Lattice determination of α_s

Rainer Sommer

based on work by ALPHA Collaboration

Mattia Bruno, Mattia Dalla Brida, Patrick Fritzsch, Tomasz Korzec, Alberto Ramos, Stefan Schaefer, Hubert Simma, Stefan Sint, RS

and simulations by CLS

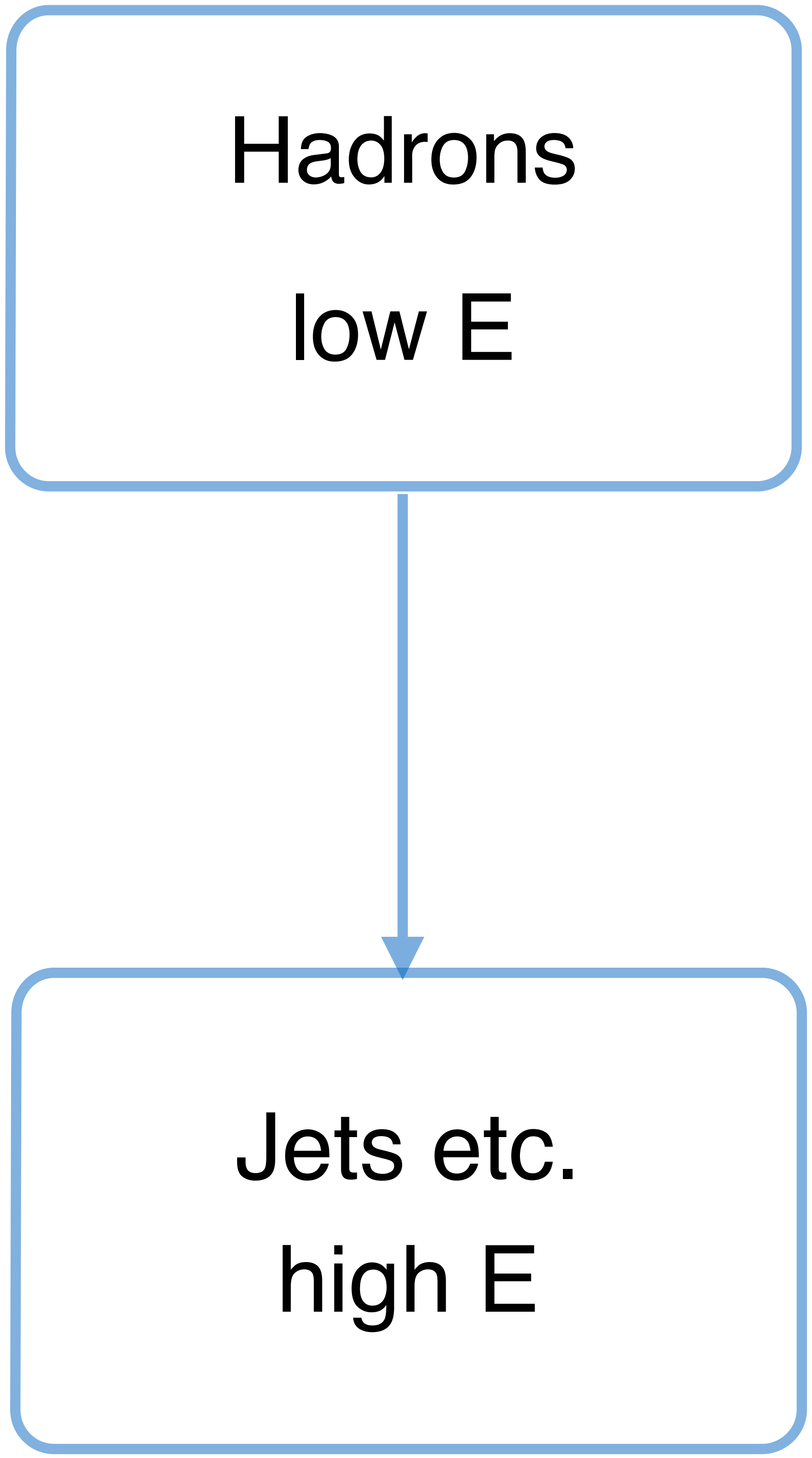
presentation at qcd@lhc16, Zürich, August 2016

HELMHOLTZ | ASSOCIATION

NIC

DESY

Overview



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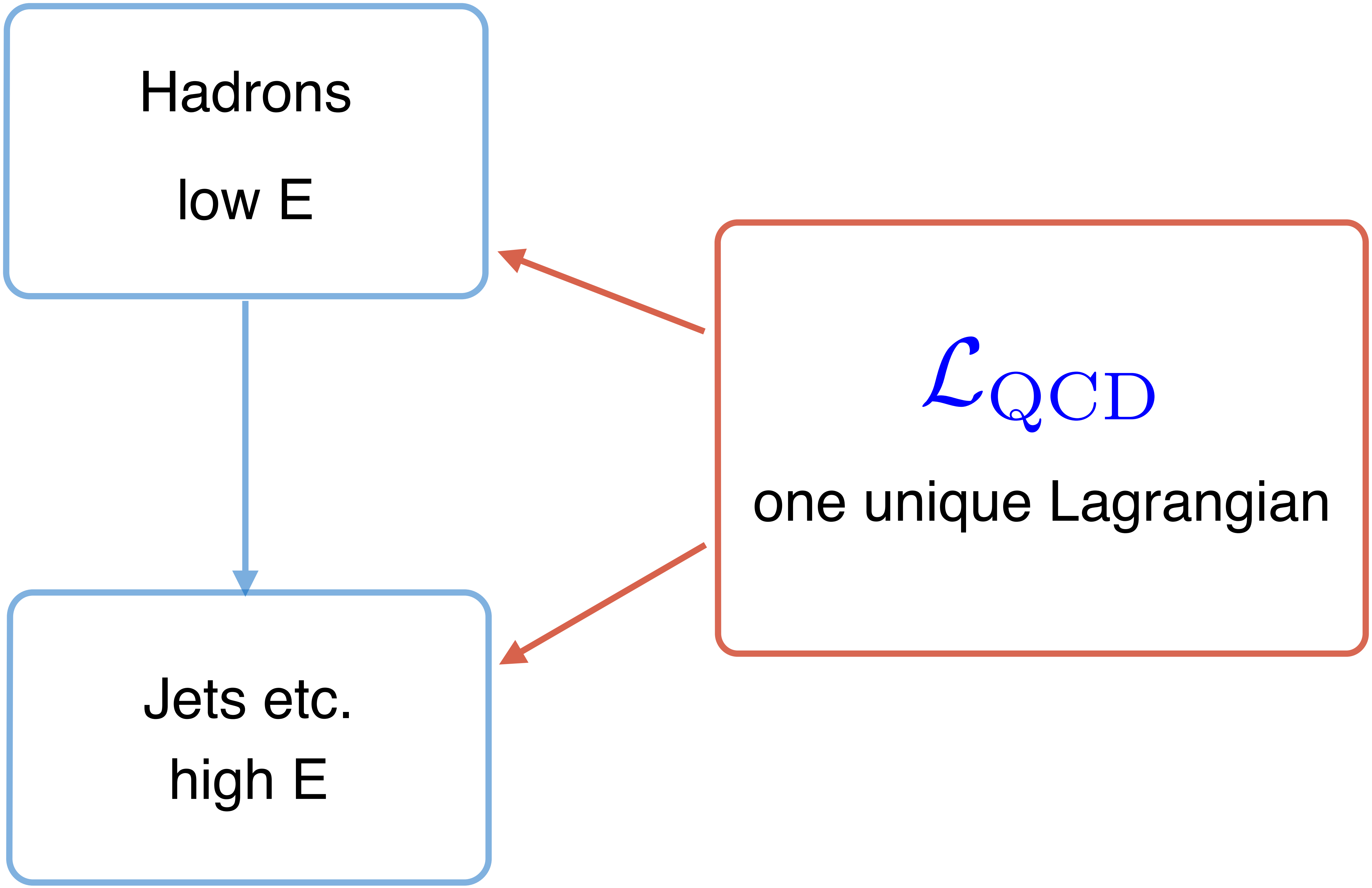
Hadrons
low E
Jets etc.
high E
$\mathcal{L}_{\mathrm{QCD}}$
one unique Lagrangian

Overview

$$m_{\mathrm{prot}},\ m_{\pi}$$
$$m_{\mathrm{K}},\ \ldots$$

$$\alpha(m_{\mathrm{Z}})$$

(and quark masses …)

Overview

$m_{\rm prot},\ m_\pi$

$m_{\rm K},\ \ldots$

$$m_{\rm prot} \sim \mathrm{e}^{-1/(2b_0 g^2)} = 0 + 0 \times g^2 + 0 \times g^4 + \ldots$$

non-perturbative

$\alpha(m_{\rm Z})$

(and quark masses …)

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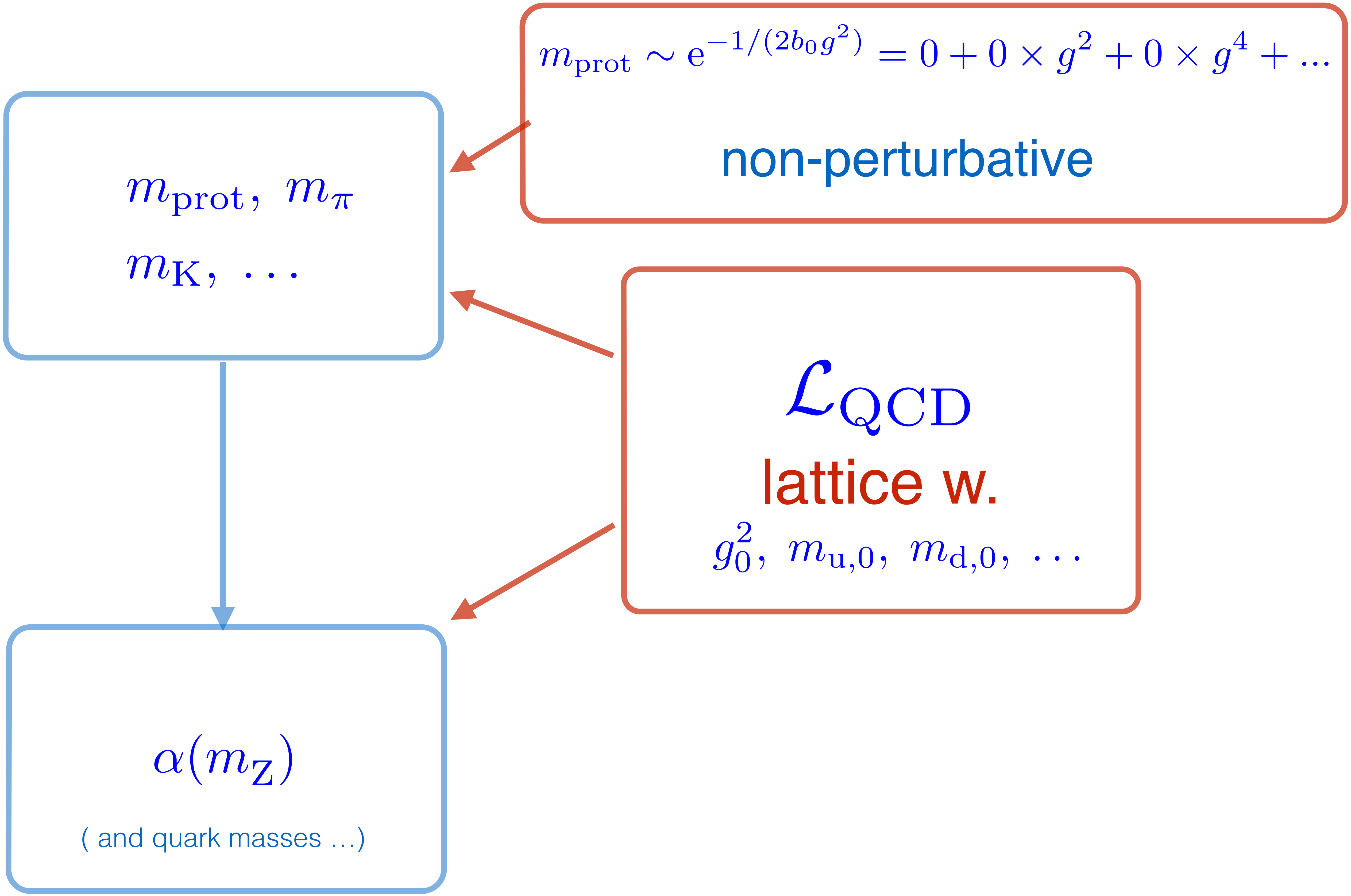
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$m_{\rm prot}\,,\;\; m_\pi$
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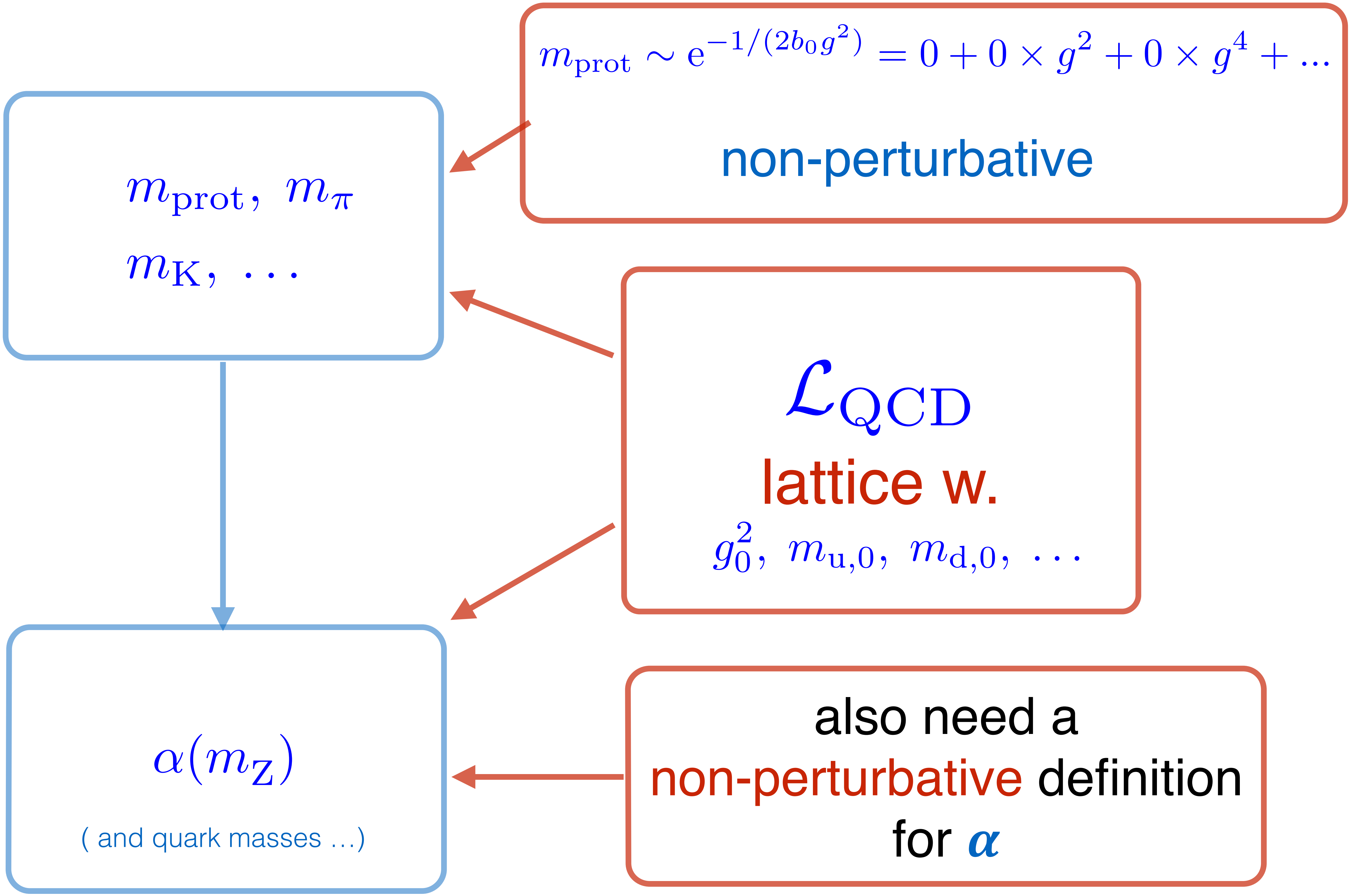
$\mathcal{L}_{\rm QCD}$
lattice w.
$g_0^2\,,\;\; m_{\rm u,0}\,,\;\; m_{\rm d,0}\,,\;\; \ldots$
$\alpha(m_{\rm Z})$
(and quark masses …)

Overview



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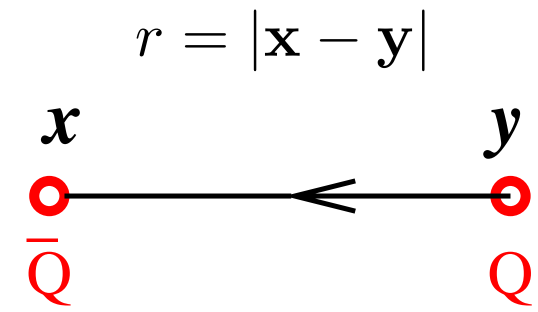
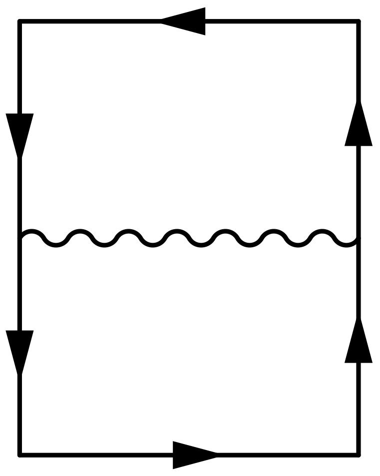
lattice w.
$g_0^2,\ m_{\rm u,0},\ m_{\rm d,0},\ \ldots$
$\alpha(m_{\rm Z})$
(and quark masses …)
also need a
non-perturbative definition
for $\boldsymbol{\alpha}$

Definition of QCD coupling (an example, non-perturbative)

$$\alpha_{\rm qq}(\mu) \equiv \frac{3r^2}{4} F_{Q\bar{Q}}(r)\,, \quad \mu = \frac{1}{r}$$





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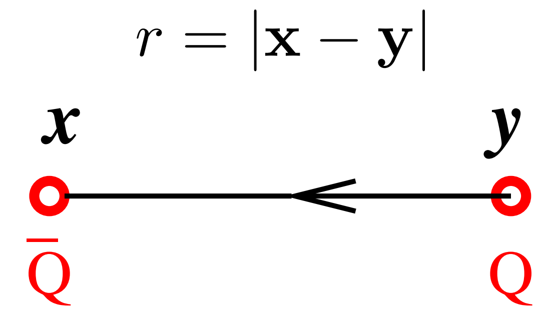
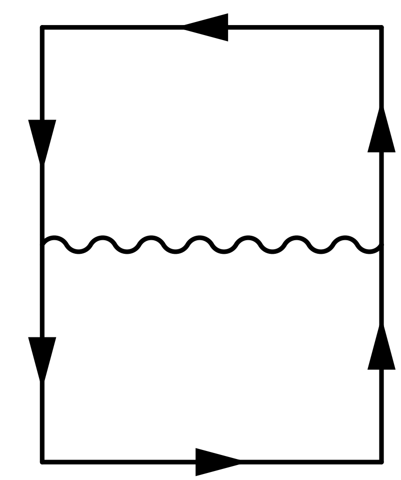




then

$$\alpha_{\mathrm{qq}}(\mu) = \alpha_{\overline{\mathrm{MS}}}(\mu) + c_1 \alpha^2_{\overline{\mathrm{MS}}}(\mu) + \ldots$$

always
(non-perturbatively)
defined
physics!

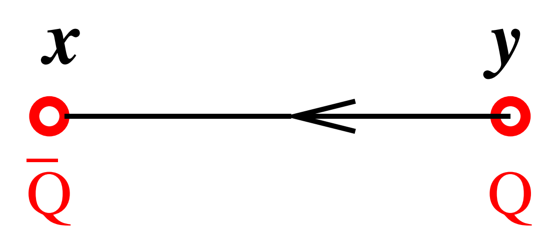
perturbatively defined
by such relations

makes sense for $\alpha \ll 1$

Definition of QCD coupling (an example, non-perturbative)

$$\alpha_{\rm qq}(\mu) \equiv \frac{3r^2}{4} F_{Q\bar{Q}}(r)\,, \quad \mu = \frac{1}{r}$$

$r = |\mathbf{x} - \mathbf{y}|$



here just one particular
short distance observable (definition)

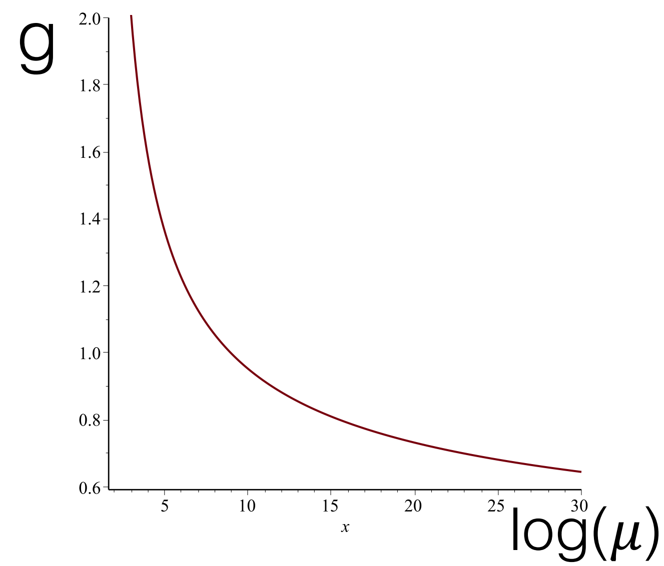
There are many definitions.
Equivalent at small α.

