What does this have to do with symmetries?

Actually, I snuck in some symmetry assumptions:

1) space & time homogeneity = symmetry under space & time translations

$$T_{(\vec{a},b)}: \varphi(\vec{x},t) \to \varphi(\vec{x}+\vec{a}, t+b) \qquad (\text{Lie group} \simeq \mathbb{R}^{d+1})$$

The ground state was invariant under translations, so unbroken. This allowed the interpretation of excitations in terms of quasi-particles via the Fourier transform.

2) Rotational symmetry, $R: \varphi(\vec{x},t) \to \varphi(R\vec{x}, t)$, where R is a $d \times d$ rotation matrix also entered because I didn't consider terms with few derivatives, like $\vec{a}\cdot\vec{\nabla}\varphi$, because a constant "$\vec{a}$" vector would violate rotational symmetry.

A basic fact in classical or quantum mechanics is that corresponding to every continuous (Lie group) symmetry is a corresponding conserved quantity. [See lagrangian reading and symmetries in quantum mechanics readings for a review.] Here translational symmetry implies energy and momentum conservation, and rotational symmetry implies angular momentum conservation.

A key point is that the lagrangian is invariant under symmetries. In fact, invariances of the lagrangian (or action) are often taken as the <u>definition</u> of the symmetries of a system.

The effective action philosophy turns this around: instead of picking a lagrangian based on a mechanical analogy as we did above, write down the most general (local, analytic) $\mathcal{L}(\varphi, \partial\varphi, \partial^2\varphi \ldots)$ which is invariant under the symmetries of the problem, keep the necessary leading terms in an expansion in the number of derivatives.

Example: Real scalar φ with space-time translational and space-rotational symmetries $\Rightarrow$

$$\mathcal{L} = -V(\varphi) + a(\varphi)\frac{\partial\varphi}{\partial t} + b(\varphi)\left(\frac{\partial\varphi}{\partial t}\right)^2 - c(\varphi)\,\vec{\nabla}\varphi\cdot\vec{\nabla}\varphi + (\text{higher derivs.})$$

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Example: Real scalar φ with space-time translational and space-rotational symmetries $\Rightarrow$

$$\mathcal{L} = -V(\varphi) + \underbrace{\cancel{a(\varphi)\frac{\partial\varphi}{\partial t}}}_{=\ \frac{\partial}{\partial t}(A(\varphi))} + b(\varphi)\left(\frac{\partial\varphi}{\partial t}\right)^2 - c(\varphi)\,\vec{\nabla}\varphi\cdot\vec{\nabla}\varphi + (\text{higher derivs.})$$

$\frac{\partial}{\partial t}(A(\varphi))$ $(A' = a)$ drops out of e.o.m.

Expand around ground state $\varphi(\vec{x},t) = \varphi_0 \Rightarrow b(\varphi) \approx b(\varphi_0)$ & $c(\varphi) \approx c(\varphi_0)$

$V(\varphi) \simeq V_0 + \frac{1}{2}m^2(\varphi-\varphi_0)^2 + \ldots$ $\Rightarrow$ We recover the previous "membrane" theory: it is the general (stable) theory for this field content & symmetry.

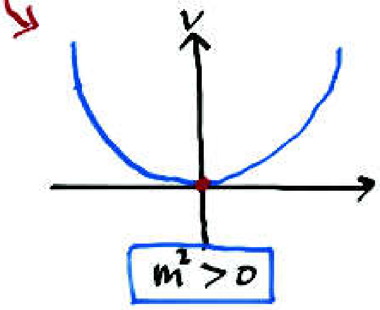
C. Real scalar with $\mathbb{Z}_2$ symmetry (Ising model universality class)

$\mathbb{Z}_2 = \{e, a\}$ with $a^2 = e$. Assume $a \in \mathbb{Z}_2$ acts as $a: \varphi \to -\varphi$.

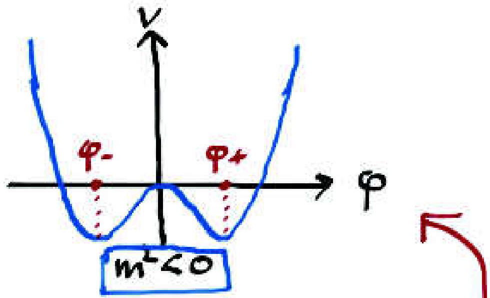
The lagrangian $\mathcal{L} = -V(\varphi) - \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{1}{2v^2}\left(\frac{\partial \varphi}{\partial t}\right)^2$ is invariant if $V(-\varphi) = V(\varphi)$

$\Rightarrow$ $V = V_0 + \frac{1}{2} m^2 \varphi^2 + \frac{1}{4}\lambda \varphi^4 + \dots$ (i.e., only even powers of φ).

