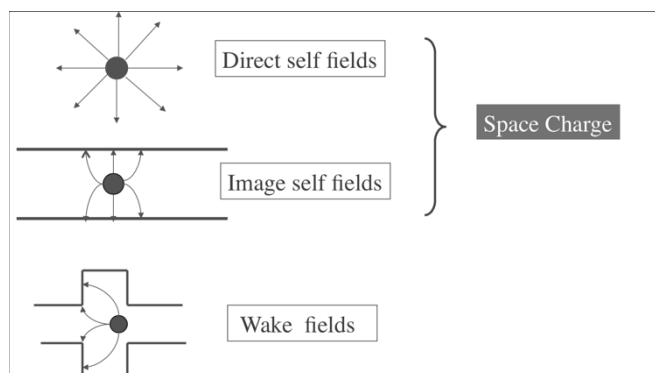


# Space Charge in Linear Machines

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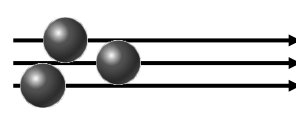
Egham – September 6<sup>th</sup> 2017



## OUTLINE

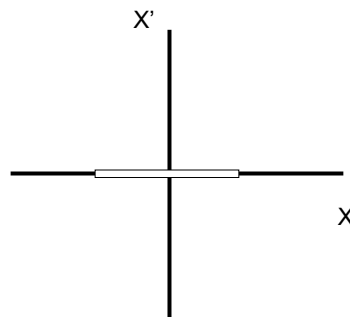
- The rms emittance concept
- rms envelope equation
- Space charge forces
- Space charge induced emittance oscillations
- Matching conditions and emittance compensation

## Trace space of an ideal laminar beam

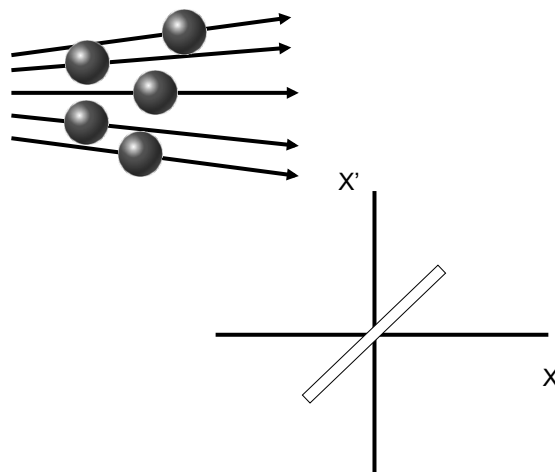


A diagram showing an ideal laminar beam. It consists of three horizontal parallel lines with arrows pointing to the right. Four black spheres are positioned along these lines: two on the top line and two on the bottom line.

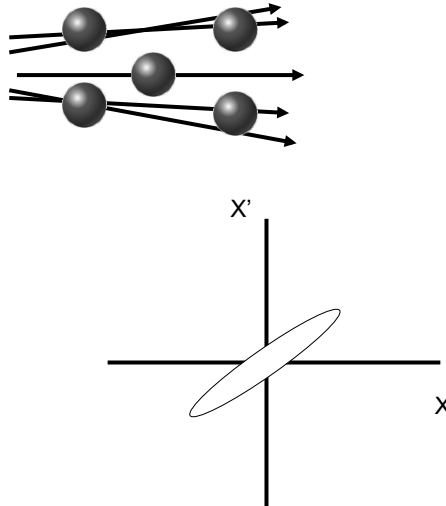
$$\begin{cases} x \\ x' = \frac{dx}{dz} = \frac{p_x}{p_z} \end{cases} \quad p_x \ll p_z$$



## Trace space of a laminar beam



# Trace space of non laminar beam



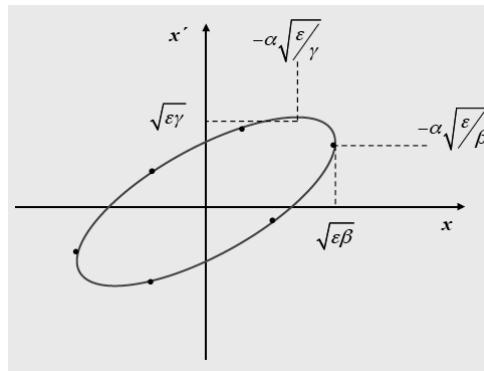
Geometric emittance:

$$\boxed{\varepsilon_g}$$

Ellipse equation:  $\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon_g$

Twiss parameters:  $\beta\gamma - \alpha^2 = 1$        $\beta' = -2\alpha$

Ellipse area:  $A = \pi\varepsilon_g$



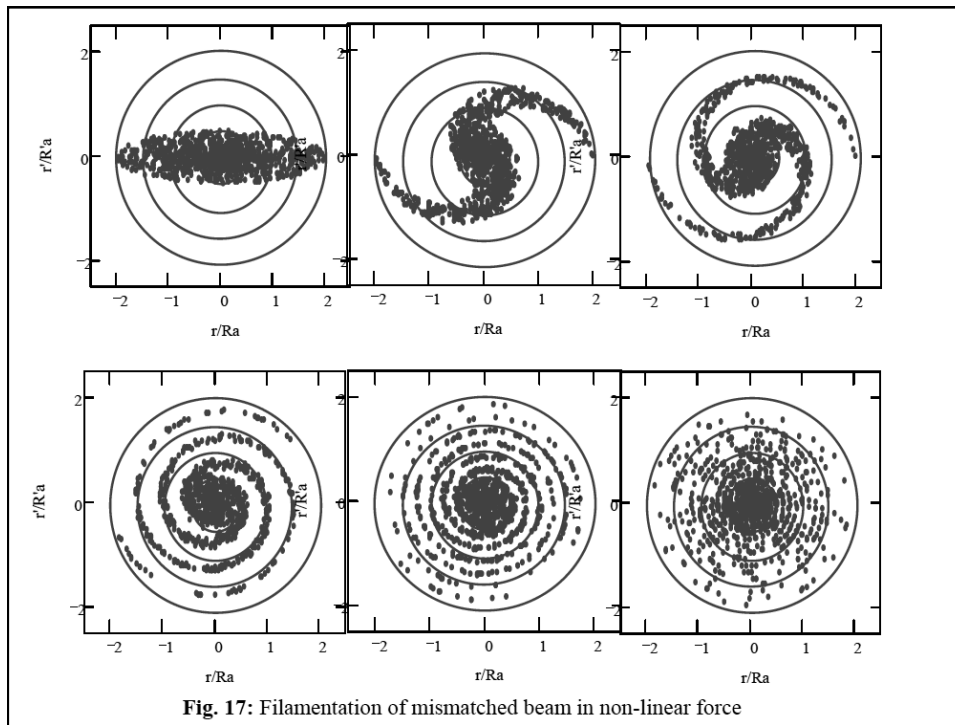


Fig. 17: Filamentation of mismatched beam in non-linear force

**rms emittance**  $\mathcal{E}_{rms}$

$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, x') dx dx' = 1$   $f'(x, x') = 0$

rms beam envelope:

$$\sigma_x^2 = \langle x^2 \rangle = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x^2 f(x, x') dx dx'$$