QCD coupling, energy dependence

$$\text{RGE:} \quad \mu\frac{\partial\bar{g}}{\partial\mu} = \beta(\bar{g}) \qquad \bar{g}(\mu)^2 = 4\pi\alpha(\mu)$$

$$\beta(\bar{g}) \overset{\bar{g}\to 0}{\sim} -\bar{g}^3\left\{b_0 + b_1\bar{g}^2 + b_2\bar{g}^4 + \ldots\right\}$$

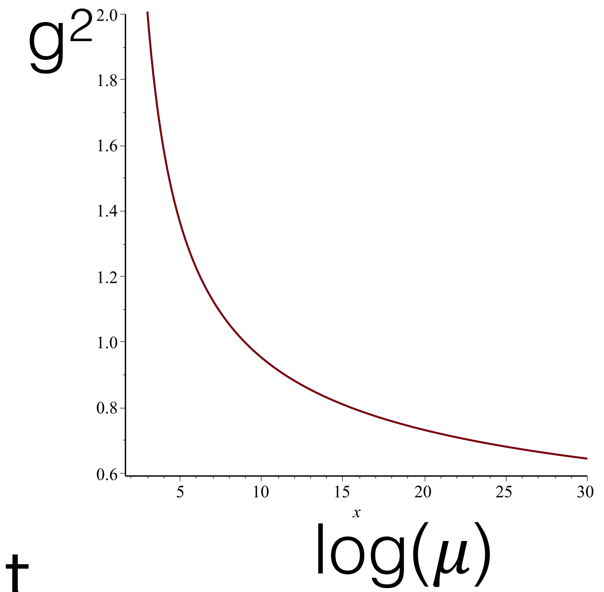
$$b_0 = \frac{1}{(4\pi)^2}\left(11 - \frac{2}{3}N_\mathrm{f}\right)$$



Asymptotic freedom

μ = energy = physical

QCD coupling, energy dependence



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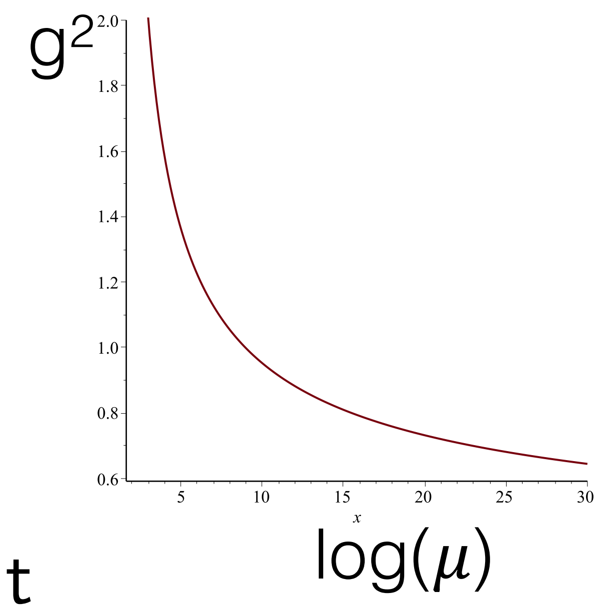
Λ-parameter $(\bar{g} \equiv \bar{g}(\mu))$ $=$ Renormalization Group Invariant
$=$ intrinsic scale of QCD = integration constant of RGE

$$\Lambda = \mu \left(b_0 \bar{g}^2\right)^{-b_1/2b_0^2} \mathrm{e}^{-1/2b_0\bar{g}^2} \exp\left\{ - \int_0^{\bar{g}} \mathrm{d}g \left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right] \right\}$$

$$\bar{g} \equiv \bar{g}(\mu)$$

QCD coupling, energy dependence



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singular behavior

QCD coupling, energy dependence

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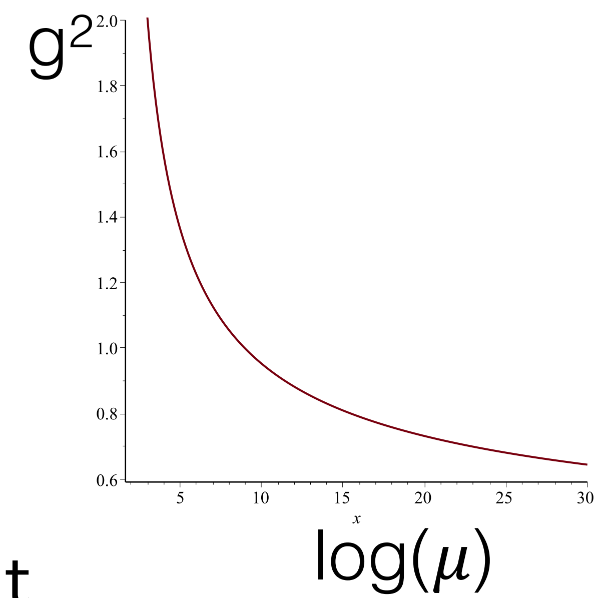

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singular behavior

convergent for g -> 0

$$\boxed{\Lambda = \mu\,(b_0\bar{g}^2)^{-b_1/2b_0^2}\mathrm{e}^{-1/2b_0\bar{g}^2}\,\exp\left\{-\int_0^{\bar{g}}\mathrm{d}g\left[\frac{1}{\beta(g)}+\frac{1}{b_0 g^3}-\frac{b_1}{b_0^2 g}\right]\right\}}$$

scheme (=definition) dependence

$\Uparrow$

exactly known

$$\boxed{\Lambda \;=\; \mu\,(b_0\bar{g}^2)^{-b_1/2b_0^2}\,\mathrm{e}^{-1/2b_0\bar{g}^2}\;\exp\left\{-\int_0^{\bar{g}}\mathrm{d}g\,[\frac{1}{\beta(g)}+\frac{1}{b_0 g^3}-\frac{b_1}{b_0^2 g}]\right\}}$$

scheme (=definition) dependence

$$\bar{g}=\bar{g}_{\overline{\mathrm{MS}}}\to\Lambda=\Lambda_{\overline{\mathrm{MS}}}\,,\qquad \bar{g}=\bar{g}_{\mathrm{qq}}\to\Lambda=\Lambda_{\mathrm{qq}}$$

$$\Lambda_{\overline{\mathrm{MS}}}/\Lambda_{\mathrm{qq}}=\exp\left(c_1/(2b_0)\right)$$

$$\alpha_{\mathrm{qq}}(\mu)=\alpha_{\overline{\mathrm{MS}}}(\mu)+c_1\alpha_{\overline{\mathrm{MS}}}^2(\mu)+\ldots$$

exactly known

$$\boxed{\Lambda \;=\; \mu\,(b_0\bar{g}^2)^{-b_1/2b_0^2}\mathrm{e}^{-1/2b_0\bar{g}^2}\;\exp\left\{-\int_0^{\bar{g}}\mathrm{d}g\left[\frac{1}{\beta(g)}+\frac{1}{b_0 g^3}-\frac{b_1}{b_0^2 g}\right]\right\}}$$

scheme (=definition) dependence

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$$\alpha_{\mathrm{qq}}(\mu)=\alpha_{\overline{\mathrm{MS}}}(\mu)+c_1\alpha_{\overline{\mathrm{MS}}}^2(\mu)+\ldots$$

exactly known

Λ is our main goal

uncertainty: — g(μ) non-perturbative

— $\boldsymbol{\beta}$(g) perturbative, (n+1)-loop

$\Longrightarrow$ $$\Delta\Lambda/\Lambda\sim[\alpha(\mu)]^n$$

Remark on perturbative errors in α (or Λ)

generally control by

high orders in PT
and large μ

Remark on perturbative errors in α (or Λ)

also relevant

Lattice

Euclidean

is an advantage

PT works

large μ by SSF method

Phenomenology

move Euclidish

by

smearing, inclusiveness

moments

Limitations of lattice computations

- Observable with energy/momentum scale μ

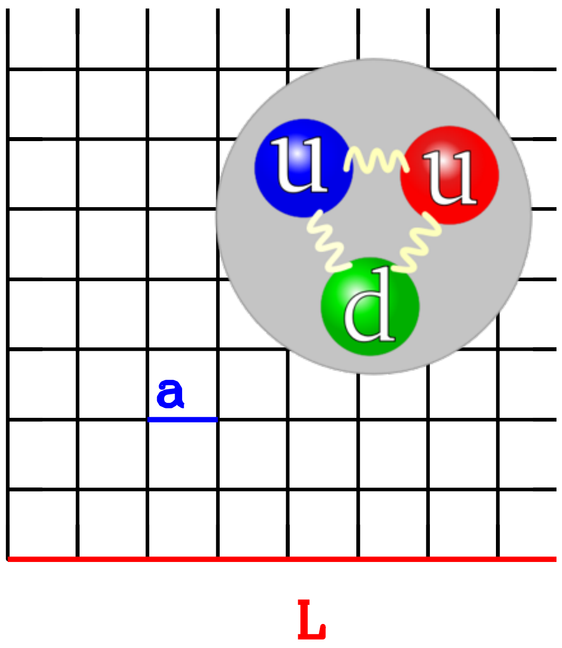
$$\mathcal{O}(\mu) \equiv \lim_{a\to 0} \mathcal{O}_{\mathrm{lat}}(a,\mu) \text{ with } \mu \text{ fixed}$$

- avoid finite size and discretization effects

$$L \gg \text{hadron size} \sim \Lambda_{\mathrm{QCD}}^{-1} \quad \text{and} \quad 1/a \gg \mu$$

or:

$$L/a \ggg \mu/\Lambda_{\mathrm{QCD}}$$

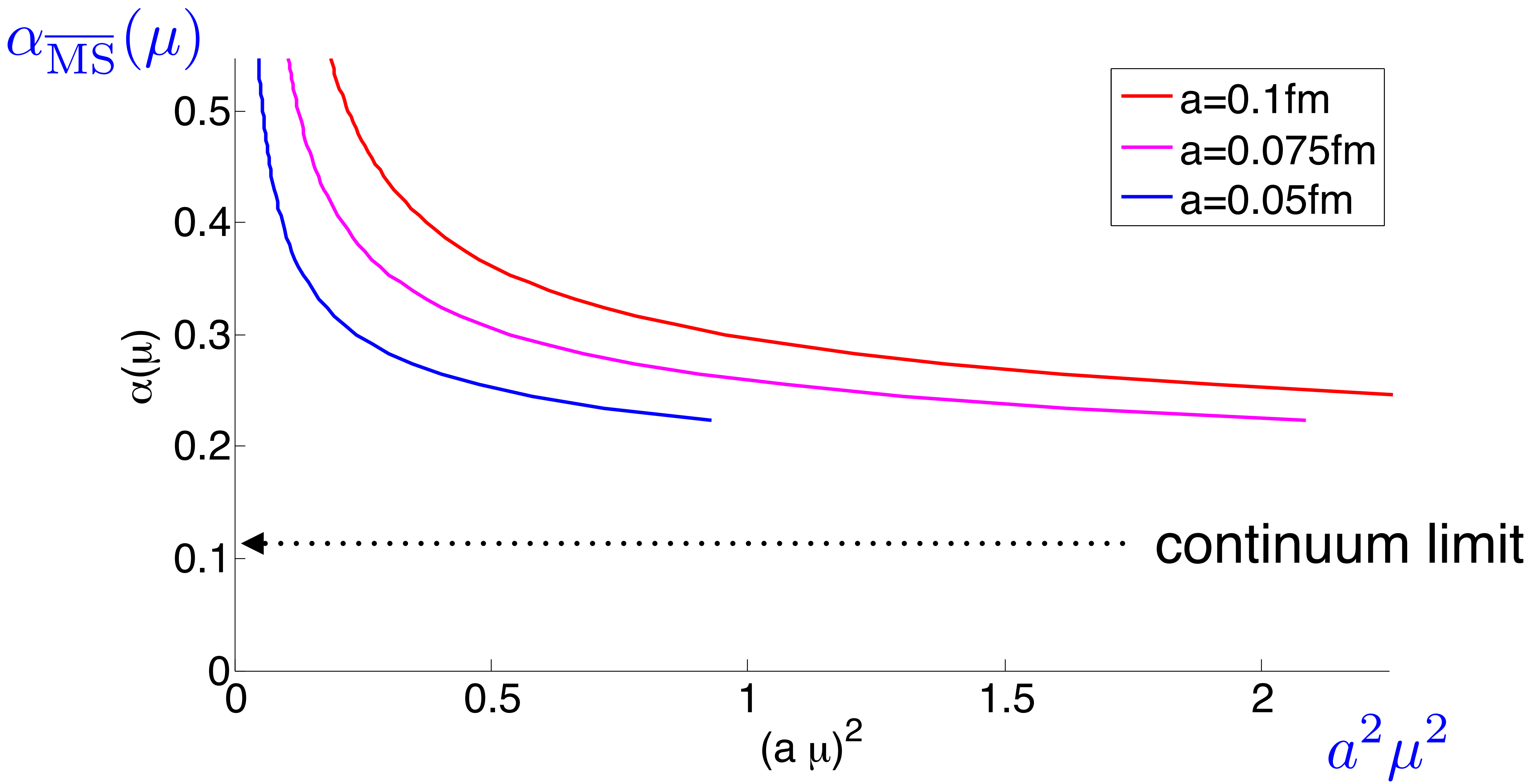


$$\mu \lll L/a \times \Lambda_{\mathrm{QCD}} \sim 5 - 20\,\mathrm{GeV}$$

$1 - 3\,\mathrm{GeV}$ at most, in conflict with $\dfrac{\Delta\Lambda}{\Lambda} \sim \{\alpha(\mu)\}^n$

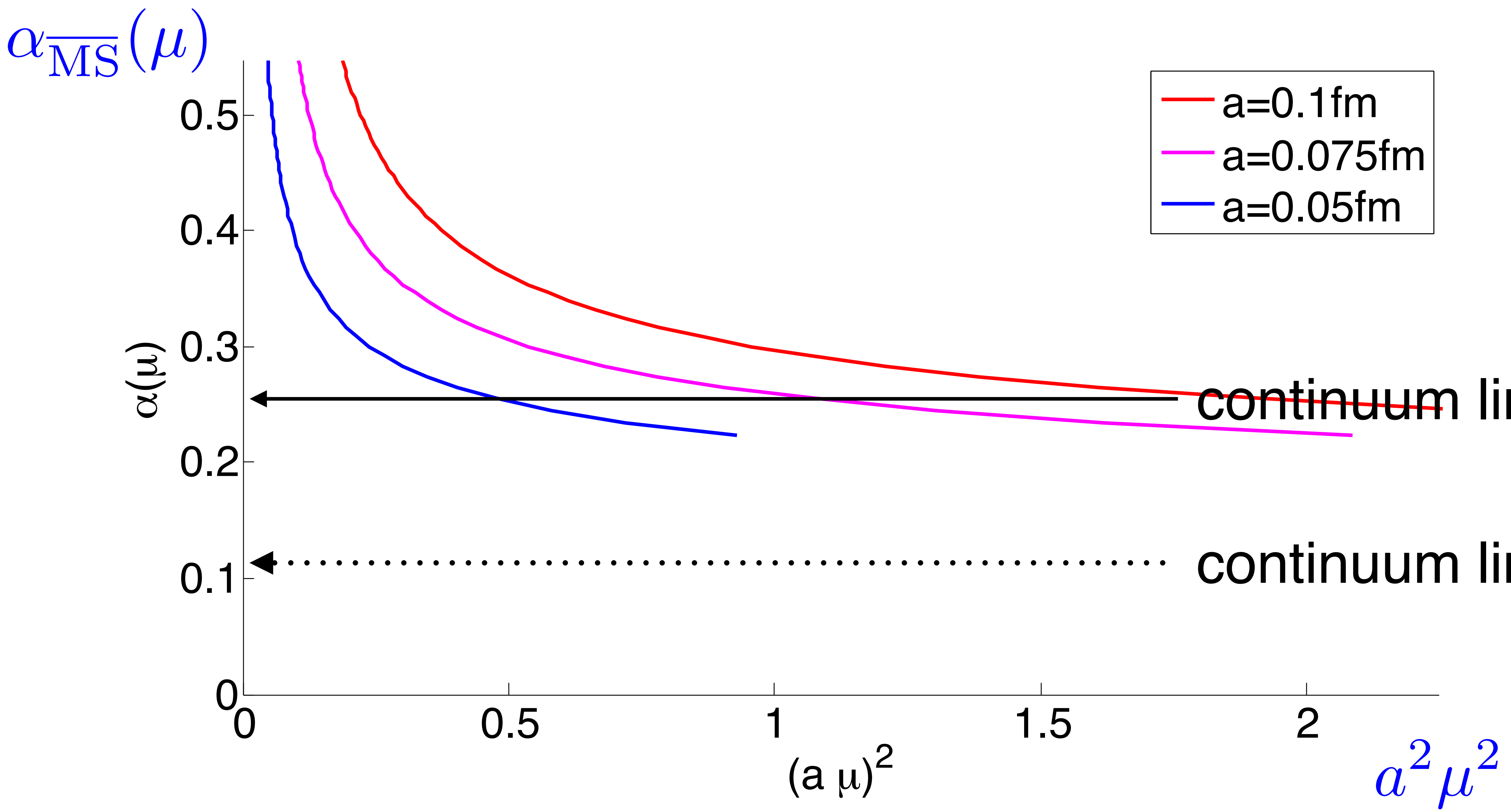
a challenge!

