

# **Supersymmetric gauge theories, conformal blocks and isomonodromic deformations**

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Plan:

- Introduction: “AGT-correspondence”, Whittaker elements and (exact?) conformal blocks;
- W-algebras and isomonodromic deformations ( $\mathfrak{sl}_N$  or  $\mathfrak{gl}_N$ , arbitrary  $\mathfrak{g}$ ?);
- Exact solution for the quasi-permutation or twist fields.

P.Gavrylenko and AM: JHEP 05 (2014)097, JHEP 02 (2016)  
181 , arXiv:1605.04554;

M.Bershtein, P.Gavrylenko and AM: in progress.

## “AGT correspondence” (LMN-2003?)

Harmonic oscillator  $[\alpha, \alpha^\dagger] = 1, H = \alpha^\dagger \alpha$

$$\alpha|0\rangle = 0, \quad \mathcal{H} = \bigoplus_{n \geq 0} \frac{(\alpha^\dagger)^n}{\sqrt{n!}} |0\rangle \quad (1)$$

$$e^{\alpha^\dagger} |0\rangle = |\Psi\rangle \in \mathcal{H} : \quad \alpha|\Psi\rangle = |\Psi\rangle$$

Whittaker matrix element ( $Z = \mathcal{B} = \mathcal{G}$ ):

$$\mathcal{G}(t) \simeq \langle \Psi | e^{t\alpha^\dagger \alpha} | \Psi \rangle = e^{et} \quad (2)$$

A free field 2d CFT

$$\begin{aligned} J(z) &= i\partial\phi(z) = \sum_{n \in \mathbb{Z}} \frac{\alpha_n}{z^{n+1}} \\ T(z) &= \frac{1}{2} :J(z)^2 := \sum_{n \in \mathbb{Z}} \frac{L_n}{z^{n+2}} \end{aligned} \tag{3}$$

so that

$$\mathcal{G}(a, t) = \langle \Psi_a | e^{tL_0} | \Psi_a \rangle = \exp \left( \frac{1}{2} t a^2 + e^t \right) \tag{4}$$

for

$$\begin{aligned} \alpha_1 |\Psi_a\rangle &= |\Psi_a\rangle, & \alpha_0 |\Psi_a\rangle &= a |\Psi_a\rangle \\ \alpha_n |\Psi_a\rangle &= 0, & n > 1 \end{aligned} \tag{5}$$

or

$$|\Psi_a\rangle = e^{\alpha_0 a} |a\rangle, \quad \alpha_n |a\rangle = 0, \quad n > 0, \quad \alpha_0 |a\rangle = a |a\rangle \tag{6}$$

- $\mathcal{G}(a, t) = \langle \Psi_a | e^{tL_0} | \Psi_a \rangle$  is Nekrasov function ( $U(1)$  gauge theory);
- $\mathcal{F} = \log \mathcal{G} = \frac{1}{2}a^2t + e^t$  solves

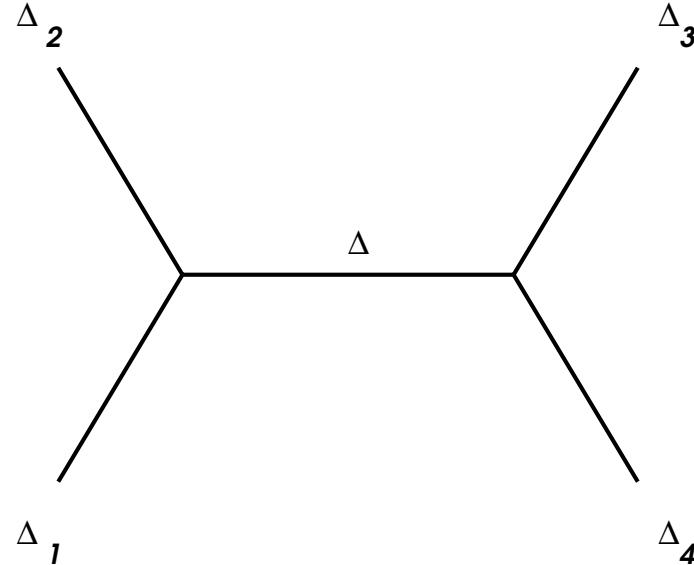
$$\frac{\partial^2 \mathcal{F}}{\partial t^2} = \exp \frac{\partial^2 \mathcal{F}}{\partial a^2} \quad (7)$$

