TMD PDFs from Monte Carlo evolution

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Motivation- why TMDs?

Why Transverse Momentum Dependent PDFs?

Goal: **full TMD PDFs**

What is Transverse Momentum Dependent (TMD) PDF?

- TMD PDF is a generalization of concept of PDF.
- TMD: depends not only on x and Q^2 but also on k_T: $TMD(x, Q^2, k_T)$

TDMs are important in studies on:

- resummation at all orders in the QCD coupling to many observables in high-energy hadronic collisions,
- nonperturbative information on hadron structure at very low k_T,
- perturbative region where QCD evolution equations (DGLAP, BFKL, CCFM) describe processes
- a proper and consistent simulation of parton showers,
- multi-scale problems in hadronic collisions,
- ...

Acta Physica Polonica B, Vol. 46 (2015), Transverse Momentum Dependent (TMD) Parton Distribution Functions: Status and Prospects

Important processes: Drell-Yan hadroproduction of electroweak gauge bosons, Higgs production...

Introduction to the method- Sudakov formalism

Sudakov formalism

DGLAP evolution equation for momentum weighted parton density $xf(x,\mu^2) = \widetilde{f}(x,\mu^2)$

$$\frac{d\widetilde{f}_a(x,\mu^2)}{d\ln\mu^2} = \sum_b \int_x^1 dz P_{ab}\left(\alpha_s(\mu^2), z\right) \widetilde{f}_b\left(\frac{x}{z},\mu^2\right) \qquad (1)$$

a, b- quark ($2N_f$ flavours) or gluon, x- longitudinal momentum fraction of the proton carried by a parton a,
$z = \frac{x_i}{x_{i-1}}$- splitting variable, μ- evolution mass scale and
a structure of a splitting function:

$$P_{ab}\left(\alpha_s(\mu^2), z\right) = D_{ab}\left(\alpha_s(\mu^2)\right)\delta(1-z) + K_{ab}\left(\alpha_s(\mu^2)\right)\frac{1}{(1-z)_+} + R_{ab}\left(\alpha_s(\mu^2), z\right), \qquad (2)$$

$\int_0^1 f(x)g(x)_+ dx = \int_0^1 f(x)g(x)dx - \int_0^1 f(1)g(x)dx$
$D_{ab}\left(\alpha_s(\mu^2)\right) = \delta_{ab} d_a\left(\alpha_s(\mu^2)\right)$, $K_{ab}\left(\alpha_s(\mu^2)\right) = \delta_{ab} k_a\left(\alpha_s(\mu^2)\right)$,
$R_{ab}\left(\alpha_s(\mu^2), z\right)$ contains logarithmic terms in $\ln(1-z)$ and has no power divergences $(1-z)^{-n}$ for $z \to 1$.

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$$\frac{d\widetilde{f}_a(x,\mu^2)}{d\ln\mu^2}=\sum_b\int_x^1 dz\left(K_{ab}\left(\alpha_s(\mu^2)\right)\frac{1}{(1-z)}+R_{ab}\left(\alpha_s(\mu^2),z\right)\right)\widetilde{f}_b\left(\frac{x}{z},\mu^2\right)+$$
$$-\sum_b\widetilde{f}_b\left(x,\mu^2\right)\int_0^1 dz\left(K_{ab}\left(\alpha_s(\mu^2)\right)\frac{1}{(1-z)}-D_{ab}\left(\alpha_s(\mu^2)\right)\delta(1-z)\right) \tag{3}$$

Sudakov formalism

$$\frac{d\widetilde{f}_a(x,\mu^2)}{d\ln\mu^2} = \sum_b \int_x^1 dz \left(K_{ab}\left(\alpha_s(\mu^2)\right) \frac{1}{(1-z)} + R_{ab}\left(\alpha_s(\mu^2), z\right) \right) \widetilde{f}_b\left(\frac{x}{z}, \mu^2\right) + \\ - \sum_b \widetilde{f}_b\left(x,\mu^2\right) \int_0^1 dz \left(K_{ab}\left(\alpha_s(\mu^2)\right) \frac{1}{(1-z)} - D_{ab}\left(\alpha_s(\mu^2)\right) \delta(1-z) \right)$$

