epsilon)\not{p}\int_0^1 dx\int\frac{d^nl}{(2\pi)^n}\frac{(1-x)}{(l^2)^2}$$

$$\frac{1}{(l^2)^2} = \underbrace{\frac{1}{(l^2)^2} - \frac{1}{(l^2-\Omega^2)^2}}_{IR} + \underbrace{\frac{1}{(l^2-\Omega^2)^2}}_{UV}$$

$$\mu^{2\epsilon}\int\frac{d^n l}{(2\pi)^n}\frac{1}{(l^2-\Omega^2)^2} = \mu^{2\epsilon}\frac{(-1)^2\, i}{(4\pi)^{\frac{n}{2}}}\frac{\Gamma(2-\frac{n}{2})}{\Gamma(2)}\left(\frac{1}{\Omega^2}\right)^{2-\frac{n}{2}}$$

$$= \frac{i}{16\pi^2}\left(\frac{\Omega^2}{4\pi\mu^2}\right)^{-\epsilon}\Gamma(\epsilon)$$

$$= \frac{i}{16\pi^2}\left\{\frac{1}{\epsilon} - \ln\frac{\Omega^2}{\bar{\mu}^2}\right\}$$

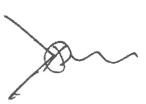
UV part of self energy:

$$= 2g_s^2\, C_F(1-\epsilon)\not{p}\int_0^1 dx\,(1-x)\,\frac{i}{16\pi^2}\left[\frac{1}{\epsilon} - \ln\frac{\Omega^2}{\bar{\mu}^2}\right]$$

$$= i\not{p}\,\frac{\alpha_s}{4\pi}C_F\left\{\frac{1}{\epsilon} - \ln\frac{\Omega^2}{\bar{\mu}^2} - 1\right\}$$

 +  finite term. MS scheme

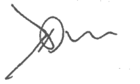
$$\delta Z = -\frac{\alpha_s}{4\pi}C_F\frac{1}{\epsilon}$$



$$= \frac{i g_w}{\sqrt{2}}\gamma^\mu P_L\cdot\delta Z = \frac{i g_w}{\sqrt{2}}\left(-\frac{\alpha_s}{4\pi}C_F\frac{1}{\epsilon}\right)\gamma^\mu P_L$$

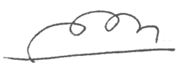
uv results:

 =  $\left\{ \frac{\alpha_s}{4\pi} C_F \left(\frac{\Lambda^2}{4\pi\mu^2}\right)^{-\epsilon} \Gamma(\epsilon)(1-\epsilon)^2 \right\}$

 =  $\left\{ -\frac{\alpha_s}{4\pi} C_F \left(\frac{\Omega}{4\pi\mu^2}\right)^{-\epsilon} \Gamma(\epsilon)(1-\epsilon) \right\}$

IR Part:

In order to calculate the cross section, we need to calculate the residue of $\frac{i}{p}$ pole.

IR  $= \mu^{2\epsilon} \int \frac{d^n l}{(2\pi)^n} \left[\frac{1}{(l^2)^2} - \frac{1}{(l^2-\Omega^2)^2} \right]$

$$= \mu^{2\epsilon} \int \frac{d^n l}{(2\pi)^n} \frac{(\Omega^4 - 2l^2\Omega^2)}{(l^2)^2 (l^2-\Omega^2)^2}$$

$$= \mu^{2\epsilon} \int_0^1 dx \int \frac{d^n l}{(2\pi)^n} \frac{(\Omega^4 - 2l^2\Lambda^2)\, 6x(1-x)}{(l^2 - x\Omega^2)^4}$$

define $\Delta = x\Omega^2$

$$= \mu^{2\epsilon} \int_0^1 dx \left[\frac{(-1)^4 i}{(4\pi)^{\frac{n}{2}}} \frac{\Gamma(4-\frac{n}{2})}{\Gamma(4)} \left(\frac{1}{\Delta}\right)^{4-\frac{n}{2}} \Omega^4 - 2\Omega^2 \frac{(-1)^3 i}{(4\pi)^{\frac{n}{2}}} \frac{n}{2} \frac{\Gamma(3-\frac{n}{2})}{\Gamma(4)} \left(\frac{1}{\Delta}\right)^{3-\frac{n}{2}} \right] 6x(1-x)$$

$$= \mu^{2\epsilon} \int_0^1 dx \frac{i}{(4\pi)^{\frac{n}{2}}} \frac{\Gamma(3-\frac{n}{2})}{\Gamma(4)} \left(\frac{1}{\Delta}\right)^{3-\frac{n}{2}} \left\{ (3-\tfrac{n}{2}) \frac{1}{\Delta} \Omega^4 + \frac{n}{2} 2\Omega^2 \right\} 6x(1-x)$$

$$= \frac{i}{16\pi^2}\frac{\mu^{2\epsilon}}{(4\pi)^{-\epsilon}}\int_0^1 dx \left\{\Gamma(1+\epsilon)\left(\frac{1}{x\Omega^2}\right)^{1+\epsilon}\left[(1+\epsilon)\frac{1}{x\Omega^2}\Omega^4 + (2-\epsilon)2\Omega^2\right] \times x(1-x)\right\}$$ B6-8

$$= \frac{i}{16\pi^2}\left(\frac{\Omega^2}{4\pi\mu^2}\right)^{-\epsilon}\Gamma(1+\epsilon)\int_0^1 dx\; x(1-x)\, x^{-1-\epsilon}\left\{(1+\epsilon)x^{-1}+4-2\epsilon\right\}$$

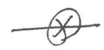
$$= \frac{i}{16\pi^2}\left(\frac{\Omega^2}{4\pi\mu^2}\right)^{-\epsilon}\Gamma(1+\epsilon)\left[(1+\epsilon)B(-\epsilon,2) + (4-2\epsilon)B(1-\epsilon,2)\right]$$

$$= -\frac{i}{16\pi^2}\left(\frac{\Omega^2}{4\pi\mu^2}\right)^{-\epsilon}\Gamma(\epsilon)$$

$$= -\frac{i}{16\pi^2}\left[\frac{1}{\epsilon} - \ln\frac{\Omega^2}{\bar{\mu}^2}\right]$$

IR for  $= i\not{p}\,(-1)\frac{\alpha_s}{4\pi}C_F\left\{\frac{1}{\epsilon} - \ln\frac{\Omega^2}{\bar{\mu}^2} - 1\right\}$

Thus the quark self-energy is

$-i\Sigma =$  $+$  $= -i\not{p}\,\frac{\alpha_s}{4\pi}C_F\frac{1}{\epsilon_{IR}}$

$$\Sigma = \not{p}\,\frac{\alpha_s}{4\pi}C_F\frac{1}{\epsilon_{IR}}$$

 $= \dfrac{i}{\not{p} - \Sigma} = \dfrac{i}{\not{p}}\left[1 + \dfrac{\alpha_s}{4\pi}C_F\dfrac{1}{\epsilon_{IR}}\right]$

thus the residue is

$$R_4 = 1 + \frac{\alpha_s}{4\pi}C_F\frac{1}{\epsilon_{IR}}$$

* IR piece of vertex correction

$$M^{IR}_{1(\square)} = i\mu^{2\epsilon}\int_0^1 dx \int_0^1 dy \; 2y \int \frac{d^n l}{(2\pi)^n} \left(-l^\nu l^\rho\right) \left[\frac{1}{[l^2-\Delta^2]^3} - \frac{1}{[l^2-\Lambda^2]^3} \right]$$

where $\Delta^2 = -y^2 x(1-x) q^2$, $l^\nu l^\rho \to \dfrac{l^2 g^{\nu\rho}}{n}$

$$= -i\mu^{2\epsilon}\frac{g^{\nu\rho}}{n}\int_0^1 dx\int_0^1 dy \; 2y \int\frac{d^n l}{(2\pi)^n}\left[\frac{1}{(l^2-\Delta^2)^2} - \frac{1}{[l^2-\Lambda^2]^2} + \frac{\Delta^2}{(l^2-\Delta^2)^3} - \frac{\Lambda^2}{(l^2-\Lambda^2)^3}\right]$$

$$= -\frac{i g^{\nu\rho}\mu^{2\epsilon}}{n}\int_0^1 dx\int_0^1 dy \; 2y \int\frac{d^n l}{(2\pi)^n}$$

$$\left\{ \underbrace{\frac{1}{[l^2-\Delta^2]^2} - \frac{1}{(l^2)^2}}_{①} + \underbrace{\frac{1}{(l^2)^2} - \frac{1}{(l^2-\Lambda^2)^2}}_{②} + \underbrace{\frac{\Delta^2}{(l^2-\Delta^2)^3}}_{③} - \underbrace{\frac{\Lambda^2}{(l^2-\Lambda^2)^3}}_{④} \right\}$$

$$① = -\frac{i g^{\nu\rho}\cancel{\mu^{2\epsilon}}}{n}\int_0^1 dx\int_0^1 dy \; 2y\cdot(-1)\left(-\frac{i}{16\pi^2}\right)\left(\frac{\Delta^2}{4\pi\mu^2}\right)^{-\epsilon}\Gamma(\epsilon)$$

$$= -\frac{i g^{\nu\rho}\cancel{\mu^{2\epsilon}}}{n}\cdot\frac{i}{16\pi^2}\Gamma(\epsilon)\left(\frac{q^2}{4\pi\mu^2}\right)^{-\epsilon} \times \int_0^1 dx\int_0^1 dy\; 2y\left[-y^2x(1-x)\right]^{-\epsilon}$$

$$= \frac{g^{\nu\rho}}{n}\frac{1}{16\pi^2}\Gamma(\epsilon)\left(\frac{q^2}{4\pi\mu^2}\right)^{-\epsilon}(-1)^{-\epsilon}\times 2\, B(2-2\epsilon,1)\, B(1-\epsilon,1-\epsilon)$$

$$= \frac{g^{\nu\rho}}{4-2\epsilon}\frac{1}{16\pi^2}\Gamma(\epsilon)\left(\frac{q^2}{4\pi\mu^2}\right)^{-\epsilon}(-1)^{-\epsilon}\times 2\;\frac{\Gamma(2-2\epsilon)}{\Gamma(3-2\epsilon)}\;\frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}$$

$$② = -\frac{i g^{\nu\rho}}{4-2\epsilon}\int_0^1 dx\int_0^1 dy\; 2y\cdot\left(-\frac{i}{16\pi^2}\right)\left(\frac{\Lambda^2}{4\pi\mu^2}\right)^{-\epsilon}\Gamma(\epsilon)$$

$$= -\frac{g^{\nu\rho}}{4-2\epsilon}\frac{1}{16\pi^2}\Gamma(\epsilon)\left(\frac{\Lambda^2}{4\pi\mu^2}\right)^{-\epsilon}$$

$$③ = \frac{-ig^{vp}\mu^{2\epsilon}}{4-2\epsilon}\int_0^1 dx\int_0^1 dy\; 2y\int\frac{d^n l}{(2\pi)^n}\frac{\Delta^2}{(l^2-\Delta^2)^3}$$

$$= -\frac{ig^{vp}}{4-2\epsilon}\mu^{2\epsilon}\int_0^1 dx\int_0^1 dy\; 2y\;\frac{(-1)^3 i}{(4\pi)^{\frac{n}{2}}}\frac{\Gamma(3-\frac{n}{2})}{\Gamma(3)}\left(\frac{1}{\Delta^2}\right)^{2-\frac{n}{2}}$$

$$= -\frac{g^{vp}}{4-2\epsilon}\mu^{2\epsilon}\frac{1}{16\pi^2}\frac{\Gamma(1+\epsilon)}{(4\pi)^{-\epsilon}\times 2}\int_0^1 dx\int_0^1 dy\; 2y\left[\frac{1}{-y^2x(1-x)q^2}\right]^{\epsilon}$$

$$= -\frac{g^{vp}}{16\pi^2}\frac{\Gamma(1+\epsilon)}{4-2\epsilon}\left[\frac{q^2}{4\pi\mu^2}\right]^{-\epsilon}(-1)^{-\epsilon}\int_0^1 dx\int_0^1 dy\; \cancel{2}y\left[\frac{1}{y^2x(1-x)}\right]^{\epsilon}$$

