

THE PENTABOX MASTER INTEGRALS WITH THE SIMPLIFIED DIFFERENTIAL EQUATIONS APPROACH

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Zurich, August 25, 2016



INTRODUCTION

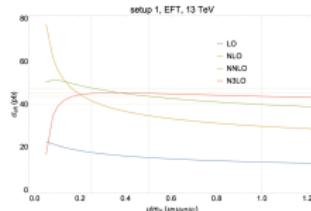


Figure 8: The dependence of the cross-section on a common renormalization and factorization scale $\mu \equiv \mu_F = \mu_R$.

$\Delta \sigma_{\text{ggF}, k}^{\text{scale}}$	
LO	($k = 0$) $\pm 14.8\%$
NLO	($k = 1$) $\pm 16.6\%$
NNLO	($k = 2$) $\pm 8.8\%$
N ³ LO	($k = 3$) $\pm 1.9\%$

Table 5: Scale variation of the cross-section as defined in eq. (3.11) for a common renormalization and factorization scale $\mu = \mu_F = \mu_R$.

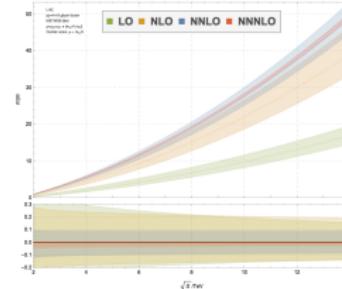
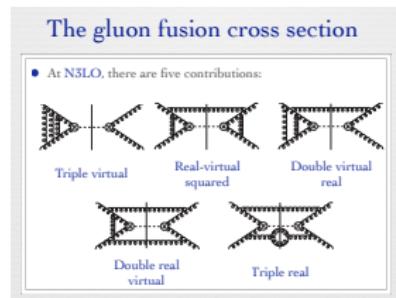


FIG. 3: The gluon fusion cross-section at all perturbative orders through N³LO in the scale interval $[m_t^\mu, m_H]$ as a function of the center-of-mass energy \sqrt{s} .

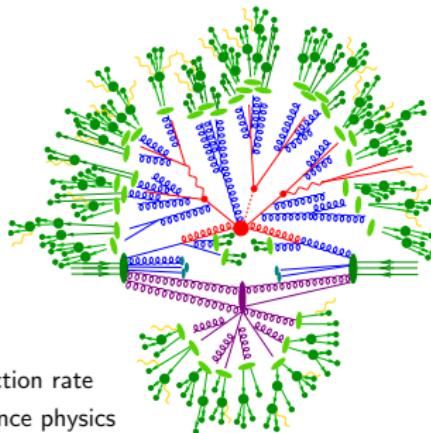


FACTORIZATION

Factorization

Collins,Soper,Sterman'85-'89

- ▶ Calculate
 - ▶ Scattering probability
 - ▶ Gluon emission probability
- ▶ Measure
 - ▶ Long distance interactions
 - ▶ Particle decay rates



Divide et Impera

- ▶ Quantity of interest: Total interaction rate
- ▶ Convolution of short & long distance physics

$$\sigma_{p_1 p_2 \rightarrow X} = \sum_{i,j \in \{q,g\}} \int dx_1 dx_2 \underbrace{f_{p_1,i}(x_1, \mu_F^2) f_{p_2,j}(x_2, \mu_F^2)}_{\text{long distance physics}} \underbrace{\hat{\sigma}_{ij \rightarrow X}(x_1 x_2, \mu_F^2)}_{\text{short distance physics}}$$

QCD as a perturbative quantum field theory: **Fixed-order calculations**

TAMING THE BEAST ...

From Feynman graphs ...

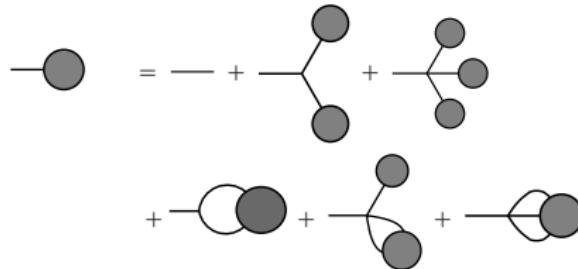
$gg \rightarrow ng$	2	3	4	5	6	7	8	9
# FG	4	25	220	2,485	34,300	559,405	10,525,900	224,449,225

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to Dyson-Schwinger recursion! Helac-Phegas



$gg \rightarrow ng$	2	3	4	5	6	7	8	9
#	5	15	35	70	126	210	330	495

TAMING THE BEAST ...

VOLUME 56, NUMBER 23

PHYSICAL REVIEW LETTERS

9 JUNE 1986

Amplitude for n -Gluon Scattering

Stephen J. Parks and T. R. Taylor

Fermi National Accelerator Laboratory, Batavia, Illinois 60510
(Received 17 March 1986)

A nontrivial squared helicity amplitude is given for the scattering of an arbitrary number of gluons to lowest order in the coupling constant and to leading order in the number of colors.

PACS numbers: 12.38.Bx

Computations of the scattering amplitudes for the vector gauge bosons of non-Abelian gauge theories, besides being interesting from a purely quantum-field-theoretical point of view (determination of the S matrix), have a wide range of important applications. In particular, within the framework of quantum chromodynamics (QCD), the scattering of vector gauge bosons (gluons) gives rise to experimentally observable multiplet production at high-energy hadron colliders. The knowledge of cross sections for the gluon scattering is crucial for any reliable phenomenology of jet physics, which holds great promise for testing QCD as well as for the discovery of new physics at present (CERN SpS and Fermilab Tevatron) and future (Superconducting Super Collider) hadron colliders.¹

In this short Letter, we give a nontrivial squared helicity amplitude for the scattering of an arbitrary number of gluons to lowest order in the coupling constant and to leading order in the number of colors. To our knowledge this is the first time in a non-Abelian gauge theory that a nontrivial, on-mass-shell, squared Green's function has been written down for an arbitrary number of external points. Our result can be

used to improve the existing numerical programs for the QCD jet production, and in particular for the studies of the four-jet production for which no analytic results have been available so far. Before presenting the helicity amplitude, let us make it clear that this result is an educated guess which we have compared to the existing computations and verified by a series of highly nontrivial and nonlinear consistency checks.

For the n -gluon scattering amplitude, there are $(n+2)/2$ independent helicity amplitudes. At the tree level, the two helicity amplitudes which must violate the conservation of helicity are zero. This is easily seen by the embedding of the Yang-Mills theory in a supersymmetric theory.^{2,3} Here we give an expression for the next helicity amplitude, also at tree level, to leading order in the number of colors in SU(N) Yang-Mills theory.

If the helicity amplitude for gluons $1, \dots, n$, of momenta p_1, \dots, p_n and helicities $\lambda_1, \dots, \lambda_n$, is $\mathcal{M}_n(\lambda_1, \dots, \lambda_n)$, where the momenta and helicities are labeled as though all particles are outgoing, then the three helicity amplitudes of interest, squared and summed over color, are

$$|\mathcal{M}_n(+-+-+\cdots)|^2 = c_n(g, N)(0 + O(g^4)), \quad (1)$$

$$|\mathcal{M}_n(-+++\cdots)|^2 = c_n(g, N)(0 + O(g^4)), \quad (2)$$

$$|\mathcal{M}_n(--++\cdots)|^2 = c_n(g, N)(p_1 \cdot p_2)^2 \times \sum_P [(p_1 \cdot p_2)(p_2 \cdot p_3)(p_3 \cdot p_4) \cdots (p_n \cdot p_1)]^{-1} + O(N^{-2}) + O(g^2)], \quad (3)$$

where $c_n(g, N) = g^{2n-4}N^{n-2}(N^2-1)/2^{n-4}n$. The sum is over all permutations P of $1, \dots, n$.