Challenge



should have $a^2\mu^2 \ll 1$ to take continuum limit

Challenge



should have $a^2\mu^2 \ll 1$ to take continuum limit

Results from

Review of lattice results concerning low-energy particle physics

July 1, 2016

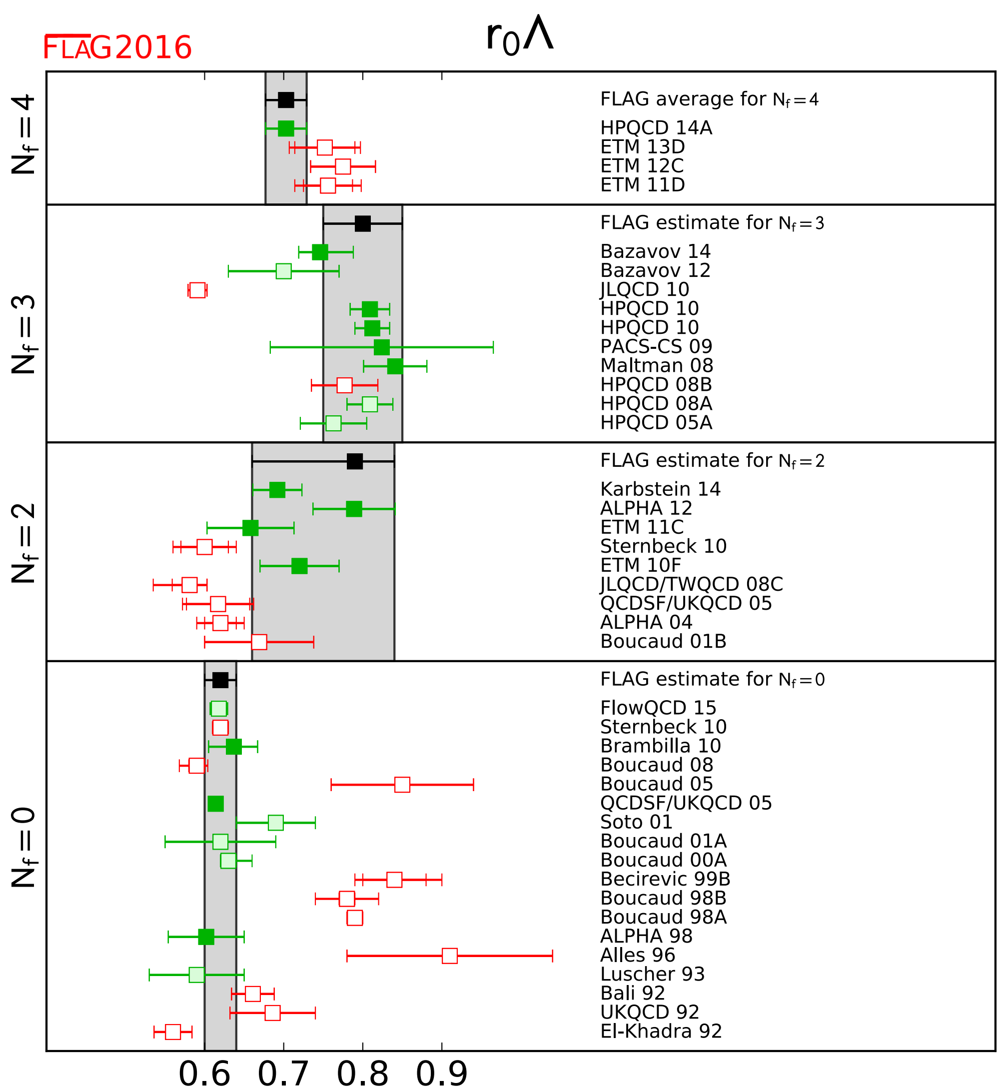
FLAG Working Group

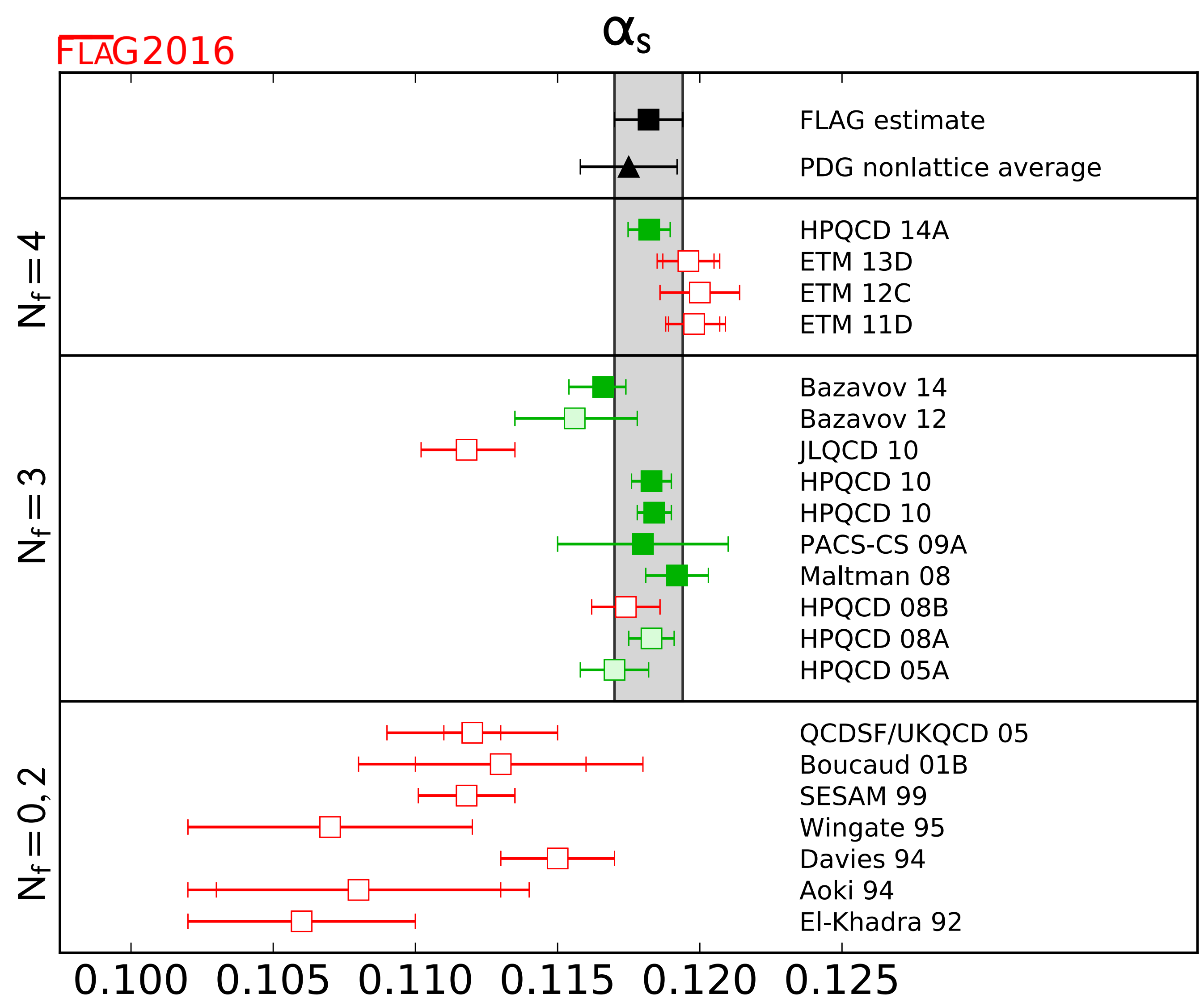
S. Aoki,[1] Y. Aoki,[2,3*] D. Bečirević,[4] C. Bernard,[5] T. Blum,[6,3] G. Colangelo,[7] M. Della Morte,[8]
P. Dimopoulos,[9] S. Dürr,[10] H. Fukaya,[11] M. Golterman,[12] Steven Gottlieb,[13] S. Hashimoto,[14,15]
U. M. Heller,[16] R. Horsley,[17] A. Jüttner,[18] T. Kaneko,[14,15] L. Lellouch,[19] H. Leutwyler,[7]
C.-J. D. Lin,[20,19] V. Lubicz,[21,22] E. Lunghi,[13] R. Mawhinney,[23] T. Onogi,[11] C. Pena,[24]
C. T. Sachrajda,[18] S. R. Sharpe,[25] S. Simula,[22] R. Sommer,[26] A. Vladikas,[27] U. Wenger,[7] H. Wittig[28]

Results reviewed by FLAG2016

arXiv:1607.00299

- Working Groups (coordinator listed first):
  - Quark masses — L. Lellouch, T. Blum, and V. Lubicz
  - V_{us}, V_{ud} — S. Simula, P. Boyle,[1] and T. Kaneko
  - LEC — S. Dürr, H. Fukaya, and U.M. Heller
  - B_K — H. Wittig, P. Dimopoulos, and R. Mawhinney
  - $f_{B_{(s)}}$, $f_{D_{(s)}}$, B_B — M. Della Morte, Y. Aoki, and D. Lin
  - $B_{(s)}$, D semileptonic and radiative decays — E. Lunghi, D. Becirevic, S. Gottlieb, and C. Pena
  - α_s — R. Sommer, R. Horsley, and T. Onogi





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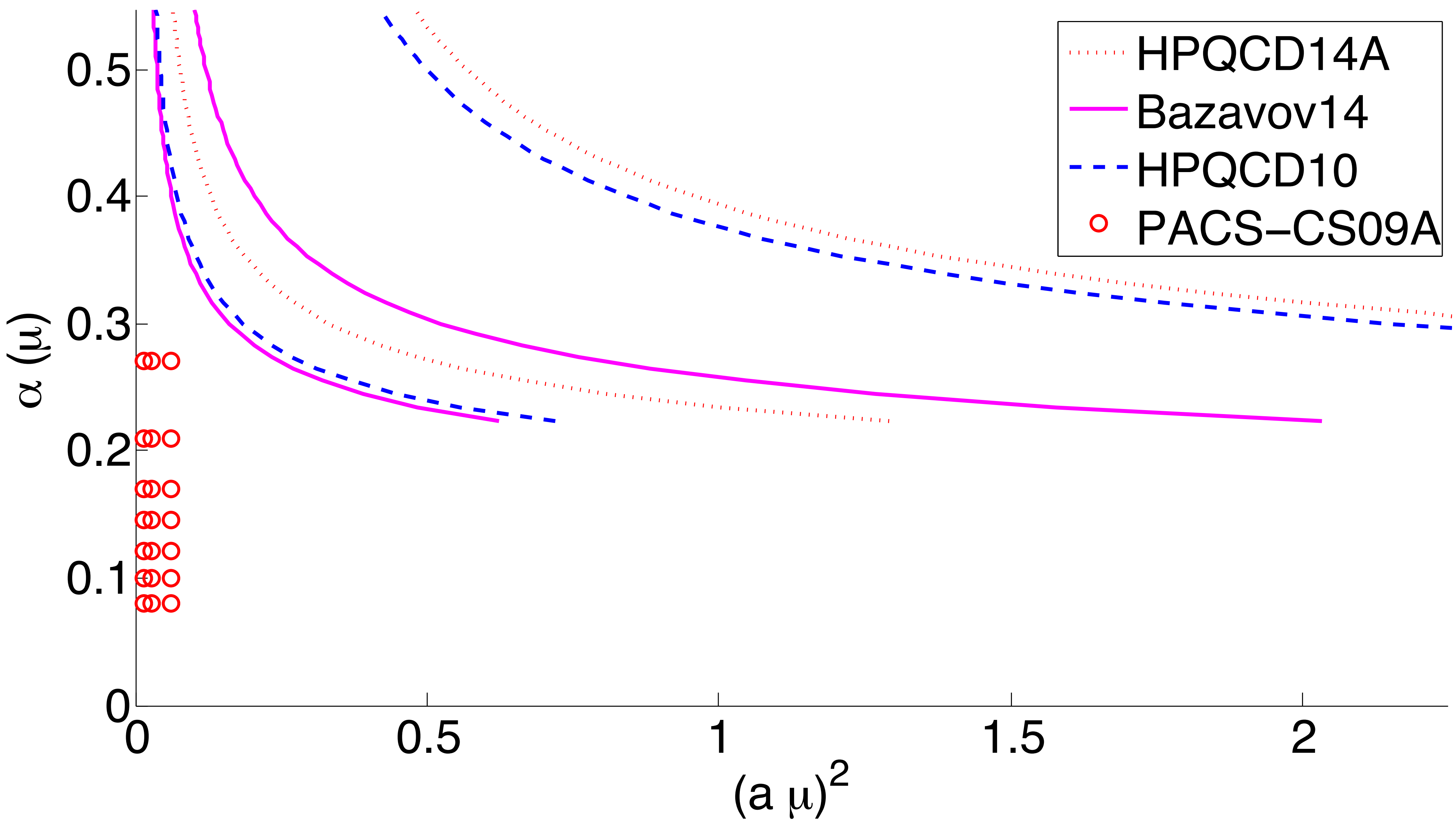

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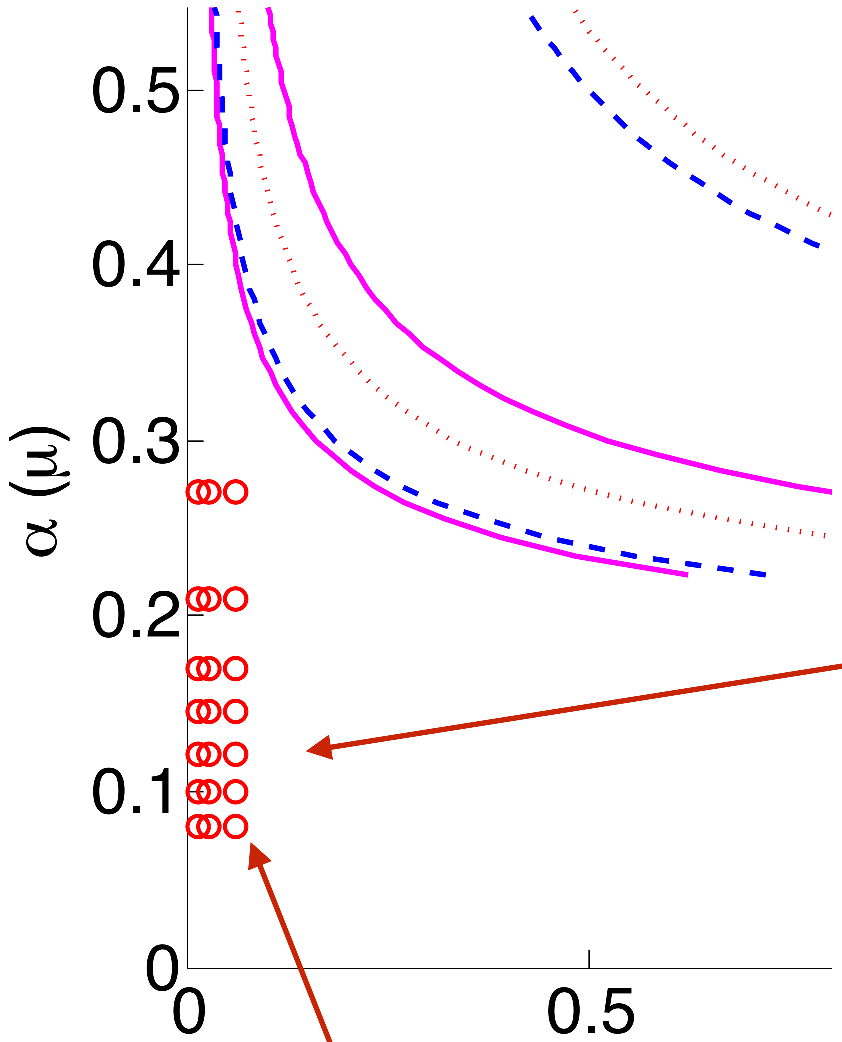
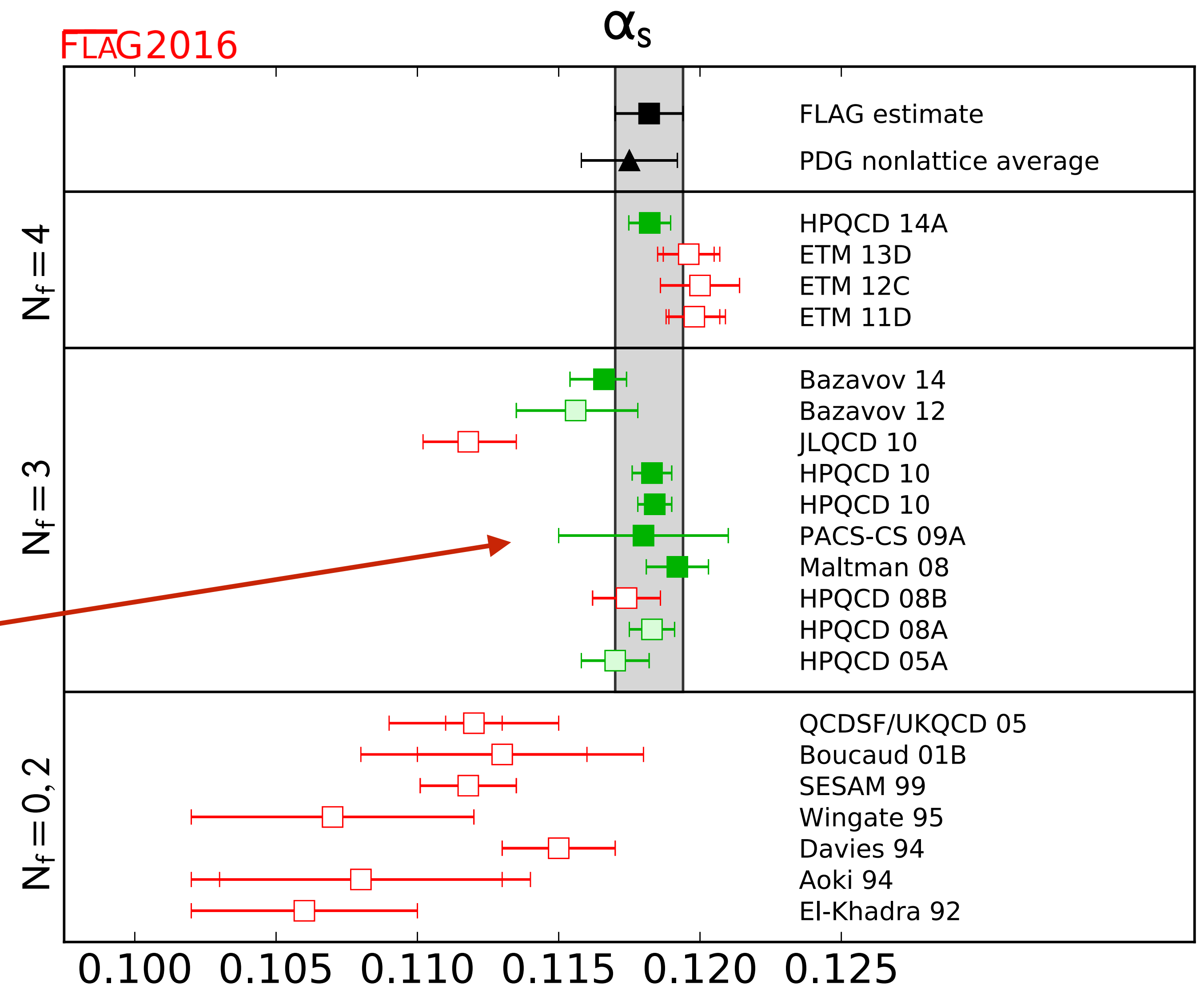
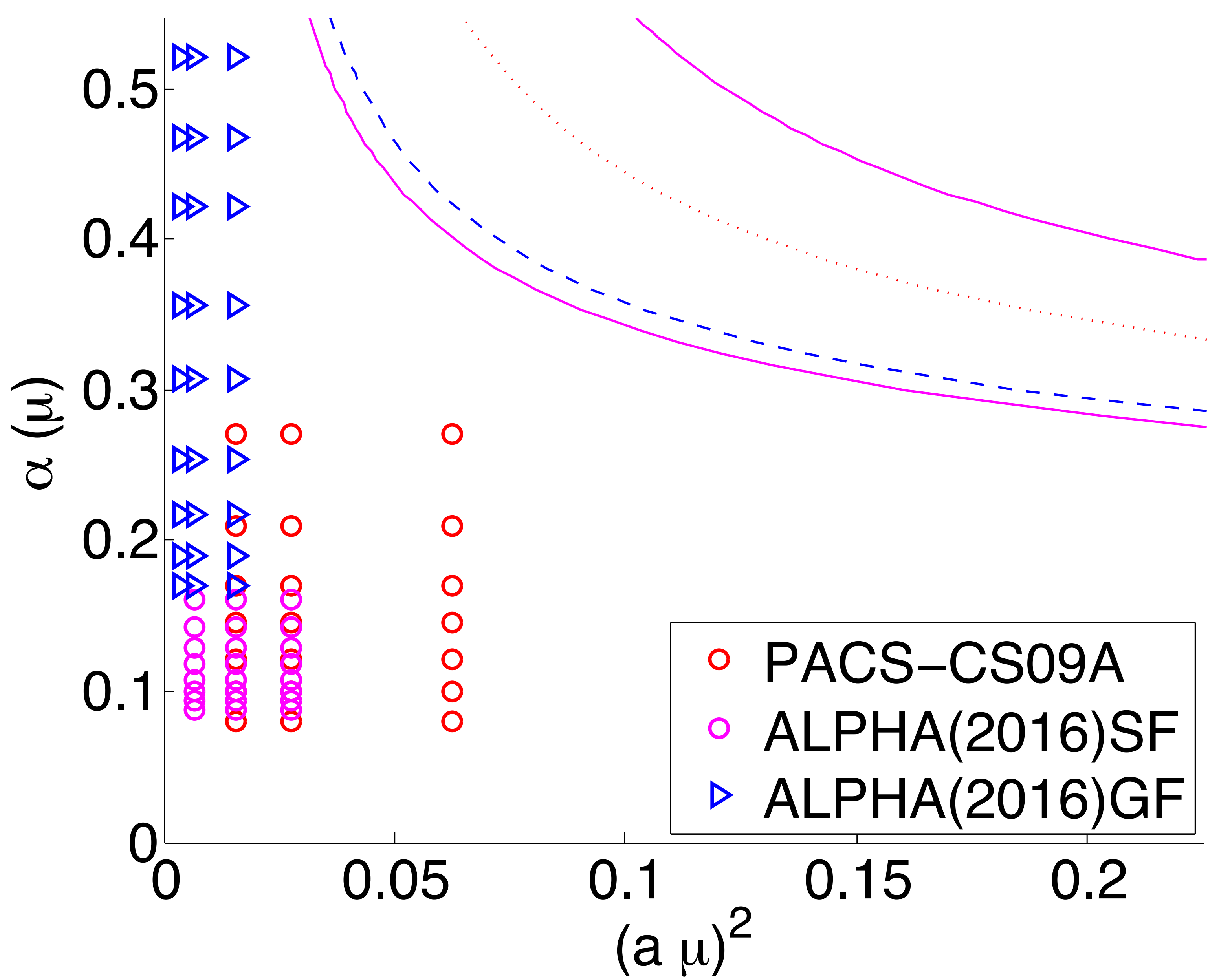




using step scaling strategy

New results: ALPHA 2016



continuum limit with good accuracy

Our Strategy to meet the Challenge

Our Strategy to meet the Challenge

ALPHA Collaboration

- finite volume: $\mu = 1/L$, with $L/a \gg 1$ get $\mu^2 a^2 \ll 1$ for any μ

finite volume (fv) as a probe of short distance

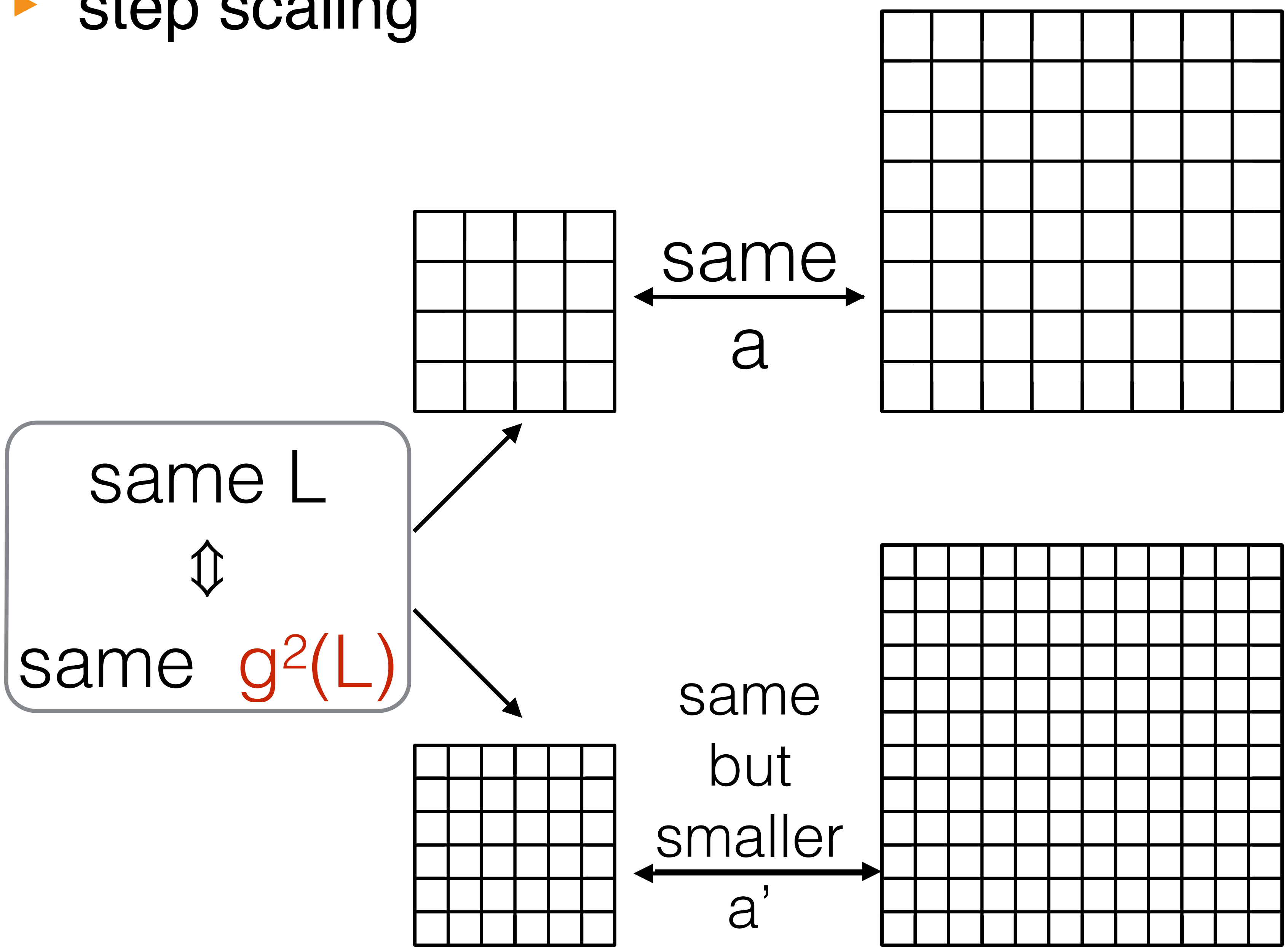
fv is essential part of the definition of the short distance observable

finite size scaling

Our Strategy to meet the Challenge

ALPHA Collaboration

- finite volume: $\mu=1/L$, with $L/a \gg 1$ get $\mu^2a^2 \ll 1$ for any μ
- step scaling