If $m^2 > 0$, ground state is $\varphi = 0$, is unique & $\mathbb{Z}_2$-invariant.



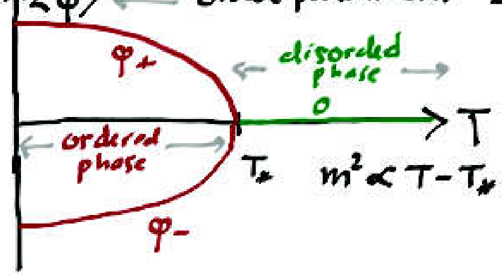
$$\boxed{V = \frac{m^2}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4}$$



But if $m^2 < 0$, now have 2 ground states, $\varphi = \varphi_\pm$, at $\varphi_\pm = \pm \frac{|m|}{\sqrt{\lambda}}$, and the $\mathbb{Z}_2$ is spontaneously broken. Look at excitations around one g.s. by expanding $\varphi \equiv \varphi_+ + \delta\varphi$ ... no longer has $\delta\varphi \to -\delta\varphi$ $\mathbb{Z}_2$ symmetry.

Say in a physical system, as you change some parameter (e.g. temperature) find that $m^2 = m^2(T)$ changes sign in the effective theory. Then you have a **phase transition**:

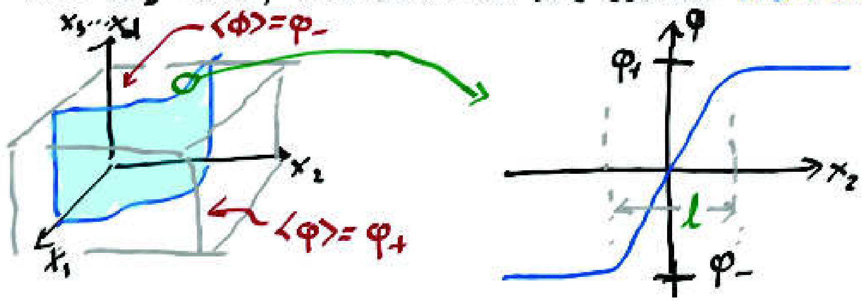
$\langle\varphi\rangle$ ← "order parameter" = value of φ in g.s. ("vacuum expectation value" = vev in particle physics)



Like ferromagnetism. As $T \to T_*$, which ground state, $\langle\varphi\rangle = \varphi_+$ or φ_-, is "chosen" is a matter of chance.

Solitons occur in spectrum of the symmetry-broken phase. A soliton is an energetic (massive) state localized in some spatial directions which can be described locally by a smooth field configuration in the effective theory.

In the $\mathbb{Z}_2$ case, the solitons are called **domain walls**:



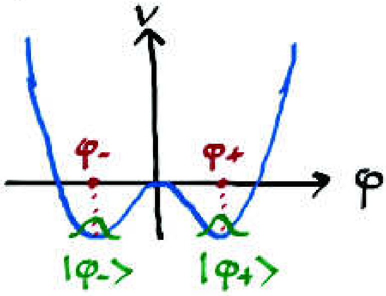
(d = # space dims)

Find:

width: $l \sim |m|$

$\frac{\text{energy}}{\text{area}}$: $\frac{E}{L^{d-1}} \sim \frac{|m|^4}{\sqrt{\lambda}}$

So far everything has been classical. In quantum mechanics, the new thing that can happen is that you can tunnel between equal energy states where classical transitions are forbidden. Thus a particle in a $\mathbb{Z}_2$-symmetric double well has only one ground state:



$|g.s.\rangle \cong \frac{1}{\sqrt{2}}(|\varphi_-\rangle + |\varphi_+\rangle)$ has energy E_0,

while

$|\psi\rangle = \frac{1}{\sqrt{2}}(|\varphi_-\rangle - |\varphi_+\rangle)$ has energy $> E_0$.

So, does QM-tunneling "wash out" the $\mathbb{Z}_2$ symmetry breaking and restore $\langle\varphi\rangle = 0$? Only for $d=1$ spatial dimensions. For $d>1$, the total energy of the domain wall energy barrier $E \sim L^{d-1} \frac{|m|^4}{\sqrt{\lambda}} \to \infty$ as $L \to \infty$, and the tunelling probability goes to 0.