Defining the *Sudakov form factor*:

$$\Delta_a(\mu^2) = \exp\left(- \int_{\ln\mu_0^2}^{\ln\mu^2} d\left(\ln\mu'^2\right) \sum_b \int_0^1 dz z P^R_{ba}\left(\alpha_s(\mu'^2), z\right) \right) \tag{4}$$

with *momentum sum rule* $\sum_c \int_0^1 \mathrm{d}z z P_{ca}\left(\alpha_s(\mu^2), z\right) = 0$:

$$\frac{d\widetilde{f}_a(x,\mu^2)}{d\ln\mu^2} = \sum_b \int_x^1 dz P^R_{ab}\left(\alpha_s(\mu^2), z\right) \widetilde{f}_b\left(\frac{x}{z}, \mu^2\right) + \widetilde{f}_a\left(x,\mu^2\right) \frac{1}{\Delta_a(\mu^2)} \frac{d\Delta_a(\mu^2)}{d\ln\mu^2}, \tag{5}$$

$P^R_{ab}\left(\alpha_s(\mu^2), z\right) = R_{ab}\left(\alpha_s(\mu^2), z\right) + K_{ab}\left(\alpha_s(\mu^2)\right) \frac{1}{1-z}$ - *real* part of the splitting function.

Sudakov formalism

After integration:

$$\widetilde{f}_a(x,\mu^2) = \widetilde{f}_a(x,\mu_0^2)\Delta_a(\mu^2) + \int_{\ln\mu_0^2}^{\ln\mu^2} d\ln\mu^{2\prime}\,\frac{\Delta_a(\mu^2)}{\Delta_a(\mu^{2\prime})}\sum_b \int_x^1 dz P_{ab}^R\left(\alpha_s(\mu^{2\prime}),z\right)\widetilde{f}_b\left(\frac{x}{z},\mu^{2\prime}\right). \quad (6)$$

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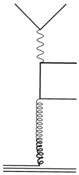
Sudakov: probability of evolving from μ_0^2 to μ^2 without any resolvable branching.



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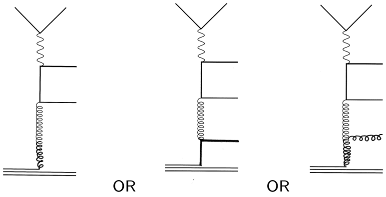

Figure : Example for a= gluon.

Sudakov formalism

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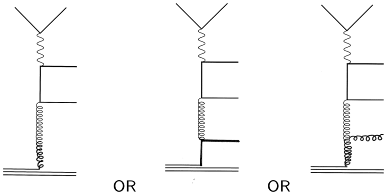

Figure : Example for a= gluon.

but! $\widetilde{f}_b\left(\frac{x}{z},\mu^{2\prime}\right)$ has it's own evolution history!

$$\widetilde{f}_b\left(\frac{x}{z},\mu^{2\prime}\right) = \widetilde{f}_b\left(\frac{x}{z},\mu_0^2\right)\Delta_b(\mu^{2\prime}) + \int_{\ln\mu_0^2}^{\ln\mu^{2\prime}} d\ln\mu^{2\prime\prime}\,\frac{\Delta_a(\mu^{2\prime})}{\Delta_a(\mu^{2\prime\prime})}\sum_c\int_x^1 dz' P_{bc}^R\left(\alpha_s(\mu^{2\prime\prime}),z'\right)\widetilde{f}_c\left(\frac{x}{zz'},\mu^{2\prime\prime}\right) \quad (7)$$

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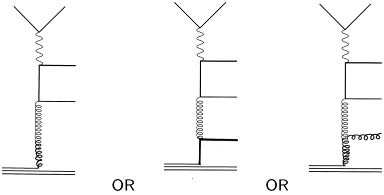

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This equation has an iterative solution which can be easily implemented in the MC code.

Results

MC code

In this presentation: MC results obtained with updated and improved ▸ uPDFevolv code:

- full coupled quark and gluon DGLAP evolution (gluon, sea and valence evolution),
- fixed flavour number scheme,
- LO in $P(z)$,
- 1-loop-α_s (but also 2-loop-α_s implemented),
- $xf(x,t)$,
- the initial distributions for $u, d, s, ..., \overline{u}, \overline{d}, \overline{s}, ...$ and gluon come from QCDNum17 (but any other parametrization can be used).

Evolution over the whole range in x, Q^2 and all kinematically allowed k_T.

https://updfevolv.hepforge.org/

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Advantage of updfevolv:

- the structure of the code suitable for usage in xFitter (to have full TMDs): structure of grids → fitting method fast.