and dual string theory (GW potential of  $\mathbb{P}^1$  or OP “isomonodromic” tau-function).

- Family of Toda curves  $\Lambda^N \left( w + \frac{1}{w} \right) = P_N(\lambda)$ ,  $d\lambda$ ,  $\frac{dw}{w}$ ,  $dS = \lambda \frac{dw}{w}$ . Solution for  $N = 1$ :  $\Lambda \left( w + \frac{1}{w} \right) = \lambda - v$ ,  $a = v$ ,  $\Lambda^2 = e^t$ .

## Conformal blocks

Pictorially, the conformal block (of Virasoro algebra) is



(a part of) 4-point function on sphere

$$\mathcal{G}(\Delta, \{\Delta_f\}; q) = \langle \Phi_{\Delta_\infty}(\infty) \Phi_{\Delta_1}(1) |_{\Delta} \Phi_{\Delta_q}(q) \Phi_{\Delta_0}(0) \rangle$$

- Free field  $\Delta_f = \frac{1}{2}\theta_f^2$ ,  $\mathcal{G}(\Delta, \{\Delta_f\}; q) = q^{\theta_0\theta_q}(1-q)^{\theta_q\theta_1}$ . After proper rescaling  $q \sim \frac{1}{\theta_q\theta_1}e^t$ ,  $(1-q)^{\theta_q\theta_1} \rightarrow e^{e^t}$ .
- Twist fields (Alesha Zamolodchikov, 80-s) at  $c = 1$ :

$$\begin{aligned} \langle \sigma(\infty)\sigma(1)|_{\Delta(a)=a^2}\sigma(q)\sigma(0)\rangle &= \\ &= \frac{16 \exp(2\pi i \mathcal{F}(a; \tau))}{q^{1/8}(1-q)^{1/8}\theta_{00}(\tau)} \end{aligned} \tag{8}$$

- $\Phi_{1/16}(q) = \sigma(q)$ : corresponding  $|\sigma\rangle$  is an eigenstate of  $L_0$ , but not of  $J_0$  - arbitrary intermediate  $\Delta(a)$ ;
- W-theory: not really defined. For the integer central charges - monodromy operator in the multi-component fermionic theory.

Old puzzle: the IR *quadratic* prepotential

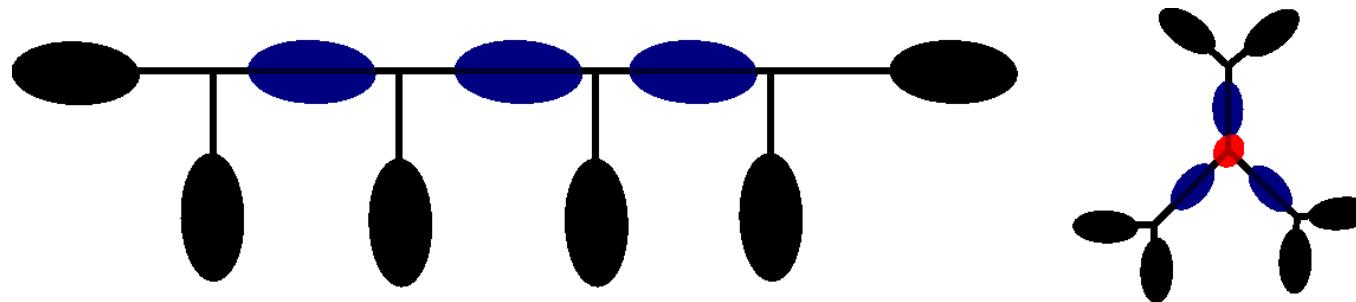
$$\begin{aligned} \mathcal{F}(a) &= \frac{1}{2\pi i} a^2 \log q_{IR} = \\ &= \frac{a^2}{2\pi i} \left( \underbrace{\log q}_{\text{classical}} - \underbrace{\log 16}_{\text{perturbative}} + \frac{q}{2} + \frac{13}{64}q^2 + \frac{23}{192}q^3 + \dots \right) \end{aligned} \quad (9)$$