Define rms emittance:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \mathcal{E}_{rms}$$

such that:  $\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta \mathcal{E}_{rms}}$

$\sigma_{x'} = \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \mathcal{E}_{rms}}$

Since:  $\alpha = -\frac{\beta'}{2}$   $\beta = \frac{\langle x^2 \rangle}{\mathcal{E}_{rms}}$

it follows:  $\alpha = -\frac{1}{2\mathcal{E}_{rms}} \frac{d}{dz} \langle x^2 \rangle = -\frac{\langle x x' \rangle}{\mathcal{E}_{rms}} = -\frac{\sigma_{xx'}}{\mathcal{E}_{rms}}$

$$\begin{aligned}\sigma_x &= \sqrt{\langle x^2 \rangle} = \sqrt{\beta \epsilon_{rms}} \\ \sigma_{x'} &= \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \epsilon_{rms}} \\ \sigma_{xx'} &= \langle xx' \rangle = -\alpha \epsilon_{rms}\end{aligned}$$

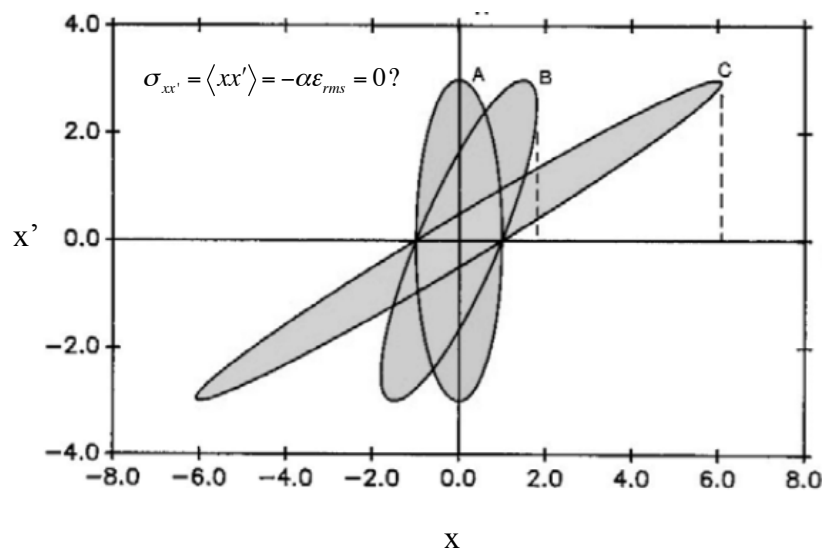
It holds also the relation:  $\gamma\beta - \alpha^2 = 1$

Substituting  $\alpha, \beta, \gamma$  we get  $\frac{\sigma_{x'}^2}{\epsilon_{rms}} \frac{\sigma_x^2}{\epsilon_{rms}} - \left( \frac{\sigma_{xx'}}{\epsilon_{rms}} \right)^2 = 1$

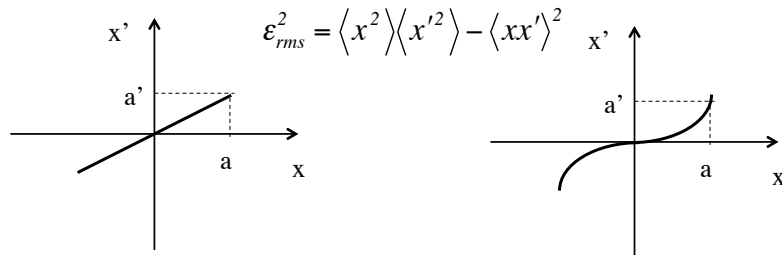
We end up with the definition of rms emittance in terms of the second moments of the distribution:

$$\epsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^2} = \sqrt{(\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2)}$$

**Which distribution has no correlations?**



What does rms emittance tell us about phase space distributions under linear or non-linear forces acting on the beam?



Assuming a generic  $x, x'$  correlation of the type:  $x' = Cx^n$

$$\epsilon_{rms}^2 = C^2 \left( \langle x^2 \rangle \langle x^{2n} \rangle - \langle x^{n+1} \rangle^2 \right)$$

When  $n = 1 \implies \epsilon_{rms} = 0$

When  $n \neq 1 \implies \epsilon_{rms} \neq 0$

### Constant under linear transformation only

$$\frac{d}{dz} \langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2 = 2 \langle xx' \rangle \langle x'^2 \rangle + 2 \langle x^2 \rangle \langle x' \rangle \langle x'' \rangle - 2 \langle xx'' \rangle \langle x' \rangle = 0$$

For linear transformations,  $x'' = -k_x^2 x$ , and the right-hand side of the equation is

$$2k_x^2 \langle x^2 \rangle \langle xx' \rangle - 2 \langle x^2 \rangle \langle xx' \rangle k_x^2 = 0,$$

so

$$\frac{d}{dz} \langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2 = 0$$

**And without acceleration:**

$$x' = \frac{p_x}{p_z}$$

### Normalized rms emittance: $\varepsilon_{n,rms}$

Canonical transverse momentum:  $p_x = p_z x' = m_o c \beta \gamma x'$   $p_z \approx p$

$$\varepsilon_{n,rms} = \frac{1}{m_o c} \sqrt{\sigma_x^2 \sigma_{p_x}^2 - \sigma_{xp_x}^2} = \frac{1}{m_o c} \sqrt{(\langle x^2 \rangle \langle p_x^2 \rangle - \langle xp_x \rangle^2)} \approx \langle \beta \gamma \rangle \varepsilon_{rms}$$

Liouville theorem: the density of particles  $n$ , or the volume  $V$  occupied by a given number of particles in phase space  $(x, p_x, y, p_y, z, p_z)$  remains invariant under conservative forces.

$$\frac{dn}{dt} = 0$$

It hold also in the projected phase spaces  $(x, p_x), (y, p_y), (z, p_z)$  provided that there are no couplings

### Limit of single particle emittance

Limits are set by Quantum Mechanics on the knowledge of the two conjugate variables  $(x, p_x)$ . According to Heisenberg:

$$\sigma_x \sigma_{p_x} \geq \frac{\hbar}{2}$$

This limitation can be expressed by saying that the state of a particle is not exactly represented by a point, but by a small uncertainty volume of the order of  $\hbar^3$  in the 6D phase space.

In particular for a single electron in 2D phase space it holds:

$$\varepsilon_{n,rms} = \frac{1}{m_o c} \sqrt{\sigma_x^2 \sigma_{p_x}^2 - \sigma_{xp_x}^2} \Rightarrow \begin{cases} = 0 & \text{classical limit} \\ \geq \frac{1}{2} \frac{\hbar}{m_o c} = \frac{\lambda_c}{2} = 1.9 \times 10^{-13} m & \text{quantum limit} \end{cases}$$

Where  $\lambda_c$  is the reduced Compton wavelength.