Equation (3) has the correct dimensions and symmetry properties for this n -particle scattering amplitude squared. Also it agrees with the known results^{4,5} for $n=4, 5$, and 6. The agreement for $n=6$ is numerical.^{5,6} More importantly, this set of amplitudes is consistent with the Altarelli and Parisi⁷ relationship for all n , when two of the gluons are made parallel. This is trivial for the first two helicity amplitudes but is a highly nontrivial statement for the last amplitude, as shown here:

$$|\mathcal{M}_n(--++\cdots)|^2 \xrightarrow[1/2]{} 0, \quad (4)$$

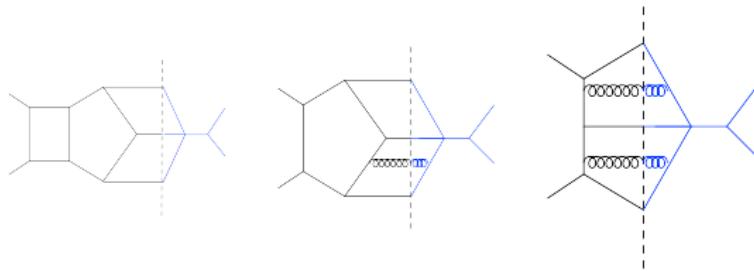
$$|\mathcal{M}_n(--++\cdots)|^2 \xrightarrow[2/3]{} 2g^2N \frac{z^4}{z(1-z)} \frac{1}{s} |\mathcal{M}_{n-1}(-++\cdots)|^2, \quad (5)$$

$$|\mathcal{M}_n(--++\cdots)|^2 \xrightarrow[3/4]{} 2g^2N \frac{1}{z(1-z)} \frac{1}{s} |\mathcal{M}_{n-1}(-++\cdots)|^2, \quad (6)$$

PERTURBATIVE QCD AT NNLO

What do we need for an NNLO calculation ?

$$p_1, p_2 \rightarrow p_3, \dots, p_{m+2}$$



PERTURBATIVE QCD AT NNLO

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$$p_1, p_2 \rightarrow p_3, \dots, p_{m+2}$$

$$\begin{aligned}\sigma_{NNLO} &\rightarrow \int_m d\Phi_m \left(2\text{Re}(M_m^{(0)*} M_m^{(2)}) + \left| M_m^{(1)} \right|^2 \right) J_m(\Phi) & \textcolor{red}{VV} \\ &+ \int_{m+1} d\Phi_{m+1} \left(2\text{Re} \left(M_{m+1}^{(0)*} M_{m+1}^{(1)} \right) \right) J_{m+1}(\Phi) & \textcolor{red}{RV} \\ &+ \int_{m+2} d\Phi_{m+2} \left| M_{m+2}^{(0)} \right|^2 J_{m+2}(\Phi) & \textcolor{red}{RR}\end{aligned}$$

$RV + RR \rightarrow$

Antenna-S, Colorfull-S, STRIPPER, q_T , N-jetiness

A. Gehrmann-De Ridder, T. Gehrmann and M. Ritzmann, JHEP 1210 (2012) 047

P. Bolzoni, G. Somogyi and Z. Trocsanyi, JHEP 1101 (2011) 059

M. Czakon and D. Heymes, Nucl. Phys. B 890 (2014) 152

S. Catani and M. Grazzini, Phys. Rev. Lett. 98 (2007) 222002

R. Boughezal, C. Focke, X. Liu and F. Petriello, Phys. Rev. Lett. 115 (2015) no.6, 062002

OPP AT TWO LOOPS

coefficients of MI \oplus spurious terms

$$\begin{aligned}\frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} &= \sum_{i_0 < i_1 < i_2 < i_3}^{m-1} \frac{d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)}{\bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2} \bar{D}_{i_3}} \\ &+ \sum_{i_0 < i_1 < i_2}^{m-1} \frac{c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)}{\bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2}} \\ &+ \sum_{i_0 < i_1}^{m-1} \frac{b(i_0 i_1) + \tilde{b}(q; i_0 i_1)}{\bar{D}_{i_0} \bar{D}_{i_1}} \\ &+ \sum_{i_0}^{m-1} \frac{a(i_0) + \tilde{a}(q; i_0)}{\bar{D}_{i_0}} \\ &+ \text{rational terms}\end{aligned}$$

OPP AT TWO LOOPS

- Write the "OPP-type" equation at two loops

$$\frac{N(l_1, l_2; \{p_i\})}{D_1 D_2 \dots D_n} = \sum_{m=1}^{\min(n, 8)} \sum_{S_{m;n}} \frac{\Delta_{i_1 i_2 \dots i_m}(l_1, l_2; \{p_i\})}{D_{i_1} D_{i_2} \dots D_{i_m}}$$

$$\sum \frac{\Delta_{i_1 i_2 \dots i_m}(l_1, l_2; \{p_i\})}{D_{i_1} D_{i_2} \dots D_{i_m}} \rightarrow \text{spurious } \oplus \text{ISP - irreducible integrals}$$

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ISP-irreducible integrals → use **IBPI** to Master Integrals

Libraries in the future: QCD2LOOP, TwOLoop

P. Mastrolia, T. Peraro and A. Primo, arXiv:1605.03157 [hep-ph].

J. Gluza, K. Kajda and D. A. Kosower, Phys. Rev. D **83** (2011) 045012

H. Ita, arXiv:1510.05626 [hep-th].

C. G. Papadopoulos, R. H. P. Kleiss and I. Malamos, PoS Corfu **2012** (2013) 019.

IBPI: THE CURRENT APPROACH

- m independent momenta / loops, $N = I(I+1)/2 + Im$ scalar products
- basis composed by $D_1 \dots D_N$, allows to express all scalar products

$$D_i = (\{k, l\} + p_i)^2 - M_i^2$$

-

$$F[a_1, \dots, a_N] = \int d^d k d^d l \frac{1}{D_1^{a_1} \dots D_N^{a_N}}$$

$$\int d^d k d^d l \frac{\partial}{\partial \{k^\mu, l^\mu\}} \left(\frac{\{k^\mu, l^\mu, v^\mu\}}{D_1^{a_1} \dots D_N^{a_N}} \right) = 0$$

- IBP Laporta: FIRE, AIR, Reduze reduce these to MI
- MI computed, Feynman parameters, Mellin-Barnes, Differential Equations
- Or numerical: SecDec, Weinzierl

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F. V. Tkachov, Phys. Lett. B 100 (1981) 65.

K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. B 192 (1981) 159.

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[S. Laporta, Int. J. Mod. Phys. A 15 \(2000\) 5087](#)

[C. Anastasiou and A. Lazopoulos, JHEP 0407 \(2004\) 046](#)

[C. Studerus, Comput. Phys. Commun. 181 \(2010\) 1293](#)

[A. V. Smirnov, Comput. Phys. Commun. 189 \(2014\) 182](#)

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Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Lett. B **302** (1993) 299.