$$= -\frac{g^{vp}}{16\pi^2}\frac{\Gamma(1+\epsilon)}{4-2\epsilon}\left(\frac{q^2}{4\pi\mu^2}\right)^{-\epsilon}(-1)^{-\epsilon}\;\cancel{2}\;B(2-2\epsilon,1)\;B(1-\epsilon,1-\epsilon)$$

$$④\;\frac{-ig^{vp}\mu^{2\epsilon}}{4-2\epsilon}\int_0^1 dx\int_0^1 dy\; 2y\int\frac{d^n l}{(2\pi)^n}\left(-\frac{\Lambda^2}{(l^2-\Lambda^2)^3}\right)$$

$$= \frac{ig^{vp}\mu^{2\epsilon}}{4-2\epsilon}\Lambda^2\frac{(-1)^3 i}{(4\pi)^{\frac{n}{2}}}\frac{\Gamma(3-\frac{n}{2})}{\Gamma(3)}\left(\frac{1}{\Lambda^2}\right)^{2-\frac{n}{2}}$$

$$= \frac{g^{vp}}{16\pi^2}\frac{\Gamma(1+\epsilon)}{4-2\epsilon}\frac{1}{2}\left[\frac{\Lambda^2}{4\pi\mu^2}\right]^{-\epsilon}$$

$$②+④ = -\frac{g\gamma^\rho}{4-2\epsilon}\frac{1}{16\pi^2}\Gamma(\epsilon)\left(\frac{\Lambda^2}{4\pi\mu^2}\right)^{-\epsilon}$$

$$+\frac{g\gamma^\rho}{16\pi^2}\frac{\Gamma(1+\epsilon)}{4-2\epsilon}\frac{1}{2}\left(\frac{\Lambda^2}{4\pi\mu^2}\right)^{-\epsilon}$$

$$= \frac{1}{16\pi^2} g\gamma^\rho\left[-\frac{1}{4-2\epsilon}+\frac{1}{4-2\epsilon}\frac{\epsilon}{2}\right]\Gamma(\epsilon)\left(\frac{\Lambda^2}{4\pi\mu^2}\right)^{-\epsilon}$$

$$= -\frac{1}{16\pi^2}\frac{g\gamma^\rho}{4}\Gamma(\epsilon)\left(\frac{\Lambda^2}{4\pi\mu^2}\right)^{-\epsilon} \qquad \text{where } \epsilon=\epsilon_{IR}$$

which has the exactly opposite Λ^2 dependence of $M_1^{\mu\nu}$.

$$①+③ = \frac{g\gamma^\rho}{4-2\epsilon}\frac{1}{16\pi^2}\Gamma(\epsilon)\left(\frac{-q^2}{4\pi\mu^2}\right)^{-\epsilon}\times 2\,\frac{\Gamma(2-2\epsilon)}{\Gamma(3-2\epsilon)}\frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}$$

$$-\frac{g\gamma^\rho}{16\pi^2}\frac{\Gamma(1+\epsilon)}{4-2\epsilon}\left(\frac{-q^2}{4\pi\mu^2}\right)^{-\epsilon} B(2-2\epsilon,1)\,B(1-\epsilon,1-\epsilon)$$

$$= \frac{1}{16\pi^2}\left(\frac{-q^2}{4\pi\mu^2}\right)^{-\epsilon}\Gamma(\epsilon)\frac{g\gamma^\rho}{2}\frac{\Gamma(2-2\epsilon)}{\Gamma(3-2\epsilon)}\frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}$$

$$= \frac{1}{16\pi^2}\left(\frac{-q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{g\gamma^\rho}{2}\Gamma(\epsilon)\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\frac{\Gamma(1-\epsilon)}{(2-2\epsilon)(1-2\epsilon)}$$

$$= \frac{1}{16\pi^2}\left(\frac{-q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{g\gamma^\rho}{4}\Gamma(\epsilon)\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[1+(\gamma_E+3)\epsilon+O(\epsilon^2)\right]$$

Now consider II term in M_1, which has only IR

$$M_{1,II}^{IR} = i\mu^{2\epsilon}\int_0^1 dx\int_0^1 dy\; 2y\int\frac{d^n l}{(2\pi)^n}\frac{(p_2-\Delta)^\nu(p_1+\Delta)^\rho}{[l^2-\Delta^2]^3}$$

where $\Delta^2 = -y^2x(1-x)q^2$

$$= i\mu^{2\epsilon}\int_0^1 dx\int_0^1 dy\; 2y\,(p_2-\Delta)^\nu(p_1+\Delta)^\rho\frac{(-1)^3 i}{(4\pi)^{\frac{n}{2}}}\frac{\Gamma(3-\frac{n}{2})}{\Gamma(3)}\left(\frac{1}{\Delta^2}\right)^{3-\frac{n}{2}}$$

$$= \mu^{2\epsilon}\int_0^1 dx\int_0^1 dy\; 2y\,(p_2-\Delta)^\nu(p_1+\Delta)^\rho\frac{1}{(4\pi)^{2-\epsilon}}\frac{\Gamma(1+\epsilon)}{2}(-1)^{-1-\epsilon}\left[y^2x(1-x)q^2\right]^{-1-\epsilon}$$

$$= (-1)^{-1+\epsilon} \frac{1}{16\pi^2} \left(\frac{q^2}{4\pi\mu^2}\right)^{-\epsilon} \Gamma(\epsilon) \frac{\epsilon}{2} \frac{1}{q^2} \int_0^1 dx \int_0^1 dy\, x$$

$$\left\{ 2y \left[y^2 x(1-x) \right]^{-1-\epsilon} (p_2-\Delta)^\nu (p_1+\Delta)^\rho \right\}$$

where $\Delta = y[x p_2 - (1-x) p_1]$

Therefore:

$$\bar{v}(p_2) \gamma^\sigma \gamma_\nu \gamma^\mu \gamma_\rho \gamma_\sigma p_2\, u(p_1) \cdot \{ p_1^\nu p_1^\rho, p_2^\nu p_2^\rho, p_1^\nu p_2^\rho, p_2^\nu p_1^\rho \}$$

$$\Rightarrow \bar{v}(p_2) \gamma^\mu p_2\, u(p_1) \{ 0, 0, -2\epsilon q^2, 2q^2 \}$$

From $\gamma^\sigma \gamma_\nu \gamma^\mu \gamma_\rho \gamma_\sigma \frac{1}{4} g^{\nu\rho} = (1-\epsilon)^2 \gamma^\mu$

thus we may just as well make the replacement in the calculation

$$\{ p_1^\nu p_1^\rho, p_2^\nu p_2^\rho, p_1^\nu p_2^\rho, p_2^\nu p_1^\rho \} \to \{0, 0, -2\epsilon q^2, 2q^2\} \frac{1}{(1-\epsilon)^2} \frac{1}{4} g^{\nu\rho}$$

$$\Rightarrow (p_2-\Delta)^\nu (p_1+\Delta)^\rho$$

$$= (p_2 - y[x p_2 - (1-x) p_1])^\nu (p_1 + y[x p_2 - (1-x) p_1])^\rho$$

$$= [(1-yx) p_2 + y(1-x) p_1]^\nu [(1-y(1-x)) p_1 + xy p_2]^\rho$$

$$= (1-yx)[1-y(1-x)] \underline{\underline{p_2^\nu p_1^\rho}} + xy(1-yx) p_2^\nu p_2^\rho$$

$$+ y(1-x)[1-y(1-x)] p_1^\nu p_1^\rho + y(1-x) xy \underline{\underline{p_1^\nu p_2^\rho}}$$

$$\Rightarrow \{ (1-yx)[1-y(1-x)] 2q^2 + y(1-x) xy (-2\epsilon) q^2 \} \frac{1}{(1-\epsilon)^2} \frac{1}{4} g^{\nu\rho}$$

$$= \{ 2(1-yx)[1-y(1-x)] - 2y^2 x(1-x)\epsilon \} \frac{1}{(1-\epsilon)^2} \frac{q^2}{4} g^{\nu\rho}$$

$$\Rightarrow \frac{1}{q^2}\int_0^1 dx \int_0^1 dy \; 2y \; [y^2 x(1-x)]^{-1-\epsilon} \; (P_2-\Delta)^\nu (P_1+\Delta)^\rho$$

$$= \int_0^1 dx \int_0^1 dy \; 2y \, [y^2 x(1-x)]^{-1-\epsilon} \; \{ 2(1-yx)[1-y(1-x)] - 2y^2 x(1-x)\epsilon \} \frac{1}{(1-\epsilon)^2}\frac{1}{4} g^{\nu\rho}$$

$$= \frac{1}{(1-\epsilon)^2}\frac{1}{4} g^{\nu\rho} \; \{ 4B(-\epsilon,-\epsilon)\, \overset{\checkmark}{B(-2\epsilon,1)} - 4B(-\epsilon,-\epsilon)\, \overset{\checkmark}{B(1-2\epsilon,1)}$$

$$-4B(2-\epsilon,\overset{\checkmark}{-\epsilon})\, B(2-2\epsilon,1) + 4B(1-\epsilon,\overset{\checkmark}{-\epsilon})\, B(2-2\epsilon,1)$$

$$-4\epsilon\, B(1-\epsilon,1-\epsilon)\, \overset{\checkmark}{B(2-2\epsilon,1)} \}$$

$$= \frac{1}{(1-\epsilon)^2} g^{\nu\rho} \; \{ \frac{\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)}\frac{\Gamma(-2\epsilon)}{\Gamma(1-2\epsilon)} - \frac{\Gamma^2(-\epsilon)\,\Gamma(1-2\epsilon)}{\Gamma(-2\epsilon)\Gamma(2-2\epsilon)} - \frac{\Gamma(2-\epsilon)\Gamma(-\epsilon)}{\Gamma(2-2\epsilon)}\frac{\Gamma(2-2\epsilon)}{\Gamma(3-2\epsilon)}$$

$$+ \frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}\frac{\Gamma(2-2\epsilon)}{\Gamma(3-2\epsilon)} - \epsilon\frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}\frac{\Gamma(2-2\epsilon)}{\Gamma(3-2\epsilon)} \}$$

$$= \frac{1}{(1-\epsilon)^2} g^{\nu\rho} \; \{ \frac{\Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)} - \frac{\Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{-2\epsilon}{1-2\epsilon}\right) - \frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)(1-\epsilon)}{\Gamma(1-2\epsilon)(2-2\epsilon)(1-2\epsilon)}$$

$$+ \frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}\frac{1}{2-2\epsilon} - \epsilon\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\frac{1}{(2-2\epsilon)(1-2\epsilon)} \}$$

$$= \frac{1}{(1-\epsilon)^2} g^{\nu\rho} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \{ \frac{\Gamma(-\epsilon)}{-\epsilon} - \frac{\Gamma(-\epsilon)}{-\epsilon}\frac{-2\epsilon}{1-2\epsilon} - \frac{\Gamma(-\epsilon)(1-\epsilon)}{(2-2\epsilon)(1-2\epsilon)}$$

$$+ \frac{\Gamma(-\epsilon)}{2-2\epsilon} - \epsilon\,\Gamma(1-\epsilon)\frac{1}{(2-2\epsilon)(1-2\epsilon)} \}$$

$$= \frac{1}{(1-\epsilon)^2} g^{\nu\rho} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \{ \cdots \}$$

$$M^{IR}_{1,II} = \frac{1}{16\pi^2}\left(\frac{-q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{g\gamma^\rho}{4(1-\epsilon)^2}\,\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left\{-\frac{2}{\epsilon^2}-\frac{4}{\epsilon}-9\cancel{+2\frac{\pi^2}{\epsilon}}-\frac{\pi^2}{3}\right\}$$