## OUTLINE

- The rms emittance concept
- rms envelope equation
- Space charge forces
- Space charge induced emittance oscillations
- Matching conditions and emittance compensation

## Envelope Equation without Acceleration

Now take the derivatives:

$$\frac{d\sigma_x}{dz} = \frac{d}{dz} \sqrt{\langle x^2 \rangle} = \frac{1}{2\sigma_x} \frac{d}{dz} \langle x^2 \rangle = \frac{1}{2\sigma_x} 2\langle xx' \rangle = \frac{\sigma_{xx'}}{\sigma_x}$$

$$\frac{d^2\sigma_x}{dz^2} = \frac{d}{dz} \frac{\sigma_{xx'}}{\sigma_x} = \frac{1}{\sigma_x} \frac{d\sigma_{xx'}}{dz} - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{1}{\sigma_x} (\langle x'^2 \rangle + \langle xx'' \rangle) - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{\sigma_x'^2 + \langle xx'' \rangle}{\sigma_x} - \frac{\sigma_{xx'}^2}{\sigma_x^3}$$

And simplify:

$$\sigma_x'' = \frac{\sigma_x'^2 + \langle xx'' \rangle}{\sigma_x} - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{\epsilon_{rms}^2}{\sigma_x^3} + \frac{\langle xx'' \rangle}{\sigma_x}$$

We obtain the rms envelope equation in which the rms emittance enters as defocusing pressure like term.

$$\sigma_x'' - \frac{\langle xx'' \rangle}{\sigma_x} = \frac{\epsilon_{rms}^2}{\sigma_x^3}$$

$$\frac{\epsilon_{rms}^2}{\sigma_x^3} \approx \frac{T}{V} \approx P$$

## Beam Thermodynamics

Kinetic theory of gases defines temperatures in each directions and global as:

$$k_B T_x = m \langle v_x^2 \rangle \quad T = \frac{1}{3} (T_x + T_y + T_z) \quad E_k = \frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} k_B T$$

Definition of beam temperature in analogy:

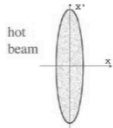

$$k_B T_{beam,x} = \gamma m_o \langle v_x^2 \rangle \quad \langle v_x^2 \rangle = \beta^2 c^2 \langle x'^2 \rangle = \beta^2 c^2 \sigma_{x'}^2 = \beta^2 c^2 \frac{\epsilon_{rms}^2}{\sigma_x^2} = \beta^2 c^2 \frac{\epsilon_{rms}}{\beta_x}$$

We get:

$$k_B T_{beam,x} = \gamma m_o \langle v_x^2 \rangle = \gamma m_o \beta^2 c^2 \frac{\epsilon_{rms}^2}{\sigma_x^2} = \gamma m_o \beta^2 c^2 \frac{\epsilon_{rms}}{\beta_x}$$

$$P_{beam,x} = n k_B T_{beam,x} = n \gamma m_o \beta^2 c^2 \frac{\epsilon_{rms}^2}{\sigma_x^2} = N_T \gamma m_o \beta^2 c^2 \frac{\epsilon_{rms}^2}{\sigma_L \sigma_x^2}$$

$$k_B T_{beam,x} = \gamma m_o \beta^2 c^2 \frac{\epsilon_{rms}}{\beta_x}$$

| Property                                                  | Hot beam                                                                            | Cold beam                                                                            |
|-----------------------------------------------------------|-------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------|
| ion mass ( $m_o$ )                                        | heavy ion                                                                           | light ion                                                                            |
| ion energy ( $\beta\gamma$ )                              | high energy                                                                         | low energy                                                                           |
| beam emittance ( $\epsilon$ )                             | large emittance                                                                     | small emittance                                                                      |
| lattice properties ( $\gamma_{xy} \approx 1/\beta_{xy}$ ) | strong focus (low $\beta$ )                                                         | high $\beta$                                                                         |
| phase space portrait                                      |  |  |

**Electron Cooling: Temperature relaxation by mixing a hot ion beam with co-moving cold (light) electron beam.**

*Particle Accelerators*  
1973, Vol. 5, pp. 61–65

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Printed in Glasgow, Scotland

### EMITTANCE, ENTROPY AND INFORMATION

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$$S = kN \log(\pi\epsilon)$$

By means of the beam temperature concept one can also define the beam emittance at the source called thermal emittance. Assuming that electrons are in equilibrium with the cathode temperature  $T_c = T_{beam}$  and  $\gamma=1$  the thermal emittance is given by:  $\epsilon_{th,rms}^{out} = \sigma_x \sqrt{\frac{k_B T_c}{m_e c^2}}$  which, per unit rms spot size at the cathode, is  $\epsilon_{th,rms} = 0.3 \mu\text{m/mm}$  at  $T_c = 2500 \text{ K}$ . For comparison in a photocathode illuminated by a laser pulse with photon energy  $\hbar\omega$  the expression for the variance of the transverse momentum of the emitted electrons is given by  $\sigma_{p_x} = \sqrt{\frac{m_e}{3}(\hbar\omega - \phi_{eff})}$  where  $\phi_{eff} = \phi_w - \phi_{Schottky}$ ,  $\phi_w$  being the material work function and  $\phi_{Schottky}$  the Schottky work function [19]. The corresponding thermal emittance is  $\epsilon_{th,rms}^{ph} = \sigma_x \sqrt{\frac{\hbar\omega - \phi_{eff}}{3m_e c^2}}$  that, with the typical parameters of a Copper photocathode illuminated by a UV laser, gives a thermal emittance per unit spot size of about  $0.5 \mu\text{m/mm}$ .

## Envelope Equation with Linear Focusing

$$\sigma_x'' - \frac{\langle xx'' \rangle}{\sigma_x} = \frac{\epsilon_{rms}^2}{\sigma_x^3}$$

Assuming that each particle is subject only to a linear focusing force, without acceleration:  $x'' + k_x^2 x = 0$

take the average over the entire particle ensemble  $\langle xx'' \rangle = -k_x^2 \langle x^2 \rangle$

$$\sigma_x'' + k_x^2 \sigma_x = \frac{\epsilon_{rms}^2}{\sigma_x^3}$$

We obtain the rms envelope equation with a linear focusing force in which, unlike in the single particle equation of motion, the rms emittance enters as defocusing pressure like term.

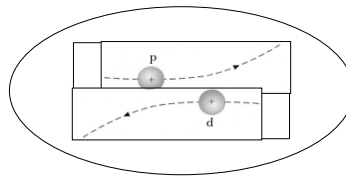
## OUTLINE

- The rms emittance concept
- rms envelope equation
- Space charge forces
- Space charge induced emittance oscillations
- Matching conditions and emittance compensation

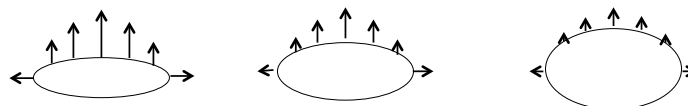
## Space Charge: what does it mean?