V. A. Smirnov, Phys. Lett. B **460** (1999) 397

T. Gehrmann and E. Remiddi, Nucl. Phys. B **580** (2000) 485 [[hep-ph/9912329](#)].

J. M. Henn, Phys. Rev. Lett. **110** (2013) 25, 251601 [[arXiv:1304.1806 \[hep-th\]](#)].

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S. Borowka, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk and T. Zirke, Comput. Phys. Commun. **196** (2015) 470

S. Becker, C. Reuschle and S. Weinzierl, JHEP **1012** (2010) 013

IBPI: THE CURRENT APPROACH

- Find a better IBP algorithm ... Generating function technique, Baikov ?

P. A. Baikov, Nucl. Instrum. Meth. A **389** (1997) 347

V. A. Smirnov and M. Steinhauser, Nucl. Phys. B **672** (2003) 199

K. J. Larsen and Y. Zhang, Phys. Rev. D **93** (2016) no.4, 041701

$$F_{a_1 \dots a_N} = \sum_{i=\text{masters}} c_{a_1 \dots a_N}^{(i)} G_i$$

- Baikov polynomial \leftrightarrow LZ construction
- Sector \leftrightarrow cut

$$\delta((k+p)^2 - m^2) \leftrightarrow \oint_{z=0} dz \frac{1}{z^{n=1}}$$

- Cut with higher powers in denominator

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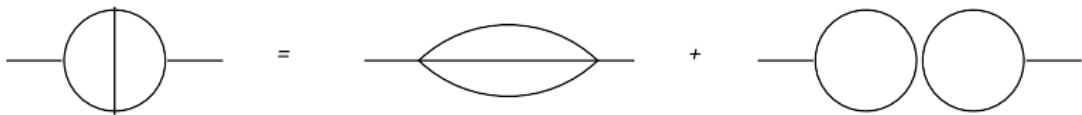
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$$F_{11111} = \frac{(3d-10)(3d-8)}{(d-4)^2(p^2)^2} F_{10011} + \frac{(3d-10)(3d-8)}{(d-4)^2(p^2)^2} F_{01101} - 2 \frac{(d-3)}{(d-4)p^2} F_{11110}$$

DIFFERENTIAL EQUATIONS APPROACH

The integral is a function of external momenta, so one can set-up differential equations by differentiating and using **IBP**

$$p_j^\mu \frac{\partial}{\partial p_i^\mu} G[a_1, \dots, a_n] \rightarrow \sum C_{b_1, \dots, b_n} F[b_1, \dots, b_n] \rightarrow \sum \textcolor{red}{C}_{a'_1, \dots, a'_n} G[a'_1, \dots, a'_n]$$

- **Find the proper parametrization:** Bring the system of equations in a form suitable to express the MI in terms of GPs

$$\begin{aligned}\partial_m f(\varepsilon, \{x_i\}) &= \varepsilon A_m(\{x_i\}) f(\varepsilon, \{x_i\}) \\ \partial_m A_n - \partial_n A_m &= 0 \quad [A_m, A_n] = 0\end{aligned}$$

★ f not MI!

J. M. Henn, Phys. Rev. Lett. 110 (2013) 25, 251601 [[arXiv:1304.1806 \[hep-th\]](#)].

- **Boundary conditions:** expansion by regions or regularity conditions.

B. Jantzen, A. V. Smirnov and V. A. Smirnov, Eur. Phys. J. C 72 (2012) 2139 [[arXiv:1206.0546 \[hep-ph\]](#)].

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DIFFERENTIAL EQUATIONS APPROACH

- **Iterated Integrals**

K. T. Chen, Iterated path integrals, Bull. Amer. Math. Soc. 83 (1977) 831

- Multiple Polylogarithms, Symbol algebra
- Goncharov Polylogarithms

$$\mathcal{G}(a_n, \dots, a_1, x) = \int_0^x dt \frac{1}{t - a_n} \mathcal{G}(a_{n-1}, \dots, a_1, t)$$

with the special cases, $\mathcal{G}(x) = 1$ and

$$\mathcal{G}\left(\underbrace{0, \dots, 0}_n, x\right) = \frac{1}{n!} \log^n(x)$$

- Shuffle algebra

DIFFERENTIAL EQUATIONS APPROACH

- Iterated Integrals
- Multiple Polylogarithms, Symbol algebra

A. B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, Phys. Rev. Lett. **105** (2010) 151605.

C. Duhr, H. Gangl and J. R. Rhodes, JHEP **1210** (2012) 075 [arXiv:1110.0458 [math-ph]].

C. Bogner and F. Brown

- Goncharov Polylogarithms

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$$\mathcal{G}\left(\underbrace{0, \dots, 0}_n, x\right) = \frac{1}{n!} \log^n(x)$$

- Shuffle algebra

DIFFERENTIAL EQUATIONS APPROACH

- Iterated Integrals
- Multiple Polylogarithms, Symbol algebra
- Goncharov Polylogarithms

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- Shuffle algebra

$$\mathcal{G}(a_1, a_2; x) \mathcal{G}(b_1; x) = \mathcal{G}(a_1, a_2, b_1; x) + \mathcal{G}(a_1, b_1, a_2; x) + \mathcal{G}(b_1, a_1, a_2; x)$$

THE SIMPLIFIED DIFFERENTIAL EQUATIONS APPROACH

C. G. Papadopoulos, JHEP 1407 (2014) 088

Making the whole procedure systematic (algorithmic) and straightforwardly expressible in terms of GPs.

- Introduce one parameter

$$G_{11\dots 1}(x) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2)(k + \cancel{x} p_1)^2 (k + p_1 + p_2)^2 \dots (k + p_1 + p_2 + \dots + p_n)^2}$$

- Factorizing external momenta dependence:

$$x : (q_1 = xp_1, q_2 = p_{12} - xp_1, \dots) \rightarrow x \otimes (q_1 = p_1, q_2 = p_2, \dots)$$

- Now the integral as a function of x , allows to define a differential equation with respect to x , schematically given by

$$\frac{\partial}{\partial x} G_{11\dots 1}(x) = -\frac{1}{x} G_{11\dots 1}(x) + xp_1^2 G_{12\dots 1} + \frac{1}{x} G_{02\dots 1}$$

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THE SIMPLIFIED DIFFERENTIAL EQUATIONS APPROACH

- and using IBPI we obtain, for instance for the one-loop 3 off-shell legs

$$\begin{aligned} m_1 x G_{121} + \frac{1}{x} G_{021} &= \left(\frac{1}{x-1} + \frac{1}{x-m_3/m_1} \right) \left(\frac{d-4}{2} \right) G_{111} \\ &\quad + \frac{d-3}{m_1 - m_3} \left(\frac{1}{x-1} - \frac{1}{x-m_3/m_1} \right) \left(\frac{G_{101} - G_{110}}{x} \right) \end{aligned}$$

- The integrating factor M is given by

$$M = x (1-x)^{\frac{4-d}{2}} (-m_3 + m_1 x)^{\frac{4-d}{2}}$$

- and the DE takes the form, $d = 4 - 2\epsilon$,

$$\frac{\partial}{\partial x} M G_{111} = c_\Gamma \frac{1}{\epsilon} (1-x)^{-1+\epsilon} (-m_3 + m_1 x)^{-1+\epsilon} \left((-m_1 x^2)^{-\epsilon} - (-m_3)^{-\epsilon} \right)$$

- Integrating factors $\epsilon = 0$ do not have branch points
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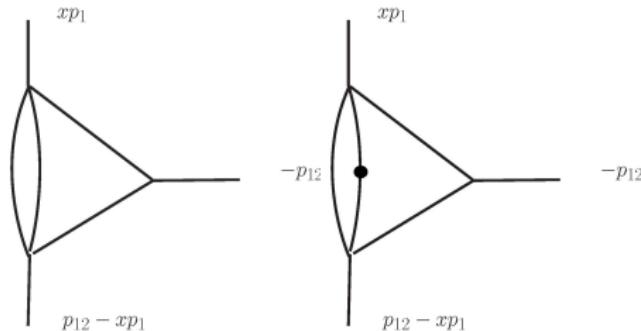
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How far we can go with the Simplified Differential Equations approach ?