$g^2(2L,a/L) = g^2(2L,1/4)$

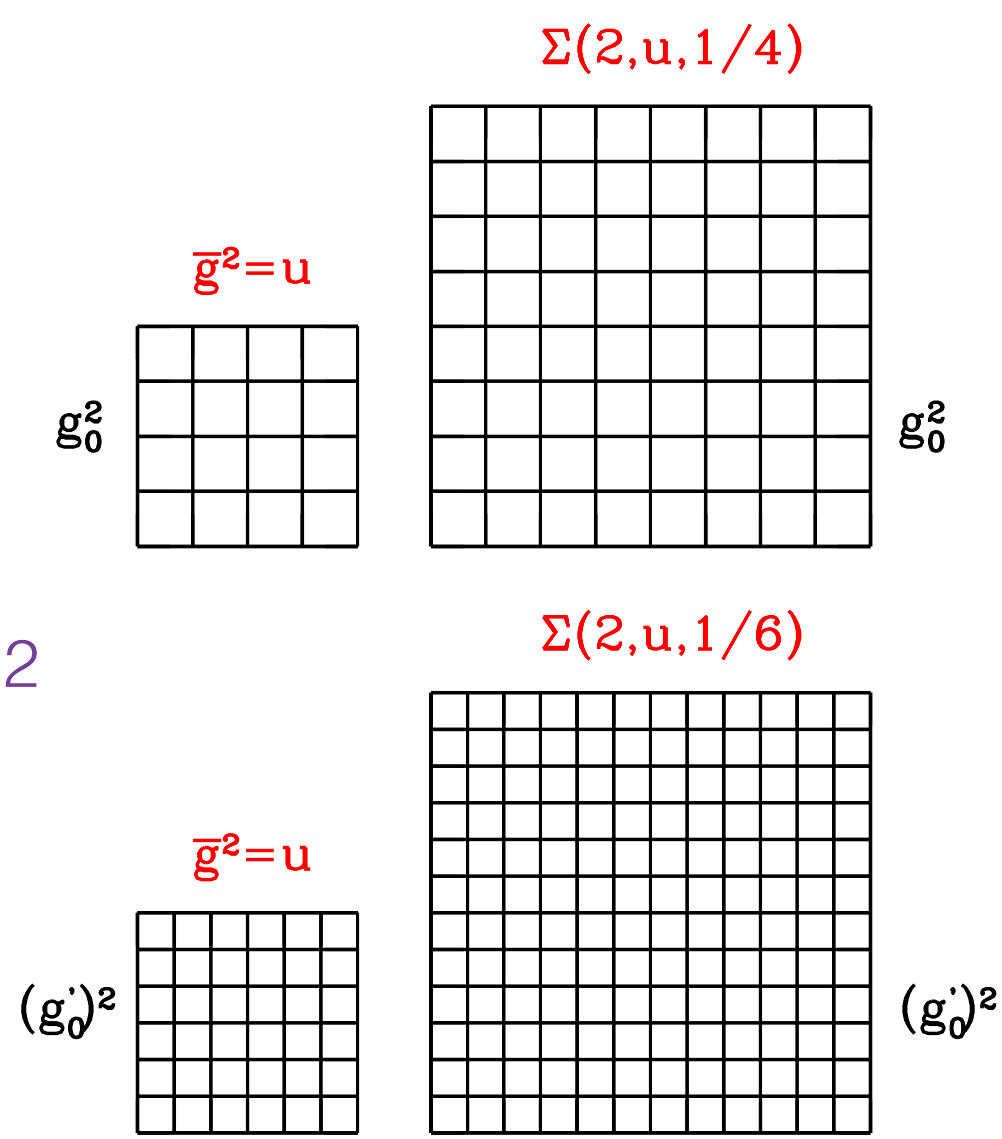
$g^2(2L,a'/L) = g^2(2L,1/6)$

⇓ extrapolate

- needs $L/a \gg 1$, not more: $g^2(2L,0)$ continuum

Our Strategy

- finite volume: μ=1/L, L/a ≫ 1 at any μ
- step scaling function (SSF): $\bar{g}^2(2L) = \sigma(\,\bar{g}^2(L)\,) = \lim_{a/L\to 0} \Sigma(2, u, a/L)$
  (discrete $\boldsymbol{\beta}$-function)



Lüscher, Weisz, Wolff, '91
Lüscher, Narayanan, Weisz, Wolff, '92
Lüscher, S., Weisz, Wolff, '94

ALPHA Collaboration

Running to (almost) any scale non-perturbatively

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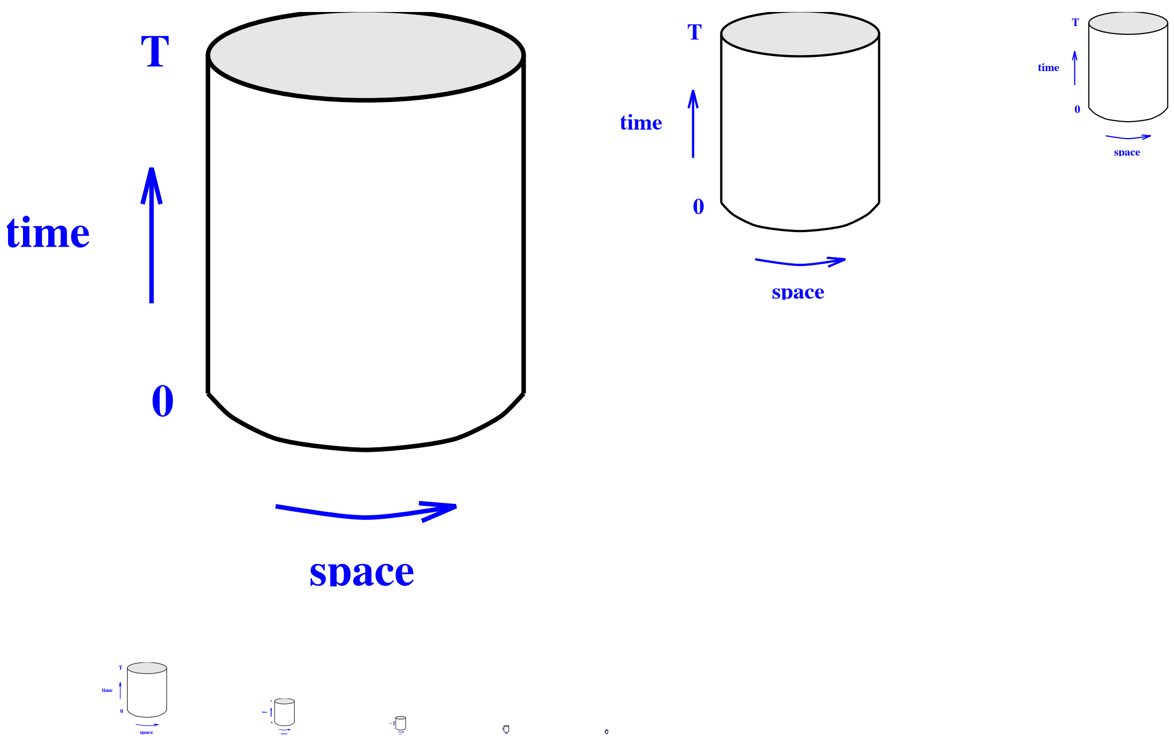
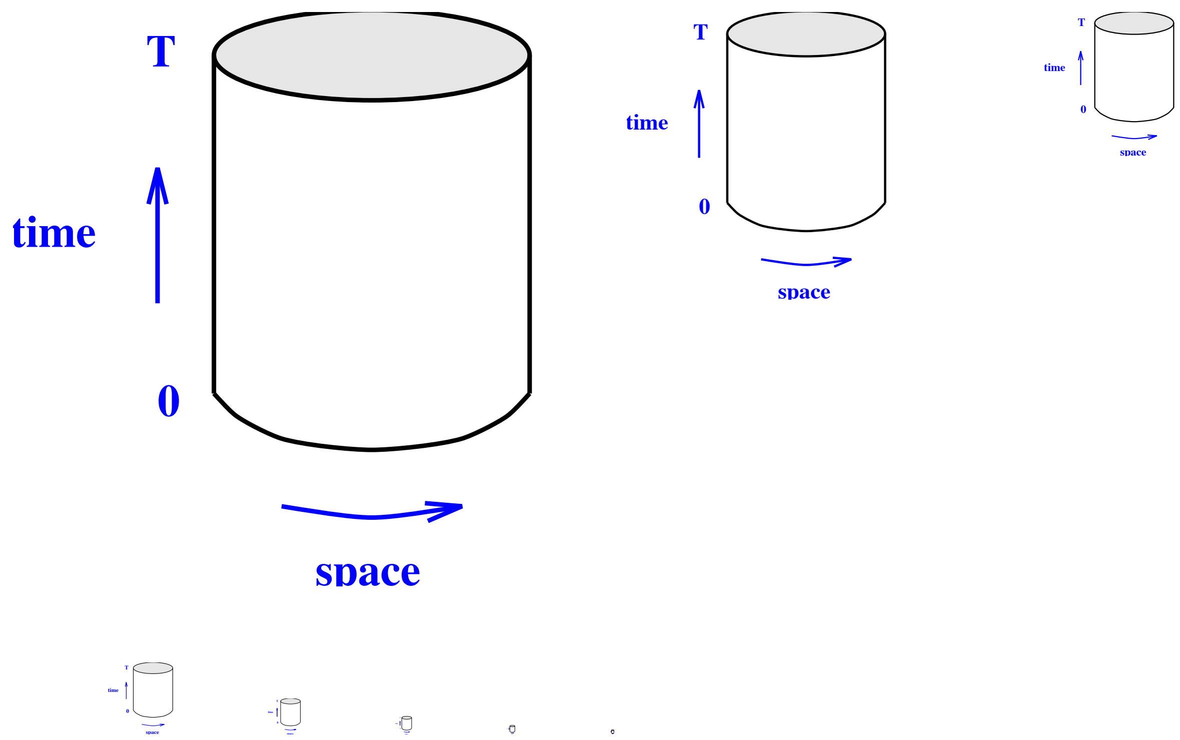
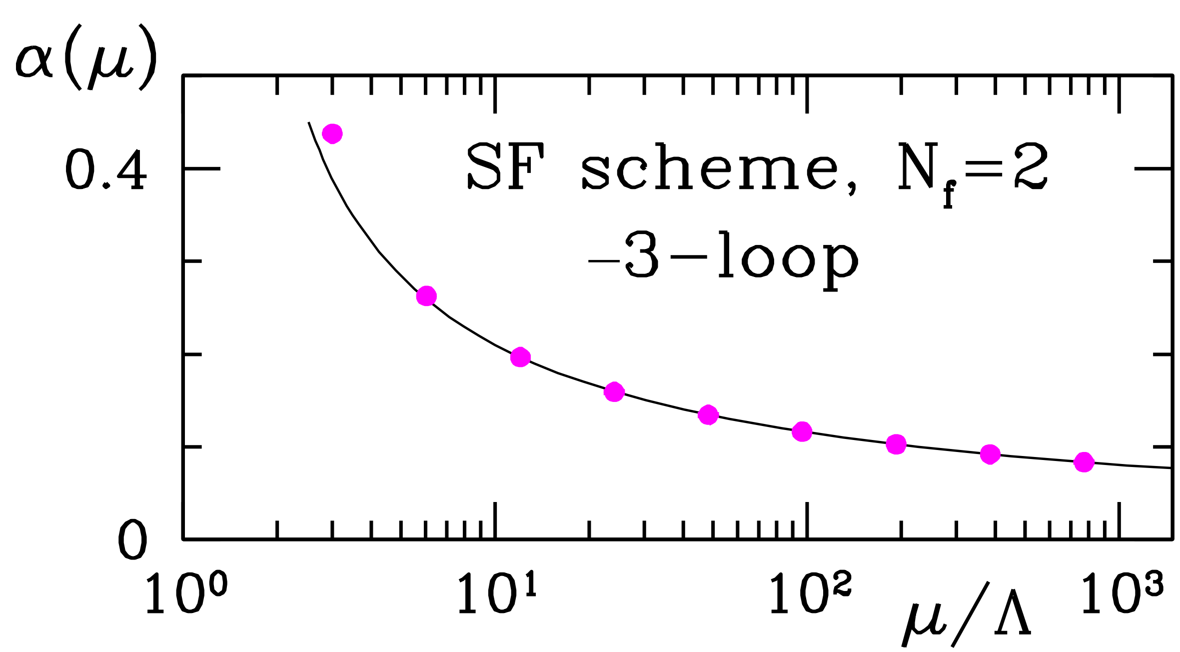
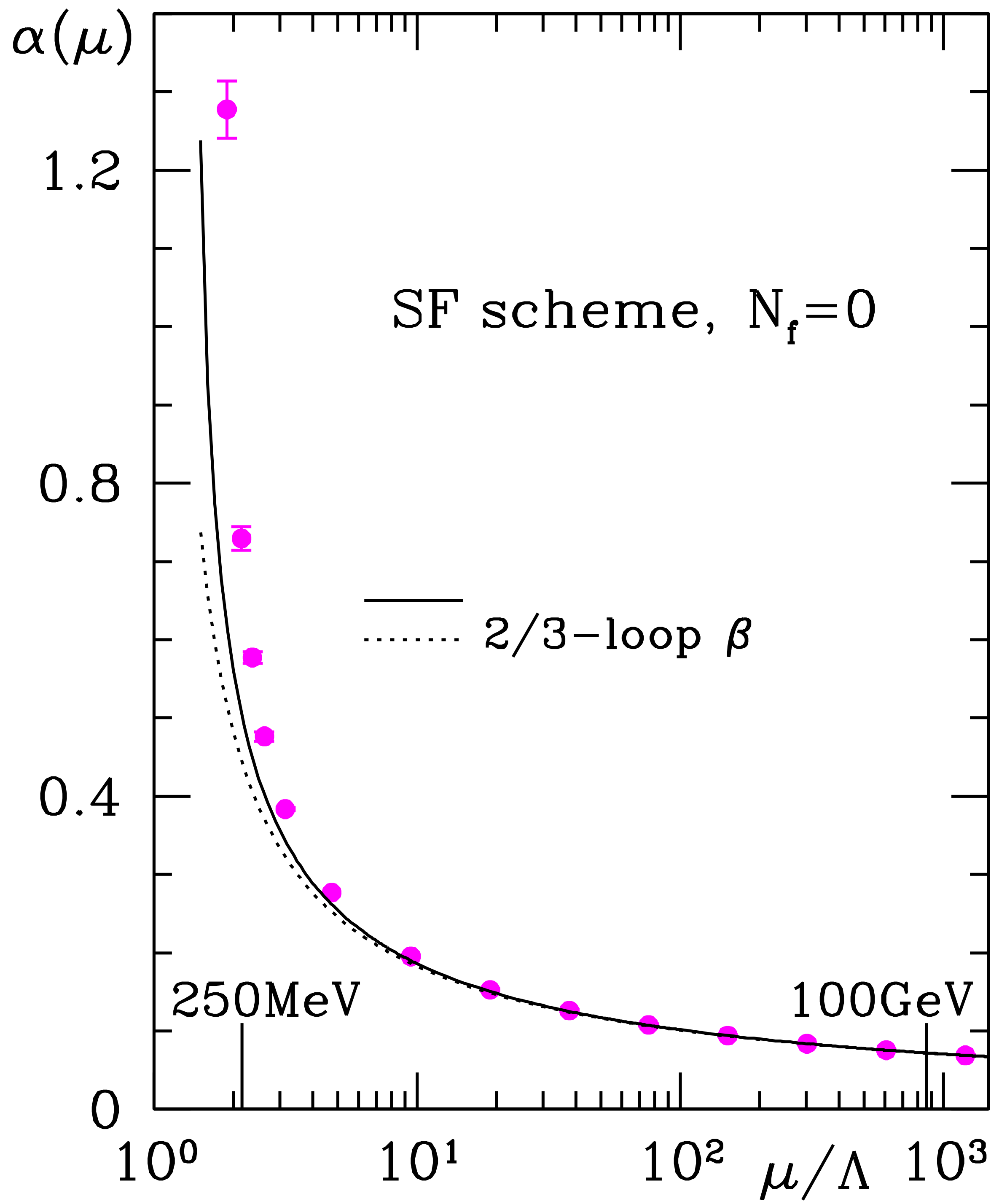


Running to (almost) any scale non-perturbatively





[ALPHA Collaboration, 2005]



[ALPHA Collaboration, 2001]

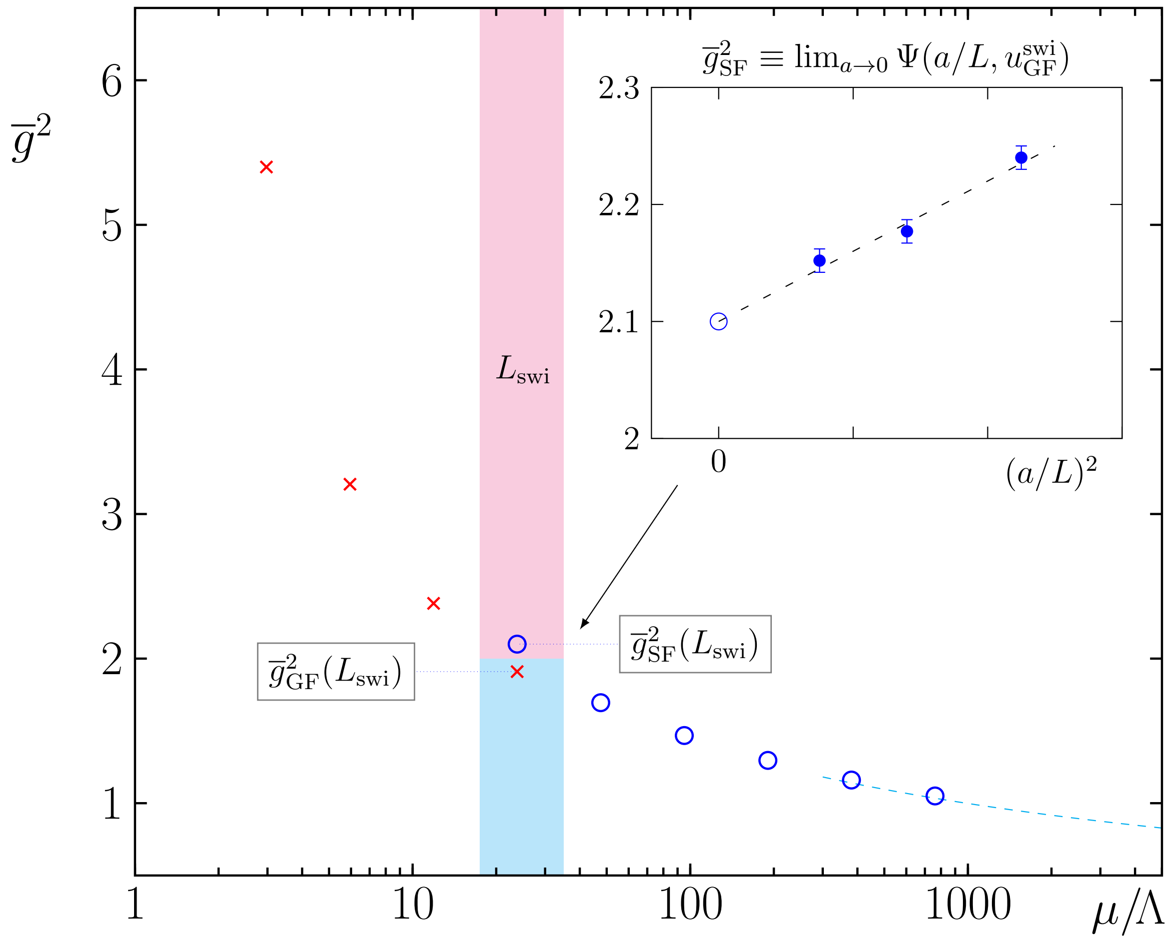
Now: N_f=3

with up, down, strange (others: see later)

two different schemes

Gradient flow
200 MeV ← 8 GeV

Schrödinger functional
4 GeV ← 200 GeV



Now: N_f=3 (hadronic world and running)

with up, down, strange; others decoupled

two different schemes

Dirichlet bc's ($\Longrightarrow$ can use massless schemes)

