EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

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Exercises for tutorial on "Non-linear Dynamics" at the CAS 2015 in Otwock

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Abstract

Exercises and solutions complementing the lectures on "Tools for Non-Linear Dynamics".

Recommendation: pick a few of the exercises and try to solve them in detail

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1 Exercise 1

1.1 Problem:

Assume a thin lens kick $f(x)$ and show that it is always symplectic (use 1D to keep it simple):

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} x_0 \\ x'_0 + f(x_0) \end{pmatrix}$$

1.2 Solution:

The Jacobian of this kick is:

$$M = \begin{pmatrix} 1 & 0 \\ f'(x_0) & 1 \end{pmatrix}$$

where $f'(x_0)$ is $\frac{\partial x'}{\partial x_0}$.

we compute:

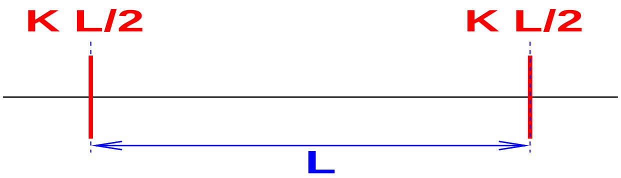
$$\begin{pmatrix} 1 & f'(x_0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ f'(x_0) & 1 \end{pmatrix} = \begin{pmatrix} f'(x_0) - f'(x_0) & 1 \\ -1 & 0 \end{pmatrix} = \mathcal{S}$$

This is the symplectic condition.

2 Exercise 2

2.1 Problem:

Assume two kicks at the end and beginning of a drift space: Compute to lumped



matrix $\mathcal{M}_{lumped}$.

2.2 Solution:

$$\mathcal{M}_{lumped} = \begin{pmatrix} 1 & 0 \\ -\frac{kL}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{kL}{2} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{2}KL^2 & L - \frac{1}{4}KL^3 \\ -K \cdot L & 1 - \frac{1}{2}KL^2 \end{pmatrix}$$

This is the same as the matrix with one kick in the centre, i.e. also a matrix with second order precision.

3 Exercise 3

3.1 Problem:

Use the function $f(x,p) \;=\; a \cdot x + b \cdot p$ (where a and b are constants) to get the maps:

$$e^{:f:}x = ?$$
$$e^{:f:}p = ?$$

What is the physical meaning ?

3.2 Solution:

The Lie transformation for $f(x,p) \;=\; a \cdot x + b \cdot p$ is:

$$e^{:f:} \;=\; e^{:a \cdot x + b \cdot p:} \;=\; e^{:a \cdot x:} e^{:b \cdot p:}$$

the latter can be easily evaluated using the solution for general monomials:

$$\begin{aligned} e^{:a \cdot x:}x &= x_0 \\ e^{:b \cdot p:}x &= -b \\ e^{:a \cdot x + b \cdot p:}x &= x_0 - b \end{aligned}$$

$$\begin{aligned} e^{:a \cdot x:}p &= a \\ e^{:b \cdot p:}p &= p_0 \\ e^{:a \cdot x + b \cdot p:}p &= p_0 + a \end{aligned}$$

The map would be:

$$\begin{aligned} x &= x_0 - b \\ p &= p_0 + a \end{aligned}$$

This map is a shift of the coordinates.

4 Exercise 4

4.1 Problem:

a)Assume a matrix M of the type:

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

described by a generator f. Use the properties of Lie transforms to evaluate the effect of this matrix on the moments x^2, xp, p^2:

$$e^{:f:}x^2 = ?$$

$$e^{:f:}p^2 = ?$$

$$e^{:f:}xp = ?$$

b) Discuss the results, considering what you learnt in previous lectures.

4.2 Solution:

a) From the matrix M we can directly write:

$$e^{:f:}x = (m_{11}x + m_{12}p)$$

and

$$e^{:f:}p = (m_{21}x + m_{22}p)$$

We know from the lecture some properties of Lie transforms (see lecture) and:

$$e^{:f:}x^2 = (e^{:f:}x)^2$$

therefore:

$$(e^{:f:}x)^2 = (m_{11}x + m_{12}p)^2$$

$$(e^{:f:}x)^2 = m_{11}^2x^2 \ + \ 2\ m_{11}m_{12}xp \ + \ m_{12}^2p^2$$

We also can compute:

$$e^{:f:}p^2 = (e^{:f:}p)^2$$

therefore:

$$(e^{:f:}p)^2 = (m_{21}x + m_{22}p)^2$$

$$(e^{:f:}p)^2 = m_{21}^2x^2 \ + \ 2\ m_{21}m_{22}xp \ + \ m_{22}^2p^2$$

also for the moment xp:

$$e^{:f:}xp = (e^{:f:}x)(e^{:f:}p) \qquad \text{(see lecture)}$$

$$e^{:f:}xp = m_{11}m_{21}x^2 \ + \ (m_{11}m_{22} \ + \ m_{12}m_{21})xp \ + \ m_{12}m_{22}p^2$$

To summarize the moments we re-write the above in matrix form:

$$\begin{pmatrix} x^2 \\ xp \\ p^2 \end{pmatrix}_{s_2} = \begin{pmatrix} m_{11}^2 & 2m_{11}m_{12} & m_{12}^2 \\ m_{11}m_{21} & m_{11}m_{22} + m_{12}m_{21} & m_{12}m_{22} \\ m_{21}^2 & 2m_{21}m_{22} & m_{22}^2 \end{pmatrix} \circ \begin{pmatrix} x^2 \\ xp \\ p^2 \end{pmatrix}_{s_1}$$

b) This is the transfer matrix for the Twiss parameters β, α, and γ. (which are basically moments !)