THE SIMPLIFIED DIFFERENTIAL EQUATIONS APPROACH

The two-loop 3-off-shell-legs triangle



THE SIMPLIFIED DIFFERENTIAL EQUATIONS APPROACH

We are interested in $G_{0101011}$. The DE involves also the MI $G_{0201011}$, so we have a system of two coupled DE, as follows:

$$\frac{\partial}{\partial x} f(x) = \frac{A_3(2-3\varepsilon)(1-x)^{-2\varepsilon} x^{-1+\varepsilon} (m_1 x - m_3)^{-2\varepsilon}}{2\varepsilon(2\varepsilon-1)} + \frac{m_1 \varepsilon (1-x)^{-2\varepsilon} (m_1 x - m_3)^{-2\varepsilon}}{2\varepsilon-1} g(x)$$

$$\frac{\partial}{\partial x} g(x) = \frac{A_3(3\varepsilon-2)(3\varepsilon-1)(-m_1)^{-2\varepsilon} (1-x)^{2\varepsilon-1} x^{-3\varepsilon} (m_1 x - m_3)^{2\varepsilon-1}}{(2\varepsilon-1)(3\varepsilon-1)(1-x)^{2\varepsilon-1} (m_1 x - m_3)^{2\varepsilon-1}} f(x)$$

where $f(x) \equiv M_{0101011} G_{0101011}$ and $g(x) \equiv M_{0201011} G_{0201011}$, $M_{0201011} = (1-x)^{2\varepsilon} x^{\varepsilon+1} (m_1 x - m_3)^{2\varepsilon}$ and $M_{0101011} = x^\varepsilon$

- Solve sequentially in ε expansion
- Reproduce limit $\varepsilon \rightarrow 0$

THE SIMPLIFIED DIFFERENTIAL EQUATIONS APPROACH

The *regularized* singularity at $x = 0$ is proportional to $x^{-1+\varepsilon}$ and can easily be integrated by the following decomposition

$$\begin{aligned} \int_0^x dt \ t^{-1+\varepsilon} F(t) &= F(0) \int_0^x dt \ t^{-1+\varepsilon} + \int_0^x dt \ \frac{F(t)-F(0)}{t} t^\varepsilon \\ &= F(0) \frac{x^\varepsilon}{\varepsilon} + \int_0^x dt \ \frac{F(t)-F(0)}{t} \left(1 + \varepsilon \log(t) + \frac{1}{2}\varepsilon^2 \log^2(t) + \dots\right) \end{aligned}$$

Reproduce correctly boundary term $x = 0$

THE SIMPLIFIED DIFFERENTIAL EQUATIONS APPROACH

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Reproduce correctly boundary term $x = 0$

Five-point one-loop integral with up to one off-shell leg at $\mathcal{O}(\varepsilon)$

SIMPLIFIED DIFFERENTIAL EQUATIONS APPROACH

- simple parametrization of external momenta based on
Triangle rule: Criterion for the x -parametrization

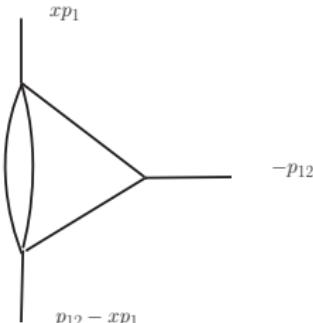


FIGURE : Required parametrization for off mass-shell triangles after possible pinching of internal line(s).

- DE in one parameter: **addressing problems with many scales**
- Boundary terms straightforwardly obtained by the DE itself, based on one-scale MI
- Expressions in terms of GP's straightforwardly obtained by expanding the DE in ε

TWO-LOOP, FOUR-POINT, TWO OFF-SHELL LEGS

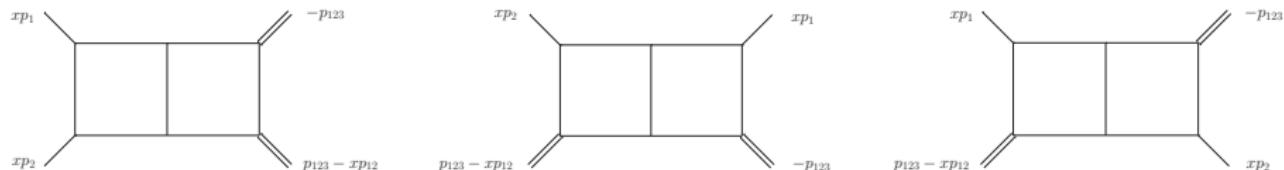


FIGURE : The parametrization of external momenta for the three planar double boxes of the families P_{12} (left), P_{13} (middle) and P_{23} (right) contributing to pair production at the LHC. All external momenta are incoming.

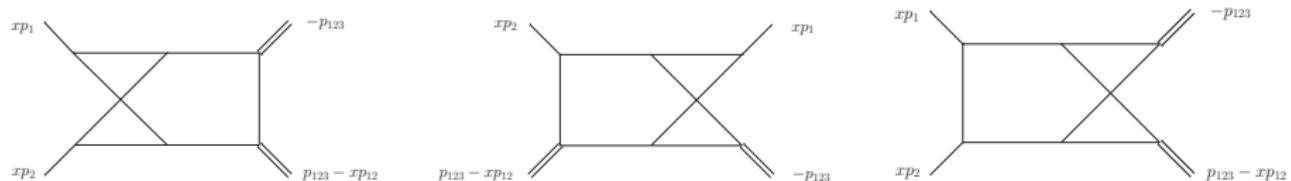


FIGURE : The parametrization of external momenta for the three non-planar double boxes of the families N_{12} (left), N_{13} (middle) and N_{34} (right) contributing to pair production at the LHC. All external momenta are incoming.

TWO-LOOP, FOUR-POINT, TWO OFF-SHELL LEGS

Planar topologies

$$G_{a_1 \dots a_9}^{P_{12}}(x, s, \epsilon) := e^{2\gamma_E \epsilon} \int \frac{d^d k_1}{i\pi^{d/2}} \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1} (k_1 + xp_1)^{2a_2} (k_1 + xp_{12})^{2a_3} (k_1 + p_{123})^{2a_4}} \\ \times \frac{1}{k_2^{2a_5} (k_2 - xp_1)^{2a_6} (k_2 - xp_{12})^{2a_7} (k_2 - p_{123})^{2a_8} (k_1 + k_2)^{2a_9}},$$

$$G_{a_1 \dots a_9}^{P_{13}}(x, s, \epsilon) := e^{2\gamma_E \epsilon} \int \frac{d^d k_1}{i\pi^{d/2}} \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1} (k_1 + xp_1)^{2a_2} (k_1 + xp_{12})^{2a_3} (k_1 + p_{123})^{2a_4}} \\ \times \frac{1}{k_2^{2a_5} (k_2 - xp_1)^{2a_6} (k_2 - p_{12})^{2a_7} (k_2 - p_{123})^{2a_8} (k_1 + k_2)^{2a_9}},$$

$$G_{a_1 \dots a_9}^{P_{23}}(x, s, \epsilon) := e^{2\gamma_E \epsilon} \int \frac{d^d k_1}{i\pi^{d/2}} \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1} (k_1 + xp_1)^{2a_2} (k_1 + p_{123} - xp_2)^{2a_3} (k_1 + p_{123})^{2a_4}} \\ \times \frac{1}{k_2^{2a_5} (k_2 - p_1)^{2a_6} (k_2 + xp_2 - p_{123})^{2a_7} (k_2 - p_{123})^{2a_8} (k_1 + k_2)^{2a_9}},$$