comes from

$$\begin{aligned} \eta^2 &= \xi(\xi - 1)(\xi - q), \quad dS \sim a \frac{d\xi}{\eta} \\ q &= \frac{\theta_{10}^4(q_{IR})}{\theta_{00}^4(q_{IR})} = 16q_{IR} \prod_{n=1}^{\infty} \left( \frac{1 + q_{IR}^{2n}}{1 + q_{IR}^{2n-1}} \right)^8 \end{aligned} \quad (10)$$

where  $q_{IR} = e^{i\pi\tau}$ .

**Quiver gauge theories:** two pictures for the  $\mathfrak{g}^{\otimes 3}$  gauge (blue!) theory



Still there exists exact 4d/2d correspondence.

For the  $\mathfrak{g} = su(2)$  quivers the curve is hyperelliptic

$$x^2 = \langle T(z) \rangle = \sum_{j=1}^n \left( \frac{\Delta_j}{(z - z_j)^2} + \frac{u_j}{z - z_j} \right) \quad (11)$$

*Statement:* for all diagrams with *massless states* at "sicilian vertices" the prepotential

$$\mathcal{F}(\mathbf{a}, \mathbf{q})|_{\bigcup_{\mathcal{V}} \sum_{i \in \mathcal{V}_i} a_i = 0} = \frac{1}{2} \sum_{\alpha, \beta=1}^{\tilde{g}} a_\alpha T_{\alpha\beta}(\mathbf{q}) a_\beta \quad (12)$$

becomes the quadratic form with period matrix  $T_{\alpha\beta}(\mathbf{q})$  of *reduced* hyperelliptic curve of genus

$$\tilde{g} = g - V = n - 3 - \left( \frac{1}{2}n - 2 \right) = \frac{1}{2}n - 1 \quad (13)$$

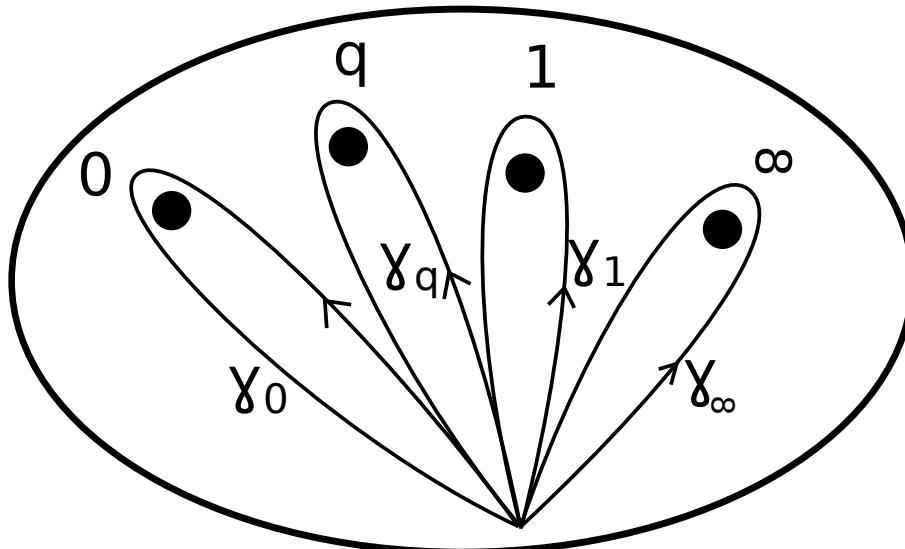
$V = |\mathcal{V}| = \frac{1}{2}n - 2$  is the number of triple vertices  $\{\mathcal{V}_i \in \mathcal{V}\}$ .