Avoiding divergences in $P(z)$ at $z \to 1$

Some of the splitting functions are divergent for $z \to 1$.

To avoid divergences:

$$\frac{d\widetilde{f}_a(x,\mu^2)}{d\ln\mu^2} = \sum_b \int_x^{1} dz P^R_{ab}\left(\alpha_s(\mu^2),z\right)\widetilde{f}_b\left(\frac{x}{z},\mu^2\right) + \widetilde{f}_a\left(x,\mu^2\right)\frac{1}{\Delta_a(\mu^2)}\frac{d\Delta_a(\mu^2)}{d\ln\mu^2} \approx$$

$$\approx \sum_b \int_x^{z_{max}} dz P^R_{ab}\left(\alpha_s(\mu^2),z\right)\widetilde{f}_b\left(\frac{x}{z},\mu^2\right) + \widetilde{f}_a\left(x,\mu^2\right)\frac{1}{\Delta_a(\mu^2)}\frac{d\Delta_a(\mu^2)}{d\ln\mu^2} \quad (8)$$

and the same cut off in *Sudakov form factor* $\Delta_a(\mu^2)$ and $\Delta_a(\mu^{2\prime})$.

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it can be shown that terms $\int_{zmax}^1$ skipped in the integral in eq. (8) are of order $\mathcal{O}(1 - z_{max})$

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Different choices of z_{max}:

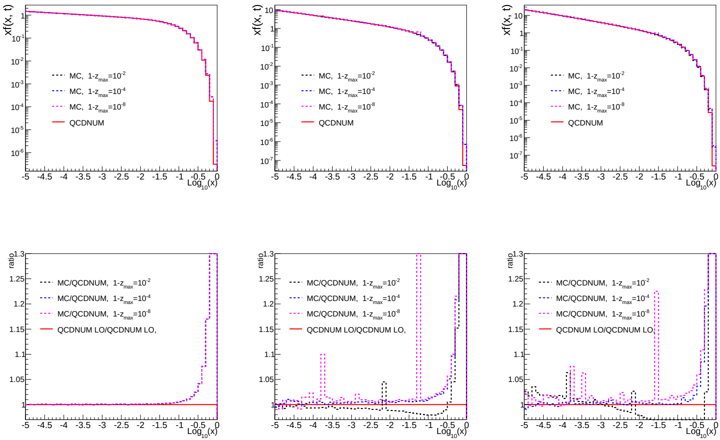
- z_{max} - fixed
- z_{max} - can change dynamically with the scale, for example: angular ordering: $z_{max} = 1 - \left(\frac{Q_0}{Q}\right)$

In this presentation: results from fixed z_{max}.

sea quarks

QCDNum, $1 - z_{max} = 10^{-2}$, $1 - z_{max} = 10^{-4}$, $1 - z_{max} = 10^{-8}$,

sea at μ^2 = 2 GeV^2 **sea at μ^2 = 1000 GeV^2** **sea at μ^2 = 100000 GeV^2**

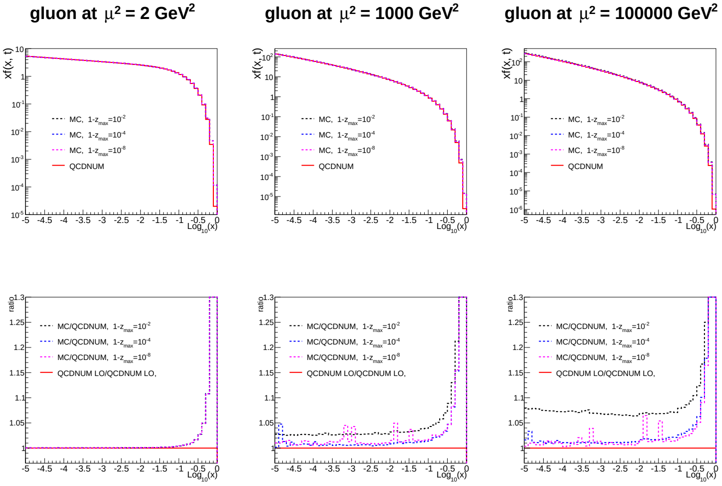


MC results close to the QCDNum results for values of z_{max} large enough $\rightarrow$ Sudakov formalism treats non-resorvable and virtual branchings to all orders.