Gradient flow

$$\frac{dB_\mu(t,x)}{dt} = D_\nu G_{\nu\mu}(t,x)\,, \qquad B_\mu(0,x) = A_\mu(x)$$

$$G_{\mu\nu}(t,x) = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

$$\bar{g}^2_{\mathrm{GF}}(1/L) = t^2 \mathcal{N}^{-1}(c) \langle \mathrm{tr}\, [G_{ij}(x_0,t) G_{ij}(x_0,t)] \Big|_{\sqrt{8t}=cL;x_0=T/2}$$

Schrödinger functional

$$A_k(x)|_{x_0=0} = C_k(\eta,\nu)\,, \quad A_k(x)|_{x_0=L} = C'_k(\eta,\nu)$$

$$C_k = \frac{i}{L}[\mathrm{diag}(-\pi/3, 0, \pi/3) + \eta(\lambda_8 + \nu\lambda_3)$$

$$C'_k = \frac{i}{L}[\mathrm{diag}(-\pi, \pi/3, 2\pi/3) - \eta(\lambda_8 - \nu\lambda_3)].$$

$$\langle \partial_\eta S|_{\eta=0} \rangle = \frac{12\pi}{\bar{g}^2_\nu} = 12\pi[\frac{1}{\bar{g}^2} - \nu\,\bar{v}]$$

- similar to Casimir effect
- non-perturbative definition of background field (BF)
  = classical solution with these bc's
  spatially constant, abelian

Now: N_f=3 (hadronic world and running)
with up, down, strange; others decoupled

ALPHA Collaboration

two different schemes

Gradient flow
200 MeV ← 8 GeV

high precision in MC

significant a^2 effects

2-loop (universal)**β**-function

Lüscher, 2010
Lüscher, Weisz, 2011
Fritzsch, Ramos, 2013

Schrödinger functional
4 GeV ← 200 GeV

high precision at small g

small a^2-effects

3-loop **β**-function

2-loop a-effects

Lüscher, Weisz, Wolff, '91
Lüscher, Narayanan, Weisz, Wolff, '92
Lüscher, Sommer, Weisz, Wolff, '93
Sint '93

First: Schrödinger functional scheme

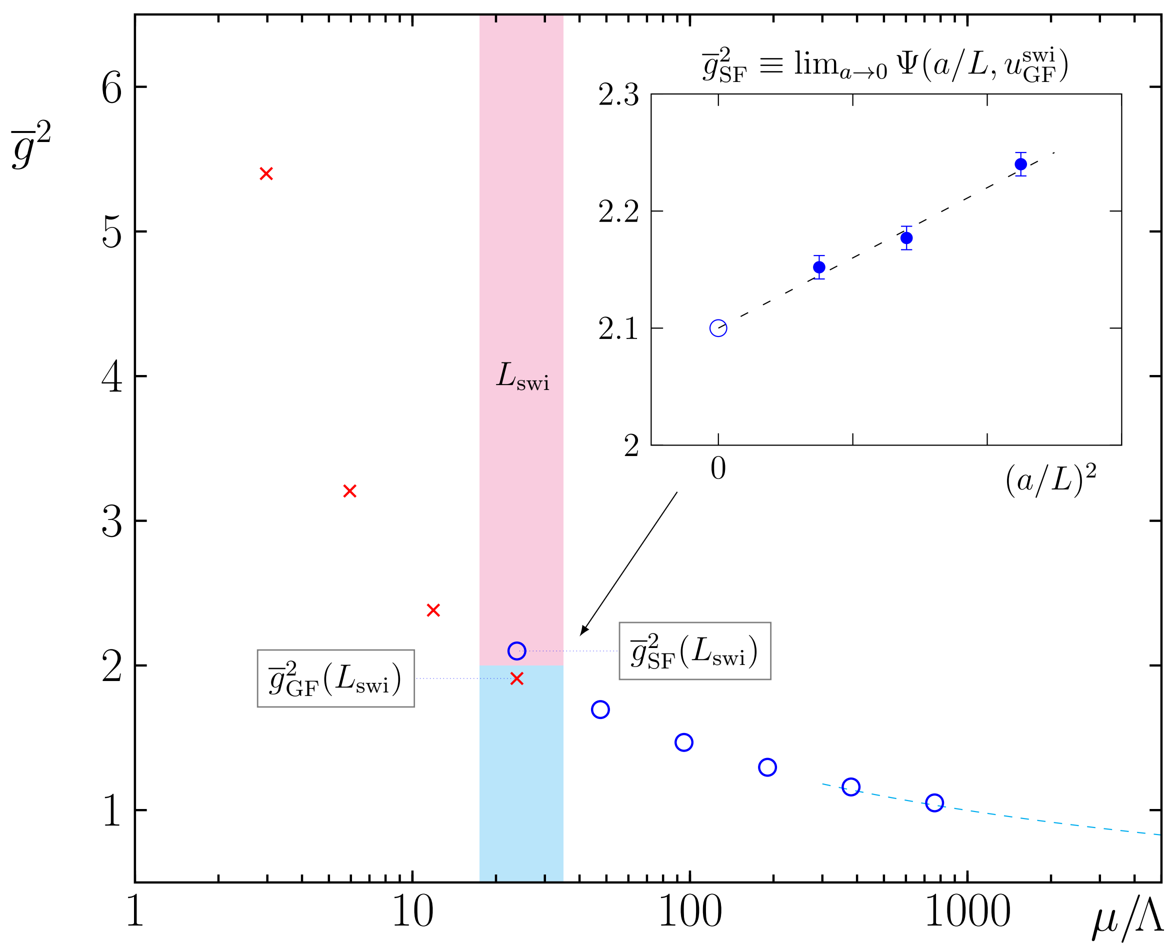
ALPHA Collaboration

arXiv:1604.06193

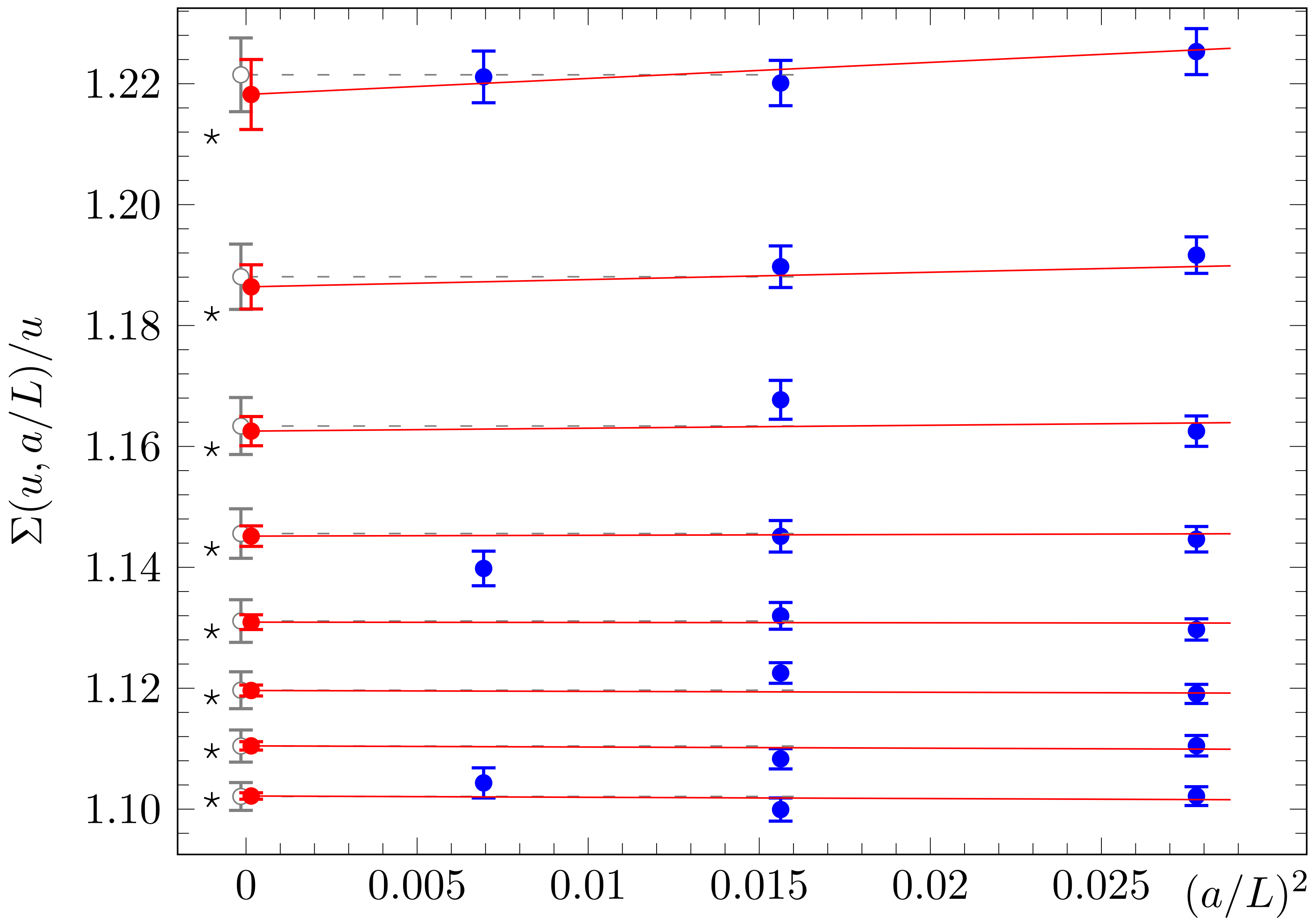
two different schemes

Gradient flow
200 MeV ← 8 GeV

Schrödinger functional
4 GeV ← 200 GeV



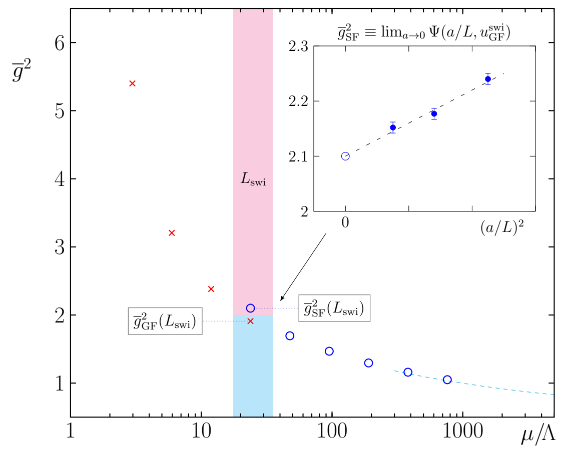
Continuum limit $\sigma(g^2) = \Sigma(g^2,0)$ in small g^2 region



- χ^2 of global fits is good - continuum limit is precise

- constant continuum extrapolation has larger errors due to propagation of boundary improvement error

Determination of Λ L_0



- step scaling (from u_0=2.012)

$$\bar{g}^2(1/L_0) = u_0\,, \quad u_k = \sigma(u_{k+1}), \quad \text{non-pert}$$

$$L_0\Lambda = 2^n \varphi^{\rm pert}(\sqrt{u_n})$$

$$\varphi_s(\bar{g}_s) = (b_0\bar{g}_s^2)^{-b_1/(2b_0^2)} \mathrm{e}^{-1/(2b_0\bar{g}_s^2)}$$

use perturbative β(g) in

$$\times \exp\left\{ -\int_0^{\bar{g}_s} \mathrm{d}x \left[\frac{1}{\beta_s(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x} \right] \right\}$$

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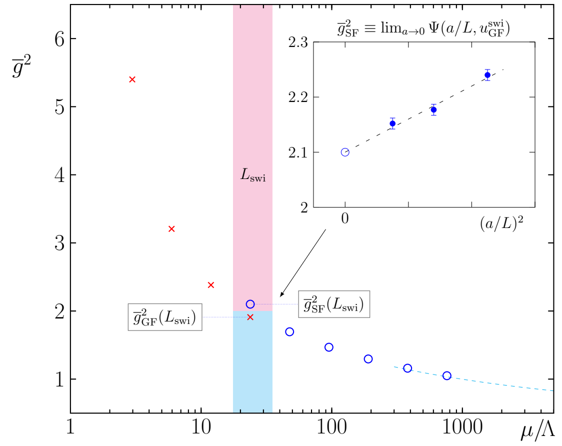

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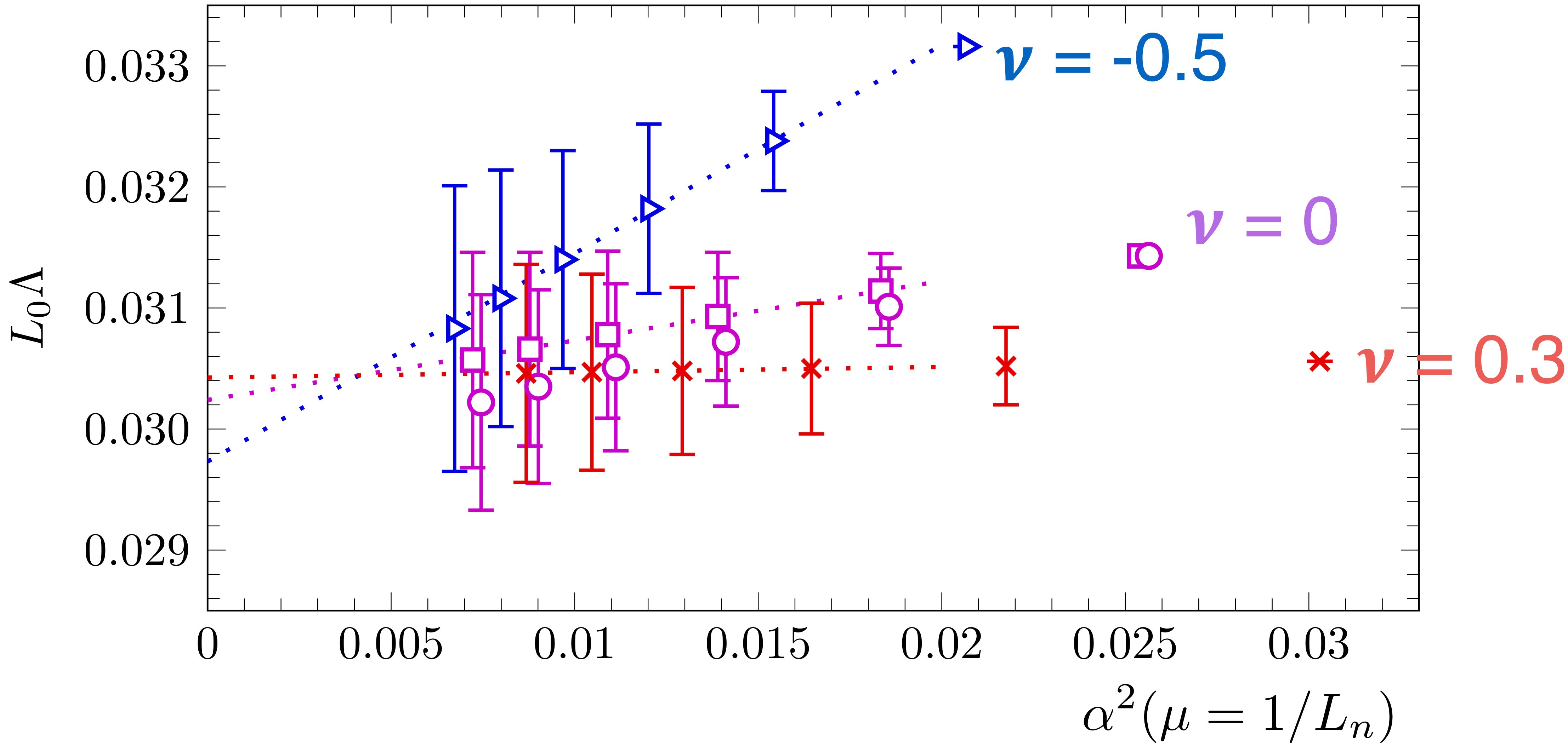
- Λ independent of n ? ⟸ excellent check of accuracy of PT

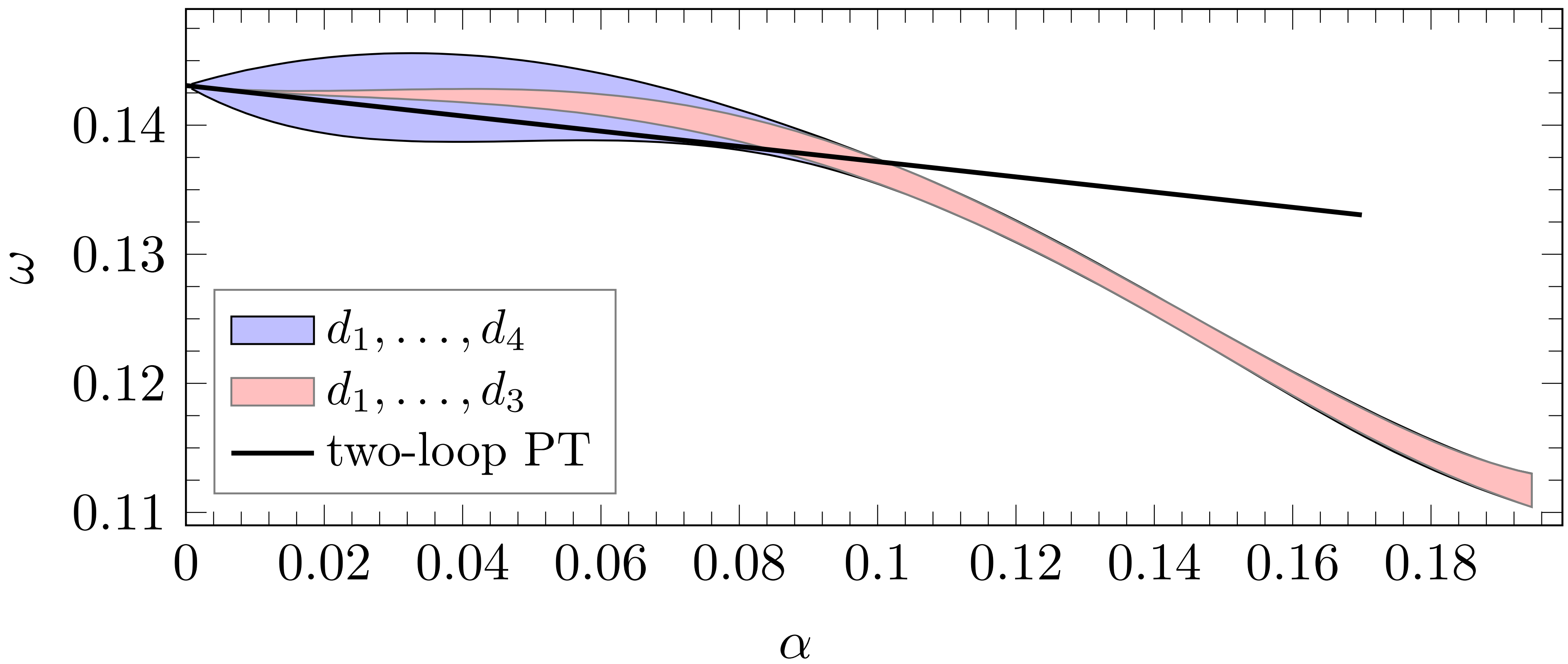
Results for Λ L_0



- 3% accuracy at α = 0.1 !
- also for non-standard schemes: ν= - 0.5, ν= 0.3
- at α = 0.2 this is not so!