Thomae (Rauch) formulas ...

Beyond  $N = 2$  or hyperelliptic case?

## Isomonodromic tau-function

Isomonodromy/CFT correspondense



with fixed monodromies  $\gamma$ :

$$M_\gamma \sim \text{diag}(\lambda_1, \dots, \lambda_N) \sim \text{diag} \left( e^{2\pi i \theta_1}, \dots, e^{2\pi i \theta_N} \right) \quad (14)$$

Isomonodromy/CFT correspondense

$$\begin{aligned} \tau_{IM}(q) = & \sum_{\mathbf{n} \in \Gamma(\mathfrak{sl}_N)} \exp(\mathbf{b}, \mathbf{n}) \cdot \\ & \cdot C_{\mathbf{n}}^{(0q)}(\theta_0, \theta_q, \mathbf{a}, \mu_{0q}, \nu_{0q}) C_{\mathbf{n}}^{(1\infty)}(\theta_1, \theta_\infty, \mathbf{a}, \mu_{1\infty}, \nu_{1\infty}) \cdot \quad (15) \\ & \cdot q^{\frac{1}{2}(a+n, a+n) - \frac{1}{2}(\theta_0, \theta_0) - \frac{1}{2}(\theta_q, \theta_q)}. \\ & \cdot \mathcal{B}_{\mathbf{n}}(\{\theta_f\}, \mathbf{a}, \mu_{0q}, \nu_{0q}, \mu_{1\infty}, \nu_{1\infty}; q) \end{aligned}$$

(GIL 2012, P.Gavrylenko 2015).

$\{a, b; \mu, \nu\}$  - parameters on moduli space of  $\mathfrak{sl}_N$  flat connections (symplectic manifold), already for 3-points its dimension  $\dim = (N-1)(N-2)$ .

Topological vertex: 5d deformation ... ???

Generic monodromy operators for W-algebras: fermionic construction

$$\mathcal{K}_{\alpha\beta}(z, w) = \frac{[\phi(z)\phi(w)^{-1}]_{\alpha\beta}}{z - w} \quad (16)$$

given in terms of the solution to the linear system

$$\frac{d}{dz}\phi(z) = \phi(z) \left( \frac{A_0}{z} + \frac{A_1}{z-1} \right) = \phi(z)A(z) \quad (17)$$

with  $A_0 \sim \boldsymbol{\sigma}$ ,  $A_1 \sim \boldsymbol{\nu}$ ,  $A_\infty \sim \boldsymbol{\theta}$  but  $[A_i, A_j] \neq 0$ .