Summary:

$$(1)+(3) = \frac{ig_w}{\sqrt{2}}\,C_F\, g_s^2\,\frac{1}{16\pi^2}\left(\frac{-q^2}{4\pi\mu^2}\right)^{-\epsilon}(1-\epsilon)^2\gamma^\mu P_L\,\Gamma(\epsilon)\,\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[1+(\beta-\beta)\epsilon+O(\epsilon^2)\right]$$

$$= \frac{ig_w}{\sqrt{2}}\gamma^\mu P_L\cdot C_F\frac{g_s^2}{16\pi^2}\left(\frac{-q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[\frac{1}{\epsilon}+1\right]$$

$$= \left[\frac{\alpha_s}{4\pi}C_F\left(\frac{-q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[\frac{1}{\epsilon}+1\right]\right]_{IR}$$

$$(2)+(4) = \frac{ig_w}{\sqrt{2}}\,C_F\, g_s^2\left(-\frac{1}{16\pi^2}\right)\cdot(1-\epsilon)^2\gamma^\mu P_L\,\Gamma(\epsilon)\left(\frac{\Lambda^2}{4\pi\mu^2}\right)^{-\epsilon}$$

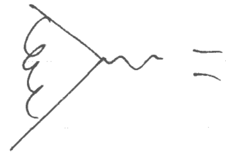
$$= \frac{ig_w}{\sqrt{2}}C_F\left(-\frac{\alpha_s}{4\pi}\right)\gamma^\mu P_L\,\Gamma(\epsilon)\left(\frac{\Lambda^2}{4\pi\mu^2}\right)^{-\epsilon}(1-\epsilon)^2$$

$$= \left[-\frac{\alpha_s}{4\pi}C_F\left(\frac{\Lambda^2}{4\pi\mu^2}\right)^{-\epsilon}\Gamma(\epsilon)(1-\epsilon)^2\right]_{IR}$$

From: $$M_{1,II} = \frac{ig_w}{\sqrt{2}}C_F g_s^2\frac{1}{16\pi^2}\left(-\frac{q^2}{4\pi\mu^2}\right)^{-\epsilon}\cancel{\epsilon}\,\gamma^\mu P_L\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left\{-\frac{2}{\epsilon^2}-\frac{4}{\epsilon}-9-\frac{\pi^2}{3}\right\}$$

$$= \left[\frac{\alpha_s}{4\pi}C_F\left(\frac{-q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left\{-\frac{2}{\epsilon^2}-\frac{4}{\epsilon}-9-\frac{\pi^2}{3}\right\}\right]_{IR}$$

Summary virtual correction:

①  $=$  $\times \Big\{ \frac{\alpha_s}{4\pi} C_F \left(\frac{\Lambda^2}{4\pi\mu^2}\right)^{-\epsilon} \Gamma(\epsilon)(1-\epsilon^2)\Big|_{UV}$

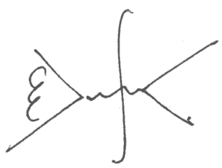
$- \frac{\alpha_s}{4\pi} C_F \left(\frac{\Lambda^2}{4\pi\mu^2}\right)^{-\epsilon} \Gamma(\epsilon)(1-\epsilon)^2 \Big|_{IR}$ ②+④

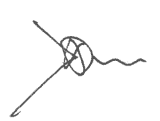
$+ \frac{\alpha_s}{4\pi} C_F \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left\{\frac{1}{\epsilon} + 1\right\}_{IR}$ ①+③

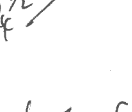
$+ \frac{\alpha_s}{4\pi} C_F \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left\{-\frac{2}{\epsilon^2} - \frac{4}{\epsilon} - 9 + \frac{2}{3}\pi^2\right\}_{IR} \Big\}$

note: $(-1)^{-\epsilon} = 1 - 2i\pi\epsilon - \frac{\pi^2}{2}\epsilon^2$ is used

only real part will survive



②  $=$  $\left\{ -\frac{\alpha_s}{4\pi} C_F \left(\frac{\Omega^2}{4\pi\mu^2}\right)^{-\epsilon} \Gamma(\epsilon)(1-\epsilon)\right\}_{UV}$

$R_4^{1/2}$  $R_4^{1/2}$ $=$ $\left\{ 1 + \frac{\alpha_s}{4\pi} C_F \left(\frac{\Omega^2}{4\pi\mu^2}\right)^{-\epsilon} \Gamma(\epsilon)(1-\epsilon)\right\}_{IR}$

chose $(\Omega^2)^{-\epsilon} = (1-\epsilon)(\Lambda^2)^{-\epsilon}$, then

note: the results not depending on the chose of Ω^2.

UV canceled with IR ① and ②

 $+$  $+$ $R_4^{1/2}$  $R_4^{1/2}$ $=$  $(1 + V_{virtual})$

with $V_{virtual} = \frac{\alpha_s}{4\pi} C_F \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left\{-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{2\pi^2}{3}\right\}$

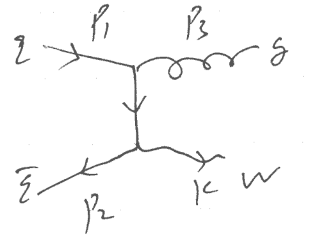
Thus: the ~~xxx~~ cross section:

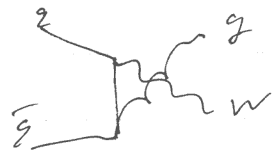
$$\hat{\sigma}_{virtual} = 2\sigma_0 \, C_F \frac{\alpha_s}{4\pi} \left(\frac{q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left\{-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{2\pi^2}{3}\right\}$$

where $\sigma_0 = \frac{\pi^2}{3} \frac{\alpha_w}{\hat{s}} (1-\epsilon)\, \delta(1-\hat{\tau})$ $\quad \hat{\tau} = \frac{M^2}{\hat{s}}$, $q^2 = M^2$

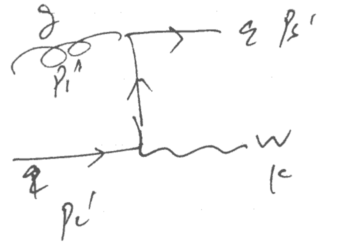
$C_F = \frac{4}{3}$

B7. Real gluon emission $q\bar{q} \to W^+$





v.s.



crossing symmetry:

$s = (p_1' + p_2')^2 \to (-p_3 + p_2)^2 = u$

$t = (p_1' - p_3')^2 \to (-p_3 + p_1)^2 = t$

$u = (p_2' - p_3')^2 \to (p_2 + p_1)^2 = s$

$$|M|^2 = -2g^2 g_s^2 \, Tr[T^A T^A] (1-\epsilon) \left\{(1-\epsilon)\left(\frac{u}{-t} + \frac{-t}{u}\right) - 2\frac{s}{tu} M_w^2 + 2\epsilon\right\}$$

$$\hat{\sigma}_R = \frac{1}{2\hat{s}} |\overline{M}|^2 \, PS$$

$$= \frac{1}{2\hat{s}} \cdot \frac{1}{4} \times \frac{1}{9} \, |M|^2 \cdot P.S.$$

$\cancel{B6-17}$

B7-2

$$\hat\sigma_R = \frac{1}{2\hat s}\,\frac{1}{4}\times\frac{1}{9}\times(-2g^2g_s^2\,4)(1-\epsilon)\left[(1-\epsilon)\left(\frac{u}{-t}+\frac{-t}{u}\right) - 2\frac{s}{tu}M_W^2 + 2\epsilon\right]$$

$$\times\frac{1}{8\pi}\left(\frac{4\pi}{M_W^2}\right)^{\epsilon}\frac{1}{\Gamma(1-\epsilon)}\left(\frac{M_W^2}{\hat s}\right)^{\epsilon}\left(1-\frac{M_W^2}{\hat s}\right)^{1-2\epsilon}\int_0^1 dy\,[y(1-y)]^{-\epsilon}\mu^{2\epsilon}$$

where $y=\frac{1}{2}(1+\cos\theta)$

$$= -\frac{g^2g_s^2\mu^{2\epsilon}}{9\hat s}(1-\epsilon)\frac{1}{8\pi}\left(\frac{4\pi}{M_W^2}\right)^{\epsilon}\frac{1}{\Gamma(1-\epsilon)}\left(\frac{M_W^2}{\hat s}\right)^{\epsilon}\left(1-\frac{M_W^2}{\hat s}\right)^{1-2\epsilon}$$

$$\int_0^1 dy\left[(1-\epsilon)\left[-\frac{y}{1-y}+\frac{-(1-y)}{y}\right] - 2\frac{M_W^2}{(1-\frac{M_W^2}{\hat s})^2 y(1-y)\hat s} + 2\epsilon\right][y(1-y)]^{-\epsilon}$$

$$= -\frac{g^2g_s^2}{9\hat s}(1-\epsilon)\frac{1}{8\pi}\left(\frac{4\pi\mu^2}{M_W^2}\right)^{\epsilon}\frac{1}{\Gamma(1-\epsilon)}\tau^{\epsilon}(1-\tau)^{1-2\epsilon}$$

$$\int_0^1 dy\left[-(1-\epsilon)\left(\frac{y}{1-y}+\frac{1-y}{y}\right) - 2\frac{\tau}{(1-\tau)^2y(1-y)} + 2\epsilon\right][y(1-y)]^{-\epsilon}$$

$$\int_0^1 dy\left\{-(1-\epsilon)\left[y^{1-\epsilon}(1-y)^{-1-\epsilon} + y^{-1-\epsilon}(1-y)^{1-\epsilon}\right] - \frac{2\tau}{(1-\tau)^2}y^{-1-\epsilon}(1-y)^{-1-\epsilon}\right.$$
$$\left. + 2\epsilon[y(1-y)]^{-\epsilon}\right\}$$

$$= -(1-\epsilon)\left[B(2-\epsilon,-\epsilon)+B(-\epsilon,2-\epsilon)\right] - \frac{2\tau}{(1-\tau)^2}B(-\epsilon,-\epsilon) + 2\epsilon B(1-\epsilon,1-\epsilon)$$

$$= \frac{-\alpha_W\alpha_s}{9\hat s}2\pi\left(\frac{4\pi\mu^2}{M_W^2}\right)^{\epsilon}(1-\epsilon)\frac{1}{\Gamma(1-\epsilon)}\tau^{\epsilon}(1-\tau)^{1-2\epsilon}$$

$$\left\{-(1-\epsilon)\frac{\Gamma(2-\epsilon)\Gamma(-\epsilon)}{\Gamma(2-2\epsilon)}\times 2 - \frac{2\tau}{(1-\tau)^2}\frac{\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)} + 2\epsilon\frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}\right\}$$

$$= \frac{-\alpha_W\alpha_s}{9\hat s}2\pi\left(\frac{4\pi\mu^2}{M_W^2}\right)^{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}(1-\epsilon)\frac{1}{\Gamma(1-\epsilon)}\tau^{\epsilon}(1-\tau)^{1-2\epsilon}$$

$$\left\{-2(1-\epsilon)\frac{1-\epsilon}{1-2\epsilon}\Gamma(-\epsilon) - \frac{2\tau}{(1-\tau)^2}\times 2\Gamma(-\epsilon) + 2\epsilon\frac{\Gamma(1-\epsilon)}{1-2\epsilon}\right\}$$

$$= -\frac{\alpha_w^{\varepsilon} \alpha_s^{\varepsilon}}{9\hat{s}} 2\pi \left(\frac{4\pi\mu^2}{M_W^2}\right)^{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} (1-\epsilon)\, \tau^{\epsilon}(1-\tau)^{1-2\epsilon}$$

$$\underbrace{\left\{ -2(1-\epsilon)\frac{1-\epsilon}{1-2\epsilon}\frac{1}{-\epsilon} - \frac{2\tau}{(1-\tau)^2} \times 2 \frac{1}{-\epsilon} + 2\epsilon \frac{1}{1-2\epsilon} \right\}}_{\frac{2}{\epsilon} + 4\epsilon + \frac{4\tau}{\epsilon(1-\tau)^2}}$$

$$\Rightarrow -\frac{\alpha_w^{\varepsilon} \alpha_s^{\varepsilon}}{9\hat{s}} 2\pi \left(\frac{4\pi\mu^2}{M_W^2}\right)^{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \tau^{\epsilon}(1-\tau)^{1-2\epsilon} (1-\epsilon) \times$$

$$\left[\frac{2}{\epsilon}\frac{\tau^2+1}{(1-\tau)^2} + 4\epsilon\right]$$

note: there is a singularity when $\tau \to 1$, $\hat{s} = m_W^2$, $k_g \to 0$, soft singularity.

plus distribution:

$$\frac{1}{(1-\tau)^{1+2\epsilon}} = \frac{1}{(1-\tau)^{1+2\epsilon}} - \left[\delta(1-\tau)\int_0^1 \frac{d\tau'}{(1-\tau')^{1+2\epsilon}} + \frac{1}{2\epsilon}\delta(1-\tau)\right]$$

note: $\displaystyle\int^1 \frac{d\tau'}{(1-\tau')^{1+2\epsilon}} \xrightarrow{\epsilon \to 0^-} -\frac{1}{2\epsilon}$

so: $$\frac{1}{(1-\tau)^{1+2\epsilon}} \equiv \left[\frac{1}{(1-\tau)^{1+2\epsilon}}\right]_+ - \frac{1}{2\epsilon}\delta(1-\tau)$$

where $$\left[\frac{1}{(1-\tau)^{1+2\epsilon}}\right]_+ \equiv \frac{1}{(1-\tau)^{1+2\epsilon}} - \delta(1-\tau)\int_0^1 \frac{d\tau'}{(1-\tau')^{1+2\epsilon}}$$

More generally we may define:

$$\left[\frac{g(z)}{1-z}\right]_+ = \frac{g(z)}{1-z} - \delta(1-z)\int_0^1 \frac{g(z')\,dz'}{1-z'} \qquad ①$$

so: $$\int_x^1 dz\, f(z)\left(\frac{g(z)}{1-z}\right)_+ = \int_x^1 dz\, \frac{f(z)\,g(z)}{1-z} - f(1)\int_0^1 \frac{g(z)\,dz}{1-z}$$

$$= \int_x^1 \frac{[f(z)-f(1)]\,g(z)}{1-z}\,dz - f(1)\int_0^x \frac{g(z)\,dz}{1-z}$$

which is finite provided that $g(z)$ is well-behaved.

