Symmetry and Order in Nature

Plan:

I. Symmetry & "symmetry-breaking"

II. Phases of matter & "symmetry-breaking order"

III*. Other kinds of order: "topological order"

This is a vast subject, a major theme of condensed matter and particle physics for the past 75 years.

I will focus on just a few key examples — superconductivity & anyons — to illustrate the ideas, but will mention many more examples in passing.

* Time permitting...

Symmetries and how they are realized at large distances are some of the most important properties of physical systems. In many cases they determine the basic behaviors of a system: solid or fluid, conductor or superconductor, magnetic behavior, ...

The mathematical language and concepts used to describe symmetries apply widely: classical mechanics, classical & quantum statistical systems, non-relativistic quantum many-body systems, relativistic quantum field theories, ...

Discovering the symmetries of a system is only half the battle: just as important, and often much more complicated, is the way the symmetry is realized in the physical system — i.e., how the symmetry "acts" on it.

There are 2 main categories of symmetry realization:

(1) linear realization = "unbroken symmetry"

(2) non-linear realization = "spontaneous symmetry-breaking"

(math name) (physics name)

When combined with locality & the low-energy limit, these give rise to various "phases" of matter (or of the universe, in the case of relativistic quantum field theory).

(1) ⇒ "trivial" phases ('gapped' excitations = massive quasi-particles, unique ground state, states in linear representations of symmetry group)

(2) ⇒ "ordered" phases (multiple ground states, often have gapless excitations = massless quasiparticles and heavy 'solitons' which are topologically stable)

This symmetry-based classification of phases & their main properties is the main thing I will try to explain and illustrate.

If time permits, I'll also try to illustrate some more modern developments (since ~ 1990), in particular the notion of

(3) "Topological orders" ("trivial" phases where non-local quantum effects are important, number of ground states depends on spatial topology and fractional-charge excitations occur.)

Examples of topological order occur in the "fractional quantum Hall effect" (seen in certain materials at low temperatures and high magnetic fields).

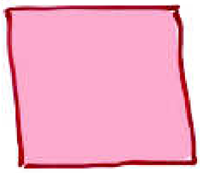
There are other types of phases, such as "symmetry-protected topological orders" (seen, for example, in "topological insulators") which I will not talk about...

Discovering the most general ways symmetries can be realized in physical quantum mechanical systems is the object of much current research in condensed matter and particle physics.

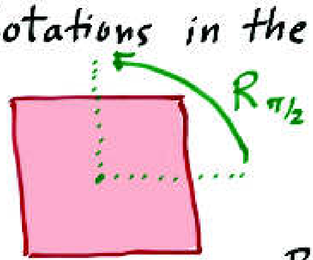
I. Symmetry and symmetry-breaking

We'll start with some simple mathematical illustrations of the basic concepts. We'll put the physics in later.

A. Finite symmetry groups.

Consider a square:  What are its symmetries?
(≈ motions which leave it unchanged)

- Rotations in the plane about its center by angles $(\dots, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \dots)$



$$\Rightarrow \{\dots, R_{-\pi/2}, R_0, R_{\pi/2}, R_\pi, R_{3\pi/2}, R_{2\pi}, \dots\}$$

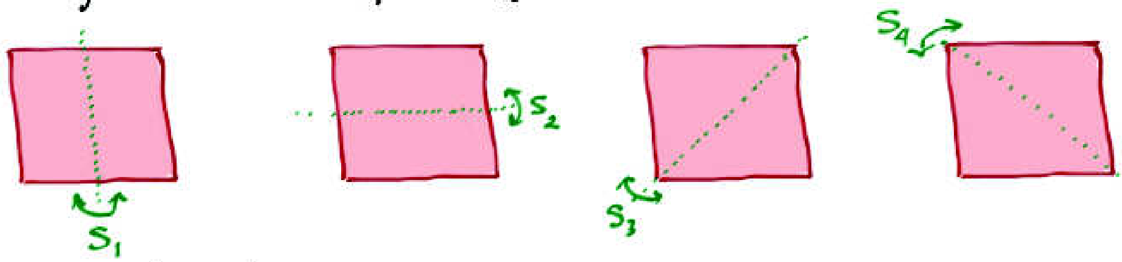
But, doing 2 successive symmetry rotations gives another one, so there are relations among them:

$$R_\beta R_\alpha = R_{\alpha+\beta}$$

(first do R_α then do R_β)

Also: R_0 does nothing = "identity transformation", so write $R_0 = e$.
$R_{2\pi}$ = same as doing nothing, so $R_{2\pi} = e$ as well.