A quasi-group element

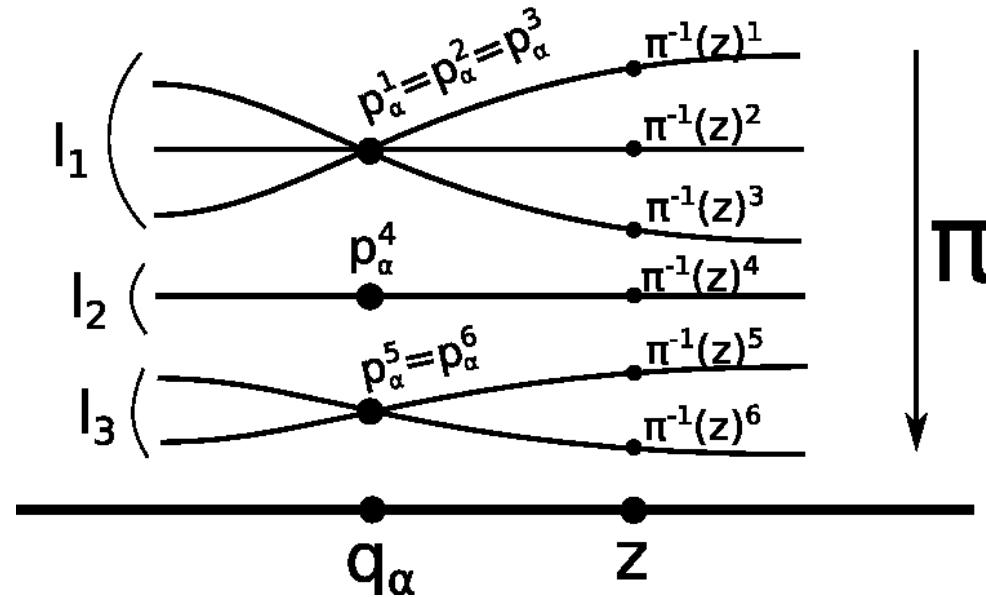
$$V_{\boldsymbol{\nu}}(t) = t^{-\Delta_{\boldsymbol{\nu}}} t^{L_0} V_{\boldsymbol{\nu}}(1) t^{-L_0}, \quad Z_{\text{bf}} \sim \langle \mathbf{Y}, \boldsymbol{\theta} | V_{\boldsymbol{\nu}} | \mathbf{Y}', \boldsymbol{\theta}' \rangle \quad (18)$$

in terms of the kernels (e.g. for  $|z| \geq 1$ ,  $|w| \leq 1$ )

$$\langle \boldsymbol{\theta} | \tilde{\psi}_{\alpha}^{\boldsymbol{\theta}}(z) V_{\boldsymbol{\nu}}(1) \psi_{\beta}^{\boldsymbol{\sigma}}(w) | \boldsymbol{\sigma} \rangle = \mathcal{K}_{\alpha\beta}(z, w) \quad (19)$$

and further application of the Wick theorem!

**Quasipermutation or twist** field for an arbitrary  $N$ -sheet cover  $\mathcal{C}$ :



with the free-field currents

$$J_i(z)J_j(z') = \frac{\delta_{ij} - \frac{1}{N}}{(z - z')^2} + \dots \quad (20)$$

$W$ -algebra generators as symmetric polynomials ( $c = N - 1$ ), e.g.

$$\begin{aligned} T(z) &= W_2(z) = \frac{1}{2} \sum_i : J_i(z)^2 : \\ W(z) &= W_3(z) = \sum_{i < j < k} : J_i(z) J_j(z) J_k(z) : \end{aligned} \quad (21)$$

Quasipermutation or twist fields

$$\gamma_q : J(z) \mathcal{O}_s(q) \mapsto s(J(z)) \mathcal{O}_s(q) \quad (22)$$

parameterized by  $s \in \mathbb{W}(sl_N) = S_N$ , with the OPE

$$\mathcal{O}_s(z') \mathcal{O}_{s^{-1}}(z) = \sum_{\theta} C_{s,\theta} (z' - z)^{\Delta(\theta) - 2\Delta(s)} (V_{\theta}(z) + \dots) \quad (23)$$

with  $V_{\theta}(z) = e^{i(\theta, \phi(z))}$  being *primaries of the current algebra*.

The (quasi-)permutation or twist fields  $\mathcal{O}$  are

- Primary fields of the W-algebra (and *not* of the current algebra!) - *non* degenerate in terms of  $W$ -representation theory;
- Their quantum numbers  $\mathcal{O}_{s,\mathbf{r}}(z) \mapsto |s, \mathbf{r}\rangle$  are symmetric functions of the Weyl vector (for each cycle in  $s = [l_1, \dots, l_k]$ ), e.g.  $\Delta \sim \text{Tr}(\log M_\gamma)^2$  or

$$\Delta = \frac{1}{2} \sum \left( \frac{\log \lambda_{i,v_i}}{2\pi i} \right)^2 = \sum_{i=1}^k \frac{l_i^2 - 1}{24l_i} + \sum_{i=1}^k \frac{1}{2} l_i r_i^2 \quad (24)$$