TWO-LOOP, FOUR-POINT, TWO OFF-SHELL LEGS

Planar topologies

$P_{12} :$ {010000011, 001010001, 001000011, 100000011, 101010010, 101010100, 101000110, 010010101,
101000011, 101000012, 100000111, 100000112, 001010011, 001010012, 010000111, 010010011,
101010110, 111000011, 101000111, 101010011, 011010011, 011010012, 110000111, 110000112,
010010111, 010010112, 111010011, 111000111, 111010111, 111m10111, 11101m111},

$P_{13} :$ {000110001, 001000011, 001010001, 001101010, 001110010, 010000011, 010101010, 010110010,
001001011, 001010011, 001010012, 001011011, 001101001, 001101011, 001110001, 001110002,
001110011, 001111001, 001111011, 001211001, 010010011, 010110001, 010110011, 011010011,
011010021, 011110001, 011110011, 011111011, m11111011},

$P_{23} :$ {001010001, 001010011, 010000011, 010000101, 010010011, 010010101, 010010111, 011000011,
011010001, 011010010, 011010011, 011010012, 011010100, 011010101, 011010111, 011020011,
012010011, 021010011, 100000011, 101000011, 101010010, 101010011, 101010100, 110000111,
111000011, 111010011, 111010111, 111m10111}.

TWO-LOOP, FOUR-POINT, TWO OFF-SHELL LEGS

Non-planar topologies

$$G_{a_1 \dots a_9}^{N_{12}}(x, s, \epsilon) := e^{2\gamma_E \epsilon} \int \frac{d^d k_1}{i\pi^{d/2}} \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1} (k_1 + xp_1)^{2a_2} (k_1 + xp_{12})^{2a_3} (k_1 + p_{123})^{2a_4}} \\ \times \frac{1}{k_2^{2a_5} (k_2 - xp_1)^{2a_6} (k_2 - p_{123})^{2a_7} (k_1 + k_2 + xp_2)^{2a_8} (k_1 + k_2)^{2a_9}},$$

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$$G_{a_1 \dots a_9}^{N_{34}}(x, s, \epsilon) := e^{2\gamma_E \epsilon} \int \frac{d^d k_1}{i\pi^{d/2}} \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1} (k_1 + xp_1)^{2a_2} (k_1 + xp_{12})^{2a_3} (k_1 + p_{123})^{2a_4}} \\ \times \frac{1}{k_2^{2a_5} (k_2 - xp_1)^{2a_6} (k_2 - p_{123})^{2a_7} (k_1 + k_2 + xp_{12} - p_{123})^{2a_8} (k_1 + k_2)^{2a_9}}.$$

TWO-LOOP, FOUR-POINT, TWO OFF-SHELL LEGS

Non-planar topologies

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$N_{13} :$ {010000110, 000110010, 001000101, 001000110, 001010001, 010110100, 001110100, 001010102, 001110002, 000110110, 001010101, 001010110, 001100110, 001110001, 001110010, 010100110, 010110101, 002010111, 001120011, 001210110, 011010102, 001110120, 001010111, 001110210, 001110011, 001110101, 001110110, 002110110, 011000111, 011010101, 011100110, 011110001, 011110110, m11010111, 010110111, m01110111, 00111m111, 001110111, 011010111, 011110101, 011110111, m11110111},

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TWO-LOOP, FOUR-POINT, TWO OFF-SHELL LEGS

GP-indices

$$I(P_{12}) = \left\{ 0, 1, \frac{q}{s_{12}}, \frac{s_{12}}{q}, \frac{q}{q - s_{23}}, 1 - \frac{s_{23}}{q}, 1 + \frac{s_{23}}{s_{12}}, \frac{s_{12}}{s_{12} + s_{23}} \right\},$$

$$I(P_{13}) = \left\{ 0, 1, \frac{q}{s_{12}}, \frac{s_{12} + s_{23}}{s_{12}}, \frac{q}{q - s_{23}}, \xi_-, \xi_+, \frac{q(q - s_{23})}{q^2 - (q + s_{12})s_{23}} \right\},$$

$$I(P_{23}) = \left\{ 0, 1, \frac{q}{s_{12}}, 1 + \frac{s_{23}}{s_{12}}, \frac{q}{q - s_{23}}, \frac{q}{s_{12} + s_{23}}, \frac{q - s_{23}}{s_{12}} \right\},$$

$$\xi_{\pm} = \frac{qs_{12} \pm \sqrt{qs_{12}s_{23}(-q + s_{12} + s_{23})}}{qs_{12} - s_{12}s_{23}}.$$

$$I(N_{12}) = I(P_{23}),$$

$$I(N_{34}) = I(P_{12}) \cup I(P_{23}) \cup \left\{ \frac{s_{12}}{q - s_{23}}, \frac{s_{12} + s_{23}}{q}, \frac{q^2 - qs_{23} - s_{12}s_{23}}{s_{12}(q - s_{23})}, \frac{s_{12}^2 + qs_{23} + s_{12}s_{23}}{s_{12}(s_{12} + s_{23})} \right\},$$

$$I(N_{13}) = I(P_{23}) \cup \left\{ \xi_-, \xi_+, 1 + \frac{q}{s_{12}} + \frac{q}{-q + s_{23}} \right\}.$$

TWO-LOOP, FOUR-POINT, TWO OFF-SHELL LEGS

Example

$$\begin{aligned} G_{011111011}^{P13}(x, \{s_{ij}\}, \epsilon) = & \frac{A_3(\epsilon)}{x^2 s_{12}(-q + x(q - s_{23}))^2} \left\{ \frac{-1}{2\epsilon^4} + \frac{1}{\epsilon^3} \left(-GP\left(\frac{q}{s_{12}}; x\right) + 2GP\left(\frac{q}{q - s_{23}}; x\right) \right. \right. \\ & + 2GP(0; x) - GP(1; x) + \log(-s_{12}) + \frac{9}{4} \Big) + \frac{1}{4\epsilon^2} \left(18GP\left(\frac{q}{s_{12}}; x\right) - 36GP\left(\frac{q}{q - s_{23}}; x\right) \right. \\ & - 8GP\left(0, \frac{q}{s_{12}}; x\right) + 16GP\left(0, \frac{q}{q - s_{23}}; x\right) + 8GP\left(\frac{s_{23}}{s_{12}} + 1, \frac{q}{q - s_{23}}; x\right) + \dots \Big) \\ & + \frac{1}{\epsilon} \left(9 \left(GP\left(0, \frac{q}{s_{12}}; x\right) + GP(0, 1; x) \right) - 4 \left(GP\left(0, 0, \frac{q}{s_{12}}; x\right) + GP(0, 0, 1; x) \right) + \dots \right) \\ & \left. \left. + 6 \left(GP(0, 0, 1, \xi_-; x) + GP(0, 0, 1, \xi_+; x) \right) - 2GP\left(0, 0, \frac{q}{q - s_{23}}, \frac{q(q - s_{23})}{q^2 - s_{23}(q + s_{12})}; x\right) + \dots \right) \right\}. \end{aligned}$$

$$A_3(\epsilon) = -e^{2\gamma_E \epsilon} \frac{\Gamma(1 - \epsilon)^3 \Gamma(1 + 2\epsilon)}{\Gamma(3 - 3\epsilon)}.$$