$\left[\frac{g(z)}{1-z}\right]_+ \Rightarrow$ "plus" distribution

Note: $$\left[\frac{\alpha g(z) + \beta h(z)}{1-z}\right]_+ = \alpha\left[\frac{g(z)}{1-z}\right]_+ + \beta\left[\frac{h(z)}{1-z}\right]_+ \qquad ①'$$

From ① $\Rightarrow$ $$\left[\frac{g(z)}{1-z}\right]_+ - \frac{g(z)}{[1-z]_+} = -\delta(1-z)\int_0^1 \frac{g(z')-g(1)}{1-z'}\,dz' \qquad ②$$

~~$\Rightarrow$ If $g(1)=0$~~

① − ② 得 $$\frac{g(z)}{(1-z)_+} = \frac{g(z)}{1-z} - \delta(1-z)\int_0^1 \frac{g(1)}{1-z'}\,dz'$$

If $g(1)=0$, then $\frac{g(z)}{(1-z)_+} = \frac{g(z)}{1-z}$

More generally, from ①', we know that

if $\phi(1)=0$, then $\phi(z)\left[\frac{g(z)}{1-z}\right]_+ = \phi(z)\left[\frac{g(z)}{1-z}\right]$

Thus: $\frac{1}{(1-\tau)^{1+2\epsilon}} \underset{\epsilon\to 0}{=} \left[\frac{1}{(1-\tau)^{1+2\epsilon}}\right]_+ - \frac{1}{2\epsilon}\delta(1-\tau)$

$$= -\frac{1}{2\epsilon}\delta(1-\tau) + \left(\frac{1}{1-\tau}\right)_+ - 2\epsilon\left[\frac{\ln(1-\tau)}{1-\tau}\right]_+ + O(\epsilon^2)$$

The cross section:

$$\sigma_{real} = \frac{-\alpha_w^2 \alpha_s^2}{9\hat{s}} 2\pi \left(\frac{4\pi\mu^2}{M_w^2}\right)^\epsilon \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} (1-\epsilon)\ (\tau^2+1)\times 2\ \frac{1}{\epsilon}$$

$$\left\{(1+\epsilon\ln\tau)\left[-\frac{1}{2\epsilon}\delta(1-\tau) + \left[\frac{1}{1-\tau}\right]_+ - 2\epsilon\left[\frac{\ln(1-\tau)}{1-\tau}\right]_+\right]\right\}$$

$$\Rightarrow -\frac{1}{2\epsilon}\delta(1-\tau) + \left(\frac{1}{1-\tau}\right)_+ - 2\epsilon\left[\frac{\ln(1-\tau)}{1-\tau}\right]_+ + \epsilon\frac{\ln\tau}{(1-\tau)_+}$$

note: $\frac{\ln\tau}{(1-\tau)_+} = \frac{\ln\tau}{1-\tau}$

$$\sigma_{real} = \frac{\alpha_w^2\alpha_s^2}{9\hat{s}} 2\pi\left(\frac{4\pi\mu^2}{M_w^2}\right)^\epsilon \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}(1-\epsilon)$$

$$\left\{\frac{2}{\epsilon^2}\delta(1-\tau) - \frac{2}{\epsilon}\frac{1+\tau^2}{(1-\tau)_+} + 4(1+\tau^2)\left[\frac{\ln(1-\tau)}{1-\tau}\right]_+ - 2\frac{1+\tau^2}{1-\tau}\ln\tau\right\}$$

Real + Virtual

$$\boxed{\begin{aligned}\sigma &= \sigma_R + \sigma_V\\ &= \frac{2\pi}{9}\frac{\alpha_s\alpha_w}{\hat{s}}\left[\left(1+\epsilon\ln\left(\frac{4\pi\mu^2}{M_w^2}\right)\right)\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}(1-\epsilon)\right]\times\\ &\qquad\left\{\frac{2}{\epsilon^2}\delta(1-\tau) - \frac{2}{\epsilon}\frac{1+\tau^2}{(1-\tau)_+} + 4(1+\tau^2)\left[\frac{\ln(1-\tau)}{1-\tau}\right]_+ - 2\frac{(1+\tau^2)}{1-\tau}\ln\tau\right\}\\ &+ \frac{2\pi}{9}\frac{\alpha_s\alpha_w}{\hat{s}}\delta(1-\tau)(1-\epsilon)\left[1-\epsilon\ln\frac{M_w^2}{4\pi\mu^2}\right]\frac{\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}\left\{-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{2\pi^2}{3}\right\}\\ &= \cdots\end{aligned}}$$

$\sigma = \sigma_{virtual} + \sigma_{real}$

$$= \sigma_0 \, \delta(1-\hat{\tau}) \, \frac{2\alpha_s}{3\pi} \left(\frac{4\pi\mu^2}{M_W^2}\right)^{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left\{ -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{2\pi^2}{3} \right\}$$

$$+ \sigma_0 \frac{2}{3\pi}\alpha_s \left(\frac{4\pi\mu^2}{M_W^2}\right)^{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left\{ \frac{2}{\epsilon^2}\delta(1-\hat{\tau}) - \frac{2}{\epsilon}\frac{1+\tau^2}{(1-\tau)_+} + 4(1+\tau^2)\left(\frac{\ln(1-\tau)}{1-\tau}\right)_+ \right.$$
$$\left. -2\frac{1+\tau^2}{1-\tau}\ln\tau \right\}$$

$$= \sigma_0 \frac{2}{3\pi}\alpha_s \left(\frac{4\pi\mu^2}{M_W^2}\right)^{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left\{ -\frac{2}{\epsilon}\left[\frac{1+\tau^2}{(1-\tau)_+} + \frac{3}{2}\delta(1-\tau)\right] - 2\frac{1+\tau^2}{1-\tau}\ln\tau \right.$$
$$\left. + 4(1+\tau^2)\left[\frac{\ln(1-\tau)}{1-\tau}\right]_+ + \left(\frac{2\pi^2}{3} - 8\right)\delta(1-\tau) \right\}$$

note: $\left[\frac{g(z)}{1-z}\right]_+ - \frac{g(z)}{[1-z]_+} = -\delta(1-z)\int_0^1 \frac{g(z')-g(1)}{1-z'}dz'$

so: $\frac{1+\tau^2}{(1-\tau)_+} + \frac{3}{2}\delta(1-\tau) = \left(\frac{1+\tau^2}{1-\tau}\right)_+$

where $\sigma_0 = \frac{\pi^2}{3}\frac{\alpha_w}{s}(1-\epsilon)$

$$\sigma = \sigma_0 \frac{2}{3\pi}\alpha_s \left(\frac{4\pi\mu^2}{M_W^2}\right)^{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left\{ -\frac{2}{\epsilon}\left(\frac{1+\tau^2}{1-\tau}\right)_+ - 2\frac{1+\tau^2}{1-\tau}\ln\tau + 4(1+\tau^2)\left(\frac{\ln(1-\tau)}{1-\tau}\right)_+ \right.$$
$$\left. + \left(\frac{2\pi^2}{3} - 8\right)\delta(1-\tau) \right\}$$

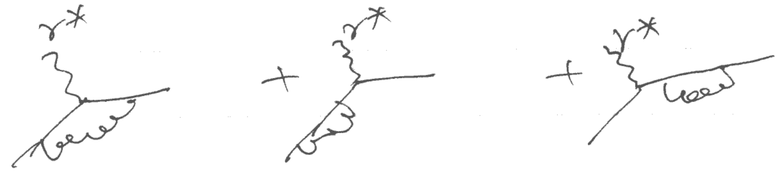
The infrared divergences, proportional to $\frac{1}{\epsilon^2}$, have cancelled. leaving the collinear divergence, proportional to $\frac{1}{\epsilon}$

This divergence will cancel with the collinear divergence present in the QCD correction to the quark distribution functions.

B-8 QCD correction to the quark PDF

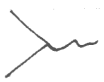
① virtual correction

v.s.



The W-boson mass replaced by the photon "mass" $q^2 = -Q^2$;

From W-boson results, we have:



$$\left\{ \left[\frac{\alpha_s}{4\pi} C_F \left(\frac{-q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[\frac{1}{\epsilon}+1\right] \right. \right.$$

$$\left. + \frac{\alpha_s}{4\pi} C_F \left(\frac{-q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[-\frac{2}{\epsilon^2} - \frac{4}{\epsilon} - 9 - \frac{\pi^2}{3}\right] \right\}$$

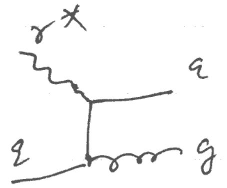
$$= \frac{\alpha_s}{4\pi} C_F \left[\frac{Q^2}{4\pi\mu^2}\right]^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 - \frac{\pi^2}{3}\right]$$

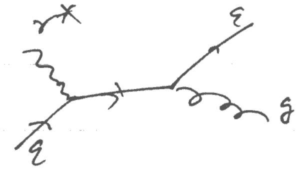
Therefore, Based on the definition of $\hat{W}_1^i$, we have

$$\hat{W}^i_{1\,(\text{virtual})} = 4\pi e_i^2 (1-\epsilon)\,\delta(1-z) \left[2\frac{\alpha_s}{4\pi} C_F \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \right] \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 - \frac{\pi^2}{3}\right]$$

$$= e_i^2 (1-\epsilon) \frac{2}{3}\alpha_s \left[\frac{4\pi\mu^2}{Q^2}\right]^{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 - \frac{\pi^2}{3}\right] \delta(1-z)$$