The net effect of the **Coulomb** interactions in a multi-particle system can be classified into two regimes:

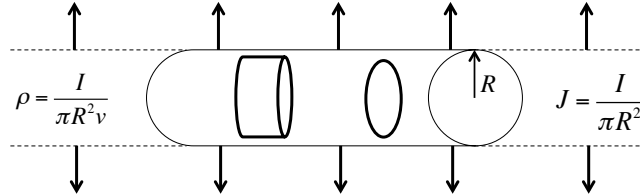
- 1) **Collisional Regime** ==> dominated by **binary collisions** caused by close particle encounters ==> **Single Particle Effects**



- 2) **Space Charge Regime** ==> dominated by the **self field** produced by the particle distribution, which varies appreciably only over large distances compare to the average separation of the particles ==> **Collective Effects**



## Continuous Uniform Cylindrical Beam Model



Gauss' s law

$$\int \epsilon_o E \cdot dS = \int \rho dV$$

$$E_r = \frac{I}{2\pi\epsilon_o R^2 v} r \quad \text{for } r \leq R$$

$$E_r = \frac{I}{2\pi\epsilon_o v} \frac{1}{r} \quad \text{for } r > R$$

$$B_\theta = \frac{\beta}{c} E_r$$

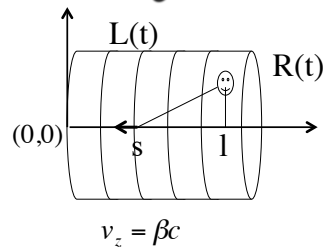
Ampere' s law

$$\int B \cdot dl = \mu_o \int J \cdot dS$$

$$B_\theta = \mu_o \frac{I r}{2\pi R^2} \quad \text{for } r \leq R$$

$$B_\theta = \mu_o \frac{I}{2\pi r} \quad \text{for } r > R$$

## Bunched Uniform Cylindrical Beam Model



Longitudinal Space Charge field in the bunch moving frame:

$$\tilde{\rho} = \frac{Q}{\pi R^2 \tilde{L}}$$

$$\tilde{E}_z(\tilde{s}, r=0) = \frac{\tilde{\rho}}{4\pi\epsilon_o} \int_0^R \int_0^{2\pi} \int_0^{\tilde{L}} \frac{(\tilde{l} - \tilde{s})}{\left[ (\tilde{l} - \tilde{s})^2 + r^2 \right]^{3/2}} r dr d\phi d\tilde{l}$$

$$\tilde{E}_z(\tilde{s}, r=0) = \frac{\tilde{\rho}}{2\epsilon_o} \left[ \sqrt{R^2 + (\tilde{L} - \tilde{s})^2} - \sqrt{R^2 + \tilde{s}^2} + (2\tilde{s} - \tilde{L}) \right]$$

Radial Space Charge field in the bunch moving frame  
by series representation of axisymmetric field:

$$\tilde{E}_r(r, \tilde{s}) \cong \left[ \frac{\tilde{\rho}}{\epsilon_0} - \frac{\partial}{\partial \tilde{s}} \tilde{E}_z(0, \tilde{s}) \right] \frac{r}{2} + [\dots] \frac{r^3}{16} +$$

$$\tilde{E}_r(r, \tilde{s}) = \frac{\tilde{\rho}}{2\epsilon_0} \left[ \frac{(\tilde{L} - \tilde{s})}{\sqrt{R^2 + (\tilde{L} - \tilde{s})^2}} + \frac{\tilde{s}}{\sqrt{R^2 + \tilde{s}^2}} \right] \frac{r}{2}$$

### **Lorentz Transformation to the Lab frame**

$$\begin{aligned} E_z &= \tilde{E}_z & \tilde{L} = \gamma L &\Rightarrow \tilde{\rho} = \frac{\rho}{\gamma} \\ E_r &= \gamma \tilde{E}_r & \tilde{s} &= \gamma s \end{aligned}$$

$$E_z(0, s) = \frac{\rho}{\gamma 2\epsilon_0} \left[ \sqrt{R^2 + \gamma^2 (L - s)^2} - \sqrt{R^2 + \gamma^2 s^2} + \gamma(2s - L) \right]$$

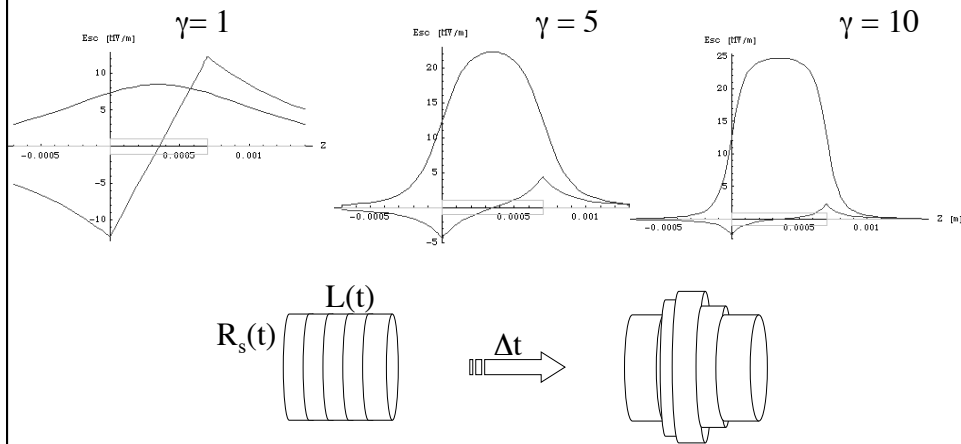
$$E_r(r, s) = \frac{\gamma \rho}{2\epsilon_0} \left[ \frac{(L - s)}{\sqrt{R^2 + \gamma^2 (L - s)^2}} + \frac{s}{\sqrt{R^2 + \gamma^2 s^2}} \right] \frac{r}{2}$$

**It is still a linear field with r but with a longitudinal correlation s**

## Bunched Uniform Cylindrical Beam Model

$$E_z(0, s, \gamma) = \frac{I}{2\pi\gamma\epsilon_0 R^2 \beta c} h(s, \gamma)$$

$$E_r(r, s, \gamma) = \frac{Ir}{2\pi\epsilon_0 R^2 \beta c} g(s, \gamma)$$



## Lorentz Force

$$F_r = e(E_r - \beta c B_\theta) = e(1 - \beta^2)E_r = \frac{eE_r}{\gamma^2}$$

is a **linear** function of the transverse coordinate

$$\frac{dp_r}{dt} = F_r = \frac{eE_r}{\gamma^2} = \frac{eIr}{2\pi\gamma^2\epsilon_0 R^2 \beta c} g(s, \gamma)$$