Formulas for the characters (logarithm up to  $\sum n_k = 0$ )

$$\Delta_{\mathbf{n}} = \frac{1}{2} \sum_k (\theta_k + n_k)^2 = \frac{N^2 - 1}{24N} + \frac{1}{2}\mathbf{n}^2 + \frac{1}{N}\rho \cdot \mathbf{n} \quad (25)$$

so that

$$\begin{aligned} \text{ch}_N(q) &= q^{\frac{N^2-1}{24N}} \frac{\prod_{k=1}^{\infty} (1-q^k)^N}{\prod_{k=1}^{\infty} (1-q^{k/N})} = q^{\frac{N^2-1}{24N}} \frac{\prod_{k=1}^{\infty} (1-q^k)^{N-1}}{\prod_{a=1}^{N-1} \prod_{k=1}^{\infty} (1-q^{k+a/N})} = \\ &= \sum_{\alpha \in \Gamma(\mathfrak{sl}_N)} q^{\frac{1}{2}(\alpha + \frac{\rho}{N}, \alpha + \frac{\rho}{N})} = q^{\frac{N^2-1}{24N}} \Theta_N(\tau\rho/2N|q) \end{aligned} \quad (26)$$

coincide with representation theory of  $\widehat{gl(N)_1}$  (Lepowsky-Wilson representations).

Similarly, e.g. for  $\mathfrak{g} = \mathfrak{so}_{2r+1}$

$$\begin{aligned} & \frac{\prod_{k \geq 0} (1 + q^{k+\frac{1}{2}})}{\prod_{n>0} (1 - q^n)^r} \sum_{\alpha \in \Gamma(\mathfrak{so}_{2r+1})} q^{\frac{1}{2}(\alpha, \alpha) + \frac{1}{2r}(\rho^\vee, \alpha)} = \\ &= 2 \prod_{n>0} (1 + q^n) \prod_{a=1}^r \frac{1}{\prod_{k \geq 0} (1 - q^{k+\frac{2a-1}{2r}})} \end{aligned} \tag{27}$$

The l.h.s. has an obvious sense of the trace

$$\begin{aligned} \text{Tr}_{\mathcal{F}_\psi^{\otimes(2r+1)}} q^{L_0 + \frac{1}{2r}(\rho^\vee, h)} &= \text{Tr}_{\mathcal{V}_0 \oplus \mathcal{V}_1} q^{L_0 + \frac{1}{2r}(\rho^\vee, h)} = \\ &= 2 \text{Tr}_{\mathcal{V}_0} q^{L_0 + \frac{1}{2r}(\rho^\vee, h)} = 2 \text{Tr}_{\mathcal{V}_1} q^{L_0 + \frac{1}{2r}(\rho^\vee, h)} \end{aligned} \tag{28}$$

A generic product formula for the lattice theta-functions (?)

$$\Theta_{\mathfrak{g}} (\tau \rho^\vee / h | q) \sim \sum_{\alpha \in \Gamma(\mathfrak{g})} q^{\frac{1}{2}(\alpha, \alpha) + \frac{1}{h}(\rho^\vee, \alpha)} \tag{29}$$

To compute the conformal block of quasimonodromy fields

$$\mathcal{G}(q) = \langle \mathcal{O}_{s_1}(q_1) \mathcal{O}_{s_1^{-1}}(q_2) \dots \mathcal{O}_{s_L}(q_{2L-1}) \mathcal{O}_{s_L^{-1}}(q_{2L}) \rangle \quad (30)$$

consider the correlation functions

$$\begin{aligned} \mathcal{G}(q) &= \langle\langle 1 \rangle\rangle_q, \quad \mathcal{G}_1^i(z|q) = \langle\langle J_i(z) \rangle\rangle_q \\ \mathcal{G}_2^{ij}(z, z'|q) &= \langle\langle J_i(z) J_j(z') \rangle\rangle_q \\ &\text{etc} \end{aligned} \quad (31)$$

so that  $\mathcal{G}_1^i(z|q)dz = \mathcal{G}_1(\pi^{-1}(z)^i|q)d\pi^{-1}(z)^i$  at the  $i$ -th preimage on  $\mathcal{C}$  etc, which are restored from the analytic properties.