② real gluon emission





$$|M|^2 = \frac{g_s^2 \mu^{2\epsilon}}{8} \times 4 q_i^2 e^2 \; 16 \times 4 \; (1-\epsilon) \left\{ (1-\epsilon)\left(-\frac{s}{t} - \frac{t}{s}\right) - \frac{2uQ^2}{st} + 2\epsilon \right\}$$

$$|\overline{M}|^2 = \frac{1}{2} \times \frac{1}{3} \, |M|^2$$

The phase space:

$$\int dPS_2 = \frac{1}{8\pi} \left(\frac{4\pi}{s}\right)^{\epsilon} \frac{1}{\Gamma(1-\epsilon)} \int_0^1 dv \; v^{-\epsilon} (1-v)^{-\epsilon}$$

$$s = \frac{Q^2}{z}(1-z) \quad , \quad t = -\frac{Q^2}{z}(1-v) \quad , \quad u = -\frac{Q^2}{z} v$$

where $v = \frac{1}{2}(1+\cos\theta)$

Therefore

$$\hat{W}_1^{v} = \int |\overline{M}|^2 dPS_2$$

$$= \frac{16}{3} g_s^2 \mu^{2\epsilon} q_i^2 \, (1-\epsilon) \, \frac{1}{8\pi} \left(\frac{4\pi}{s}\right)^{\epsilon} \frac{1}{\Gamma(1-\epsilon)}$$

$$\int_0^1 dv \; v^{-\epsilon} (1-v)^{-\epsilon} \left\{ (1-\epsilon)\left(-\frac{s}{t} - \frac{t}{s}\right) + \frac{2uQ^2}{st} + 2\epsilon \right\}$$

$$= \frac{2}{3\pi} g_s^2 q_i^2 \, (1-\epsilon) \left(\frac{4\pi\mu^2}{s}\right)^{\epsilon} \frac{1}{\Gamma(1-\epsilon)}$$

$$\int_0^1 dv \; v^{-\epsilon} (1-v)^{-\epsilon} \left\{ (1-\epsilon)\left(\frac{1-z}{1-v} + \frac{1-v}{1-z}\right) + \frac{2z}{1-z}\frac{v}{1-v} + 2\epsilon \right\}$$

$$= \frac{2}{3\pi} g_s^2 q_i^2 \, (1-\epsilon) \left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon} \left(\frac{z}{1-z}\right)^{\epsilon} \frac{1}{\Gamma(1-\epsilon)} \Big\{ (1-\epsilon)\big[(1-z) B(1-\epsilon, -\epsilon)$$

$$+ \frac{1}{1-z} B(1-\epsilon, 2-\epsilon) + \frac{2z}{1-z} B(2-\epsilon, -\epsilon) + 2\epsilon B(1-\epsilon, 1-\epsilon) \Big\}$$

$$\hat{W}_T^i(\text{real}) = \frac{2}{3\pi} g_s^2 e_i^2 (1-\epsilon)\left[\frac{4\pi\mu^2}{Q^2}\right]^\epsilon \left[\frac{z}{1-z}\right]^\epsilon \frac{1}{\Gamma(1-\epsilon)}$$

$$\Big\{ (1-\epsilon)\Big[(1-z)\frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)} + \frac{1}{1-z}\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)} + \frac{2z}{1-z}\frac{\Gamma(2-\epsilon)\Gamma(-\epsilon)}{\Gamma(2-2\epsilon)}$$

$$+ 2\epsilon\frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}\Big\}$$

$$= \frac{2}{3\pi} g_s^2 e_i^2 (1-\epsilon)\left[\frac{4\pi\mu^2}{Q^2}\right]^\epsilon \left[\frac{z}{1-z}\right]^\epsilon \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \times$$

$$\Big\{ \frac{1}{\epsilon}\frac{z^2+1}{z-1} + \frac{-2z^2+8z-3}{2(z-1)} + \epsilon\frac{12z-5}{2(z-1)}\Big\}$$

$$= \frac{2}{3\pi} g_s^2 e_i^2 (1-\epsilon)\left[\frac{4\pi\mu^2}{Q^2}\right] \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} [1+\epsilon\ln z]$$

$$\left[\left(\frac{1}{1-z}\right)_+ - \epsilon\left(\frac{\ln(1-z)}{1-z}\right)_+ - \frac{1}{\epsilon}\delta(1-z)\right] \times \left[-\frac{1}{\epsilon}(z^2+1) - \frac{1}{2}(-2z^2+8z-3) - \epsilon\,\frac{12z-5}{2}\right]$$

where
$$\left(\frac{1}{1-z}\right)\left(\frac{1}{1-z}\right)^\epsilon = \left[\frac{1}{(1-z)^{1+\epsilon}}\right]_+ - \frac{1}{\epsilon}\delta(1-z)$$

$$= \left[\frac{1}{1-z}\right]_+ [1-\epsilon\ln(1-z)]_+ - \frac{1}{\epsilon}\delta(1-z)$$

$$= \left[\frac{1}{1-z}\right]_+ - \epsilon\left[\frac{\ln(1-z)}{1-z}\right]_+ - \frac{1}{\epsilon}\delta(1-z)$$

$$\hat{W}_T^i(\text{real}) = \frac{2}{3\pi} g_s^2 e_i^2 (1-\epsilon)\left[\frac{4\pi\mu^2}{Q^2}\right] \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \times$$

$$\Big\{ \frac{2}{\epsilon^2}\delta(1-z) - \frac{1}{\epsilon}\frac{z^2+1}{(1-z)_+} + \frac{1}{\epsilon}\frac{3}{2}\delta(1-z) + (1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+$$

$$- \frac{1+z^2}{(1-z)_+}\ln z - \frac{1}{2}(-2z^2+8z-3)\left(\frac{1}{1-z}\right)_+ + \frac{7}{2}\delta(1-z)\Big\}$$

if $g(1)=0$, then $\frac{g(z)}{(1-z)_+} = \frac{g(z)}{1-z}$, therefore: $\frac{1+z^2}{(1-z)_+}\ln z = \frac{1+z^2}{1-z}\ln z$

$$\frac{1}{2}(-2z^2+8z-3)\frac{1}{(1-z)_+}$$

$$= -\frac{1}{2}(2z^2-8z+3+3-3)\frac{1}{(1-z)_+}$$

$$= \frac{3}{2}\frac{1}{(1-z)_+} - \cancel{(z-1)(z-3)} \quad \frac{(z-1)(z-3)}{(1-z)_+}$$

$$= \frac{3}{2}\frac{1}{(1-z)_+} + (z-3)$$

$$\hat{w}_q^i(\text{real}) = \frac{2}{3\pi} g_s^2 e_i^2 (1-\epsilon)\left[\frac{4\pi\mu^2}{Q^2}\right] \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \times$$

$$\Big\{ \frac{2}{\epsilon^2}\delta(1-z) - \frac{1}{\epsilon}\frac{z^2+1}{(1-z)_+} + \frac{1}{\epsilon}\frac{3}{2}\delta(1-z) + (1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+$$

$$- \frac{1+z^2}{1-z}\ln z - \frac{3}{2}\frac{1}{(1-z)_+} - z+3 + \frac{7}{2}\delta(1-z)\Big\}$$

$$\hat{w}_q^i = \hat{w}_q^i(\text{virtual}) + \hat{w}_q^i(\text{real})$$

$$= e_i^2 (1-\epsilon)\frac{8}{3}\alpha_s \left[\frac{4\pi\mu^2}{Q^2}\right]^\epsilon \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \times$$

$$\Big\{ -\frac{1}{\epsilon}\frac{z^2+1}{(1-z)_+} - \frac{3}{2\epsilon}\delta(1-z) + (1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - \frac{1+z^2}{1-z}\ln z$$

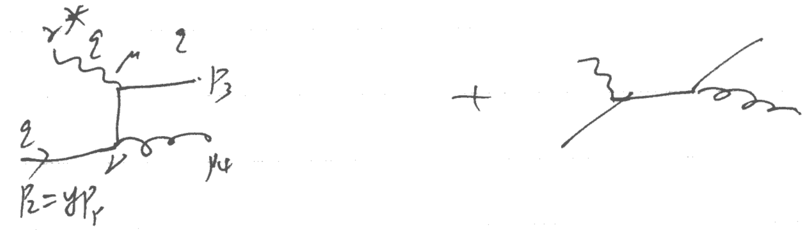
$$- \frac{3}{2}\frac{1}{(1-z)_+} - z+3 - \left(\frac{9}{2}+\frac{\pi^2}{3}\right)\delta(1-z)\Big\}$$

$$= e_i^2 \cancel{(1-\epsilon)} \frac{8}{3}\alpha_s (1-\epsilon)\left[1-\epsilon\ln\frac{Q^2}{4\pi\mu^2}\right] \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

$$\Big\{ -\frac{1}{\epsilon}\frac{z^2+1}{(1-z)_+} - \frac{3}{2\epsilon}\delta(1-z) + (1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - \frac{1+z^2}{1-z}\ln z$$

$$- \frac{3}{2}\frac{1}{(1-z)_+} - z+3 - \left(\frac{9}{2}+\frac{\pi^2}{3}\right)\delta(1-z)\Big\}$$

Longitudinal structure function



$$p_2^\mu M_\mu = i q_i e\, i g_s \mu^\epsilon\, \bar u(p_3)\left[\slashed{p}_2 \frac{i}{\slashed{p}_2-\slashed{p}_4}\gamma^\nu T^A + \gamma^\nu T^A \frac{i}{\slashed{q}+\slashed{p}_2}\slashed{p}_2\right] u(p_2)\,\epsilon_\nu(p_4)$$

$$= -q_i e g_s \mu^\epsilon\, \bar u(p_3)\, \frac{i\slashed{p}_2(\slashed{p}_2-\slashed{p}_4)\gamma^\nu T^A}{t}\, u(p_2)\,\epsilon_\nu(p_4)$$

$$= -q_i e g_s \mu^\epsilon\, \bar u(p_3)\, \frac{-i\slashed{p}_2\slashed{p}_4\gamma^\nu T^A}{t}\, u(p_2)\,\epsilon_\nu(p_4)$$

$$= -q_i e g_s \mu^\epsilon\, \bar u(p_3)\,(-i)\frac{[2p_2\cdot p_4 - \slashed{p}_4\slashed{p}_2]\gamma^\nu T^A}{t}\, u(p_2)\,\epsilon_\nu(p_4)$$

$$= -q_i e g_s \mu^\epsilon\, \bar u(p_3)\,(-i)\frac{[2p_2\cdot p_4 - \slashed{p}_4\, 2p_2^\nu]\, T^A}{t}\, u(p_2)\,\epsilon_\nu(p_4)$$

$$p_2^\mu p_2^\nu M_\mu M_\nu^* = q_i^2 e^2 g_s^2 \mu^{2\epsilon}\, 8u$$

$$\overline{p_2^\mu p_2^\nu M_\mu M_\nu^*} = \frac{4}{3} q_i^2 e^2 g_s^2 \mu^{2\epsilon}\, u$$

Therefore: $\hat{W}_L^i = \int \overline{p_2^\mu p_2^\nu M_\mu M_\nu^*}\ p.s$

$$= \frac{4}{3} q_i^2 \cancel{e^2} g_s^2 \mu^{2\epsilon}\, u\, \frac{1}{8\pi}\left(\frac{4\pi}{s}\right)^\epsilon \frac{1}{\Gamma(1-\epsilon)}\int_0^1 dv\, v^{-\epsilon}(1-v)^{-\epsilon}$$

$$= \frac{4}{3} q_i^2 g_s^2 \mu^{2\epsilon}\, \frac{1}{8\pi}\left(\frac{4\pi}{s}\right)^\epsilon \frac{1}{\Gamma(1-\epsilon)}\int_0^1 dv\, v^{-\epsilon}(1-v)^{-\epsilon}\left(-\frac{Q^2}{z}v\right)$$

$$= \frac{q_i^2}{6\pi} g_s^2 \left(\frac{4\pi\mu^2}{Q^2}\frac{z}{1-z}\right)^\epsilon \left(-\frac{Q^2}{z}\right)\frac{1}{2} + O(\epsilon)$$

$$= \frac{q_i^2}{6\pi} g_s^2 \left(-\frac{Q^2}{z}\right)\frac{1}{2} + O(\epsilon)$$

$$(1-\epsilon)\frac{1}{M_p}F_2 = xW_T + 4\frac{x^3}{Q^2}(3-2\epsilon)W_L$$

$$= \frac{x}{4\pi M_p}\sum_i \int_x^1 \frac{dy}{y} q(y)\; q_i^2 \frac{8}{3}\alpha_s (1-\epsilon)\left[1-\epsilon \ln\frac{Q^2}{4\pi\mu^2}\right]\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

$$\left\{-\frac{1}{\epsilon}\frac{z^2+1}{(1-z)_+} - \frac{3}{2\epsilon}\delta(1-z) + (1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - \frac{1+z^2}{1-z}\ln z\right.$$

$$\left. -\frac{3}{2}\left(\frac{1}{1-z}\right)_+ - z + 3 - \left(\frac{9}{2}+\frac{\pi^2}{3}\right)\delta(1-z)\right\}$$

$$+ \frac{4x^3}{Q^2}\frac{3-2\epsilon}{4\pi M_p}\sum_i \int_x^1 \frac{dy}{y^2} q(y)\frac{q_i^2}{6\pi} g_s^2 \left(-\frac{Q^2}{z}\right)\frac{1}{2}$$

$$\Rightarrow q(x,Q^2) = q_0(x) + \frac{\alpha_s}{2\pi}\frac{4}{3}\frac{1}{1-\epsilon}\int_x^1 \frac{dy}{y} q_0(y)\,(1-\epsilon)\times$$

$$\left[\left(\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right)\left[-\frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} + \ln\frac{Q^2}{4\pi\mu^2}\right]\right.$$

$$+ (1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - \frac{3}{2}\frac{1}{(1-z)_+} - \frac{1+z^2}{1-z}\ln z$$

$$\left. + 3 + 2z - \left(\frac{9}{2}+\frac{\pi^2}{3}\right)\delta(1-z)\right]$$

$$\frac{d}{d\ln Q^2} q(x,Q^2) = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dy}{y} q_0(y)\, P_{qq}\left(\frac{x}{y}\right)$$

where the splitting function

see B7-6

$$P_{qq}(z) = \frac{4}{3}\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right] = \frac{4}{3}\left(\frac{1+z^2}{1-z}\right)_+$$

