# Hands on Quantum Mechanics 2016 – HonQM16

## Running of fundamental constants and gauge coupling unification

Our Universe is ruled by four fundamental interactions very different in strength: the electromagnetic, strong and weak interactions and gravity.

**Properties of the Interactions**

The strengths of the interactions (forces) are shown relative to the strength of the electromagnetic force for two u quarks separated by the specified distances.

| Property | Gravitational Interaction | Weak Interaction (Electroweak) | Electromagnetic Interaction | Strong Interaction |
|---|---|---|---|---|
| Acts on: | Mass – Energy | Flavor | Electric Charge | Color Charge |
| Particles experiencing: | All | Quarks, Leptons | Electrically Charged | Quarks, Gluons |
| Particles mediating: | Graviton (not yet observed) | $W^+$ $W^-$ $Z^0$ | $\gamma$ | Gluons |
| Strength at $10^{-18}$ m | $10^{-41}$ | 0.8 | 1 | 25 |
| Strength at $3\times10^{-17}$ m | $10^{-41}$ | $10^{-4}$ | 1 | 60 |

Fig. 1 – Relative strength of the four known fundamental interactions with respect to the electromagnetic force.

Aside from gravity, which is described by the classical theory of General Relativity, the three remaining forces are described by a quantum theory, which is known as the Standard Model (SM). The SM is one of the most successful and well-tested theories in modern physics. However, it does not offer an answer to several fundamental questions like:

**WHY ARE THE FUNDAMENTAL INTERACTIONS SO DIFFERENT?**

The answer to this question may be that perhaps, at some point in the History of our Universe (when it was more energetic), the four interactions were the same, i.e. **they were unified**. So why are they so different today?

The SM predicts that the values of the fundamental constants change with energy. This can be only understood in the light of a Quantum Theory as the SM. Aside from gravity (for which there is not a quantum description), the *running* [1] of the fundamental constants $g_i$ $(i = 1,2,3)$ which describe the electromagnetic, weak and strong interactions is given by simple mathematical equations, called **renormalization group equations** (RGEs), of the form:

$$\frac{dg_i(t)}{dt} = \frac{\beta_{g_i}^{(1)}(t)}{16\pi^2}\ , \qquad \beta_{g_i}^{(1)}(t) = g_i^3(t)\, B_i\ , \qquad t = \ln(\mu/\mu_0)\,. \qquad \text{Eq. (1)}$$

at lowest order in quantum effects (things get actually more complicated if we go to higher orders in quantum effects). The functions $\beta_{g_i}^{(1)}(t)$ are called the one-loop beta functions for the gauge couplings. In the above equation, $\mu$ is

[1] From now on *running* means the variation of some parameter with energy.

the energy scale at which we want to compute $g_i$ and $\mu_0$ is a reference scale such that $g_i(0)$ are known. Therefore, the gauge coupling constants will be functions of $t$, i.e. $g_i(t)$. The quantities $B_i$ are constants and depend on the model in which we are calculating the *running* of the *gauge couplings* $g_i$. The computation of the $B_i$ is highly non trivial and requires knowledge of advanced Quantum Field Theory. Therefore, we will not say anything about how these quantities are obtained. Notice that, it is sometimes convenient to work with the couplings $\alpha_i = g_i^2/(4\pi)$.

## PART I – DO GAUGE COUPLINGS UNIFY IN THE SM AT LOWEST ORDER IN QUANTUM EFFECTS?

In order to answer this question, take the following steps:

**I.1)** Solve Eq. (1) to obtain an analytical expression for $g_i^2(t)$ and $\alpha_i^{-1}(t)$.

The several couplings referred up to now are related through the following relations:

$$\alpha_i = \frac{g_i^2}{4\pi} \quad , \quad \alpha_{em} = \frac{g_2^2 \sin^2\theta_W}{4\pi} \quad , \sin^2\theta_W = \frac{g_Y^2}{g_Y^2 + g_2^2} \quad , \qquad g_1^2 = k_Y g_Y^2 \,. \qquad \text{Eq. (2)}$$

Experimentally, it is known that, at the reference scale of the Z boson mass $m_Z = 91.2$ GeV:

$$\alpha_{em}^{-1}(m_Z) = 127.916 \pm 0.015,$$

$$\sin^2\theta_W(m_Z) = 0.23116 \pm 0.00013, \qquad \text{Eqs. (3)}$$

$$\alpha_3(m_Z) = 0.1184 \pm 0.0007 \,.$$

For the SM:

$$k_Y = \frac{5}{3} \quad \text{and} \quad B_i = \left(\frac{41}{6k_Y}\, , -\frac{19}{6}\, , -7\right) \qquad \text{Eq. (4)}\,.$$

**I.2)** You can now start testing gauge-coupling unification in the SM. For that:

i) Find (analitically) the expression for the energy scale $\mu_{12}$ for which $g_1$ and $g_2$ unify and, taking into account the experimental values given above, find the common value of $\alpha_1$ and $\alpha_2$ at $\mu_{12}$.

ii) Imposing now $\alpha_1(\mu_{12}) = \alpha_2(\mu_{12}) = \alpha_3(\mu_{12})$ , make a prediction for $\alpha_3(m_Z)$ and compare it with the measured value. **Is there gauge coupling unification in the SM?**

iii) Make a plot of $\alpha_i^{-1}(\mu)$ varying $\mu$ from $m_Z$ to $10^{18}$ GeV to observe that, in fact, there is no SM gauge coupling unification at lowest order in quantum effects, **even if one considers the experimental errors**.