D. Complex scalar with U(1) symmetry (superfluid universality class)

$\varphi = \varphi_1 + i\varphi_2$ with $\varphi_{1,2} \in \mathbb{R}$, so complex scalar is just 2 real scalars.
$U(1) = \{R_\theta,\ 0 \le \theta < \pi\}$ acts as $R_\theta : \varphi(x,t) \to e^{i\theta}\varphi(x,t)$ "phase rotation"
Then invariant effective Lagrangian is

$$\mathcal{L} = -V - \vec{\nabla}\varphi \cdot \vec{\nabla}\varphi^* + \frac{1}{v^2}\frac{d\varphi}{dt}\frac{d\varphi^*}{dt} \quad \text{with} \quad V(\varphi,\varphi^*) = m^2\varphi\varphi^* + \frac{1}{2}\lambda\varphi^2\varphi^{*2}$$

$m^2 < 0 \Rightarrow \langle\varphi\rangle = \frac{|m|}{\sqrt{\lambda}}e^{i\theta}$ $\qquad$ $m^2 > 0 \Rightarrow \langle\varphi\rangle = 0$

$\varphi = r e^{i\theta}$

Spectrum $\begin{cases} \delta\varphi_a : & \omega^2 = v^2(k^2+m^2) \\ \delta\varphi_b : & \omega^2 = v^2 k^2 \end{cases}$

$\Rightarrow$ 1 massive & 1 massless quasiparticle.

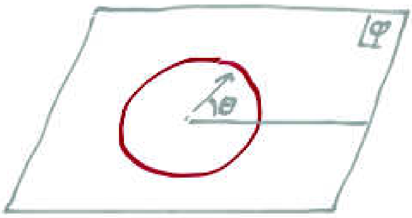
Spectrum $\begin{cases} \delta\varphi_1 : & \omega^2 = v^2(k^2+m^2) \\ \delta\varphi_2 : & \omega^2 = v^2(k^2+m^2) \end{cases}$

$\Rightarrow$ 2 massive quasiparticles

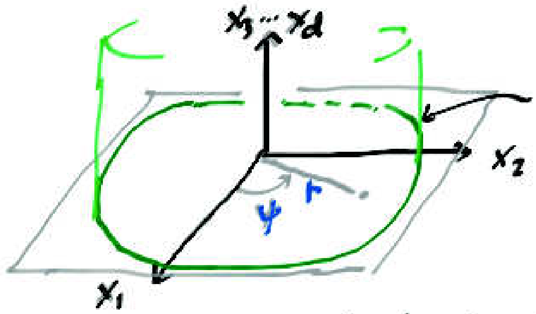
This massless q-particle is massless by virtue of the spontaneous breaking of the U(1) symmetry. It is called a Nambu Goldstone boson.

Phase transition? <u>Yes</u>, similar to the $\mathbb{Z}_2$ case...

Solitons? <u>No</u>. There is a topologically stable "vortex" field configuration



$$\langle\varphi\rangle = \frac{|m|}{\sqrt{\lambda}}\, e^{i\theta}$$



circle in $x_1 x_2$ plane at ∞ ($r \gg 1$).

Cylinder coordinates in space $(r, \psi, x_3, \ldots x_d)$

$\Rightarrow$ write $\varphi(x) = |\varphi(x)|\, e^{i\theta(x)}$

and want $\lim_{|x|\to\infty} |\varphi(x)| = \frac{|m|}{\sqrt{\lambda}}$.

Make an n-vortex field configuration by choosing boundary conditions

$$\begin{cases} \lim_{r\to\infty} |\varphi(x)| = \frac{|m|}{\sqrt{\lambda}} & \text{(so is in ground state at } \infty \text{ in } x_{1,2} \text{ directions)} \\ \lim_{r\to\infty} \theta(x) = n\psi & \text{(so phase of } \langle\varphi\rangle \text{ at } \infty \text{ winds } n \text{ times around circle in } x_{1,2} \text{ plane)} \end{cases}$$

Solution $\varphi(x) = f(r)\, e^{in\psi}$:

looks like a 'string' along $x_3 \cdots x_d$ directions at $x_1 = x_2 \approx 0$.