Very high precision quantity: ω

$$\frac{1}{\bar{g}_{\nu}^2} = \frac{1}{\bar{g}^2} - \nu \times \omega(\bar{g}^2)$$



- deviation from PT at $\boldsymbol{\alpha}$ = 0.19 :

$$(\omega(\bar{g}^2) - v_1 - v_2\bar{g}^2)/v_1 = -3.7(2)\,\alpha^2$$

- not small, does not look perturbative
- statistically very significant

Now: Gradient flow scheme

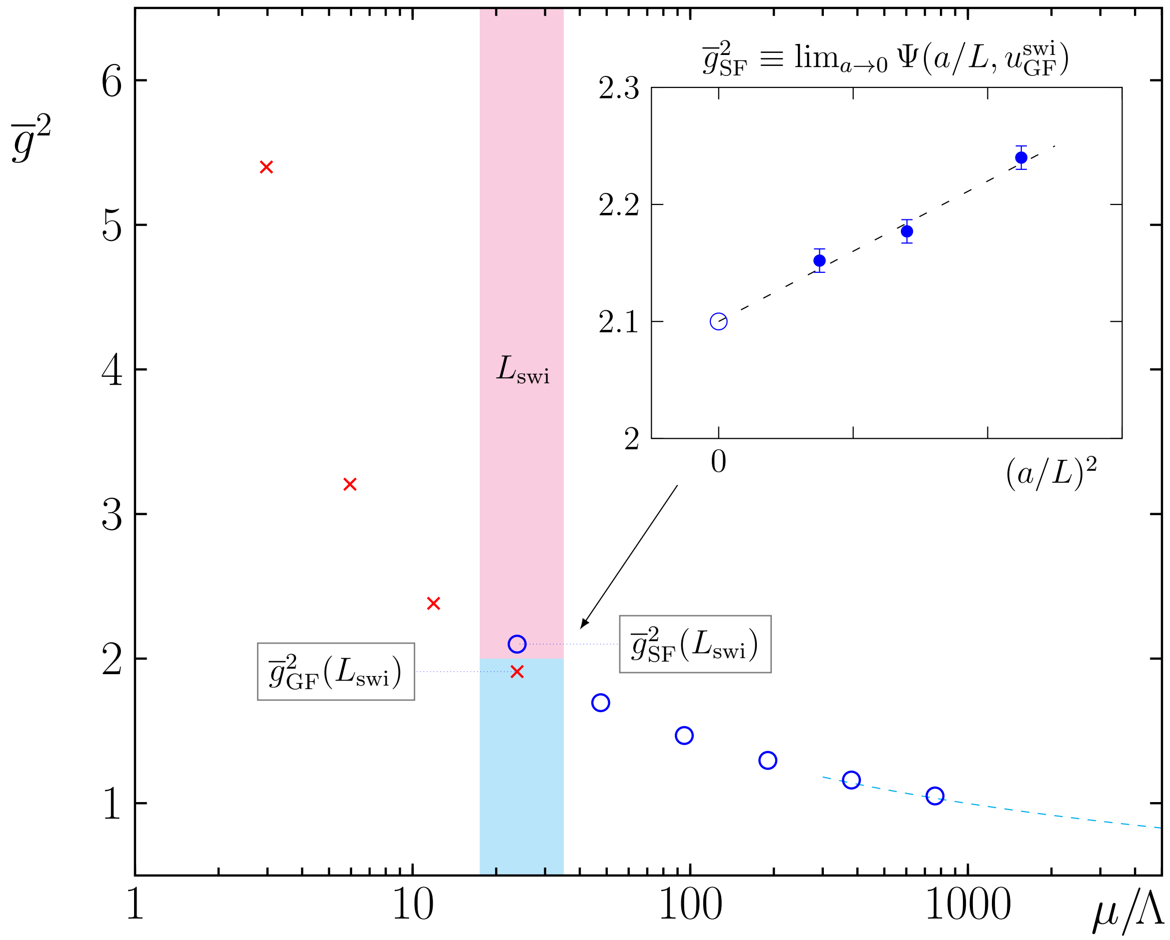
arXiv:1607.06423

ALPHA Collaboration

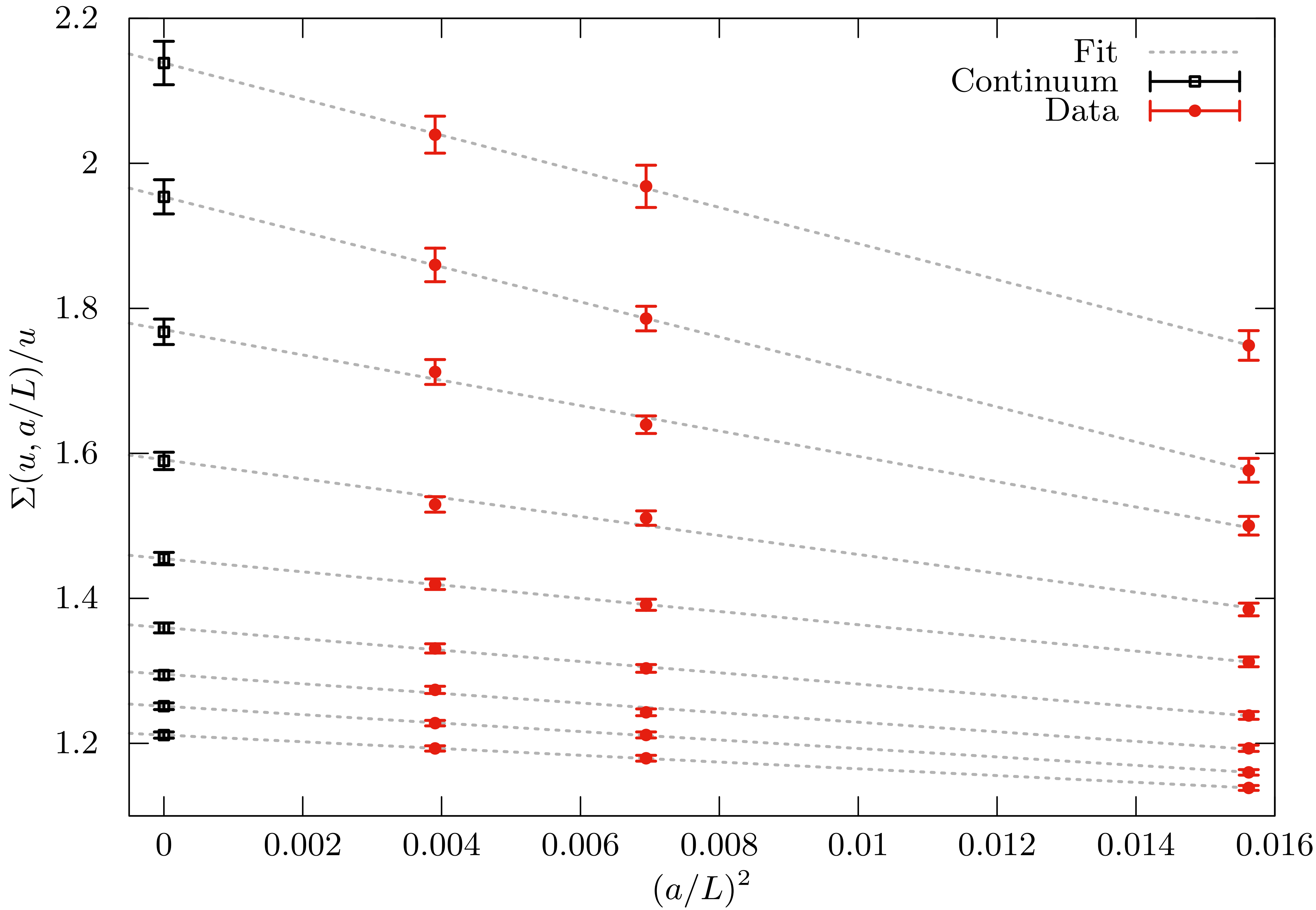
two different schemes

Gradient flow
200 MeV ← 8 GeV

Schrödinger functional
4 GeV ← 200 GeV



Continuum limit $\sigma(g^2) = \Sigma(g^2,0)$ in large g^2 region



- χ^2 of global fits is good - continuum limit is precise
  ~ 12 pages discussion in the paper (slopes are significant!)

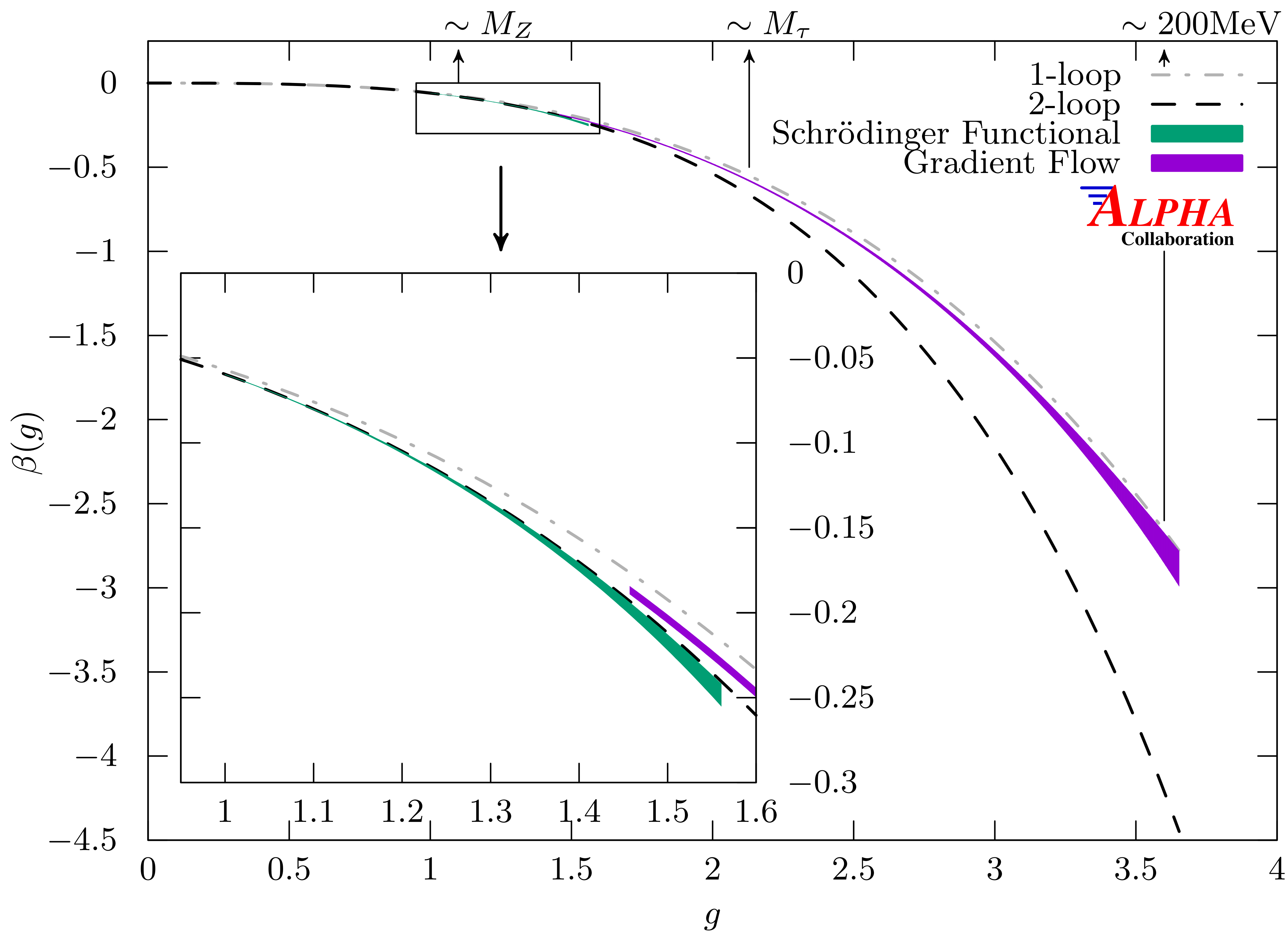
$\boldsymbol{\beta}$-function from $\boldsymbol{\sigma}$(u)= Σ(u,0) (u=g²)

$$\log(2) = -\int_{\sqrt{u}}^{\sqrt{\sigma(u)}} \frac{\mathrm{d}x}{\beta(x)} = \int_{\sqrt{u}}^{\sqrt{\sigma(u)}} \mathrm{d}x \frac{P(x^2)}{x^3}$$

$$= -\frac{p_0}{2}\left[\frac{1}{\sigma(u)} - \frac{1}{u}\right] + \frac{p_1}{2}\log\left[\frac{\sigma(u)}{u}\right] + \sum_{n=1}^{n_{\max}} \frac{p_{n+1}}{2n}\left[\sigma^n(u) - u^n\right],$$

- fit directly to a parameterization $P(x^2)$

Results (1): the non-perturbative β-functions



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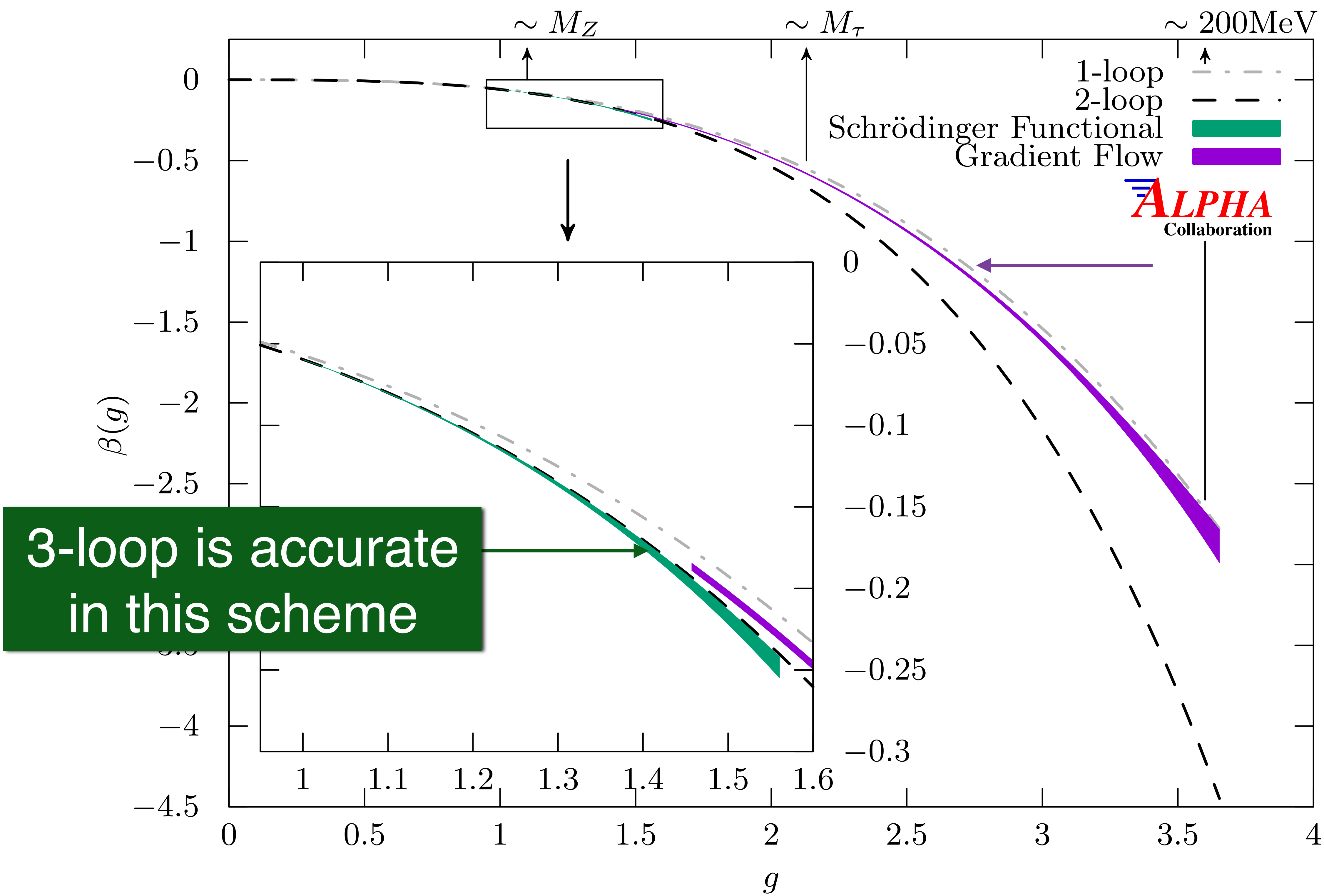

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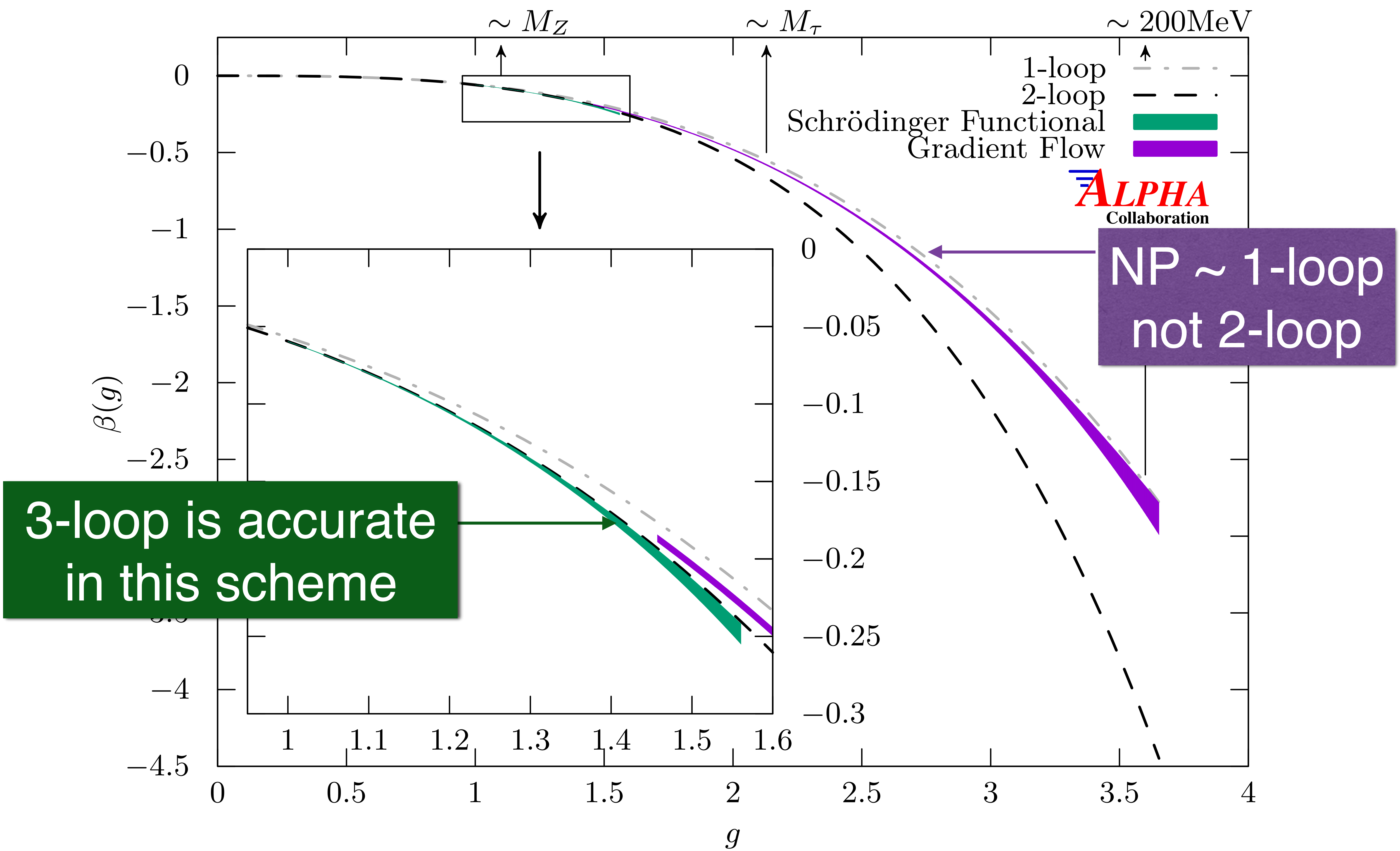
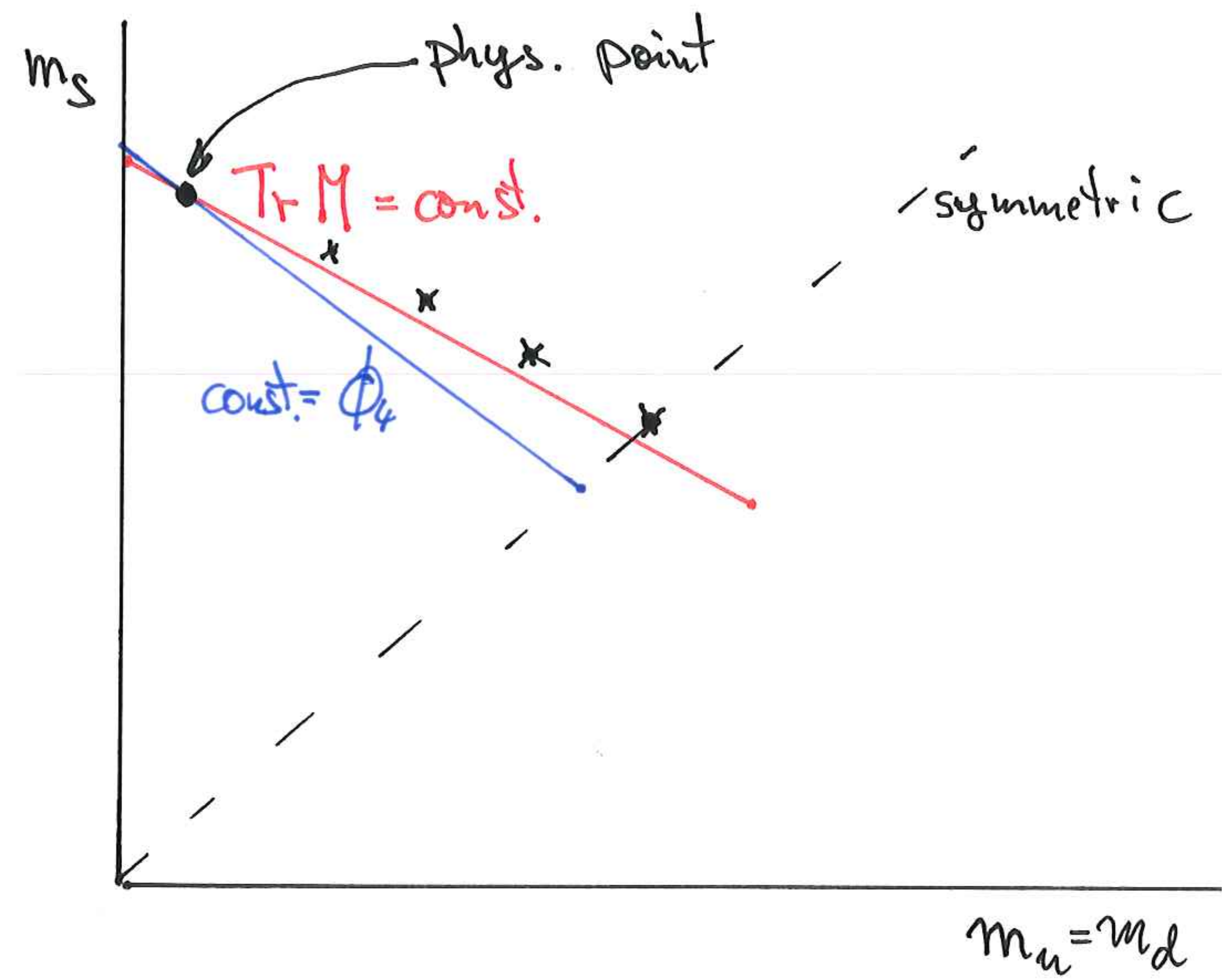


Connection to hadronic world: CLS Ensembles large volume!