A quark coming from quark splitting $q \to qg$

The full DGLAP equation for the quark distribution function, including both $g \to q\bar{q}$ and $q \to q g$

$$\frac{d}{d\ln Q^2} q(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{dy}{y} \left[g(y, Q^2) P_{qg}(z) + q(y, Q^2) P_{qq}(z) \right]$$

where $P_{qg}(z) = \frac{1}{2}\left[z^2 + (1-z)^2 \right]$

$$P_{qq}(z) = \frac{4}{3}\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \right]$$

③ $pp \to W^+ + X$ virtual and Real gluon emission

$$\hat{\sigma}(\text{real} + \text{virtual}) = \sigma_0 \frac{2}{3\pi} \alpha_s \left(\frac{4\pi\mu^2}{M_W^2} \right)^{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

$$\left\{ -\frac{2}{\epsilon} \left(\frac{1+\tau^2}{1-\tau}\right)_+ - 2\frac{1+\tau^2}{1-\tau} \ln\tau + 4(1+\tau^2) \left(\frac{\ln(1-\tau)}{1-\tau} \right)_+ \right.$$

$$\left. + \left(\frac{2\pi^2}{3} - 8 \right) \delta(1-\tau) \right\}$$

where $\sigma_0 = \frac{\pi^2}{3} \frac{\alpha_W}{\hat{s}} (1-\epsilon)$

pp collider cross section

$$\sigma^{(1)} = \sum_{i,j} \int_{\tau_0}^1 dx_1 \int_{\tau_0/x_1}^1 dx_2 \left[q_i^0(x_1) \bar{q}_j^0(x_2) + \bar{q}_i^0(x_1) q_j^0(x_2) \right] \hat{\sigma}(\text{real} + \text{virtual})$$

$$q_0(x) = q(x, Q^2) - \frac{\alpha_s}{2\pi} \frac{4}{3} \frac{1}{1-\epsilon} \int_x^1 \frac{dy}{y} q_0(y) (1-\epsilon) \times \{ \ldots \}$$

$$\sigma^{(1)} = \sum_{i,j} \int_{\tau_0}^1 dx_1 \int_{\tau_0/x_1}^1 dx_2 \left[q_i(x_1, Q^2) \bar{q}_j(x_2, Q^2) + \bar{q}_i(x_1, Q^2) q_j(x_2, Q^2) \right]$$

$$\sigma_0 \frac{2}{3\pi} \alpha_s \left(\frac{4\pi\mu^2}{M_W^2} \right)^{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left\{ -\frac{2}{\epsilon} \left(\frac{1+\tau^2}{1-\tau} \right)_+ - 2\frac{1+\tau^2}{1-\tau} \ln\tau + 4(1+\tau^2) \left(\frac{\ln(1-\tau)}{1-\tau} \right)_+ \right.$$

$$\left. + \left(\frac{2\pi^2}{3} - 8 \right) \delta(1-\tau) \right\}$$

Leading order cross section:

$$\sigma_{Born} = \frac{\pi^2}{3}\alpha_w(1-\epsilon)\sum_{i,j}\int_{\tau_0}^1 dx_1 \int_{\tau_0/x_1}^1 dx_2 \,[q_i(x_1)\bar{q}_j(x_2)+\bar{q}_i(x_1)q_j(x_2)]\,\delta(x_1x_2s-M_W^2)$$

where $\hat{s}=x_1x_2 s$, $\tau_0=\frac{M_W^2}{s}$

$$\sigma_{Born} = \frac{\pi^2}{3}\frac{\alpha_w}{s}(1-\epsilon)\sum_{i,j}\int_{\tau_0}^1 \frac{dx_1}{x_1}[q_{0i}(x_1)\bar{q}_{0j}(\tau_0/x_1)+\bar{q}_{0i}(x_1)q_{0j}(\tau_0/x_1)]$$

QCD correction due to the quark PDF

only focus on a particular distribution function, say $\bar{q}_{0j}(\tau_0/x_1)$

$$\sigma^{(1)} = \frac{\pi^2}{3}\frac{\alpha_w}{s}(1-\epsilon)\sum_{i,j}\int_{\tau_0}^1 \frac{dx_1}{x_1} q_{0i}(x_1,Q^2)\Big[-\frac{\alpha_s}{2\pi}\frac{4}{3}\frac{1}{1-\epsilon}\int_{\tau_0/x_1}^1\frac{dx_2}{x_2}\bar{q}(x_2)\,(1-\epsilon)$$

$$\Big\{\Big[\frac{1+z^2}{(1-z)_+}+\frac{3}{2}\delta(1-z)\Big]\Big[-\frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}+\ln\frac{Q^2}{4\pi\mu^2}\Big]$$
$$+(1+z^2)\Big(\frac{\ln(1-z)}{1-z}\Big)_+ -\frac{3}{2}\frac{1}{(1-z)_+}-\frac{1+z^2}{1-z}\ln z+3+2z-\Big(\frac{9}{2}+\frac{\pi^2}{3}\Big)\delta(1-z)\Big\}$$

where $z=\frac{\tau_0/x_1}{x_2}=\frac{\tau_0}{x_1x_2}=\frac{M_W^2}{x_1x_2s}=\frac{M_W^2}{\hat{s}}=\tau$

$$\sigma^{(1)} = \frac{\pi^2}{3}\frac{\alpha_w}{s}(1-\epsilon)\sum_{i,j}\int_{\tau_0}^1\frac{dx_1}{x_1}\Big(-\frac{\alpha_s}{2\pi}\Big)\frac{4}{3}\frac{1}{1-\epsilon}\int_{\tau_0/x_1}^1\frac{dx_2}{x_2}\, q_i(x_1,Q^2)\,\bar{q}(x_2)\,(1-\epsilon)$$
$$\{\ldots\}$$

$$=\frac{\pi^2}{3}\alpha_w(1-\epsilon)\sum_{i,j}\int_{\tau_0}^1\frac{dx_1}{x_1}\Big(-\frac{\alpha_s}{2\pi}\frac{4}{3}\Big)\frac{1}{1-\epsilon}\int_{\tau_0/x_1}^1\frac{dx_2}{x_2 s}\, q_i(x_1,Q^2)\,\bar{q}(x_2)(1-\epsilon)$$
$$\{\ldots\}$$

$$=\sum_{i,j}\int_{\tau_0}^1 dx_1\int_{\tau_0/x_1}^1 dx_2\; q_i(x_1,Q^2)\,\bar{q}(x_2,Q^2)\,\sigma_0$$
$$-\frac{2}{3\pi}\alpha_s\Big\{\Big[\frac{1+z^2}{(1-z)_+}+\frac{3}{2}\delta(1-z)\Big]\Big[-\frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}+\ln\frac{Q^2}{4\pi\mu^2}\Big]+$$
$$(1+z^2)\Big(\frac{\ln(1-z)}{1-z}\Big)_+-\frac{3}{2}\frac{1}{(1-z)_+}-\frac{1+z^2}{1-z}\ln z+3+2z-\Big(\frac{9}{2}+\frac{\pi^2}{3}\Big)\delta(1-z)\Big\}$$

$$\sigma^{(1)} = \sum_{i,j} \int_{\tau_0}^{1} dx_1 \int_{\tau_0/x_1}^{1} dx_2\, q_i(x_1, \mu^2)\, \bar{q}(x_2, \mu^2)\, \hat{\sigma}_0$$

$$-\frac{2}{3\pi}\alpha_s \Big\{ \left(\frac{1+\tau^2}{1-\tau}\right)_+ \left(-\frac{1}{\epsilon} + \ln\frac{\mu^2}{4\pi\mu^2}\right) + (1+\tau^2)\left(\frac{\ln(1-\tau)}{1-\tau}\right)_+$$

$$-\frac{3}{2}\frac{1}{(1-\tau)_+} - \frac{1+\tau^2}{1-\tau}\ln\tau + 3 + 2\tau - \left(\frac{9}{2} + \frac{\pi^2}{3}\right)\delta(1-\tau) \Big\}$$

$$\sigma^{(1)}_{(\text{real}+\text{virtual})} = \sum_{i,j} \int_{\tau_0}^{1} dx_1 \int_{\tau_0/x_1}^{1} dx_2\, [q_i(x_1,\mu^2)\, \bar{q}_j(x_2,\mu^2) + \bar{q}_i(x_1,\mu^2)\, q_j(x_2,\mu^2)]\, \hat{\sigma}_0$$

$$\frac{2}{3\pi}\alpha_s \Big\{ \left(\frac{1+\tau^2}{1-\tau}\right)_+ \left(-\frac{2}{\epsilon} - 2\ln\frac{4\pi\mu^2}{M_W^2}\right) - 2\frac{1+\tau^2}{1-\tau}\ln\tau$$

$$+ 4(1+\tau^2)\left(\frac{\ln(1-\tau)}{1-\tau}\right)_+ + \left(\frac{2\pi^2}{3} - 8\right)\delta(1-\tau) \Big\}$$

It shows the collinear divergences have cancelled.

$$2\sigma^{(1)} + \sigma^{(1)}_{(\text{real}+\text{virtual})}$$

$$= \sum_{i,j} \int_{\tau_0}^{1} dx_1 \int_{\tau_0/x_1}^{1} dx_2\, [q_i(x_1,\mu^2)\, \bar{q}_j(x_2,\mu^2) + \bar{q}_i(x_1,\mu^2)\, q_j(x_2,\mu^2)]\, \hat{\sigma}_0$$

$$\frac{2}{3\pi}\alpha_s \Big\{ \left(\frac{1+\tau^2}{1-\tau}\right)_+ 2\ln\frac{M_W^2}{\mu^2} + \left(1 + \frac{4}{3}\pi^2\right)\delta(1-\tau) + \frac{3}{(1-\tau)_+}$$

$$-6 - 4\tau + 2(1+\tau^2)\left(\frac{\ln(1-\tau)}{1-\tau}\right)_+ \Big\}$$

$$\text{where } \left(\frac{1+\tau^2}{1-\tau}\right)_+ = \frac{1+\tau^2}{(1-\tau)_+} + \frac{3}{2}\delta(1-\tau), \quad \tau = \frac{M_W^2}{\hat{s}}$$

The large log term $\ln\frac{M_W^2}{\mu^2}$ can be eliminated via the DGLAP equation.