C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP 1501 (2015) 072

5BOX - ONE LEG OFF-SHELL: ALL FAMILIES

C. G. Papadopoulos, D. Tommasini and C. Wever, arXiv:1511.09404 [hep-ph].

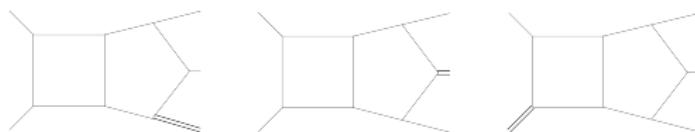


FIGURE : The three planar pentaboxes of the families P_1 (left), P_2 (middle) and P_3 (right) with one external massive leg.

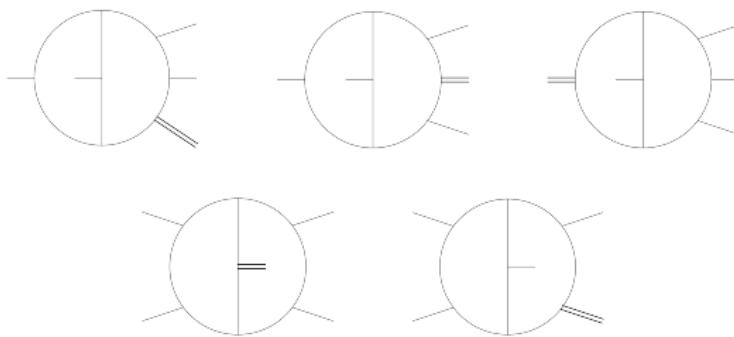


FIGURE : The five non-planar families with one external massive leg.

5BOX - ONE LEG OFF-SHELL: P1

$$p(q_1)p'(q_2) \rightarrow V(q_3)j_1(q_4)j_2(q_5), \quad q_1^2 = q_2^2 = 0, \quad q_3^2 = M_3^2, \quad q_4^2 = q_5^2 = 0.$$

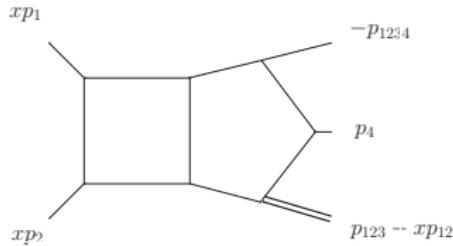


FIGURE : The parametrization of external momenta in terms of x for the planar pentabox of the family P_1 . All external momenta are incoming.

$$s_{12} := p_{12}^2, \quad s_{23} := p_{23}^2, \quad s_{34} := p_{34}^2, \quad s_{45} := p_{45}^2 = p_{123}^2, \quad s_{51} := p_{15}^2 = p_{234}^2,$$

$$q_1^2 = q_2^2 = q_4^2 = q_5^2 = 0 \quad q_3^2 = (s_{45} - s_{12}x)(1-x) \\ q_{12}^2 = s_{12}x^2 \quad q_{23}^2 = s_{45}(1-x) + s_{23}x \quad q_{34}^2 = (s_{34} - s_{12}(1-x))x \quad q_{45}^2 = s_{45} \quad q_{51}^2 = s_{51}x$$

5BOX - ONE LEG OFF-SHELL: P1

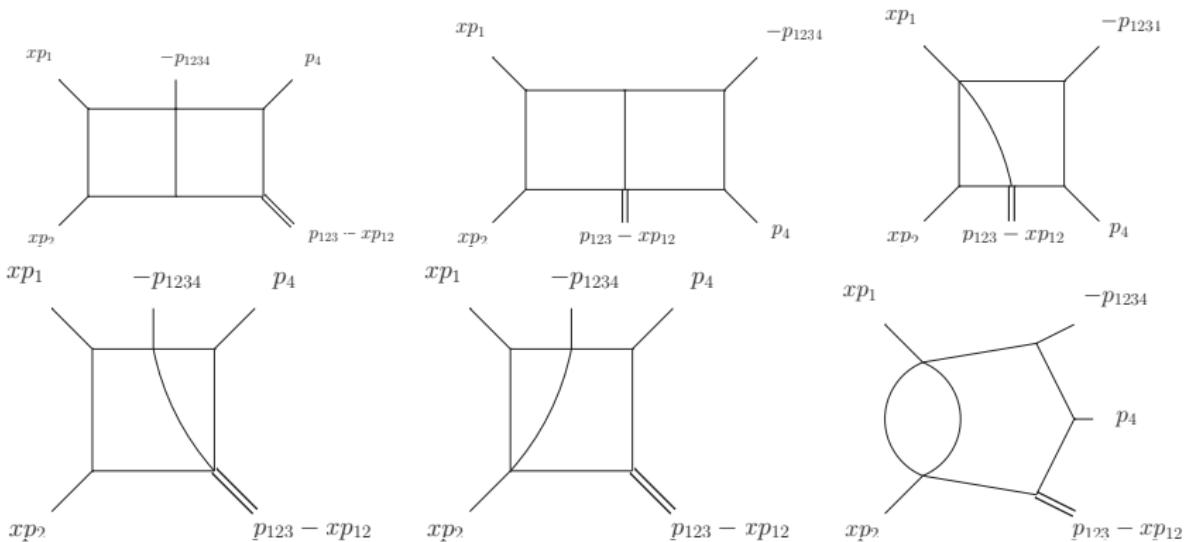


FIGURE : The five-point Feynman diagrams, besides the pentabox itself in Figure 4, that are contained in the family P_1 . All external momenta are incoming.

5BOX - ONE LEG OFF-SHELL: P1

$$G_{a_1 \dots a_{11}}^{P_1}(x, s, \epsilon) := e^{2\gamma_E \epsilon} \int \frac{d^d k_1}{i\pi^{d/2}} \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1} (k_1 + xp_1)^{2a_2} (k_1 + xp_{12})^{2a_3} (k_1 + p_{123})^{2a_4}} \\ \times \frac{1}{(k_1 + p_{1234})^{2a_5} k_2^{2a_6} (k_2 - xp_1)^{2a_7} (k_2 - xp_{12})^{2a_8} (k_2 - p_{123})^{2a_9} (k_2 - p_{1234})^{2a_{10}} (k_1 + k_2)^{2a_{11}}},$$

$P_1 :$ {10000000101, 01000000101, 00100000101, 10000001001, 01000000011, 00100000011, 10100001100, 10100001010, 10100101000, 01000101001, 10100100100, 10100000102, 10100000101, 10100000011, 10000001102, 10000001101, 10000001011, 01000100101, 01000001101, 01000001011, 00100100102, 00100100101, 11100000101, 11100000011, 11000001102, 11000001101, 11000001012, 11000001011, 11000000111, 10100000112, 10000001111, 01100100102, 01100100101, 01100100011, 01100000111, 01000101102, 01000101101, 01000101011, 01000100111, 01000001111, 00100100111, 10100101100, 10100100101, 10100001101, 10100001011, 10100000111, 111m0000111, 110000m1111, 11000001111, 10100101110, 10100100111, 10100001111, 011001m0111, 01100100111, 010m0101111, 01000101111, 11100100101, 11100001101, 11100001011, 11100000111, 111m0101101, 111001m1101, 11100101101, 1110m1010111, 11100101011, 111m0100111, 11100100111, 111000m1111, 111m0001111, 11100001111, 111001m0111, 11100101111, 111001m1111, 111m0101111},