- simulations: $\mathrm{Tr}\, M_q = \text{const.}$
- other trajectories
  - $\phi_4 = 8\, t_0\, (m_\mathrm{k}^2 + \frac{1}{2} m_\pi^2) = \text{const.}$
  - $\dfrac{m_\mathrm{k}^2 + \frac{1}{2} m_\pi^2}{f_{\pi K}^2} = \text{const.}$
  - shift there (small shifts in masses)



- physical point:

$$m_\pi^2 / f_{\pi K}^2 = phys. \quad m_\mathrm{K}^2 / f_{\pi K}^2 = phys. \quad \rightarrow \quad \Phi_4 = 1.11(2)$$

- …
- $[8 t_0^{\text{symm}}] = 0.4130(45)\, \text{fm}$ GF scale [Lüscher '10]

Connection to hadronic world: …

- Φ_4=1.11 trajectory
- Continuum extrapolation fit

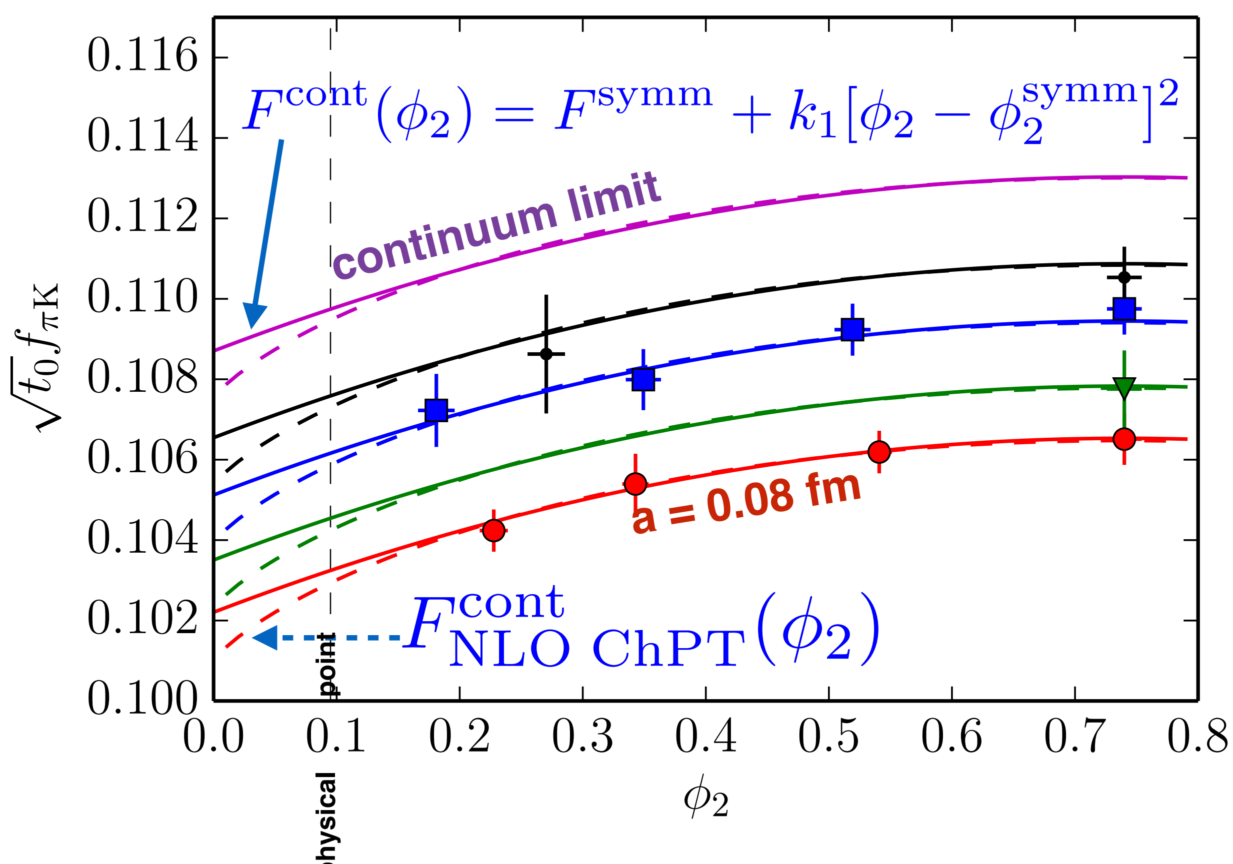
$$\phi_2 = 8\,t_0\,m_\pi^2$$

$$\phi_4 = 8\,t_0\,(m_\mathrm{k}^2 + \frac{1}{2} m_\pi^2)$$

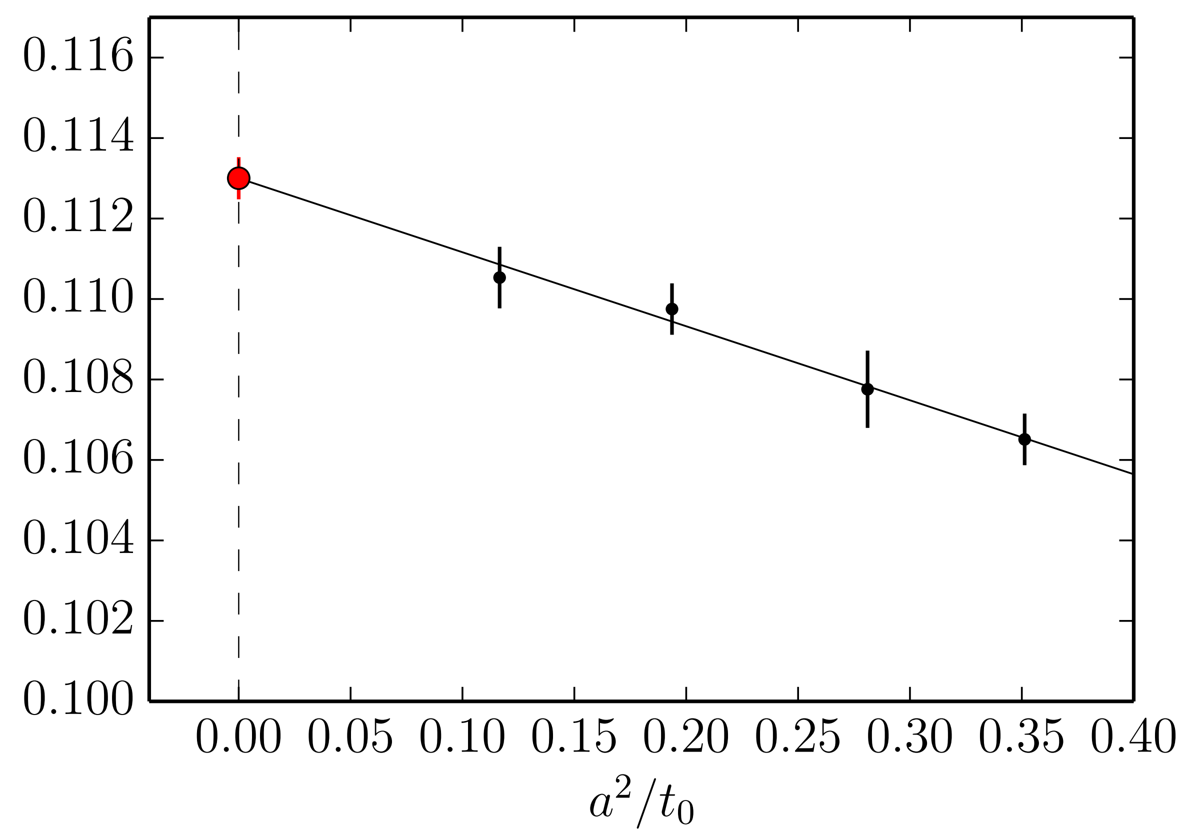
$$\sqrt{t_0} f_{\pi K} = F^\mathrm{cont}(\phi_2) + c \frac{a^2}{t_0^\mathrm{sym}}$$

$$f_{\pi K} = \frac{2}{3}(f_\mathrm{K} + \frac{1}{2} f_\pi)$$

light quark mass dependence



a-dependence at symm. point



Connection to hadronic world

$$\Lambda_{\overline{\mathrm{MS}}} = \frac{\Lambda_{\overline{\mathrm{MS}}}}{\Lambda_{\mathrm{SF}}} \times \Lambda_{\mathrm{SF}} L_0 \times \frac{L_{\mathrm{max}}}{L_0} \times \frac{\sqrt{t_0^{\mathrm{sym}}}}{L_{\mathrm{max}}} \times \frac{1}{\sqrt{t_0^{\mathrm{sym}}}}$$

$\sqrt{8t_0^{\mathrm{sym}}} = 0.4130(45)\,\mathrm{fm} \;\longleftarrow\; (f_K + f_\pi/2)^{\mathrm{PDG}}$

$\bar{g}^2(L_{\mathrm{max}}) = 11.31$ definition

$\bar{g}^2(L_0) = 2.012$ definition

$L_{\mathrm{max}}/\sqrt{t_0^{\mathrm{sym}}}$

$\longrightarrow \Lambda^{(3)}_{\overline{\mathrm{MS}}} = 332(14)\mathrm{MeV}$

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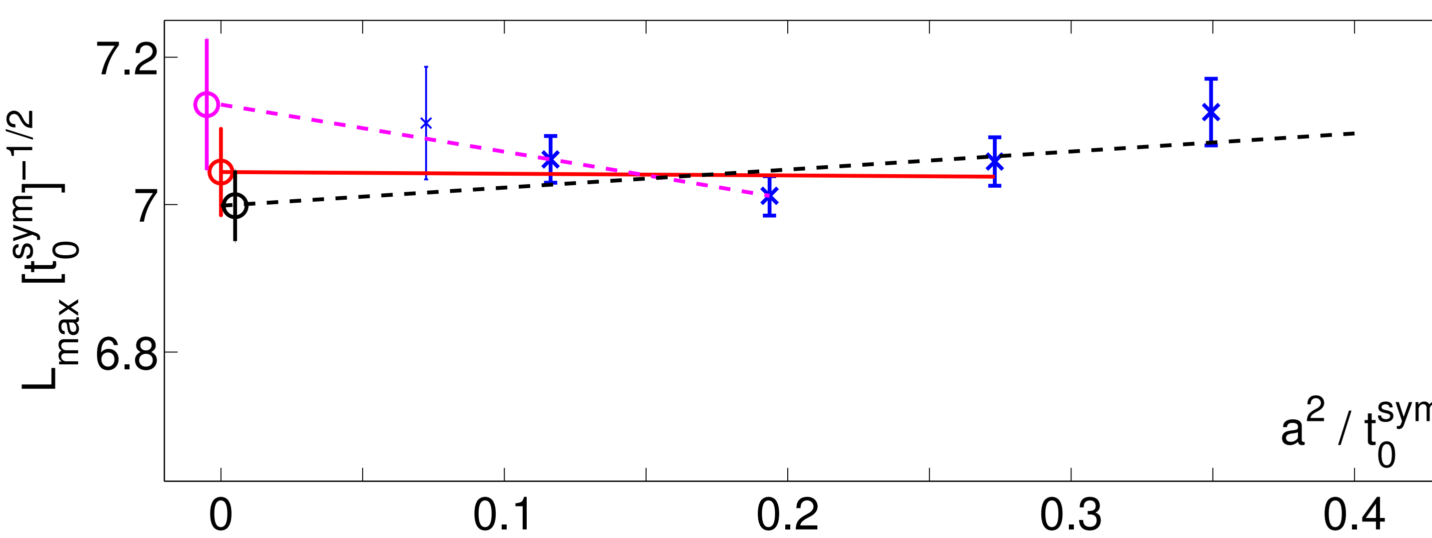

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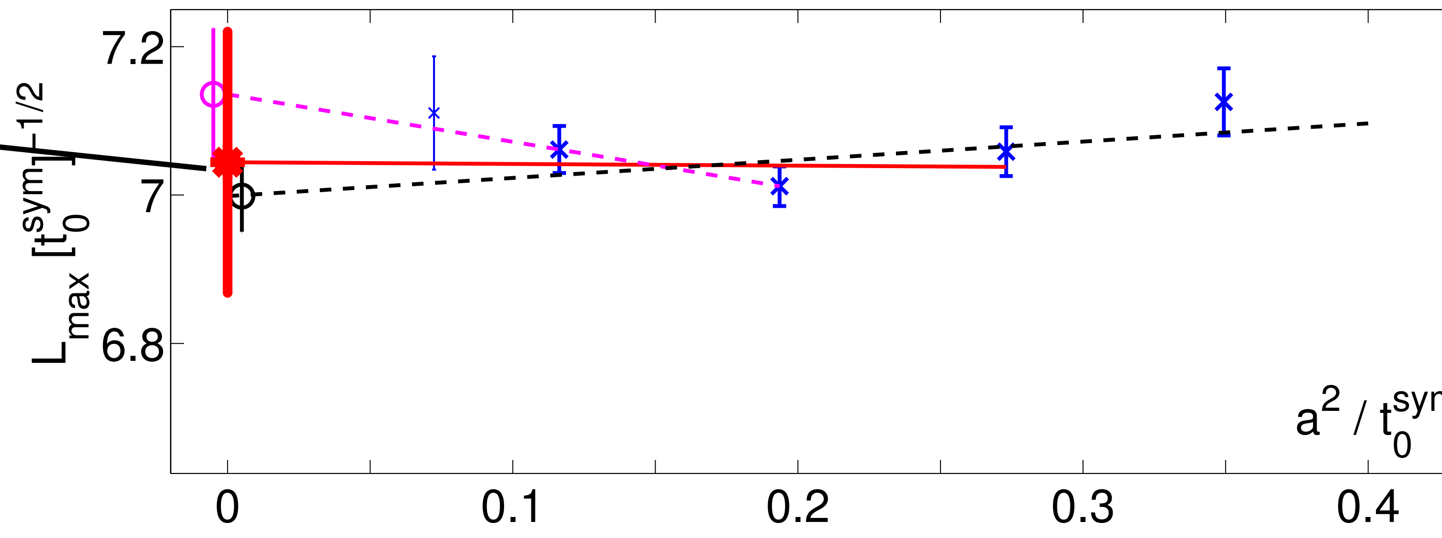

Results (2): the value of $\alpha_s(m_Z)$

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Perturbative conversion $N_f=3 \rightarrow N_f=5$

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Perturbative conversion $N_f=3 \rightarrow N_f=5$

- Use relation of Λ-parameters [formulae: cf. Bruno et al., PoS LATTICE2015 (2016) 256]
  input: $m_c(m_c)$, $m_b(m_b)$ from PDG

$$\Longrightarrow \quad \Lambda^{(4)}_{\overline{\rm MS}} = 289(14)\,\mathrm{MeV}, \quad \Lambda^{(5)}_{\overline{\rm MS}} = 207(11)\,\mathrm{MeV},$$

$$\alpha_{\overline{\rm MS}}(m_Z) = 0.1179(10)(2)$$

- Error estimate

n (= loops)	α_n	α_n - α_{n-1}
2	0.11670	-
3	0.11771	0.00109
4	0.11787	0.00016
5-loop β(g)	0.11794	0.00007

P.A. Baikov (SINP, Moscow), K.G. Chetyrkin, J.H. Kühn (KIT, Karlsruhe, TTP)
arXiv:1606.08659

Preliminary result for α

- $\Lambda^{(3)}_{\overline{\mathrm{MS}}} = 332(14)\mathrm{MeV}$
- $\alpha_{\overline{\mathrm{MS}}}(m_{\mathrm{Z}}) = 0.1179(10)(2)$
- using 3-flavor theory (decoupling; $N_f=3 \rightarrow N_f=5$ from PT)
- error budget:

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quantity	value	error	rel. err.	comment
$\Lambda^{(3)}_{\overline{\mathrm{MS}}} L_0$	0.0791	0.0021	0.026	arXiv:1604.06193
$L_{2.6723}/(2L_0)$	1	0.0080	0.008	scheme change arXiv:1607.06423
$s(11.31, 2.6723)$	10.93	0.21	0.019	scale factor
$t^{1/2}_{0,\mathrm{symm}}/L_{11.31}$	0.1420	0.0036	0.025	preliminary, Lat16
$[8t_{0,\mathrm{symm}}]^{1/2}$ [fm]	0.4130	0.0045	0.011	at $\phi_4 = 1.11$ preliminary, Lat16
$t^{-1/2}_{0,\mathrm{symm}}$ [GeV]	1.3514	0.0146	0.0108	at $\phi_4 = 1.11$
$\Lambda^{(3)}_{\overline{\mathrm{MS}}}$ [GeV]	0.332	0.014	0.042	preliminary, Lat16
$\alpha(m_\mathrm{Z})$	**0.1179**	**0.0010**	**0.009**	$\pm$ **0.00016** = conversion error
$\alpha(m_\mathrm{Z})$	0.1177	0.0010	0.0085	3-loop conversion
$\alpha(m_\mathrm{Z})$	0.1179	0.0009	0.0085	5-loop β-function
$\Lambda^{(3)}_{\overline{\mathrm{MS}}}$ [GeV]	0.336	0.019		FLAG3 [arXiv:1607.00299]

Preliminary result for α

- $\Lambda^{(3)}_{\overline{\mathrm{MS}}} = 332(14)\mathrm{MeV}$
- $\alpha_{\overline{\mathrm{MS}}}(m_{\mathrm{Z}}) = 0.1179(10)(2)$
- using 3-flavor theory (decoupling; N_f=3 → N_f=5 from PT)
- agrees well with PDG non-lattice: 0.1175(17)
- agrees well with FLAG16 (lattice): 0.1182(12)
- also with FLAG13 (lattice): 0.1184(12)

Conclusions

- errors of (asymptotic) series expansions are difficult to assess

- at α=0.2: we have examples where α=0.2 does not lead to an accurate perturbative result
  — more generally, this may be a reason for differences in determinations in $\alpha(m_z)$
  — also a reason for caution in some phenomenological uses of PT, eg. in flavor physics

- at α=0.1: PT is accurate

  - SSF technology allows to get there

  - very accurate prediction for LHC

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- at α=0.1: PT is accurate
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Euclidean
one scale observable
lowest power correction
$(\Lambda / \mu)^{3.8}$

Thank you

Backup

Change of N_f

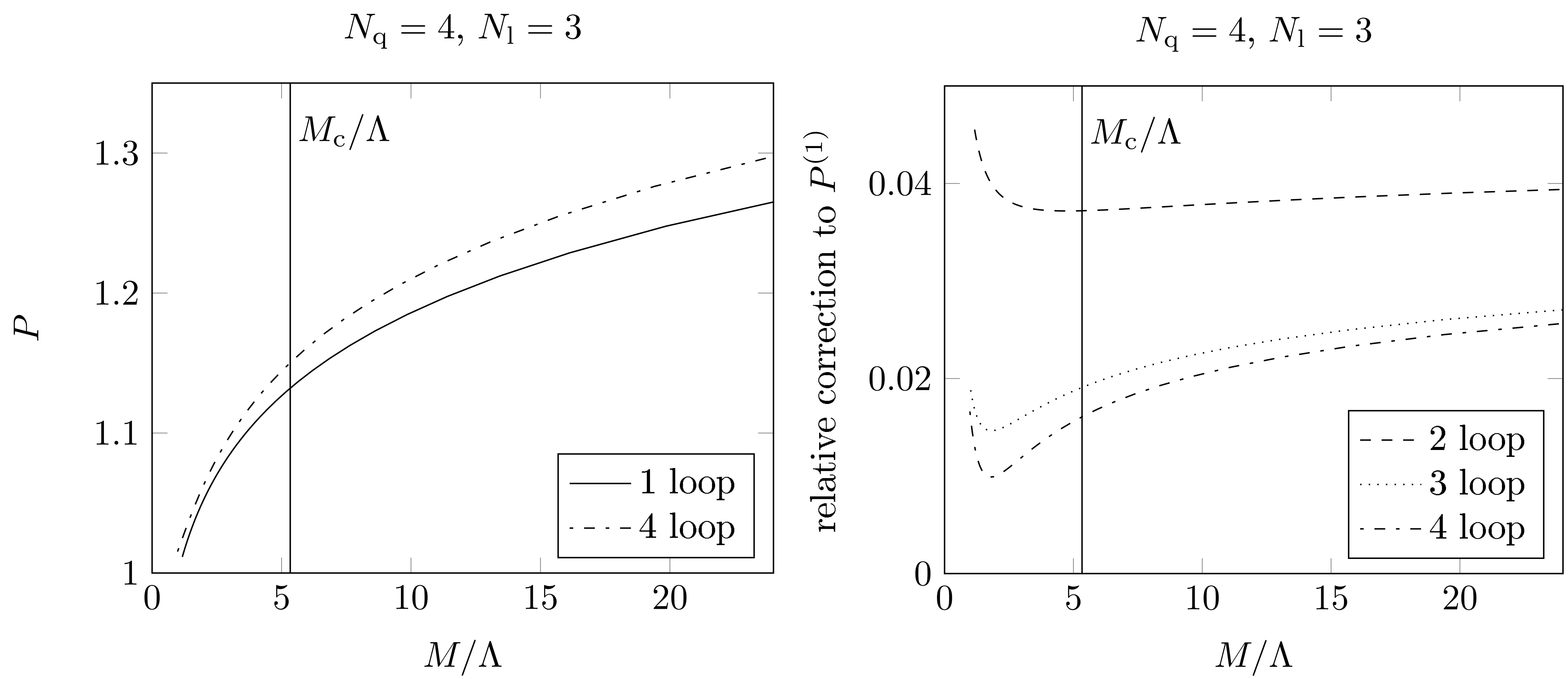


Figure 6: The mass-dependence P at 1-loop formula and at 4-loop (left) as well as 2,3,4-loop correction normalised to the 1-loop approximation (right) for the case $N_q = 4, N_l = 3$.