$$\frac{d}{d\ln Q^2} q(x, Q^2) = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dy}{y}\, q_0(y)\, P_{qq}\left(\frac{x}{y}\right)$$

$$\Rightarrow q(x, M_W^2) = q(x, Q^2) + \frac{\alpha_s}{2\pi} \ln \frac{M_W^2}{Q^2} \int_x^1 \frac{dy}{y}\, q(y)\, P_{qq}\left(\frac{x}{y}\right)$$

$$\sigma_{Born} = \frac{\pi^2}{3} \frac{\alpha_W}{s} (1-\epsilon) \sum_{i,j} \int_{\tau_0}^1 \frac{dx_1}{x_1} \left[q_{0i}(x_1)\, \bar{q}_j\left(\frac{\tau_0}{x_1}, Q^2\right) + \bar{q}_i(x_1, Q^2)\, q_j\left(\frac{\tau_0}{x_1}, Q^2\right) \right]$$

we focus on the first term as an example

$$q_i(x_1, Q^2)\, \bar{q}_j\left(\frac{\tau_0}{x_1}, Q^2\right)$$

$$\Rightarrow q_i(x_1, M_W^2) \left[-\frac{\alpha_s}{2\pi} \ln \frac{M_W^2}{Q^2} \right] \int_{\frac{\tau_0}{x_1}}^1 \frac{dx_2}{x_2}\, \bar{q}_j(x_2)\, \frac{4}{3} \left(\frac{1+z^2}{1-z}\right)_+$$

$$\Rightarrow \sigma_{Born} \supset \frac{\pi^2}{3} \frac{\alpha_W}{s} (1-\epsilon) \sum_{i,j} \int_{\tau_0}^1 \frac{dx_1}{x_1} \int_{\tau_0/x_1}^1 \frac{dx_2}{x_2}\, q_i(x_1, M_W^2)\, \bar{q}_j(x_2, M_W^2)$$
$$\left(-\frac{\alpha_s}{2\pi} \ln \frac{M_W^2}{Q^2}\right) \frac{4}{3} \left(\frac{1+z^2}{1-z}\right)_+$$

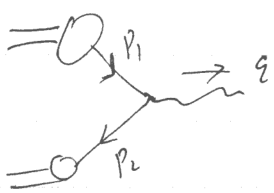
$$= \sum_{i,j} \int_{\tau_0}^1 dx_1 \int_{\tau_0/x_1}^1 dx_2\, q_i(x_1, M_W^2)\, \bar{q}_j(x_2, M_W^2)\, \sigma_0$$
$$-\frac{2\alpha_s}{3\pi} \ln \frac{M_W^2}{Q^2} \left(\frac{1+z^2}{1-z}\right)_+$$

$$= \sum_{i,j} \int_{\tau_0}^1 dx_1 \int_{\tau_0/x_1}^1 dx_2\, q_i(x_1, M_W^2)\, \bar{q}_j(x_2, M_W^2)\, \sigma_0$$
$$-\frac{2\alpha_s}{3\pi} \ln \frac{M_W^2}{Q^2} \left(\frac{1+z^2}{1-z}\right)_+$$

The total cross section

$$\sigma = \sum_{i,j} \int_{\tau_0}^1 dx_1 \int_{\tau_0/x_1}^1 dx_2 \left[q_i(x_1, M_W^2)\, \bar{q}_j(x_2, M_W^2) + \bar{q}_j(x_1, M_W^2)\, q_j(x_2, M_W^2) \right] \sigma_0$$
$$\frac{2}{3\pi}\alpha_s \left\{ \left(1 + \frac{4}{3}\pi^2\right) \delta(1-z) + \frac{3}{(1-z)_+} - 6 - 4z + 2(1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ \right\}$$

Summary: $pp \to W^+ + X$



(1) Total cross section (Hadron level)

$$\sigma(pp\to W^+) = \sum_{i,j} \int_{\tau_0}^{1} dx_1 \int_{\tau_0/x_1}^{1} dx_2 \left[q_i(x_1)\bar{q}_j(x_2) + \bar{q}_i(x_1) q_j(x_2)\right] \hat{\sigma}(q_i\bar{q}_j \to W)$$

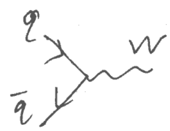
where $\tau_0 = \frac{M_W^2}{s}$, $\sqrt{s}$ the c.m. energy of pp

$$\hat{s} = (p_1+p_2)^2 = x_1 \cdot x_2\, s$$

$$\tau = \frac{M_W^2}{\hat{s}} = \frac{M_W^2}{x_1 x_2 s} = \frac{\tau_0}{x_1 x_2}$$

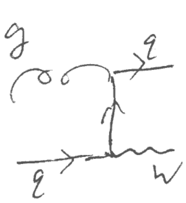
(2) Subprocess cross section: ($\hat{\sigma}$)

a. Parton level:



$$\hat{\sigma}_{Born} = \frac{\pi^2}{3}\frac{\alpha_W}{\hat{s}}(1-\epsilon)\,\delta(1-\tau) = \sigma_0\,\delta(1-\tau)$$

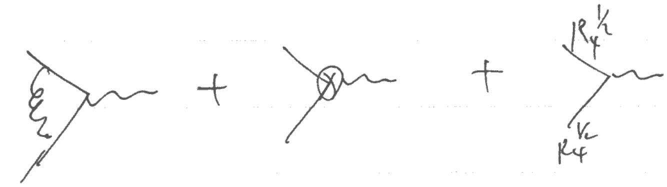
b. Gluon PDF contribution $gq \to W + q$



$$\hat{\sigma}^{(1)}_{g\to qq} = \frac{\alpha_s}{4\pi}\frac{\pi}{12}\frac{g^2}{\hat{s}}\left\{ \left[\tau^2+(1-\tau)^2\right]\left[-\frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} + \ln\frac{M_W^2}{4\pi\mu^2}\frac{(1-\tau)^2}{\tau}\right] + \frac{9}{2} + \tau - \frac{3}{2}\tau^2 \right\}$$

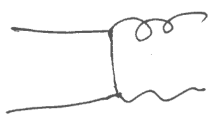
Collinear divergence

c. virtual correction



$$\hat{\sigma}^{(1)}_{vir} = \frac{2\alpha_s}{3\pi}\sigma_0\,\delta(1-\tau)\left(\frac{M_W^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left\{-\frac{2}{\epsilon^2}-\frac{3}{\epsilon}-8+\frac{2\pi^2}{3}\right\}$$

d. Real gluon emission:



$$\hat{\sigma}^{(1)}_{real} = \frac{2}{3}\frac{\alpha_s}{\pi}\sigma_0\frac{\left(\frac{4\pi\mu^2}{M_W^2}\right)^{\epsilon}\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left\{\frac{2}{\epsilon^2}\delta(1-\tau)-\frac{2}{\epsilon}\frac{1+\tau^2}{(1-\tau)_+}+4(1+\tau^2)\left[\frac{\ln(1-\tau)}{1-\tau}\right]_+ -2\frac{1+\tau^2}{1-\tau}\ln\tau\right\}$$

c+d. virtual and Real gluon emission

$$\hat{\sigma}^{(1)}_{vir}+\hat{\sigma}^{(1)}_{real} = \sigma_0\frac{2}{3\pi}\alpha_s\left(\frac{4\pi\mu^2}{M_W^2}\right)^{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\times$$

$$\left\{-\frac{2}{\epsilon}\left(\frac{1+\tau^2}{1-\tau}\right)_+ -2\frac{1+\tau^2}{1-\tau}\ln\tau + 4(1+\tau^2)\left(\frac{\ln(1-\tau)}{1-\tau}\right)_+ + \left(\frac{2\pi^2}{3}-8\right)\delta(1-\tau)\right\}$$

(3) Redefine the parton Distribution function.

a1 quark PDF from gluon contribution

$$q(x,Q^2) = q_0(x) + \frac{\alpha_s}{4\pi}\frac{1}{1-\epsilon}\int_x^1 \frac{dy}{y} g(y) \times$$

$$\left\{ [z^2+(1-z)^2]\left[-\frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} + \ln\left(\frac{Q^2}{4\pi\mu^2}\frac{1-z}{z}\right)\right] + 6z(1-z)\right\}$$

where $z = \frac{x}{y}$

a2. DGLAP equation

$$\frac{d}{d\ln Q^2} q(x,Q^2) = 2\frac{\alpha_s}{4\pi}\int_x^1 \frac{dy}{y} g(y) P_{qg}\left(\frac{x}{y}\right)$$

where $P_{qg}(z) = \frac{1}{2}[z^2+(1-z)^2]$ splitting function.

$$q(x,M_W^2) = q(x,Q^2) + \frac{\alpha_s}{4\pi}\ln\frac{M_W^2}{Q^2}\int_x^1 \frac{dy}{y} g(y) [z^2+(1-z)^2]$$

b1. Quark PDF (virtual and real gluon emission)

$$q(x,Q^2) = q_0(x) + \frac{\alpha_s}{2\pi}\frac{4}{3}\frac{1}{1-\epsilon}\int_x^1 \frac{dy}{y} q_0(y)(1-\epsilon)\times$$

$$\left\{ \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right]\left[-\frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} + \ln\frac{Q^2}{4\pi\mu^2}\right] \right.$$

$$+ (1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - \frac{3}{2}\frac{1}{(1-z)_+} - \frac{1+z^2}{1-z}\ln z + 3 + 2z$$

$$\left. - \left(\frac{9}{2} + \frac{\pi^2}{3}\right)\delta(1-z) \right\}$$

b2. DGLAP equation

$$\frac{d}{d\ln Q^2} q(x,Q^2) = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dy}{y} q_0(y) P_{qq}\left(\frac{x}{y}\right)$$

where $P_{qq}(z) = \frac{4}{3}\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right]$

$$q(x,M_W^2) = q(x,Q^2) + \frac{\alpha_s}{2\pi}\ln\frac{M_W^2}{Q^2}\int_x^1 \frac{dy}{y} q(y) P_{qq}\left(\frac{x}{y}\right)$$

c. Total DGLAP equation for the quark PDF

$$\frac{d}{d\ln Q^2} q(x,Q^2) = \frac{\alpha_s(Q^2)}{2\pi}\int_x^1 \frac{dy}{y}\left[g(y,Q^2) P_{qg}\left(\frac{x}{y}\right) + q(y,Q^2) P_{qq}\left(\frac{x}{y}\right)\right]$$

(4) Total cross section @ Hadron collider.

After the DGLAP equation, the $\ln\frac{M_W^2}{Q^2}$ is eliminated

a. leading order:

$$\sigma^0 = \frac{\pi^2}{3}\frac{\alpha_W}{s}\sum_{i,j}\int_{\tau_0}^1 dx_1 \int_{\tau_0/x_1}^1 dx_2\, \delta(x_1 x_2 - \tau_0) \times$$
$$\left[q_i(x_1,M_W^2)\bar{q}_j(x_2,M_W^2) + \bar{q}_i(x_1,M_W^2) q_j(x_2,M_W^2)\right]$$

b. gluon PDF contribution

$$\sigma^{0,1}_{g\to q\bar{q}} = \frac{\pi}{12}\alpha_s\alpha_W\sum_i\int_{\tau_0}^1 dx_1\int_{\tau_0/x_1}^1 dx_2\left[q_i(x_1)g(x_2) + \bar{q}_i(x_1)g(x_2) + (x_1\leftrightarrow x_2)\right]$$
$$\times\frac{1}{s}\left\{\left[z^2+(1-z)^2\right]\ln(1-z) + \frac{3}{2} - 5z + \frac{9}{2}z^2\right\}$$

(c). virtual and real emission

$$\sigma^{(1)}_{(\text{real}+\text{virtual})} = \sum_{i,j} \int_{\tau_0}^{1} dx_1 \int_{\tau_0/x_1}^{1} dx_2 \left[q_i(x_1, M_W^2)\, \bar{q}_j(x_2, M_W^2) + \bar{q}_j(x_1, M_W^2)\, q_i(x_2, M_W^2) \right]$$

$$\times \sigma_0 \frac{2}{3\pi}\alpha_s \left\{ \left(1+\frac{4}{3}\pi^2\right)\delta(1-\tau) + \frac{3}{(1-\tau)_+} - 6 - 4\tau \right.$$

$$\left. + 2(1+\tau^2)\left(\frac{\ln(1-\tau)}{1-\tau}\right)_+ \right\}$$

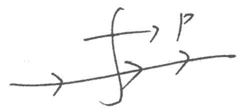
The total cross section (a)+(b)+(c)

C. Cut diagram notation

Handbook of Perturbative QCD

* For each cut, replace $\frac{1}{p^2-m^2+i\epsilon} \to -2\pi i\,\delta^+(p^2-m^2)$ on each cut propagator

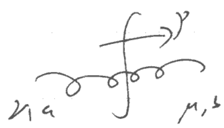
① Quark line



$$= 2\pi\,\delta^+(p^2-m^2)\,(\not p+m)$$

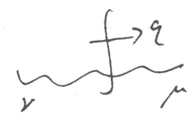
$$\delta^+(p^2-m^2) = \delta(p^2-m^2)\,\theta(p_0)$$

② Gluon line



$$= 2\pi\,\delta^+(k^2)\,(-g_{\mu\nu})\,\delta_{ab}$$

③ W-boson line



$$= 2\pi\,\delta^+(q^2-M^2)\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{M^2}\right)$$

(a) Note that the cut-diagram notation only valid for the final state.