• More symmetries: reflections across 4 lines:



They satisfy relations too: e.g., $S_i S_i = e$ $\quad i = 1,2,3,4$

& $S_2 R_{\pi/2} = S_4$ & $R_{\pi/2} S_2 = S_3$ ...

So, abstractly, symmetries form a group "G" with elements $g, h, \ldots \in G$ which can be "multiplied": $gh \in G$, associatively: $(gh)k = g(hk)$. Also, there is an identity element "e": $eg = ge = g$ for all $g \in G$. Finally, for each element there is an inverse g^{-1} such that $gg^{-1} = g^{-1}g = e$.

[See the group theory reading for more details.]

Note: group multiplication doesn't have to be commutative: $gh \neq hg$ generally. Commutative groups are called "abelian groups".

Symmetries of the square is called the "dihedral group D_8".

Some simple groups.

- Simplest: $G = \{e\}$, only 1 element = identity. "trivial group".

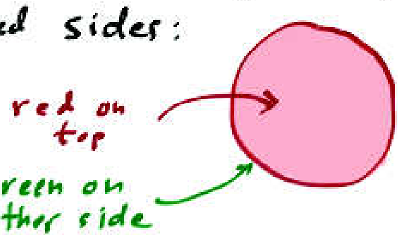
- 2-element group: $G = \{e, a\}$ & only possible multiplication rules:
  ($ee = e$, $ea = a$, $ae = a$, $aa = e$). This is equivalent to the group of addition of integers mod 2: ($0+0=0$, $0+1=1$, $1+0=1$, $1+1=0$ (mod 2)) } so called "$\mathbb{Z}_2$"

- Addition mod N = $\mathbb{Z}_N$, abelian group with N elements.

But infinitely many other possibilities... All finite groups have (essentially) been classified by mathematicians.

B. Lie groups.

What is the symmetry group of a circular disk with differently colored sides:



- Rotation in plane about center by <u>any</u> angle, θ, is a symmetry: R_θ.
- $R_0 = e = R_{2\pi}$, so take $0 \le \theta < 2\pi$.
- $R_{\theta_1} R_{\theta_2} = R_{\theta_1+\theta_2} = R_{\theta_2} R_{\theta_1} \Rightarrow$ abelian
- Reflections are <u>not</u> symmetries.

So is abelian group with an in<u>finite</u> number of elements $\{R_\theta,\ 0 \le \theta < 2\pi\}$ which $\underbrace{\text{depend continuously}}_{\Downarrow\ \text{"Lie group"}}$ on $\underbrace{\text{1 real parameter}}_{\Downarrow\ \text{"1-dimensional"}}$ (θ).

This group is called $U(1)$ = 1×1 unitary matrices $\{e^{i\theta}\}$ under multiplication.

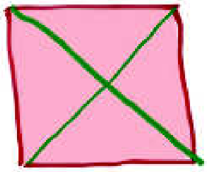
Another 1-dim'l Lie group is $\mathbb{R}$ = real numbers $\{-\infty < x < \infty\}$ under addition.
All finite-dimensional Lie groups have (essentially) been classified...
Famous examples: $U(N)$, $SO(N)$, E_8, $Spin(N)$, Lorentz, Poincaré, ...

C. Spontaneous symmetry-breaking.

Reconsider the square with symmetry group D_8, and ask the question:
What is the shortest network of paths that connects the 4 corners?

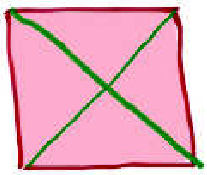
C. Spontaneous symmetry-breaking.

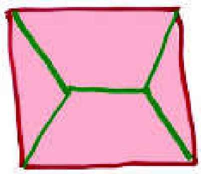
Reconsider the square with symmetry group D_8, and ask the question:
What is the shortest network of paths that connects the 4 corners?

Symmetric answer:  is not correct!

C. Spontaneous symmetry-breaking.

Reconsider the square with symmetry group D_8, and ask the question:
What is the shortest network of paths that connects the 4 corners?

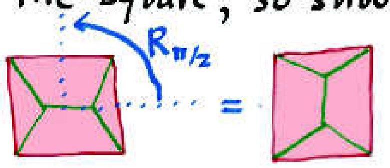
Symmetric answer:  is not correct!

Correct answer:  is not D_8-symmetric!

"Spontaneous symmetry breaking" is when a question has more symmetry than its answer.

What is symmetry of the answer? $\{e, R_\pi, S_1, S_2\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$
So answer has smaller symmetry than question: $D_8 \supset \mathbb{Z}_2 \times \mathbb{Z}_2$.