The differences between MC and QCDNum at large x are an artefact of the histogram binning.

gluon

QCDNum, $1 - z_{max} = 10^{-2}$, $1 - z_{max} = 10^{-4}$, $1 - z_{max} = 10^{-8}$,



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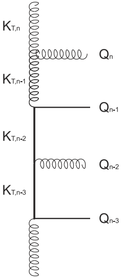
k_T dependence

MC method: for every branching Q is generated and Q_x and Q_y are calculated
$\rightarrow$ The information about k_T is available for every branching.



two ways of defining k_T :

Definition 1: $\vec{k}_{T,n} = \vec{k}_{T,n-1} + \vec{Q}_{T,n}$.

or :

Definition 2: $\vec{k}_{T,n} = \vec{k}_{T,n-1} + (1-z)\vec{Q}_{T,n}$ - angular ordering.

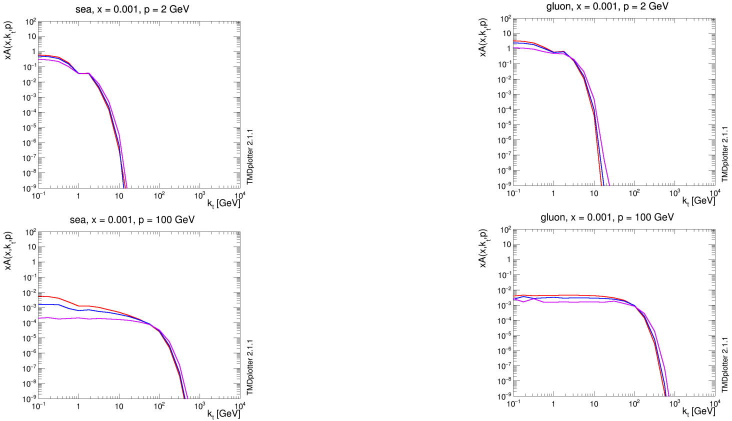
k_T contains the whole history of the evolution.

In this method k_T is treated properly from the beginning of the evolution- no extra reshuffling at the end is required.

TMD PDFs for Definition 1 for different z_{max} values

Definition 1: $\overrightarrow{k}_{T,n} = \overrightarrow{k}_{T,n-1} + \overrightarrow{Q}_{T,n}$.

$1 - z_{max} = 10^{-4}$, $1 - z_{max} = 10^{-5}$, $1 - z_{max} = 10^{-8}$

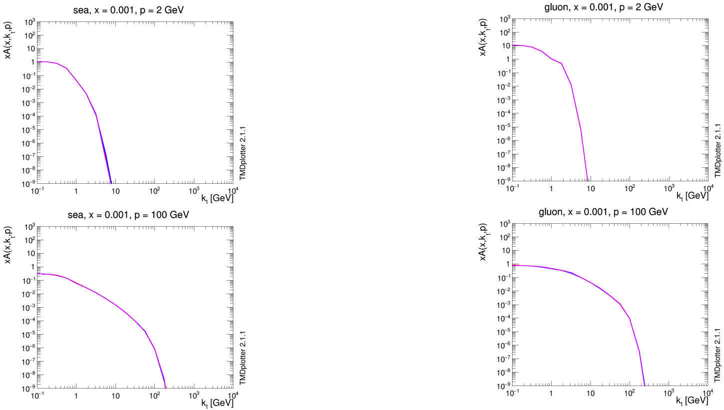


We observe k_T tails, which can be larger than the evolution scale.
Different z_{max} values give different large k_T tails: the bigger the value of z_{max} the larger the k_T tail:
Larger z_{max} → Sudakov form factor smaller → the probability of evolving without any resolvable branching smaller → more branchings (soft gluons!), larger k_T accumulated during the evolution process.

TMD PDFs for angular ordering for different z_{max} values

Definition 2: $\vec{k}_{T,n} = \vec{k}_{T,n-1} + (1-z)\vec{Q}_{T,n}$- angular ordering.

$1 - z_{max} = 10^{-4}$, $1 - z_{max} = 10^{-5}$, $1 - z_{max} = 10^{-8}$



We observe k_T tails, which can be larger than the evolution scale.