5 Exercise 5

5.1 Problem:

Assume a function:

$$f(x) = \frac{1}{x + \frac{1}{x}}$$

Compute the first derivative $f'(x)$ fo $x = 2$ using the Automatic Differentiation algorithm.

5.2 Solution:

For comparison, school calculus gives us:

$$f'(x) = -\frac{1 - \frac{1}{x^2}}{(x + \frac{1}{x})^2}$$

For $x = 2$ we get therefore: $f'(2) = -\frac{3}{25}$

To use differential Algebra, we replace x by the pair $(a, \delta) = (2, 1) = (q_0, q_1)$
From the lecture we know the inverse of this pair:

$$(q_0, q_1)^{-1} = \left(\frac{1}{q_0}, -\frac{q_1}{q_0^2}\right)$$

and

$$(q_0, q_1) + (r_0, r_1) = (q_0 + r_0, q_1 + r_1)$$

Substituting it into the function $f(x)$:

$$f(2, 1) = \frac{1}{(2, 1) + \frac{1}{(2,1)}} = \frac{1}{(2, 1) + (\frac{1}{2}, -\frac{1}{4})} = \frac{1}{(\frac{5}{2}, \frac{3}{4})} = (\frac{2}{5}, -\frac{3}{25})$$

So we have $f(2) = \frac{2}{5}$ and $f'(2) = -\frac{3}{25}$

6 Exercise 6

6.1 Problem:

a) Compute the map:

$$X(L) = ?$$
$$P(L) = X'(L) = ?$$

for a thick sextupole (1D) (length L, strength k) with the equation of motion:

$$x'' = -k \cdot x^2$$

up to order $\mathcal{O}(L^2)$, using the symplectic integration method.

b) Compute the map:

$$X(L) = ?$$

for a thick sextupole (2D) with the Hamiltonian (to give the equation of motion above):

$$H = \frac{1}{3}k(x^3 - 3xy^2) + \frac{1}{2}(p_x^2 + p_y^2)$$

using the Lie transformation method, compare with the solution from a).

6.2 Solution:

a) A solution to order $\mathcal{O}(L^2)$ is given by a thin lens approximation with a single kick in the centre of the element. The map can be written as a "leap-frog" integration:

$$\begin{aligned} x(L) &\approx x_0 + \frac{L}{2}(x'_0 + x'(L)) \\ x'(L) &\approx x'_0 + Lf(x_0 + \frac{L}{2}x'_0) \end{aligned}$$

For the sextupole with:

$$x'' = -k \cdot x^2 = f(x)$$

using the thin lens approximation (type D in the lecture) gives:

$$\begin{aligned} x(L) &= x_0 + x'_0 L - \frac{1}{2}kx_0^2 L^2 - \frac{1}{2}kx_0 x'_0 L^3 - \frac{1}{8}kx_0'^2 L^4 \\ x'(L) &= x'_0 - kx_0^2 L - kx_0 x'_0 L^2 - \frac{1}{4}kx_0'^2 L^3 \end{aligned}$$

Map for thick sextupole of length L in thin lens approximation, accurate to $\mathcal{O}(L^2)$

b) In the case an element is described by a Hamiltonian H, the Lie map of an element of length L and the Hamiltonian H is:

$$e^{-L:H:} = \sum_{i=0}^{\infty} \frac{1}{i!}(-L : H :)^i \tag{1}$$

For example, the Hamiltonian for a thick sextupole is:

$$H = \frac{1}{3}k(x^3 - 3xy^2) + \frac{1}{2}(p_x^2 + p_y^2) \tag{2}$$

To find the transformation we search for:

$$e^{-L:H:}x \quad \text{and} \quad e^{-L:H:}p_x \quad \text{i.e. for} \tag{3}$$

$$X(L) \;=\; e^{-L:H:}x = \sum_{i=0}^{\infty} \frac{1}{i!}(-L : H :)^i x \tag{4}$$

We can compute:

$$: H :^i x \tag{5}$$

for sufficiently large i:

$$: H :^0 x = x \tag{6}$$

$$: H :^1 x = \left(\frac{\partial H}{\partial x}\frac{\partial x}{\partial p_x} - \frac{\partial H}{\partial p_x}\frac{\partial x}{\partial x}\right) \quad = \quad -\, p_x \tag{7}$$

$$: H :^2 x = : H :(-p_x) = \left(\frac{\partial H}{\partial x}\frac{\partial(-p_x)}{\partial p_x} - \frac{\partial H}{\partial p_x}\frac{\partial(-p_x)}{\partial x}\right) \quad = \quad -\, k(x^2 - y^2) \tag{8}$$

$$: H :^3 x = : H :(-k(x^2 - y^2)) = \tag{9}$$

$$\left(\frac{\partial H}{\partial x}\frac{\partial(-k(x^2 - y^2))}{\partial p_x} - \frac{\partial H}{\partial p_x}\frac{\partial(-k(x^2 - y^2))}{\partial x}\right) \quad = \quad 2kxp_x$$

The same for y to get $2kyp_y$ and we have:

$$: H :^3 x = 2k(xp_x - yp_y) \tag{10}$$

$$.... \tag{11}$$

then we obtain:

$$X(L) \;=\; e^{-L:H:}x = x + p_x L - \frac{1}{2}kL^2(x^2 - y^2) - \frac{1}{3}kL^3(xp_x - yp_y) + ... \tag{12}$$

Comparison with the leap-frog algorithm shows deviation of order $\mathcal{O}(L^3)$.