The attractive magnetic force, which becomes significant at high velocities, tends to compensate for the repulsive electric force. Therefore space charge defocusing is primarily a non-relativistic effect. Using  $R=2\sigma_x$  for a uniform distribution:

$$F_x = \frac{eIx}{8\pi\gamma^2\epsilon_0\sigma_x^2\beta c} g(s, \gamma)$$

## Envelope Equation with Space Charge

Single particle transverse motion:

$$\frac{dp_x}{dt} = F_x \quad p_x = p \quad x' = \beta\gamma m_o c x'$$

$$\frac{d}{dt}(p x') = \beta c \frac{d}{dz}(p x') = F_x$$

$$x'' = \frac{F_x}{\beta c p}$$

$$F_x = \frac{e I x}{2\pi\gamma^2 \epsilon_0 \sigma_x^2 \beta c} g(s, \gamma)$$

$$x'' = \frac{k_{sc}(s, \gamma)}{\sigma_x^2} x$$

$$k_{sc} = \frac{2I}{I_A} g(s, \gamma)$$

$$I_A = \frac{4\pi\epsilon_o m_o c^3}{e}$$

Now we can calculate the term  $\langle x x'' \rangle$  that enters in the envelope equation

$$\sigma_x'' = \frac{\epsilon_{rms}^2}{\sigma_x^3} + \frac{\langle x x'' \rangle}{\sigma_x}$$

$$x'' = \frac{k_{sc}}{\sigma_x^2} x$$

$$\langle x x'' \rangle = \frac{k_{sc}}{\sigma_x^2} \langle x^2 \rangle = k_{sc}$$

Including all the other terms the envelope equation reads:

Space Charge De-focusing Force

$$\sigma_x'' + k^2 \sigma_x = \frac{\epsilon_n^2}{(\beta\gamma)^2 \sigma_x^3} + \frac{k_{sc}}{\sigma_x}$$

Emittance Pressure

External Focusing Forces

Laminarity Parameter:

$$\rho = \frac{(\beta\gamma)^2 k_{sc} \sigma_x^2}{\epsilon_n^2}$$

### The beam undergoes two regimes along the accelerator

$$\sigma_x'' + k^2 \sigma_x = \frac{\cancel{\varepsilon_n^2}}{\cancel{(\beta\gamma)^2} \sigma_x^3} + \frac{k_{sc}}{\sigma_x}$$

$\rho \gg 1$

Laminar Beam

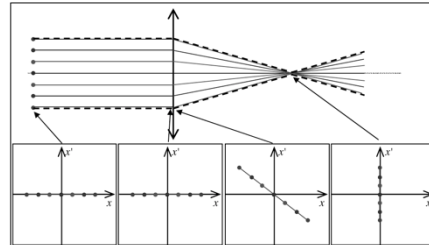


Fig. 10: Particle trajectories in laminar beam

$$\sigma_x'' + k^2 \sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2 \sigma_x^3} + \cancel{\frac{k_{sc}}{\sigma_x}}$$

$\rho \ll 1$

Thermal Beam

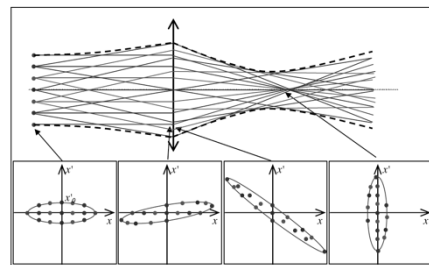
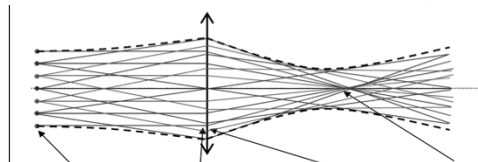
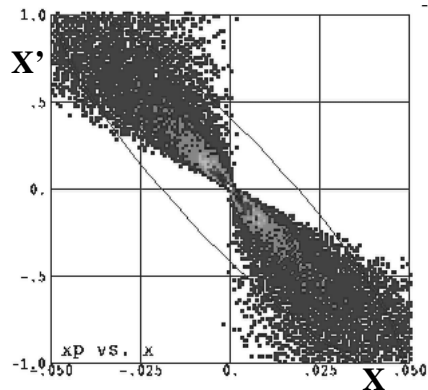


Fig. 11: Particle trajectories in non-zero emittance beam

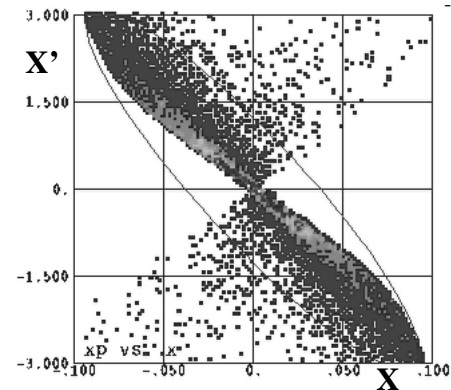
### Trace space evolution



No space charge =&gt; cross over



With space charge =&gt; no cross over

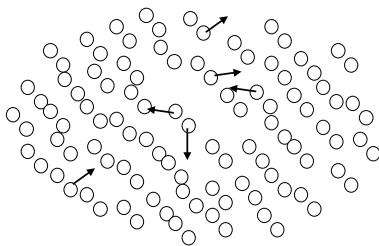


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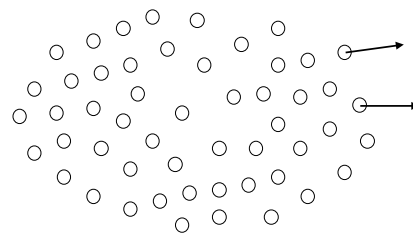
### Neutral Plasma

- Oscillations
- Instabilities
- EM Wave propagation



### Single Component Cold Relativistic Plasma

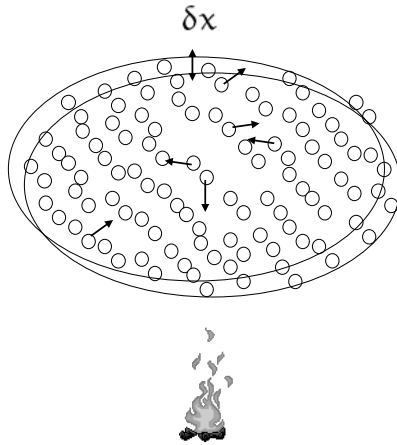
Magnetic focusing



Magnetic focusing

Surface charge density

$$\sigma = e n \delta x$$



Surface electric field

$$E_x = -\sigma/\epsilon_0 = -e n \delta x/\epsilon_0$$

Restoring force

$$m \frac{d^2 \delta x}{dt^2} = e E_x = -m \omega_p^2 \delta x$$

Plasma frequency

$$\omega_p^2 = \frac{n e^2}{\epsilon_0 m}$$

Plasma oscillations

$$\delta x = (\delta x)_0 \cos(\omega_p t)$$

$$\sigma'' + k_s^2 \sigma = \frac{k_{sc}(s, \gamma)}{\sigma}$$

Equilibrium solution:

$$\sigma_{eq}(s, \gamma) = \frac{\sqrt{k_{sc}(s, \gamma)}}{k_s}$$

Small perturbation:

$$\sigma(\xi) = \sigma_{eq}(s) + \delta\sigma(s)$$

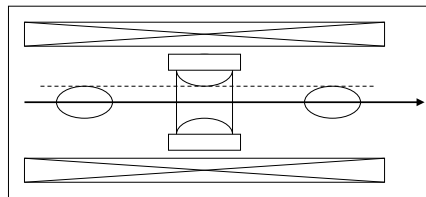
$$\delta\sigma''(s) + 2k_s^2 \delta\sigma(s) = 0$$

Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes:

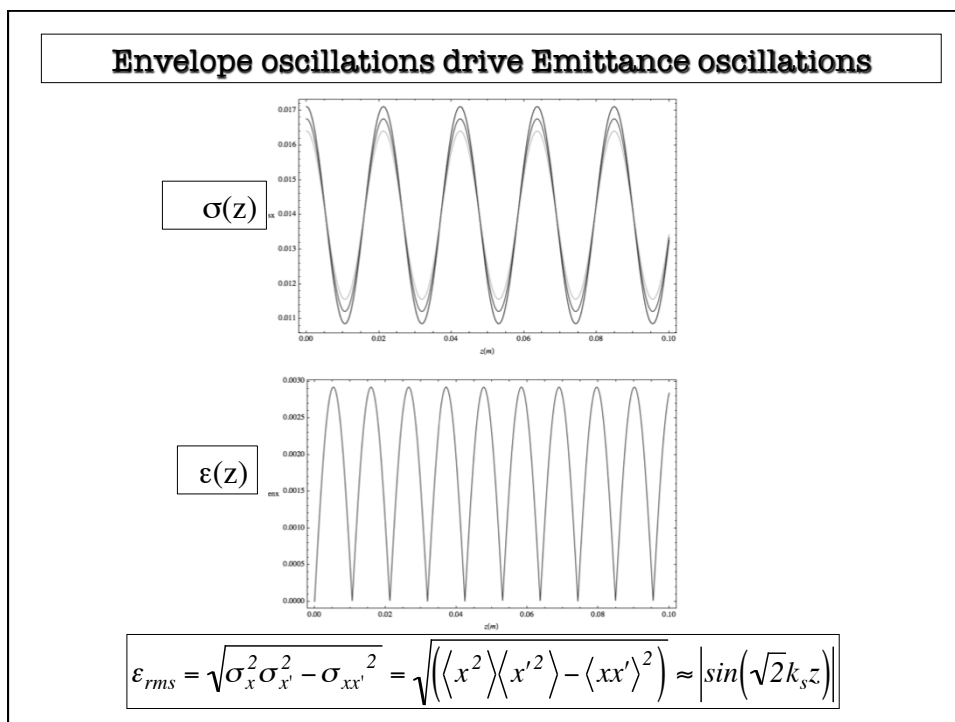
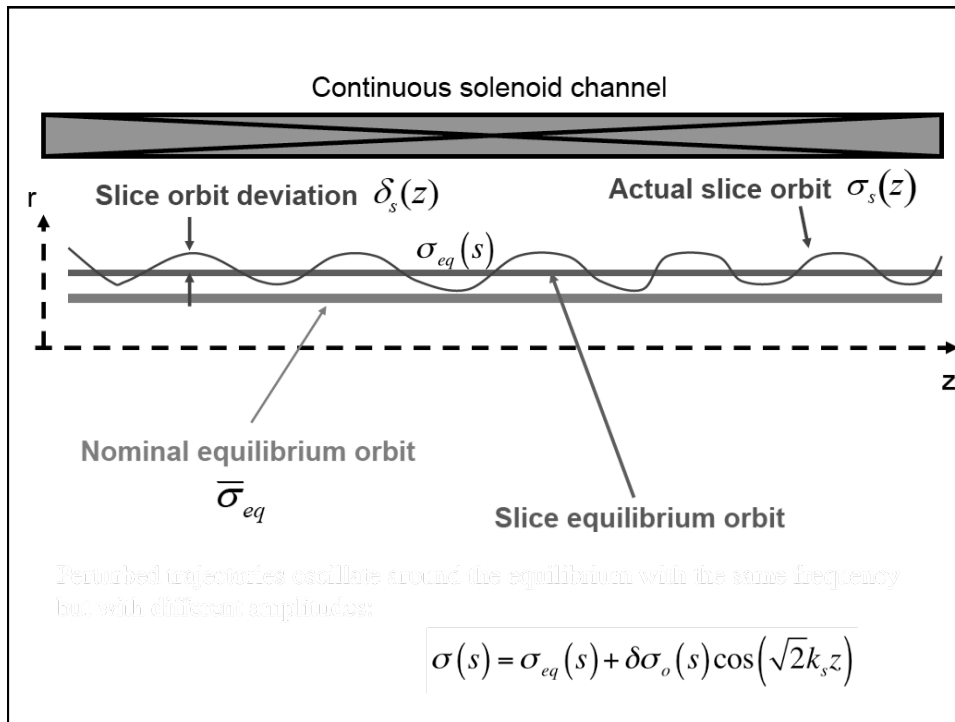
$$\sigma(s) = \sigma_{eq}(s) + \delta\sigma_o(s) \cos(\sqrt{2} k_s z)$$

Single Component  
Relativistic Plasma

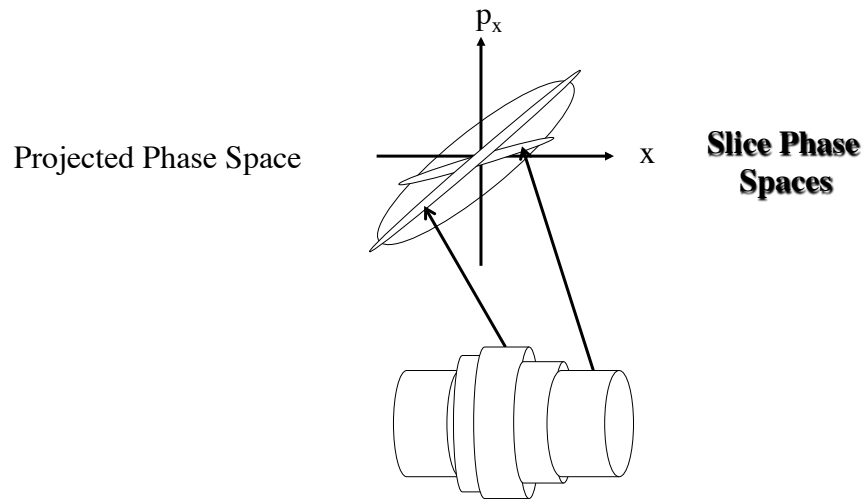
$$k_s = \frac{qB}{2mc\beta\gamma}$$



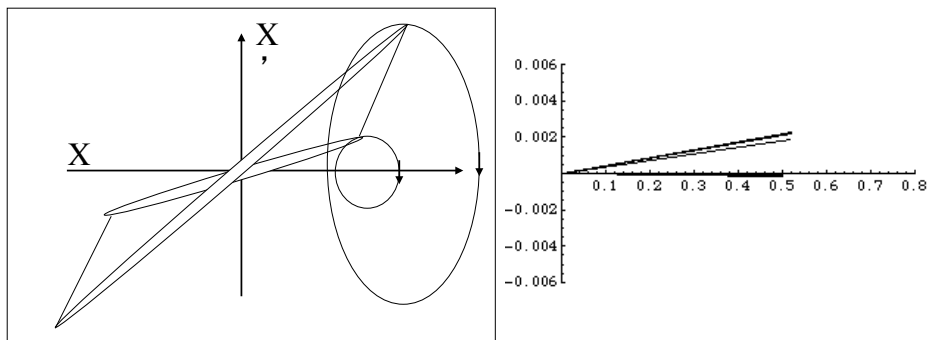
$$\delta\sigma(s) = \delta\sigma_o(s) \cos(\sqrt{2} k_s z)$$