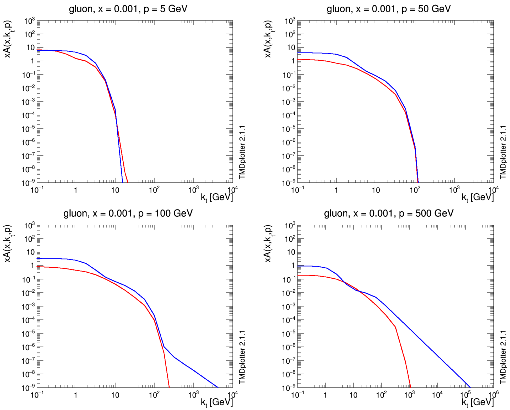
Different z_{max} values give the same large k_T tails!

When z_{max} large $1 - z_{max}$ small so the contribution from large z suppressed- soft gluons suppressed by angular ordering!

Comparison of different TMD sets

Definition 2: $\overrightarrow{k}_{T,n} = \overrightarrow{k}_{T,n-1} + (1-z)\overrightarrow{Q}_{T,n}$- angular ordering.

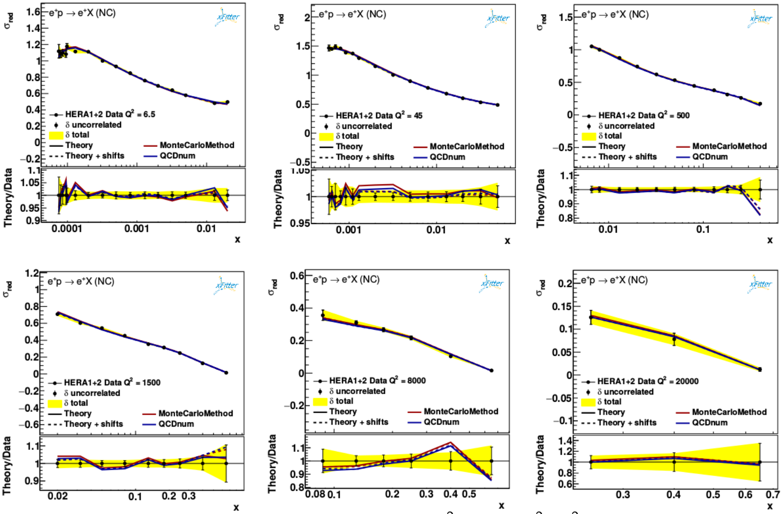
updfevolv for DGLAP with angular ordering, $1 - z_{max} = 10^{-4}$, ccfm-JH-2013-set2



ccfm-JH-2013-set2 for small scales gives very simillar results to DGLAP with angular ordering.

First fit of full integrated TMDs to HERA H1 and Zeus data

Integrated TMDs for gluon, valence and sea from MC code were used in xFitter to fit F_2.
QCDNum convolution of integrated TMDs with collinear ME was used to obtain the structure function.



Fits work reasonably well for the whole x range and $Q^2 > 5\text{GeV}^2$ ($\chi^2/ndf \approx 1$).

Summary

Summary

New approach to solve coupled gluon and quark DGLAP evolution equation with MC method was shown.

Advantages:

- a full TMD pdf evolution including gluon, sea and valence quarks over the full range in x and Q^2 with the k_T dependence in the whole kinematically available range (not limited to the small k_T),
- reproduce semi-analytical solution (results consistent with QCDNum),
- direct usage in PS matched calculation.

TMDs are implemented in the preliminary version of xFitter.

New results of fitting integrated TMD pdfs to F_2 with xFitter were shown:
gluon and quark are fitted for $Q^2 > 5\mathrm{GeV}^2$ for all x with $\chi^2/ndf \approx 1$

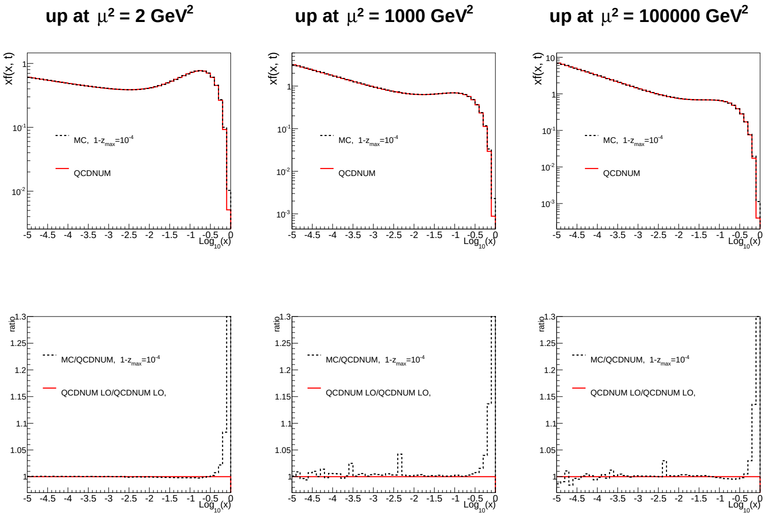
Thank you!