**I.3)** In some more elaborated theories like Grand-Unified or String Theories, the $k_Y$ factor may be different from that of the SM and, in principle, gauge coupling unification could be possible. In order to test whether this is the case find the values of $k_Y$ for which the three gauge couplings unify at a certain scale $\mu_U$. Determine the sets $(k_Y, \mu_U)$. Verify the result graphically.

I.4) **Gauge coupling unification at two-loops? (leave this to the end)**

We have just seen that the gauge couplings do not unify in the SM. This conclusion was based on the analysis of the RGEs at lowest order in quantum effects (one-loop). One may wonder if gauge-coupling unification occurs once we consider higher-order effects or, in other words, if one takes into account the *two-loop* RGEs. In order to analyse this, you must accept that all parameters in the SM run and that the running of a certain parameter depends on the other parameters of the theory. In general, for a parameter $X$ we have, at second-order in quantum effects:

$$\frac{dX(t)}{dt} = \frac{\beta_X^{(1)}(t)}{16\pi^2} + \frac{\beta_X^{(2)}(t)}{(16\pi^2)^2}, \qquad \text{Eq. (5)}$$

where $\beta_X^{(1)}(t)$ and $\beta_X^{(2)}(t)$ are the one and two-loop beta functions for the parameter $X$, respectively. We will consider a simplified version in which the relevant parameters for the gauge coupling beta functions are:

- The gauge couplings themselves
- The top quark Yukawa coupling $y_t$, which is nothing but the value of the coupling between the top quark and the Higgs field in the SM.
- The quartic Higgs coupling $\lambda$, which is related with the Higgs self-interacting term $\lambda\phi^4$ present in the SM lagrangian.

The relevant beta functions for our next exercise are:

- The $\beta_{g_i}^{(1)}(t)$ given in Eq. (1)
- $\beta_{g_i}^{(2)}(t) = \sum_{k=1}^{3} b_{ki}\, g_k^2 g_i^3 - 2g_i^3 y_t^2\ ,\ b_{ki} = \begin{pmatrix} \frac{199}{50} & \frac{9}{10} & \frac{11}{10} \\ \frac{27}{10} & \frac{35}{6} & \frac{9}{2} \\ \frac{44}{5} & 12 & -26 \end{pmatrix}$
- $\beta_{y_t}^{(1)}(t) = \frac{9}{2} y_t^3 - 8g_3^2\, y_t$
- $\beta_{y_t}^{(2)}(t) = -\frac{15}{4} y_t^5 + \frac{3}{2}\lambda^2 y_t - 6\lambda y_t^3 + 36 g_3^2 y_t^3 - 109 g_3^4 y_t$
- $\beta_{\lambda}^{(1)}(t) = 12\lambda + 12 y_t\lambda - 12 y_t^4$
- $\beta_{\lambda}^{(2)}(t) = -78\,\lambda^3 - 64 g_3^2 y_t^4 + 80\lambda g_3^2 y_t^2 - 72\lambda^2 y_t^2 - 3y_t^4\lambda + 60 y_t^6$

Consider moreover that $y_t(m_Z) = 1.0$ and $\lambda(m_Z) = 0.5$.

Solve (numerically) the system of differential equations for $g_i(t), y_t(t)$ and $\lambda(t)$, and plot the energy dependence of $\alpha_i^{-1}(t)$. Compare with the one-loop case and answer the question: **is there gauge coupling unification in the SM at the two-loop level?**

## PART II – RESCUED BY SUSY (SUPERSYMMETRY)

One of the most appealing theories beyond the SM is supersymmetry (SUSY) which is related with a new fundamental symmetry of Nature. The minimal supersymmetric SM extension is called the *Minimal Supersymmetric Standard Model* (MSSM). Due to the structure of the theory and the presence of new degrees of freedom (each known fundamental particle has its superpartner), the fundamental constants in the MSSM run in a different way. For instance, for the gauge couplings $g_i$ the RGEs have the same structure as in Eq. (1) but the parameters $B_i$ are now:

$$k_Y = \frac{5}{3} \quad \text{and} \quad B_i = \left(\frac{11}{k_Y}\ ,1\ ,-3\right).$$

**II.1)** Repeat the steps I.2.ii and I.2.ii for the MSSM. What do you conclude? Isn't it amazing?!!

**II.2)** In II.1 you have considered that SUSY is valid from $m_Z$ up to higher scales. However, this is not actually true, otherwise we should have observed SUSY particles already at LEP or at the LHC. So, most likely, SUSY is only a true symmetry of Nature for energy scales $\mu > \mu_{\text{SUSY}}$, while for $\mu < \mu_{\text{SUSY}}$, the valid theory is the SM.

Taking the above into account, give an expression for $\alpha_i^{-1}(\mu)$, for $\mu > \mu_{\text{SUSY}}$. Make a plot of $\alpha_i^{-1}(\mu)$ considering that SUSY is broken at $\mu_{\text{SUSY}} = 1$ TeV. Is unification of the gauge couplings preserved?