E.g.

$$\frac{\mathcal{G}_1(\xi|q)d\xi}{\mathcal{G}(q)} = \sum_{I=1}^g a_I d\omega_I + \sum_{\alpha=1}^{2L} d\Omega_{\mathbf{r}^\alpha} = dS \quad (32)$$

1-form on the cover, decomposed over the basis of Abelian differentials.

- Holomorphic part

$$a_I = \oint_{A_I} dS = \oint_{A_I} \frac{d\xi \mathcal{G}_1(\xi|q)}{\mathcal{G}(q)}, \quad I = 1, \dots, g \quad (33)$$

is fixed by intermediate channel's charges, the number comes from RH formula

$$g = \sum_{j=1}^L \sum_{\alpha=1}^{k_\alpha} (l_i - 1) - N + 1 \quad (34)$$

- Meromorphic part

$$d\Omega_{\mathbf{r}^\alpha} \underset{p_{\alpha,i}}{\sim} r_i^\alpha \frac{d\xi_{\alpha,i}}{\xi_{\alpha,i}}, \quad \pi(p_{\alpha,i}) = q_\alpha, \quad \oint_{\mathbf{A}} d\Omega_{\mathbf{r}^\alpha} = 0 \quad (35)$$

is fixed by  $U(1)^{\sum k_\alpha - 1}$  charges for reducible permutations

$$\sum_i r_i^\alpha = 0, \quad \alpha = 1, \dots, 2L$$

Similarly

$$\frac{\mathcal{G}_2(p', p)}{\mathcal{G}} d\xi_{p'} d\xi_p = dS(p') dS(p) + K(p', p) - \frac{1}{N} K_0(p', p) \quad (36)$$

in terms of

$$K(p', p) = d\xi_{p'} d\xi_p \log E(p', p) = \frac{d\xi_{p'} d\xi_p}{(\xi_{p'} - \xi_p)^2} + 2t_\xi(p) d\xi_p^2 + \dots$$

$$\oint_{A_I} K(p', p) = 0 \quad (37)$$

which leads to

$$\begin{aligned} & \frac{\langle T(z) \mathcal{O}_{s_1}(q_1) \mathcal{O}_{s_1^{-1}}(q_2) \dots \mathcal{O}_{s_L}(q_{2L-1}) \mathcal{O}_{s_L^{-1}}(q_{2L}) \rangle}{\langle \mathcal{O}_{s_1}(q_1) \mathcal{O}_{s_1^{-1}}(q_2) \dots \mathcal{O}_{s_L}(q_{2L-1}) \mathcal{O}_{s_L^{-1}}(q_{2L}) \rangle} = \\ &= \sum_{\pi(p)=z} \left( t_z(p) + \frac{1}{2} \left( \frac{dS(p)}{dz} \right)^2 \right) \end{aligned} \quad (38)$$

In local co-ordinate  $\xi = z^{1/l}$

$$t_z(p) = t_\xi(p) \left( \frac{d\xi}{dz} \right)^2 + \frac{1}{12} \{\xi, z\} = t_\xi(p) z^{2/l-2} + \frac{l^2 - 1}{24l^2} \frac{1}{z^2} \quad (39)$$

it gives the dimensions  $\sum_{i=1}^k \frac{l_i^2 - 1}{24l_i} + \sum_{i=1}^k l_i \frac{r_i^2}{2}$ .