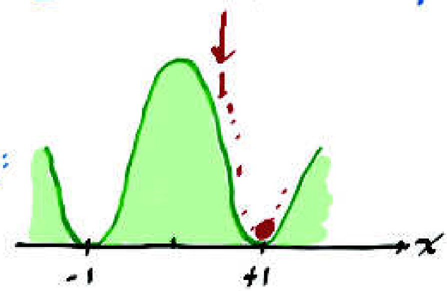
But the "missing" or "broken" symmetries, e.g. $R_{\pi/2}$, still takes the square to the square, so should take solution to solution:



} 2 distinct solutions $\quad 2 = \dfrac{|D_8|}{|\mathbb{Z}_2 \times \mathbb{Z}_2|} = \dfrac{8}{4}$.

- $\mathbb{Z}_2$ & $U(1)$ examples: drop a marble on a landscape and let it come to rest...

$\mathbb{Z}_2$:

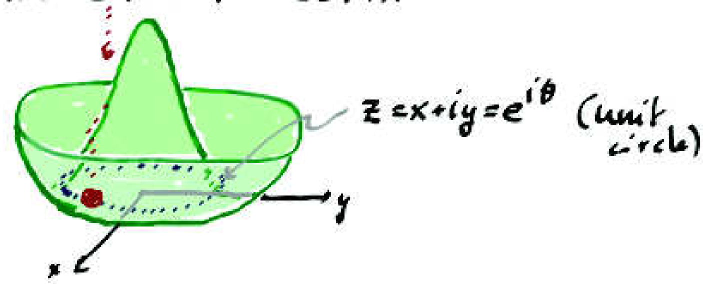


Comes to rest at either $x=+1$ or $x=-1$

$\Rightarrow$ "Breaks" $\mathbb{Z}_2 \to \{e\}$

$\Rightarrow$ 2 solutions.

$U(1)$:



Comes to rest at $z = e^{i\theta}$ for some $\theta \in [0, 2\pi)$

$\Rightarrow$ Breaks $U(1) \to \{e\}$

$\Rightarrow$ Infinite number (whole circle) of solutions.

- In the rest of these lectures I'll focus on these two examples: $\mathbb{Z}_2$ & $U(1)$.

II. Phases of matter and "symmetry-breaking orders"

A. The physical setting.

The basic question: Given a microscopic (or "fundamental") description of a system, what are its properties at late times and long distances? In other words, what are its macroscopic properties after it has settled down to a steady state ($\approx$"ground state", or "vacuum")?

Example: (condensed matter physics) A bunch of atoms in a large box is described by quantum mechanics and electro-dynamics. Let it cool down. What will you see? Solid? Fluid? Conductor? Insulator? ... What are the possibilities? As you vary the types of atoms, density, pressure, ... what transitions between these phases do you expect?

Example: (particle physics) In the theory of quarks and gluons interacting via the SU(3) "color" gauge theory (QCD), what are the properties of the lightest states (particles) above the ground state (vacuum)? Don't see quarks & gluons! Instead, see color-neutral pions, mesons, and baryons interacting via a short-range "strong" force. As you vary the number, masses, & charges of quarks & gluons, what other phases occur?

This basic question focusses on the ground state/vacuum physics and its low-energy excitations. One finds that relatively few parameters are needed to describe the long-distance 'phase' of a given microscopic system. This is called

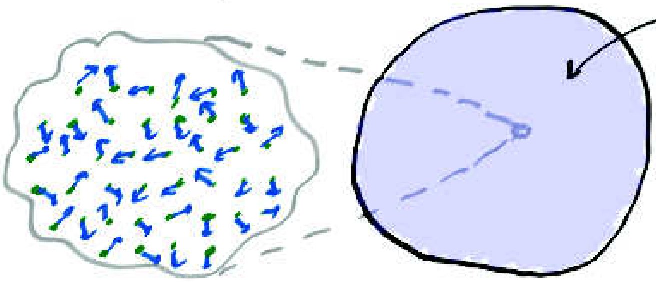
"Universality": infinitely-many micro-systems give rise to the same macro-phases.

Example: many different kinds of atoms are in fluid phase for large ranges of pressures and temperatures.

Example: many different proposed theories of "beyond the Standard Model" particle physics all agree with SM predictions at low energies.

Universality — the existence of only a finite number of relevant parameters in physical systems — can be explained by the renormalization group and the notion of an effective theory (K. Wilson). [See the reading on effective theories by S. Caroll for more details.]