**Emittance Oscillations are driven by space charge differential defocusing in core and tails of the beam**



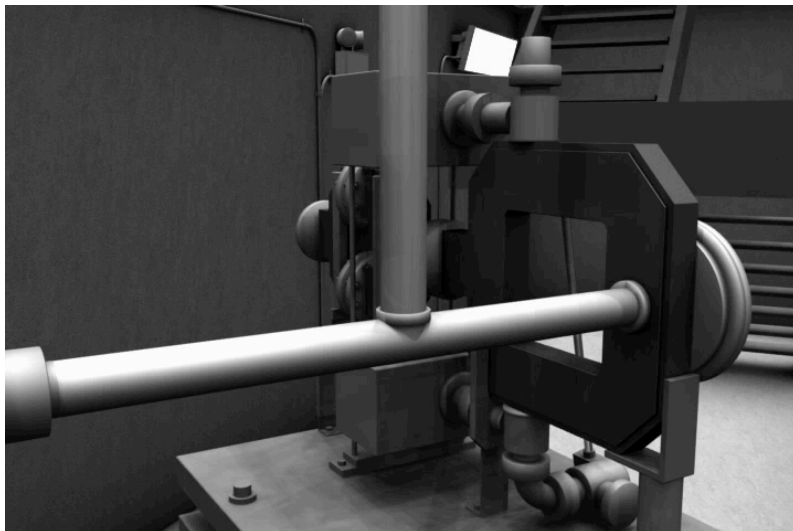
**Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes**



## OUTLINE

- The rms emittance concept
- rms envelope equation
- Space charge forces
- Space charge induced emittance oscillations
- Matching conditions and emittance compensation

## High Brightness Photo-Injector



### Envelope Equation with Acceleration

$$\frac{dp_x}{dt} = \frac{d}{dt}(px') = \beta c \frac{d}{dz}(px') = 0$$

$$p = \beta\gamma m_o c$$

$$x'' + \frac{p'}{p} x' = 0$$

$$x'' = -\frac{(\beta\gamma)'}{\beta\gamma} x'$$

$$\sigma_x'' = \frac{\varepsilon_{rms}^2}{\sigma_x^3} + \frac{\langle xx'' \rangle}{\sigma_x}$$

$$\langle xx'' \rangle = -\frac{(\beta\gamma)'}{\beta\gamma} \langle xx' \rangle = -\frac{(\beta\gamma)'}{\beta\gamma} \sigma_{xx'} = -\frac{(\beta\gamma)'}{\beta\gamma} \sigma_x \sigma_x'$$

Space Charge De-focusing Force

$$\sigma_x'' + \frac{(\beta\gamma)'}{\beta\gamma} \sigma_x' + k^2 \sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2 \sigma_x^3} + \frac{k_{sc}}{\sigma_x}$$

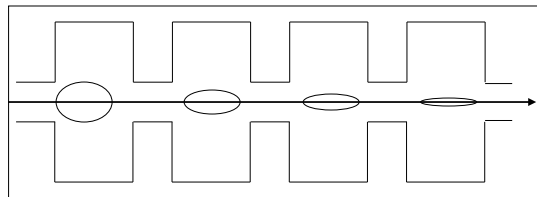
Adiabatic Damping

Emittance Pressure

Other External Focusing Forces

$$\varepsilon_n = \beta\gamma \varepsilon_{rms}$$

### Beam subject to strong acceleration



$$\sigma_x'' + \frac{\gamma'}{\gamma} \sigma_x' + \frac{k_{RF}^2}{\gamma^2} \sigma_x = \frac{\varepsilon_n^2}{\gamma^2 \sigma_x^3} + \frac{k_{sc}^o}{\gamma^3 \sigma_x}$$

We must include also the RF focusing force:

$$k_{RF}^2 = \frac{\gamma'^2}{2}$$

$$k_{sc}^o = \frac{2I}{I_A} g(s, \gamma)$$

$$\sigma_x'' + \frac{\gamma'}{\gamma} \sigma_x' + \frac{k_{RF}^2}{\gamma^2} \sigma_x = \frac{\varepsilon_n^2}{\gamma^2 \sigma_x^3} + \frac{k_{sc}^o}{\gamma^3 \sigma_x}$$

$$\boxed{\gamma = 1 + \alpha z} \quad \Rightarrow \quad \boxed{\gamma'' = 0}$$

Looking for an "equilibrium" solution  $\boxed{\sigma_{inv} = \sigma_o \gamma^n}$

$\Rightarrow$  all terms must have the same dependence on  $\gamma$

$$\boxed{\sigma_{inv}' = n \sigma_o \gamma^{n-1} \gamma'}$$

$$\boxed{\sigma_{inv}'' = n(n-1) \sigma_o \gamma^{n-2} \gamma'^2}$$

$$n(n-1) \sigma_o \gamma^{n-2} \gamma'^2 + n \sigma_o \gamma^{n-2} \gamma'^2 + k_{RF}^2 \sigma_o \gamma^{n-2} = \frac{k_{sc}^o}{\sigma_x} \gamma^{-3-n}$$

$$\boxed{n-2 = -3-n \Rightarrow n = -\frac{1}{2}}$$

$$\sigma_x'' + \frac{\gamma'}{\gamma} \sigma_x' + \frac{k_{RF}^2}{\gamma^2} \sigma_x = \frac{\varepsilon_n^2}{\gamma^2 \sigma_x^3} + \frac{k_{sc}^o}{\gamma^3 \sigma_x}$$

$$\boxed{\gamma = 1 + \alpha z} \quad \Rightarrow \quad \boxed{\gamma'' = 0}$$

Looking for an "equilibrium" solution  $\boxed{\sigma_{inv} = \sigma_o \gamma^n}$

$\Rightarrow$  all terms must have the same dependence on  $\gamma$

Laminar beam  $\boxed{\rho \gg 1 \Rightarrow n = -\frac{1}{2}}$

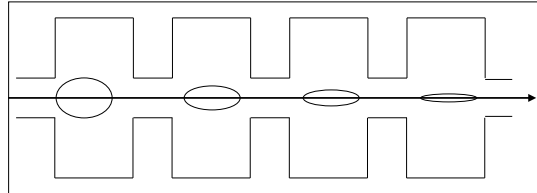
$$\boxed{\sigma_q = \frac{\sigma_o}{\sqrt{\gamma}}}$$