Choosing m= -1 or 2

5BOX P1 - DE

$$\partial_x \mathbf{G} = \mathbf{M} (\{s_{ij}\}, \varepsilon, x) \mathbf{G}$$

$$\partial_x \mathbf{G}' = \mathbf{M}' \mathbf{G}' \quad \mathbf{M}' = \mathbf{T} \mathbf{M} \mathbf{T}^{-1} + (\partial_x \mathbf{T}) \mathbf{T}^{-1} \quad \mathbf{G}' = \mathbf{T} \mathbf{G}$$

$$(M_D)_{IJ} = \delta_{IJ} M_{II} (\varepsilon = 0), I, J = 1 \dots 74$$

$\mathbf{G} \rightarrow \mathbf{S}^{-1} \mathbf{G}$, $\mathbf{S} = \exp(\int dx \mathbf{M}_D)$ and $\mathbf{M} \rightarrow \mathbf{S}^{-1} (\mathbf{M} - \mathbf{M}_D) \mathbf{S}$.

$$M_{IJ} = N_{IJ}(\varepsilon) \left(\sum_{i=1}^{20} \sum_{j=1}^2 \sum_{k=0}^1 \frac{C_{IJ;ijk} \varepsilon^k}{(x - l_i)^j} + \sum_{j=0}^1 \sum_{k=0}^1 \tilde{C}_{IJ;jk} \varepsilon^k x^j \right).$$

Letters (20):

$$\begin{aligned}
& 0, \quad 1, \quad \frac{s_{45}}{s_{45}-s_{23}}, \quad \frac{s_{45}}{s_{12}}, \quad 1 - \frac{s_{34}}{s_{12}}, \quad 1 + \frac{s_{23}}{s_{12}}, \\
& 1 - \frac{s_{34}-s_{51}}{s_{12}}, \quad \frac{s_{45}-s_{23}}{s_{12}}, \quad -\frac{s_{51}}{s_{12}}, \quad \frac{s_{45}}{-s_{23}+s_{45}+s_{51}}, \quad \frac{s_{45}}{s_{34}+s_{45}}, \\
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& \frac{s_{12}s_{23}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51} \pm \sqrt{\Delta_1}}{2s_{12}(s_{23}+s_{34}-s_{51})}, \quad \frac{s_{12}s_{45} \pm \sqrt{\Delta_3}}{s_{12}s_{34}+s_{12}s_{45}}, \quad \frac{s_{45}}{s_{12}+s_{23}},
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$$\int_0^x dt \frac{1}{(t - a_n)^2} \mathcal{G}(a_{n-1}, \dots, a_1, t) \quad \quad \quad \int_0^x dt \ t^m \ \mathcal{G}(a_{n-1}, \dots, a_1, t)$$

Fuchsian

$$N_{IJ}(\varepsilon) = n_J(\varepsilon) / n_I(\varepsilon), \quad G_I \rightarrow n_I(\varepsilon) \ G_I$$

$$M_{IJ} = \left(\sum_{i=1}^{20} \sum_{j=1}^2 \sum_{k=0}^1 \frac{C_{IJ;ijk}\varepsilon^k}{(x - l_i)^j} + \sum_{j=0}^1 \sum_{k=0}^1 \tilde{C}_{IJ;jk}\varepsilon^k x^j \right).$$

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$$\partial_x \mathbf{G} = \left(\varepsilon \sum_{a=1}^{19} \frac{\mathbf{M}_a}{(x - l_a)} \right) \mathbf{G}$$

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$\mathbf{M}(\varepsilon = 0)$ contains $(x - l_i)^{-2}$ and x^0

$$(\mathbf{K}_1)_{IJ} = \begin{cases} \int dx (\mathbf{M}(\varepsilon = 0))_{IJ} & I, J \neq 69, 74 \\ 0 & I, J = 69, 74 \end{cases}$$

$$(\mathbf{K}_2)_{IJ} = \begin{cases} \int dx (\mathbf{M}(\varepsilon = 0))_{IJ} & I, J \neq 74 \\ 0 & I, J = 74 \end{cases}$$

$$(\mathbf{K}_3)_{IJ} = \int dx (\mathbf{M}(\varepsilon = 0))_{IJ}$$

M.A. Barkatou and E.Pflügel, Journal of Symbolic Computation, 44 (2009), 1017

$$\partial_x \mathbf{G} = \left(\varepsilon \sum_{a=1}^{19} \frac{\mathbf{M}_a}{(x - l_a)} \right) \mathbf{G}$$

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Fuchsian

$$N_{IJ}(\varepsilon) = n_J(\varepsilon)/n_I(\varepsilon), \quad G_I \rightarrow n_I(\varepsilon) G_I$$

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5BOX P1 - SOLUTION

- Solution:

$$\begin{aligned}\mathbf{G} &= \varepsilon^{-2} \mathbf{b}_0^{(-2)} + \varepsilon^{-1} \left(\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{b}_0^{(-2)} + \mathbf{b}_0^{(-1)} \right) \\ &+ \varepsilon^0 \left(\sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{b}_0^{(-1)} + \mathbf{b}_0^{(0)} \right) \\ &+ \varepsilon \left(\sum \mathcal{G}_{abcd} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{M}_d \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-1)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)} \right) \\ &+ \varepsilon^2 \left(\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(0)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)} \right)\end{aligned}$$

$\mathbf{b}_0^{(k)}$, $k = -2, \dots, 2$ representing the x -independent boundary terms in the limit $x = 0$ at order ε^k

$$\mathbf{G} \underset{x \rightarrow 0}{\sim} \sum_{k=-2}^2 \varepsilon^k \sum_{n=0}^{k+2} \mathbf{b}_n^{(k)} \log^n(x) + \text{subleading terms.}$$

$\mathcal{G}_{a,b,\dots} = \mathcal{G}(l_a, l_b, \dots; x)$ with $a, b, c, d = 1, \dots, 19$.

- Uniform transcendental: UT multi- vs one-parameter DE

\mathbf{M}_a depend on kinematics, but eigenvalues not: $(x - l_a)^{-n_a \varepsilon}$, n_a positive integers, $x \rightarrow l_a$.