An effective low-energy, long-distance description averages over the microscopic details:



locally uniform medium described by some slowly-varying fields:

$$\varphi(\vec{x},t) \quad \psi_\alpha(\vec{x},t) \quad \vec{A}(\vec{x},t) \quad \ldots$$

universality $\Rightarrow$ only a few fields needed
averaging $\Rightarrow$ locality in space & time
relaxation to equilibrium $\Rightarrow$ homogeneity in space & time

(plausibility argument, not rigorous)

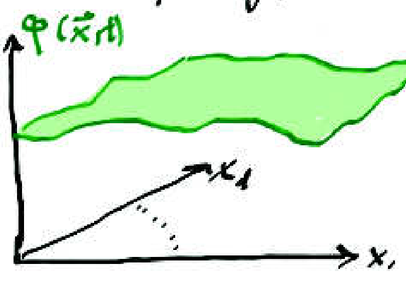
$\Rightarrow$ can describe effective theory in terms of a (semi-)classical lagrangian as

$$S = \underbrace{\int d^d x \, dt}_{\text{space-time translation invariance}} \; \mathcal{L}\Big[\underbrace{\varphi(\vec{x},t),\ \vec{\nabla}\varphi(\vec{x},t),\ \tfrac{\partial}{\partial t}\varphi(\vec{x},t),\ \nabla^2\varphi(\vec{x},t), \ldots}_{\text{locality} \Rightarrow \text{ all at same space-time point } (\vec{x},t).}\Big]$$

$d \equiv$ # space dimensions

Expect that the low-energy deviations from a homogeneous ground state will be slowly-varying in time & space. Thus successive derivatives ($\frac{\partial}{\partial t}$, $\vec{\nabla}$) will be small. So for a low-energy effective lagrangian, we should keep only the terms with the fewest derivatives (necessary to capture the response to a given observable).

For simplicity, let's focus for now on a single scalar field



can visualize as a "membrane" over space. This analogy $\Rightarrow$

$\vec{\nabla}\varphi$ ~ measures stretching of membrane.

$\frac{\partial\varphi}{\partial t}$ ~ measures velocity of membrane.

B. Scalar field effective theory

Recall [see the reading on Lagrangians] that $\mathcal{L} \sim \begin{pmatrix}\text{Kinetic}\\ \text{energy}\end{pmatrix} - \begin{pmatrix}\text{Potential}\\ \text{energy}\end{pmatrix}$.

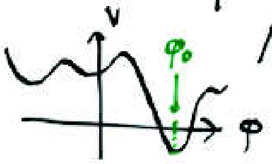
$$\mathcal{L} = -\Big(\underbrace{V(\varphi)}_{\text{"external potential"}} + \underbrace{\tfrac{1}{2}\vec{\nabla}\varphi \cdot \vec{\nabla}\varphi}_{\text{"stretching" energy}} \Big) + \underbrace{\frac{1}{2v^2}\left(\frac{\partial \varphi}{\partial t}\right)^2}_{\text{Kinetic energy}}$$

$[v]$ = velocity (to get dimension right).

Euler-Lagrange equation of motion: $\displaystyle \frac{1}{v^2}\frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi = -\frac{\partial V}{\partial \varphi}$

Ground state = stable & stationary: $\displaystyle \frac{\partial \varphi}{\partial t} = \vec{\nabla}\varphi = \frac{\partial V}{\partial \varphi} = 0$

$\Rightarrow$ $\varphi(\vec{x},t) = \varphi_0$ = constant, and φ_0 is minimum of V:



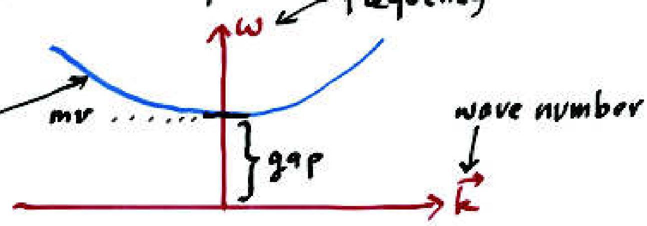
Near minimum, write $\varphi(\vec{x},t) = \varphi_0 + \delta\varphi(\vec{x},t)$.

$V(\varphi) \approx V_0 + \frac{1}{2}m^2(\delta\varphi)^2 + \cdots$

E.O.M. $\approx$ $\displaystyle \frac{\partial^2 \delta\varphi}{\partial t^2} = v^2(\nabla^2 - m^2)\,\delta\varphi$

Fourier transform: $\delta\varphi \sim e^{i(\omega t + \vec{k}\cdot\vec{x})} \Rightarrow$

$$\omega^2 = v^2(k^2 + m^2)$$



In quantum mechanics:

$\hbar\omega$ = energy
$\hbar\vec{k}$ = momentum

of "quasi-particle" = quantum of excitation above ground state. (Gap = mass.)