Thermal beam  $\boxed{\rho \ll 1 \Rightarrow n = 0}$

$$\boxed{\sigma_\varepsilon = \sigma_o}$$

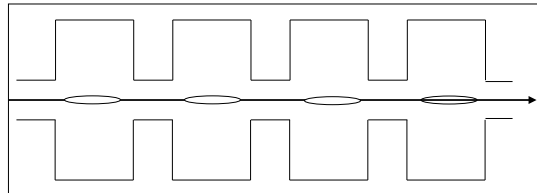
### Space charge dominated beam (Laminar)

$$\sigma_q = \frac{I}{\gamma'} \sqrt{\frac{2I}{I_A \gamma}}$$

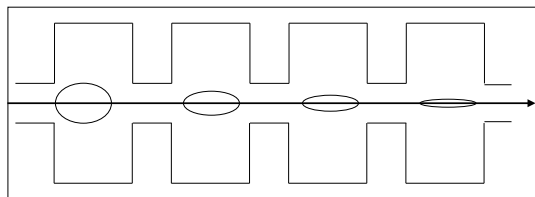


### Emittance dominated beam (Thermal)

$$\sigma_\varepsilon = \sqrt{\frac{2\varepsilon_n}{\gamma'}}$$



$$\sigma_q = \frac{I}{\gamma'} \sqrt{\frac{2I}{I_A \gamma}}$$



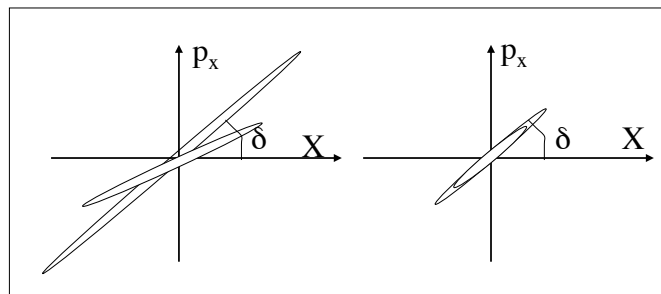
**This solution represents a beam equilibrium mode that turns out to be the transport mode for achieving minimum emittance at the end of the emittance correction process**

### An important property of the laminar beam

$$\sigma_q = \frac{1}{\gamma'} \sqrt{\frac{2I}{I_A \gamma}}$$

$$\sigma'_q = -\sqrt{\frac{2I}{I_A \gamma^3}}$$

Constant phase space angle:  $\delta = \frac{\gamma \sigma'_q}{\sigma_q} = -\frac{\gamma'}{2}$

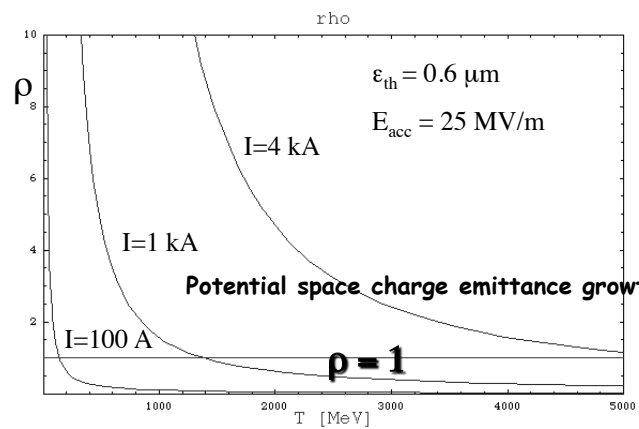


### Laminarity parameter

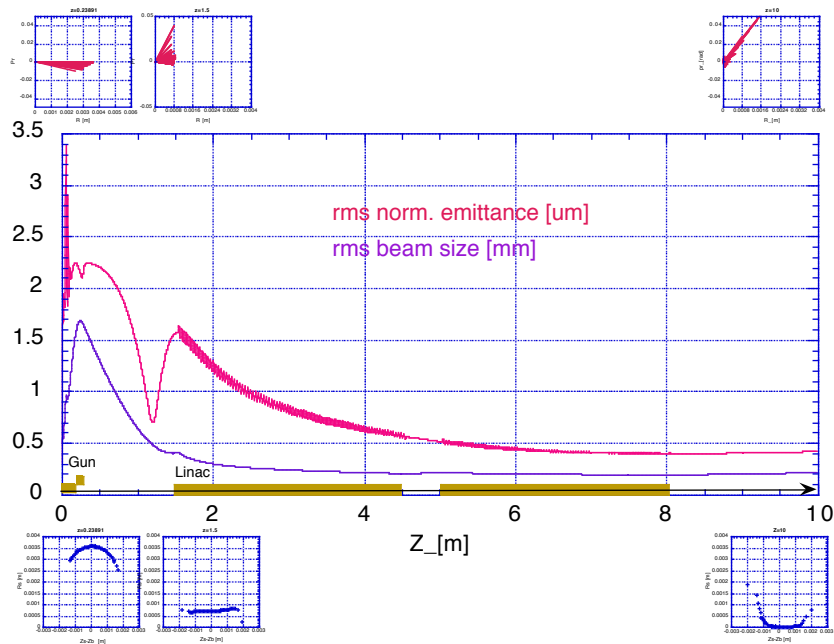
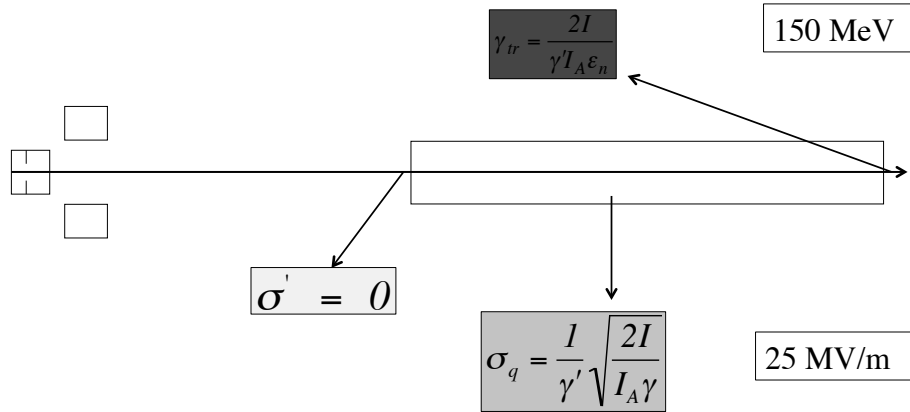
$$\rho = \frac{2I\sigma^2}{\gamma I_A \varepsilon_n^2} \equiv \frac{2I\sigma_q^2}{\gamma I_A \varepsilon_n^2} = \frac{4I^2}{\gamma'^2 I_A^2 \varepsilon_n^2 \gamma^2}$$

### Transition Energy ( $\rho=1$ )

$$\gamma_{tr} = \frac{2I}{\gamma' I_A \varepsilon_n}$$



## Matching Conditions with a TW Linac



**Emittance Compensation for a SC dominated beam:**  
**Controlled Damping of Plasma Oscillations**

- $\epsilon_n$  oscillations are driven by Space Charge
- propagation close to the laminar solution allows control of  $\epsilon_n$  oscillation “phase”
- $\epsilon_n$  sensitive to SC up to the transition energy

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