Back up

Results for integrated TMDs

up

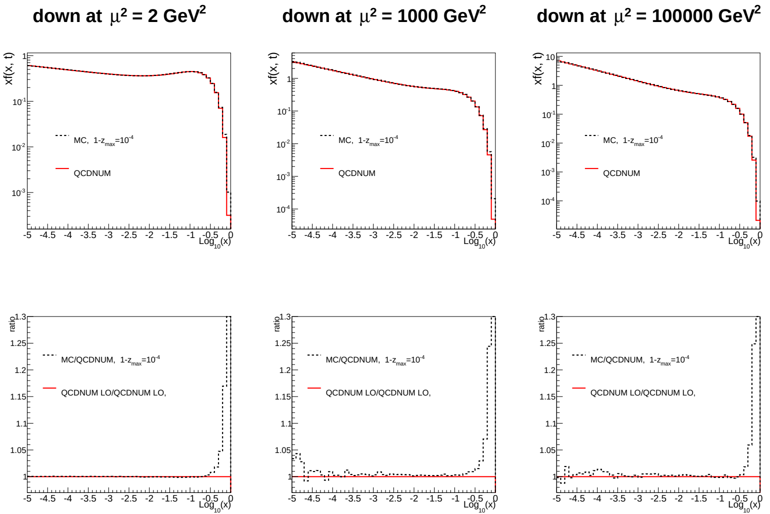
QCDNum, 1-zmax=10^{-4}



MC results close to the QCDNum results.
The differences between MC and QCDNum at large x are an artefact of the histogram binning.

down

QCDNum, 1-zmax=10^{-4}



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MC solution - forward evolution

Goal: to solve DGLAP with MC method

$$\widetilde{f}_a(x,\mu^2) = \widetilde{f}_a(x,\mu_0^2)\Delta_a(\mu^2) + \int dx_{i-1} \int_{\ln\mu_0^2}^{\ln\mu^2} d\ln\mu^{2\prime} \frac{\Delta_a(\mu^2)}{\Delta_a(\mu^{2\prime})} \sum_b \int_x^1 dz z P_{ab}^R\left(\alpha_s(\mu^{2\prime}), z\right) \widetilde{f}_b\left(x_{i-1},\mu_0^2\right)\Delta_b(\mu^{2\prime})\delta(zx_{i-1}-x) + \ldots$$

Forward evolution: at the beginning of the evolution we don't know what we will have at the end $\longrightarrow$ is a a gluon or a quark?

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Forward evolution: at the beginning of the evolution we don't know what we will have at the end $\rightarrow$ is a a gluon or a quark?

Example:
Initial parton is a gluon: b =gluon :
$\rightarrow \tilde{f}_b\left(x_{i-1},\mu_0^2\right) = \tilde{g}\left(x_{i-1},\mu_0^2\right)$. and we don't have $\sum_b$:

$$\tilde{f}_a(x,\mu^2) = \tilde{f}_a(x,\mu_0^2)\Delta_a(\mu^2) + \int dx_{i-1} \int_{\ln\mu_0^2}^{\ln\mu^2} d\ln\mu^{2\prime} \frac{\Delta_a(\mu^2)}{\Delta_a(\mu^{2\prime})} \int_x^1 dz z P_{ag}^R\left(\alpha_s(\mu^{2\prime}),z\right) \tilde{g}\left(x_{i-1},\mu_0^2\right)\Delta_g(\mu^{2\prime})\delta(zx_{i-1}-x) + \ldots$$

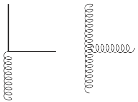
MC solution - forward evolution

Goal: to solve with MC method

$$\widetilde{f}_a(x,\mu^2) = \widetilde{f}_a(x,\mu_0^2)\Delta_a(\mu^2) + \int dx_{i-1} \int_{\ln \mu_0^2}^{\ln \mu^2} d\ln \mu^{2\prime} \frac{\Delta_a(\mu^2)}{\Delta_a(\mu^{2\prime})} \int_x^1 dz z P_{ag}^R\left(\alpha_s(\mu^{2\prime}), z\right) \widetilde{g}\left(x_{i-1}, \mu_0^2\right)\Delta_g(\mu^{2\prime})\delta(zx_{i-1} - x) + \ldots.$$

Forward evolution: at the beginning of the evolution we don't know what we will have at the end $\rightarrow$ is a a gluon or a quark?