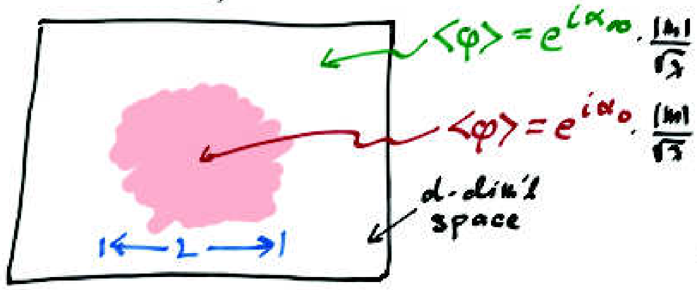
What goes wrong? We are not __exactly__ in a ground state even at $r \to \infty$ because $\langle \varphi \rangle \sim e^{in\psi}$ is spatially varying. The energy per unit length of the vortex is

$$\frac{E}{L^{d-2}} > \int d^2x_\perp \, |\vec{\nabla}\varphi|^2 = \int^\infty r\,dr \int_0^{2\pi} d\psi \, |\vec{\nabla} e^{in\psi}|^2 = \int^\infty r\,dr \int_0^{2\pi} d\psi \cdot \frac{n^2}{r^2}$$

$$= 2\pi n^2 \int^\infty \frac{dr}{r} = 2\pi n^2 \ln r \Big|^\infty \to \infty .$$

So takes an infinite energy per unit length to create this vortex, which means it does not exist as a physical excitation.

Do quantum (or thermal) fluctuations "wash out" the symmetry-broken phase? (No longer need to tunnel since there is no barrier between vacua!)



$$\text{Energy cost}^{*} \sim \int d^dx \, |\vec{\nabla}\varphi|^2$$

$$\sim L^d \, \frac{|e^{i\alpha_\infty} - e^{i\alpha_0}|}{L^2} \sim L^{d-2} (\Delta\alpha)^2$$

To change vacuum, take $L \to \infty$
$\Rightarrow$ energy $\to \infty$ for $d > 2$.

(* I am cheating here.)

$$\Rightarrow \begin{cases} d > 2: & \text{circle of vacua are stable and there exists massless NGB.} \\ d \le 2: & \text{fluctuations average over all vacua, } U(1) \text{ symm. restored, no NGB.} \end{cases}$$

E. Charged scalar electromagnetism (superconductor universality class)

Electromagnetism (Maxwell's equations) is described in the Lagrangian formalism in terms of the potential fields $(\vec{A}, \Phi)$ related to the electric & magnetic fields by $\begin{cases} \vec{E} = -\vec{\nabla}\Phi + \frac{\partial}{\partial t}\vec{A} \\ \vec{B} = \vec{\nabla}\times\vec{A} \end{cases}$. Then $\mathcal{L}_{EM} = \frac{1}{2}E^2 - \frac{1}{2}B^2$

A charged scalar field is a complex φ as above, but with the substitutions $\begin{cases} \vec{\nabla}\varphi \to \vec{D}\varphi \equiv (\vec{\nabla} - iq\vec{A})\varphi \\ \frac{\partial}{\partial t}\varphi \to D_t\varphi \equiv (\frac{\partial}{\partial t} - iq\Phi)\varphi . \end{cases}$ (q = charge of φ)

So the effective theory is described by

$$\mathcal{L}_{\varphi EM} = -V(\varphi\varphi^*) - (\vec{D}\varphi)\cdot(\vec{D}\varphi)^* + \frac{1}{v^2}(D_t\varphi)(D_t\varphi)^* + \mathcal{L}_{EM} .$$

What are its symmetries?

Looks like $U(1)$: $\left\{\begin{array}{l} \varphi \to e^{i\alpha}\varphi \\ \vec{A} \to \vec{A} \\ \Phi \to \Phi \end{array}\right\}$ is still a symmetry. We will call this $U(1)_{global}$.

But $\mathcal{L}_{\varphi EM}$ has a much (infinitely) larger symmetry group called "U(1) gauge invariance": ($U(1)_{gauge}$ for short).

$$U(1)_{gauge} \begin{cases} \varphi(x) \to e^{iq\alpha(x)}\varphi(x) \\ \vec{A}(x) \to \vec{A}(x) + \vec{\nabla}\alpha(x) \\ \Phi(x) \to \Phi(x) + \frac{\partial}{\partial t}\alpha(x). \end{cases}$$

(You should check that it leaves $\mathcal{L}_{\varphi EM}$ invariant.)

Here the group parameter α is now $\alpha(x)$ = which is an arbitrary function of space-time. So $U(1)_{gauge}$ is an ∞-dimensional Lie group! Note that $U(1)_{global} \subset U(1)_{gauge}$.

In classical E&M, $\vec{E}$ and $\vec{B}$ are physical observables, not $\vec{A}$ or Φ. This means that $U(1)_{gauge}$ relates different $(\varphi, \vec{A}, \Phi)$ field configurations which are physically identical.

This is true quantum-mechanically as well. In fact it is required to have a consistent quantum interpretation (e.g., no negative probabilities).

Therefore $U(1)_{gauge}$ is not a symmetry because it does not relate distinct physical configurations!