$$P = \frac{\Lambda^{(N_f - 1)}}{\Lambda^{(N_f)}}$$

it is harmless in perturbation theory

The SF scheme - basic definition

M. Lüscher, R. Narayanan, P. Weisz, and U. Wolff, Nucl. Phys. **B384**, 168 (1992), arXiv:hep-lat/9207009 [hep-lat].

M. Lüscher, R. Sommer, P. Weisz, and U. Wolff, hep-lat/9309005

- Dirichlet bc's

$$A_k(x)|_{x_0=0} = C_k(\eta,\nu)\,, \quad A_k(x)|_{x_0=L} = C'_k(\eta,\nu)$$

$$C_k = \frac{i}{L}[\mathrm{diag}(-\pi/3, 0, \pi/3) + \eta(\lambda_8 + \nu\lambda_3)$$

$$C'_k = \frac{i}{L}[\mathrm{diag}(-\pi, \pi/3, 2\pi/3) - \eta(\lambda_8 - \nu\lambda_3)].$$

$$\langle \partial_\eta S|_{\eta=0}\rangle = \frac{12\pi}{\bar{g}_\nu^2} = 12\pi[\frac{1}{\bar{g}^2} - \nu\,\bar{v}]$$

- similar to Casimir effect

- non-perturbative definition of background field (BF)
  = classical solution with these Dirichlet bc's
  spatially constant, abelian

- each value of ν : a different scheme

The GF scheme - basic definition

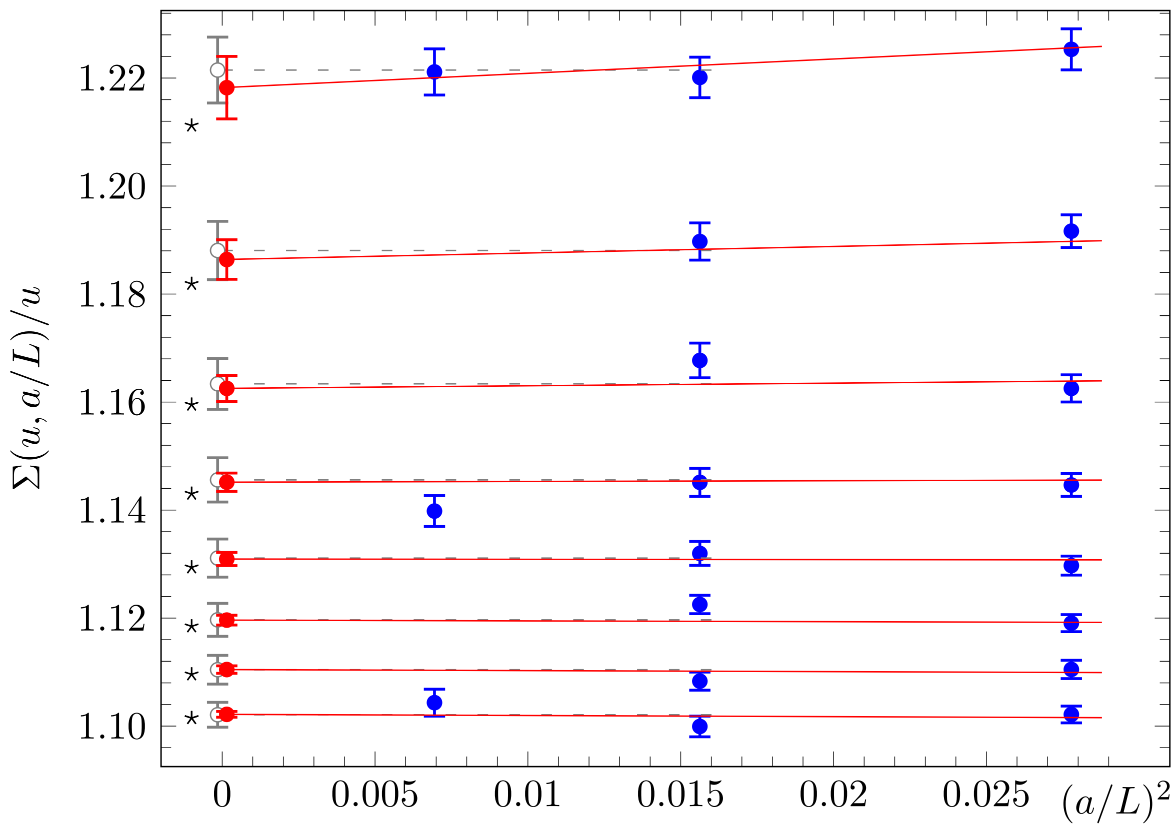
$$\frac{dB_\mu(t,x)}{dt} = D_\nu G_{\nu\mu}(t,x)\,, \qquad B_\mu(0,x) = A_\mu(x)$$

$$G_{\mu\nu}(t,x) = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

$$\bar{g}^2_{\rm GF}(1/L) = t^2 \mathcal{N}^{-1}(c)\langle \mathrm{tr}\,[G_{ij}(x_0,t)G_{ij}(x_0,t)] \Big|_{\sqrt{8t}=cL;x_0=T/2}$$

Continuum limit of Σ

N_f=3 from now on



- linear in a/L discretisation errors suppressed by Symanzik improvement (boundary terms)
  - 2-loop coefficients
  - in weak coupling region
  - taking $1 + c_1 g^2 + (c_2 \pm c_2)\, g^4 \quad (g = g_0)$
- extrapolate with $O((a/L)^2)$

Properties of the SF scheme

- $\Delta_{\mathrm{stat}}\bar{g}_\nu^2 = s(a/L)\bar{g}_\nu^4 + \mathrm{O}(\bar{g}_\nu^6)$. good accuracy for small g

- no μ^{-1}, μ^{-2} renormalons (infrared cutoff)

  instead: secondary minimum of the action

$$\exp(-2.62/\alpha) \sim (\Lambda/\mu)^{3.8}$$

- 3-loop **β**

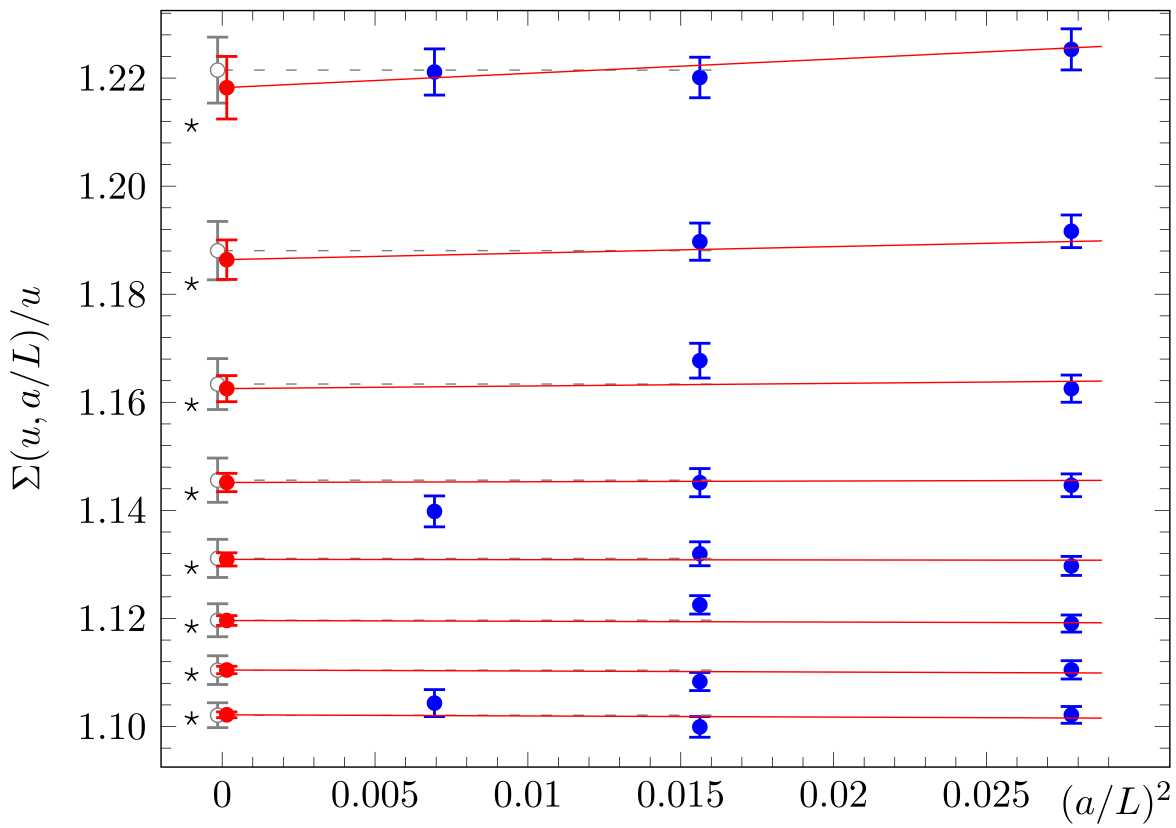
$$(4\pi)^3 \times b_{2,\nu} = -0.06(3) - \nu \times 1.26\,, \;\; (N_\mathrm{f} = 3)$$

- small discretisation effects (a^4 at LO PT)
  we also subtract them including 2-loop terms

  [hep-lat/9911018 Bode, Weisz, Wolff]

- but O(a) discretisation effects due to boundary terms

Continuum limit of Σ



- use perturbative improvement (i=1,2)

$$\Sigma^{(i)}(u,a/L) = \frac{\Sigma(u,a/L)}{1+\sum_{k=1}^{i}\delta_k(a/L)\,u^k}\,,$$

- and global fit

$$\Sigma_\nu^{(i)}(u,a/L) = \sigma_\nu(u) + \rho_\nu^{(i)}(u)\,(a/L)^2$$

- with

$$\rho_\nu^{(i)}(u) = \sum_{k=1}^{n_\rho^{(i)}} \rho_{\nu,k}^{(i)} u^{i+1+k}\,, \quad \sigma_\nu(u) = u + u^2\sum_{k=0}^{3} s_k u^k$$

Continuum limit of Σ

- was also tested carefully in pure gauge theory

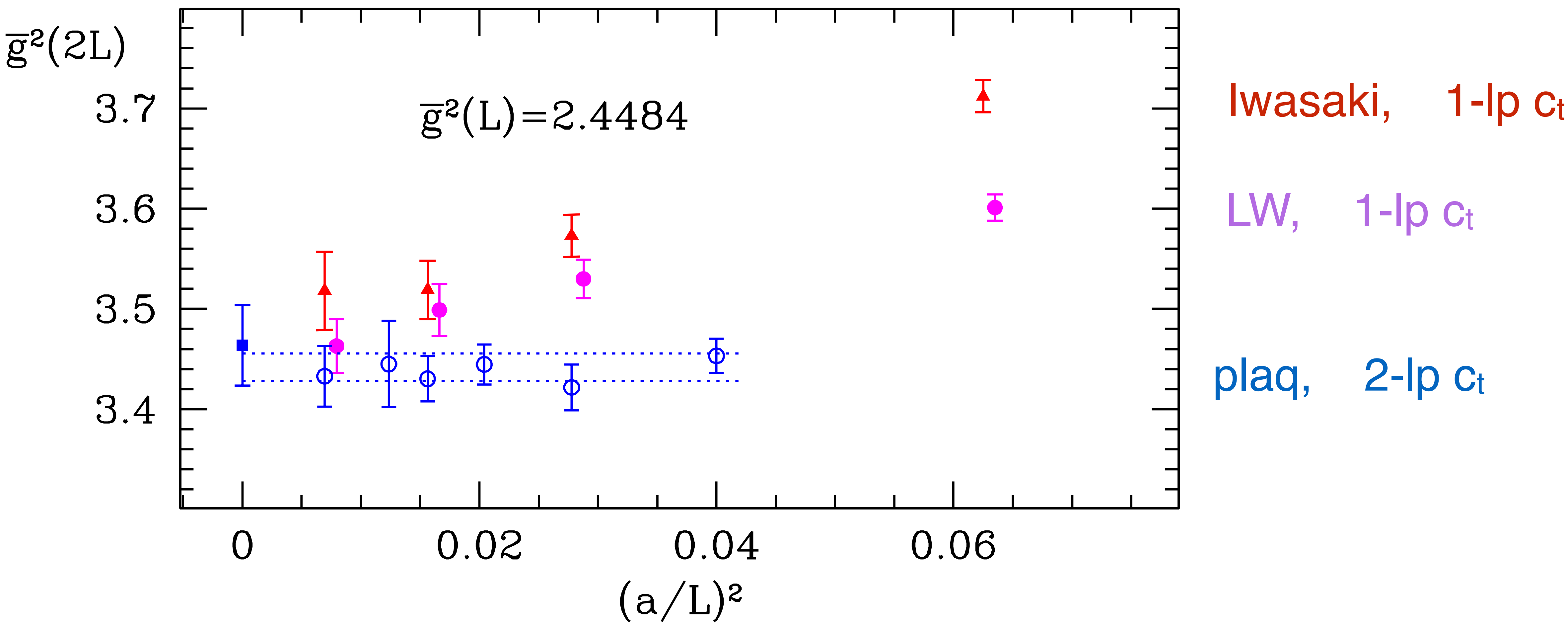
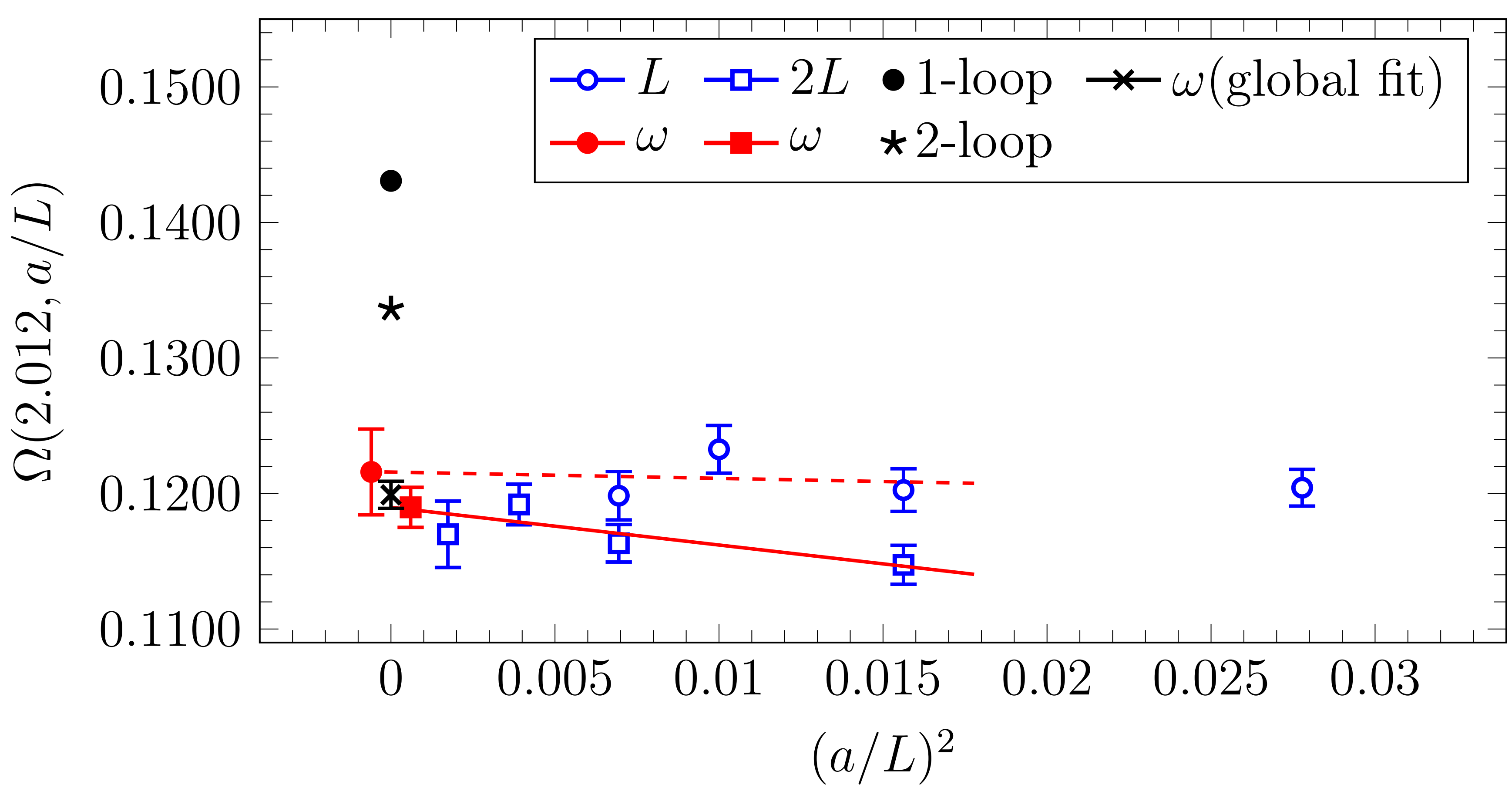


Figure 8: A test of the continuum extrapolations with different actions for $N_f = 0$. The data from top (triangles) to bottom (open circles) are for the Iwasaki, the tree level Lüscher Weisz and the Wilson gauge action. Both the boundary improvement of the action and the improvement of the observables have been included. At present this is possible at the 2-loop level for the Wilson gauge action only, and at the 1-loop level in the two other cases. Figure from [29] based on data from [63, 64].

Continuum limit of Ω

$$\Omega(u, a/L) = \bar{v}|_{\bar{g}^2(L)=u} \quad \omega(u) = \Omega(u, 0)$$

- Global fits, similar to Σ
- but with L/a=6,8,10,12 (“L”) and L/a=12,16,24 (“2L”)
- a-effects different for “L” vs. “2L” (different def. of m=0)



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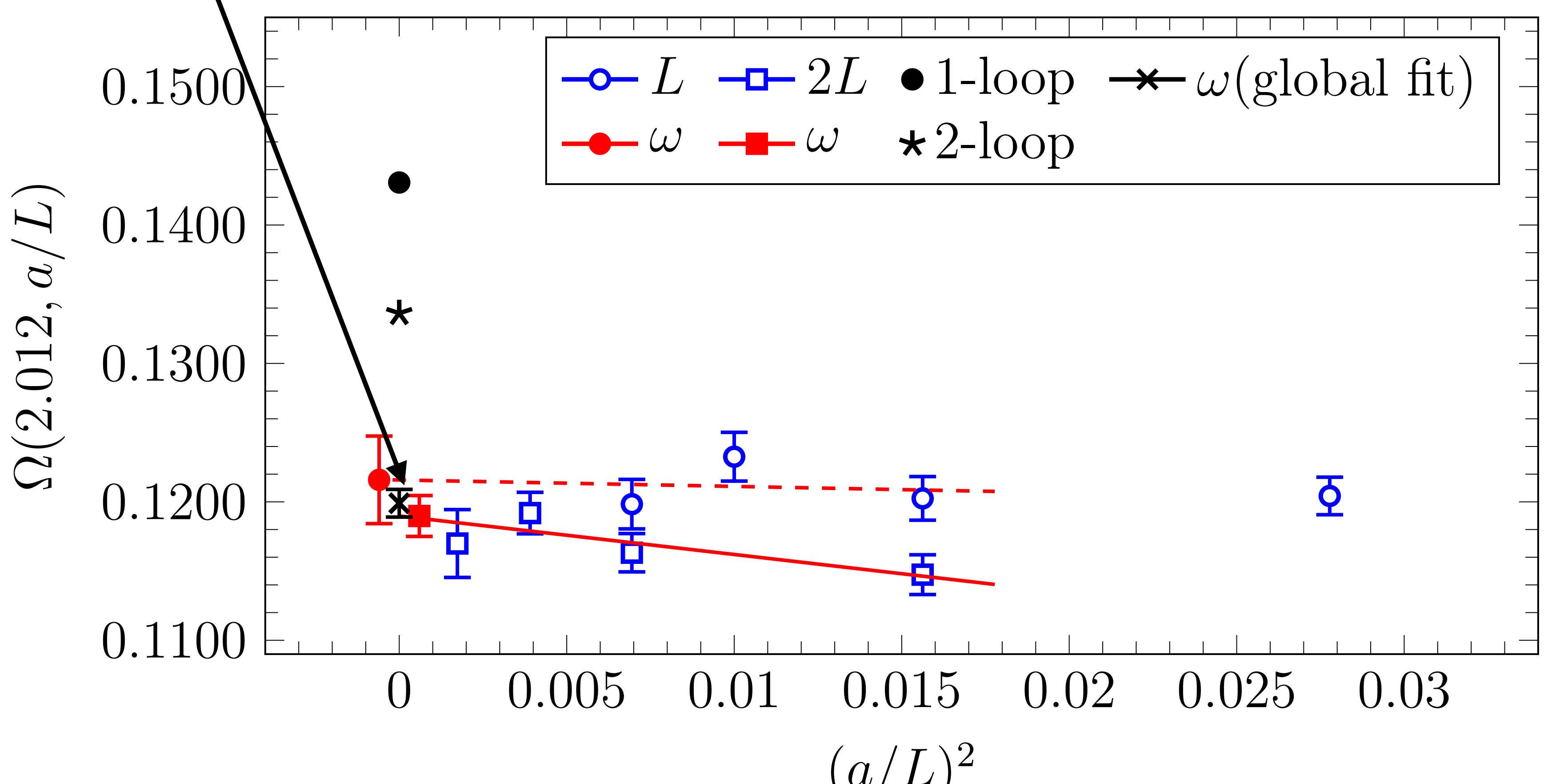

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