- Generate $\mu^{2\prime}$ according to $\Delta_g(\mu^{2\prime}) = R_1$
  where R_1- random number in the interval $(0, 1)$.
  Check:
  If $\mu^{2\prime} > \mu^2$ then $x_{i-1} = x$, a =gluon we put to the grid $\widetilde{g}\left(x, \mu_0^2\right) \Delta_g(\mu^2)$ and we don't have any branching.
  If $\mu^{2\prime} < \mu^2$ (and let's assume it is this case) then we have the branching and we don't consider any more $\widetilde{f}_a(x,\mu_0^2)\Delta_a(\mu^2)$.



- Generate z according to
  $\int_{z_{min}}^{z} dz' \left(\widehat{P}_{gg}(z') + \widehat{P}_{qg}(z')\right) = R_2 \int_{z_{min}}^{z_{max}} dz' \left(\widehat{P}_{gg}(z') + \widehat{P}_{qg}(z')\right)$
  where R_2- random number in the interval $(0, 1)$.
  Calculate $x_i = z x_{i-1} \rightarrow$ fulfil $\delta(zx_{i-1} - x)$!

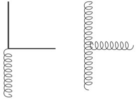
$$\widetilde{f}_a(x,\mu^2) = \int dx_{i-1} \int_{\ln \mu_0^2}^{\ln \mu^2} d\ln \mu^{2\prime} \frac{\Delta_a(\mu^2)}{\Delta_a(\mu^{2\prime})} \int_x^1 dz z P_{ag}^R\left(\alpha_s(\mu^{2\prime}), z\right) \widetilde{g}\left(x_{i-1}, \mu_0^2\right)\Delta_g(\mu^{2\prime}) + \ldots.$$

MC solution - forward evolution

Goal: to solve with MC method

$$\widetilde{f}_a(x,\mu^2) = \int dx_{i-1} \int_{\ln\mu_0^2}^{\ln\mu^2} d\ln\mu^{2\prime} \frac{\Delta_a(\mu^2)}{\Delta_a(\mu^{2\prime})} \int_x^1 dz z P_{ag}^R\left(\alpha_s(\mu^{2\prime}), z\right) \widetilde{g}\left(x_{i-1},\mu_0^2\right) \Delta_g(\mu^{2\prime}) + \ldots.$$

Moreover! if in the next step:



- $\mu^{2\prime\prime} > \mu^2$ the evolution will be stopped with only one branching,
- $\mu^{2\prime\prime} < \mu^2$ than evolution will continue with second branching (we will be in the "..." piece).

$\rightarrow \frac{\Delta_a(\mu^2)}{\Delta_a(\mu^{2\prime})}$ fulfilled by constraction!

$$\widetilde{f}_a(x,\mu^2) = \int dx_{i-1} \int_{\ln\mu_0^2}^{\ln\mu^2} d\ln\mu^{2\prime} \int_x^1 dz z P_{ag}^R\left(\alpha_s(\mu^{2\prime}), z\right) \widetilde{g}\left(x_{i-1},\mu_0^2\right) \Delta_g(\mu^{2\prime}) + \ldots.$$

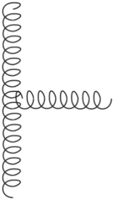
MC solution - forward evolution

Goal: to solve with MC method

$$\widetilde{f}_a(x,\mu^2) = \int dx_{i-1} \int_{\ln \mu_0^2}^{\ln \mu^2} d \ln \mu^{2\prime} \int_x^1 dz z P_{ag}^R \left(\alpha_s(\mu^{2\prime}), z\right) \widetilde{g}\left(x_{i-1}, \mu_0^2\right) \Delta_g(\mu^{2\prime}) + \ldots .$$