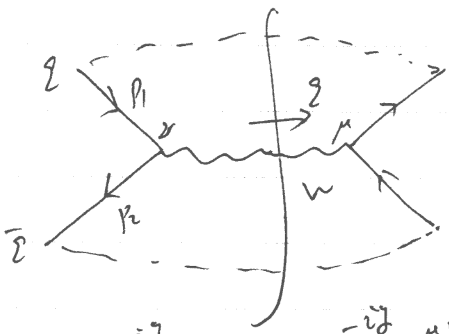
(b) For each cut, there is a loop integral factor

$\Downarrow$

it is corresponding to the phase space

Example:

(a) Born cross section



$\frac{ig}{\sqrt{2}}\gamma^\nu P_L I_{3\times 3}$ $\quad$ $\frac{-ig}{\sqrt{2}}\gamma^\mu P_L \cdot \underbrace{I_{3\times 3}}_{\to \text{color}}$

Note: $\sum u\bar{u} = \not{p}_1$

$\sum v\bar{v} = \not{p}_2$

$$\sum_{\text{spin}} \text{cut} = \int \frac{d^n q}{(2\pi)^n} (2\pi)^n \delta^n(q-p_1-p_2) \left[2\pi\delta^+(q^2-M^2)\left(-g^{\mu\nu} + \frac{q_\mu q_\nu}{M^2}\right)\right]$$

$$\text{tr}\left[\not{p}_1 \frac{-ig}{\sqrt{2}}\gamma^\mu P_L \not{p}_2 \frac{ig}{\sqrt{2}}\gamma^\nu P_L\right] \text{tr}[I^2_{3\times 3}]$$

Note: $\cancel{\int \frac{d^n q}{(2\pi)^n}\delta^n(q-p_1-p_2)\frac{1}{2q_0} = \int d^n q\, \delta^4(q-p_1-p_2)\,\delta^+(q^2-M^2)}$

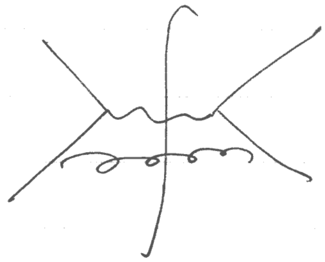
$$\int \frac{d^3 p}{(2\pi)^3}\frac{1}{2E_p} = \int \frac{d^4 p}{(2\pi)^4}(2\pi)\,\delta(p^2-M^2)\Big|_{p^0>0}$$

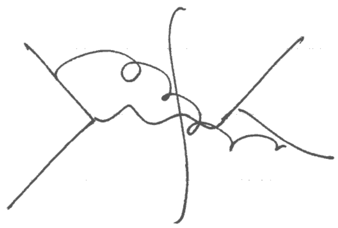
$$\sum_{\text{spin}} \text{cut} = \underbrace{\int \frac{d^n q}{(2\pi)^n}(2\pi)^n \delta^n(q-p_1-p_2)\, 2\pi\,\delta^+(q^2-M^2)}_{} = \int dPS_1$$

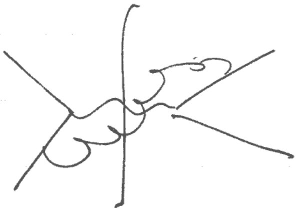
$$\left[-g^{\mu\nu} + \frac{q_\mu q_\nu}{M^2}\right] \text{tr}\left[\not{p}_1 \frac{-ig}{\sqrt{2}}\gamma^\mu P_L \not{p}_2 \frac{ig}{\sqrt{2}}\gamma^\nu P_L\right] \text{tr}[I^2_{3\times 3}]$$

(b) Real Emission

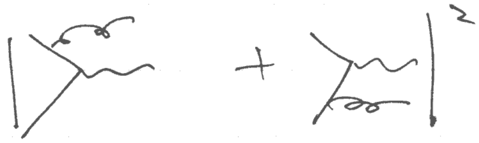
①



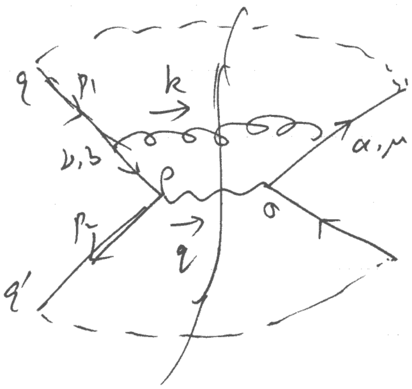




~~cuts~~ corresponding to



we only focus on ① as an example.

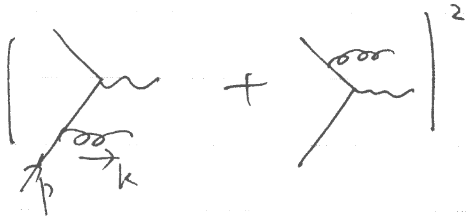


$$\sum_{spin} \text{cut} = \underbrace{\int \frac{d^n k}{(2\pi)^n}\frac{d^n q}{(2\pi)^n}(2\pi)^n\delta^n(k+q-p_1-p_2)\, 2\pi\delta^+(k^2)\, 2\pi\delta^+(q^2-M^2)}_{\int dPS_2}$$

$$-g_{\mu\nu}\delta_{ab}\left(-g_{\rho\sigma}+\frac{q_\rho q_\sigma}{M^2}\right)\mathrm{tr}[T^a T^b] \quad g_s^2\,\frac{g_W^2}{2}$$

$$\mathrm{tr}\left[\not p_1\,\gamma^\mu\,\frac{-i(\not p_1-\not k)}{(p_1-k)^2}\,\gamma^\sigma P_L\,\not p_2\,\gamma^\rho P_L\,\frac{i(\not p_1-\not k)}{(p_1-k)^2}\,\gamma^\nu\right]$$

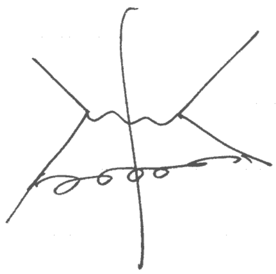
Infrared singularities.



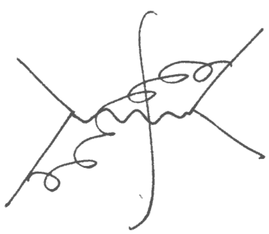
$$\frac{1}{(p-k)^2-m^2} = \frac{1}{-2p\cdot k}$$

① $k^\mu \to 0$ soft divergence

② $k^\mu \parallel p^\mu$ collinear divergence



$\sim \frac{1}{\epsilon}$ collinear



$\sim \frac{1}{\epsilon^2}$ soft and collinear

Soft limit. the wave function long enough to know both the initial state

KLN theorem : all the soft singularities cancel

only collinear singularity

From Handbook of perturbative QCD

(CTEQ)

B: Cut Diagram Notation

A convenient technique for organizing calculations of $|\mathcal{M}|^2$ in cross sections is through *cut diagrams*, which combine contributions to $\mathcal{M}$ and $\mathcal{M}^*$ into a single diagram for $|\mathcal{M}|^2$ with slightly modified Feynman rules.

The form of cut diagrams is derived in Fig. 50, for the annihilation of a fermion pair of momenta k_1 and k_2 into a set of n final state lines, of which only a fermion with momentum p_1 and an antifermion of momentum p_n are exhibited.

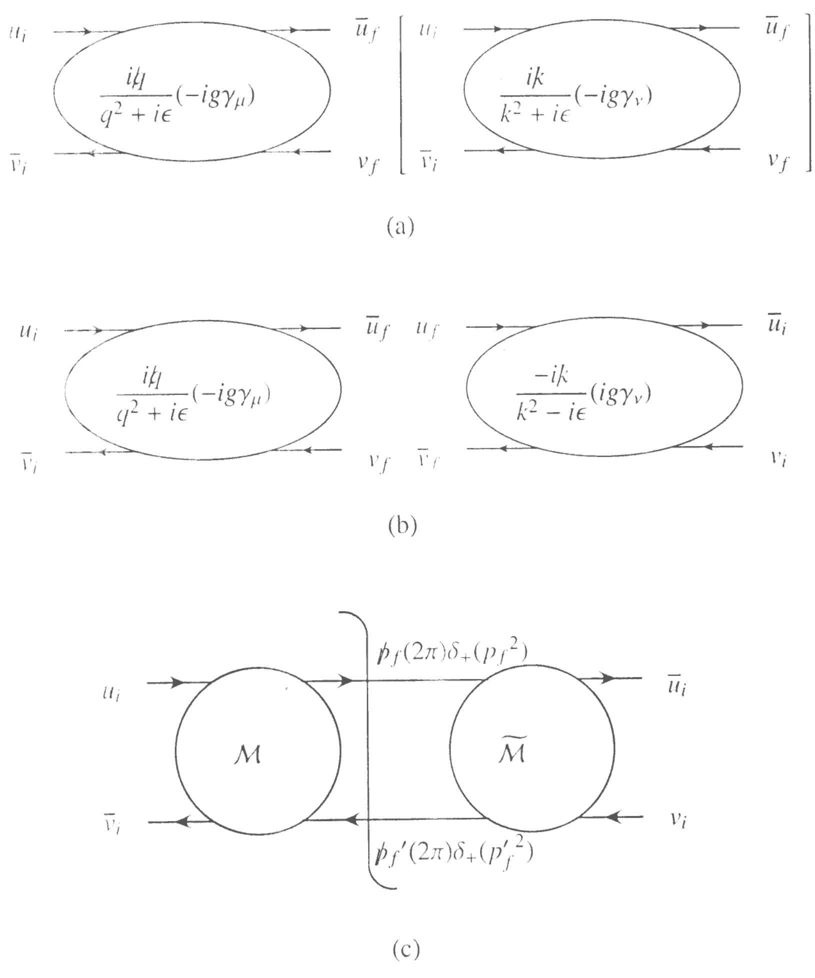


Figure 50: Cut diagram identities.

The underlying identity for these manipulations is

$$[\bar{w}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\sigma^{\alpha\beta}\cdots\gamma^{\nu}\gamma_5\cdots)w']^*$$
$$= \bar{w}'(\cdots\gamma^{\nu}\gamma_5\cdots\sigma^{\alpha\beta}\cdots\gamma^{\mu_2}\gamma^{\mu_1})w\ , \tag{2.1}$$

where w and w' are any two Dirac spinors.

Fig. 50a shows a typical fermion propagator and vertex in $\mathcal{M}$ and $\mathcal{M}^*$. Fig. 50b shows the application of Eq. (2.1) to Fig. 50a. The diagram in $\mathcal{M}^*$ has been flipped over, all arrows on fermion lines have been reversed, *and* all momenta have been reversed in sign. This leaves the sign of momenta in fermion propagators the same, as shown. Color sums can be reversed in the same manner as spinor sums, because the color generators are hermitian.

Fig. 50c exhibits the cut diagram notation, in which the contribution of any final state is a modified forward scattering diagram. The final-state lines are indicated by a vertical line (the "cut"). Cut lines are represented in the integral corresponding to the cut diagram by factors

$$(\not{p}_i + m_i)(2\pi)\delta_+(p_i^2 - m_i^2)\ , \tag{2.2}$$

for fermions or antifermions, after a spin sum. For polarized fermions or for vectors, the usual spin projections replace $(\not{p}_i + m_i)$. The Feynman rules for $\mathcal{M}$ are the normal ones, and those for $\widetilde{\mathcal{M}}$ differ only in the sign of *explicit* factors of i at vertices and in propagators. The three-gluon vertex also changes sign in $\widetilde{\mathcal{M}}$, because of the reversal of momenta.