The coefficients at the first-order poles give

$$\begin{aligned} \partial_{q_\alpha} \log \mathcal{G}(q_1, \dots, q_{2L}) &= \sum_{\pi(p_{\alpha,i})=q_\alpha} \text{res}_{p_{\alpha,i}} t_z dz + \\ &+ \frac{1}{2} \sum_{\pi(p_{\alpha,i})=q_\alpha} \text{res}_{p_{\alpha,i}} \frac{(dS)^2}{dz} \end{aligned} \quad (40)$$

where the last term is a formula from extended SW theory.

Result:

$$\mathcal{G}(q) = \langle\langle 1 \rangle\rangle_q = \tau_{SW}(a, q) \cdot \tau_B(q) \quad (41)$$

where

$$\partial_{q_\alpha} \log \tau_{SW}(a; q) = \frac{1}{2} \sum_{\pi(p_{\alpha,i})=q_\alpha} \text{res}_{p_{\alpha,i}} \frac{(dS)^2}{dz} \quad (42)$$

and

$$\partial_{q_\alpha} \log \tau_B(q) = \sum_{\pi(p_{\alpha,i})=q_\alpha} \text{res}_{p_{\alpha,i}} t_z dz \quad (43)$$

together with the SW equations

$$\partial_{a_I} \log \tau_{SW} = \oint_{B_I} dS = \sum_J \mathcal{T}_{IJ} a_J + \Omega_I \quad (44)$$

where:

- $\mathcal{T}_{IJ} = \oint_{B_I} d\omega_J$  is the period matrix of the cover in fixed the basis in homologies  $H_1(\Sigma)$ ,

$$\frac{\partial \mathcal{T}_{IJ}}{\partial q_\alpha} = \sum_{\pi(p_\alpha^i) = q_\alpha} \text{Res}_{p_\alpha^i} \frac{d\omega_I d\omega_J}{dz} \quad (45)$$

- The linear pieces (here  $A(p)$  is the Abel map):

$$\Omega_J = \sum_{\alpha} \oint_{B_J} d\Omega_{\mathbf{r}^\alpha} = \sum_{\alpha, j} r_j^\alpha A_J(p_{\alpha, j}), \quad J = 1, \dots, g \quad (46)$$

Consistency is proven by RBI.

The SW part can be computed

$$\log \tau_{SW} = \frac{1}{2} \sum_{IJ} a_i \mathcal{T}_{IJ} a_J + \sum_I \Omega_I a_I + Q(\mathbf{r}) \quad (47)$$

where the quadratic form

$$Q(\mathbf{r}) = \sum_{\substack{p_\alpha^i \neq p_\beta^j}} r_\alpha^i r_\beta^j \log \Theta_*(A(p_\alpha^i) - A(p_\beta^j)) - \left. - \sum_{p_\alpha^i} (r_\alpha^i)^2 l_\alpha^i \log \frac{d(z(p) - q_\alpha)^{1/l_\alpha^i}}{h_*^2(p)} \right|_{p=p_\alpha^i} \quad (48)$$

Summing up these conformal blocks

$$\begin{aligned}\tau_{IM}(q|\mathbf{a}, \mathbf{b}) &= \sum_{\mathbf{n} \in \mathbb{Z}^g} \mathcal{G}(q|\mathbf{a} + \mathbf{n}) e^{2\pi i(\mathbf{n}, \mathbf{b})} = \\ &= \tau_B(q) \exp\left(\frac{1}{2}Q(\mathbf{r})\right) \Theta \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}(\Omega)\end{aligned}\tag{49}$$

one gets the exact isomonodromic tau-function. The most nontrivial step here is

$$\mathcal{G}(q, \bullet) = q^\# CC \mathcal{B}(q, \bullet)\tag{50}$$

where  $\mathcal{B}(q, \bullet) = 1 + O(q)$  corresponds to original BPZ normalization, and

$$CC \approx 4^{-\sum a_I^2 - (\sum a_I)^2}\tag{51}$$

comes from degenerate period matrix (in hyperelliptic example, and can be extended).