Decision about the splitting using R- random number in interval $(0, 1)$
if

$$R \leq \frac{\int_x^1 \mathrm{d}z z P_{gg}^R \left(\alpha_s(\mu^{2\prime}), z\right)}{\int_x^1 \mathrm{d}z z \sum_c P_{cg}^R \left(\alpha_s(\mu^{2\prime}), z\right)}$$



continue as gluon $\longrightarrow$ now we know that a=gluon

$$\widetilde{g}(x,\mu^2) = \int dx_{i-1} \int_{\ln \mu_0^2}^{\ln \mu^2} d \ln \mu^{2\prime} \sum_c \int_x^1 dz z P_{cg}^R \left(\alpha_s(\mu^{2\prime}), z\right) \widetilde{g}\left(x_{i-1}, \mu_0^2\right) \Delta_g(\mu^{2\prime}) + \ldots .$$

Goal: to solve with MC method

$$\widetilde{g}(x,\mu^2) = \int dx_{i-1} \int_{\ln \mu_0^2}^{\ln \mu^2} d\ln \mu^{2\prime} \sum_c \int_x^1 dz z P_{cg}^R\left(\alpha_s(\mu^{2\prime}), z\right) \widetilde{g}\left(x_{i-1}, \mu_0^2\right) \Delta_g(\mu^{2\prime}) + \ldots .$$

when we generate z: $weight_z$

$$\widetilde{g}(x,t) = \int dx_{i-1} \int_{\ln \mu_0^2}^{\ln \mu^2} d\ln \mu^{2\prime} \sum_c \int_x^1 dz z P_{cg}^R\left(\alpha_s(\mu^{2\prime}), z\right) \widetilde{g}\left(x_{i-1}, \mu_0^2\right) \Delta_g(\mu^{2\prime}) weight_z$$

Goal: to solve with MC method

$$\widetilde{g}(x,\mu^2) = \int dx_{i-1} \int_{\ln\mu_0^2}^{\ln\mu^2} d\ln\mu^{2\prime} \sum_c \int_x^1 dz z P_{cg}^R\left(\alpha_s(\mu^{2\prime}),z\right)\widetilde{g}\left(x_{i-1},\mu_0^2\right)\Delta_g(\mu^{2\prime}) + \ldots.$$

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Now the procedure starts from beginning.
The starting distribution: $\widetilde{g}(x,t)$.
Generate $\mu^{2\prime\prime}$ according to $\Delta_g(\mu^{2\prime\prime},\mu^{2\prime})$.
If $\mu^{2\prime\prime} > \mu \rightarrow$ only one branching & put to the grid:

$$\widetilde{g}(x,t) = \sum_c \int_x^1 dz z P_{cg}^R\left(\alpha_s(\mu^{2\prime}),z\right)\widetilde{g}\left(x_{i-1},\mu_0^2\right)\Delta_g(\mu^{2\prime}) weight_z$$

if $\mu^{2\prime\prime} < \mu^2 \rightarrow$ we have second branching, we are in ... piece.

Goal: to solve with MC method

$$\widetilde{g}(x, \mu^2) = \int dx_{i-1} \int_{\ln \mu_0^2}^{\ln \mu^2} d \ln \mu^{2\prime} \sum_c \int_x^1 dz z P_{cg}^R \left(\alpha_s(\mu^{2\prime}), z\right) \widetilde{g}\left(x_{i-1}, \mu_0^2\right) \Delta_g(\mu^{2\prime}) + \dots .$$

when we generate z: $weight_z$

$$\widetilde{g}(x, t) = \int dx_{i-1} \int_{\ln \mu_0^2}^{\ln \mu^2} d \ln \mu^{2\prime} \sum_c \int_x^1 dz z P_{cg}^R \left(\alpha_s(\mu^{2\prime}), z\right) \widetilde{g}\left(x_{i-1}, \mu_0^2\right) \Delta_g(\mu^{2\prime}) weight_z$$

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if $\mu^{2\prime\prime} < \mu^2 \rightarrow$ we have second branching, we are in ... piece.

integrals $\int \mathrm{d}x_{i-1}$ and $\int_{\ln \mu_0^2}^{\ln \mu^2} d \ln \mu^{2\prime}$ solved by MC integration method:

$$\int_a^b f(x)\mathrm{d}x = (b-a)\frac{1}{N}\sum_{i=1}^{N} f(x_i) \tag{9}$$