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5BOX - BOUNDARY TERMS

- Resummed

$$G_{res} = \lim_{x \rightarrow 0} G = \sum_j c_j x^{i_0 + j\epsilon} + d_j x^{i_0 + 1 + j\epsilon} + \mathcal{O}(x^{i_0 + 2}),$$

- DE: using the above and equating terms $x^{i+j\epsilon}$, linear equations for c_i and d_i
- *bottom-up*: MI with homogeneous DE treated exactly
- MI needing special treatment (20)
 - Expansion by regions (11)
 - Shifted boundary point (6)
 - Extraction from known integrals (3)

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$$\{(10100000101), (10100000102), (11000001012), (11000001011), (01000101011), (10100100111), \\ (10100001111), (111m0100111), (111000m1111), (11100001111), (111001m0111)\}.$$

- Shifted boundary point (6)

$$\infty : \quad \{(10100000011), (10000001011), (11100000011), (01100100011), (10100100111)\} \\ (s_{12} - s_{34} + s_{51})/s_{12} : \quad \{(01000001011)\}$$

- Extraction from known integrals (3)

$$\begin{aligned} G_{11100001011}(x, s_{12}, s_{34}, s_{51}) &= G_{11100100101}(x' = 1, s'_{12}, s'_{23}, s'_{45}), \\ G_{11100101011}(x, s_{12}, s_{34}, s_{51}) &= G_{11100101101}(x' = 1, s'_{12}, s'_{23}, s'_{45}), \\ G_{111m0101011}(x, s_{12}, s_{34}, s_{51}) &= G_{111m0101101}(x' = 1, s'_{12}, s'_{23}, s'_{45}), \\ s'_{12} &= x^2 s_{12}, \quad s'_{23} = x s_{51}, \quad s'_{45} = -x s_{12} + x s_{34} + x^2 s_{12}. \end{aligned} \tag{1}$$

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- Resummed

$$G_{res} = \lim_{x \rightarrow 0} G = \sum_j c_j x^{i_0 + j\epsilon} + d_j x^{i_0 + 1 + j\epsilon} + \mathcal{O}(x^{i_0 + 2}),$$

- DE: using the above and equating terms $x^{i+j\epsilon}$, linear equations for c_i and d_i
- *bottom-up*: MI with homogeneous DE treated exactly
- MI needing special treatment (20)
 - Expansion by regions (11)
 - Shifted boundary point (6)
 - Extraction from known integrals (3)

Systematic approach: combining information from the expansion by regions technique (asy2) and the DE itself

Mellin-Barnes, XSummer

5BOX - ON-SHELL

All planar one-shell 5box by taking the limit $x \rightarrow 1$.

- $x = 1$ corresponds to I_2

$$\mathbf{G} = \sum_{n \geq -2} \varepsilon^n \sum_{i=0}^{n+2} \frac{1}{i!} \mathbf{c}_i^{(n)} \log^i (1-x)$$

- with \mathbf{M}_2 the residue matrix at $x = 1$ and
- $\mathbf{G}_{trunc} \equiv \mathbf{G}_{reg}(x = 1)$

$$\mathbf{G}_{x=1} = \left(\mathbf{I} + \frac{3}{2} \mathbf{M}_2 + \frac{1}{2} \mathbf{M}_2^2 \right) \mathbf{G}_{trunc}$$

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$$\mathbf{G}_{reg} = \sum_{n \geq -2} \varepsilon^n \mathbf{c}_0^{(n)}.$$

characteristic polynomial: $x^{61}(1+x)^9(2+x)^4$

$$\mathbf{G} = \mathbf{G}_{reg} + \frac{\left((1-x)^{-2\varepsilon} - 1\right)}{(-2\varepsilon)} \mathbf{X} + \frac{\left((1-x)^{-\varepsilon} - 1\right)}{(-\varepsilon)} \mathbf{Y}$$

$$\mathbf{X} = \sum_{n \geq -1} \varepsilon^n \mathbf{X}^{(n)} \quad \mathbf{Y} = \sum_{n \geq -1} \varepsilon^n \mathbf{Y}^{(n)}.$$

$$(-1)^n \mathbf{M}_2^n = \mathbf{M}_2^2 (2^{n-1} - 1) + \mathbf{M}_2 (2^{n-1} - 2), \quad n \geq 1.$$

minimal polynomial: $x(x+1)(x+2)$

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- $\mathcal{O}(3,000)$ GPs for all 74 MI
- Directly computed by using **GiNaC**
- All invariants negative (Euclidean): perfect agreement with SecDec
- $\mathcal{O}(10)$ secs.
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[J. Vollinga and S. Weinzierl, Comput. Phys. Commun. 167 \(2005\) 177](#)

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E. Panzer, Comput. Phys. Commun. 188 (2014) 148

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- ➋ IBP: better understanding
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- ➍ Complete massless MI with up to 8 denominators, at least 3 of-shell legs
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- ➏ Feynman parametrization - MB vs DE: pros and cons
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O. Gituliar and V. Magerya, arXiv:1607.00759 [hep-ph].

Christoph Meyer, Loops and Legs 2016

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- ① Understanding QFT and provide precise calculations for analysis of experimental data
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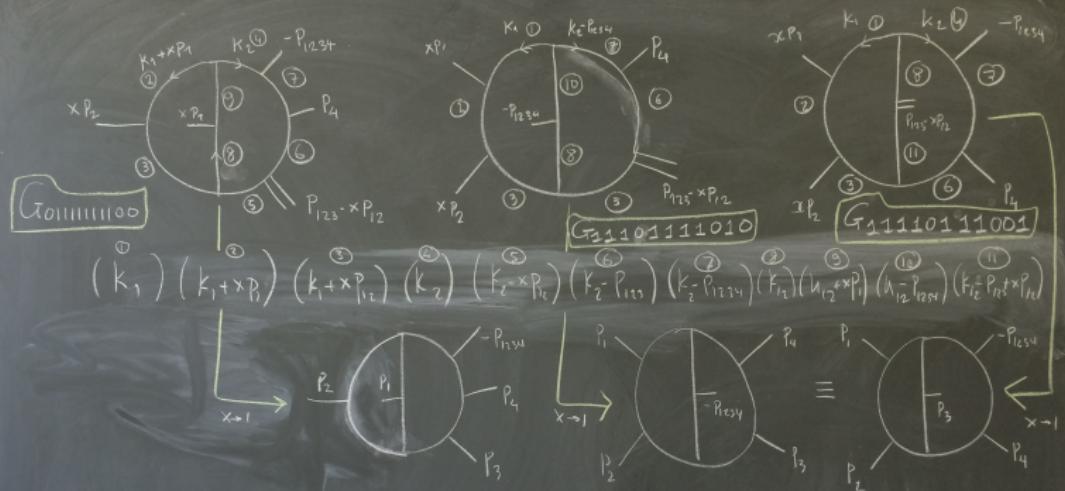
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Backup Slides

TWO-LOOP, FOUR-POINT, TWO OFF-SHELL LEGS

Physical region

$$S > \left(\sqrt{M_3^2} + \sqrt{M_4^2} \right)^2, \quad T < 0, \quad U < 0,$$

$$M_3^2 > 0, \quad M_4^2 > 0, \quad q_\perp^2 = \frac{TU - M_3^2 M_4^2}{S} > 0,$$

$$x > 1, \quad \frac{q - s_{12}}{s_{23}} > 1, \quad xs_{12} > q, \quad q > 0.$$

$$x > 1, \quad \begin{cases} s_{23} < 0, & s_{12} + s_{23} > q, \quad q > 0 \\ s_{23} > 0, & s_{12} + s_{23} < q, \quad s_{12} > q/x. \end{cases}$$

TWO-LOOP, FOUR-POINT, TWO OFF-SHELL LEGS

Analytic continuation

- Feynman propagator

$$D \rightarrow D + i\epsilon$$

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s_{ij} (s_{12} , s_{23} and q in the present study) and the parameter x ,
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$$G_{001000011}^{P12} \sim (-(-1+x)(-q+s_{12}x))^{1-2\epsilon} \sim (1-x)^{1-2\epsilon} (1 - xs_{12}/q)^{1-2\epsilon} (-q)^{1-2\epsilon}$$

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