What does this mean for the phases of this theory?

Same potential $V(\varphi\varphi^*)$ as before, so order parameters look the same:

$m^2>0 \Rightarrow \langle\varphi\rangle=0$

Spectrum $\begin{cases}\delta\varphi_1: & \omega^2=v^2(k^2+m^2)\\ \delta\varphi_2: & \omega^2=v^2(k^2+m^2)\end{cases}$

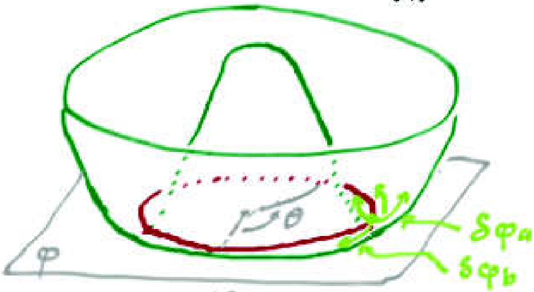
$\Rightarrow$ 2 massive scalar quasiparticles that transform under $U(1)_{global}$.

and $(\vec{A},\Phi)/U(1)_{gauge}$ in usual way

$\Rightarrow$ 1 massless photon

"Coulomb phase" with 'emergent' $U(1)_{global}$ symm.

$m^2<0 \Rightarrow \langle\varphi\rangle=\frac{|m|}{\sqrt{\lambda}}e^{i\theta}$



$\varphi=re^{i\theta}$

Spectrum $\begin{cases}\delta\varphi_a\propto\delta r: & \omega^2=v^2(k^2+m^2)\\ \delta\varphi_b\propto\delta\theta: & \text{can set } \delta\theta=0 \text{ by } U(1)_{gauge}\end{cases}$ since $|\varphi|\neq0$. So no q-part.

$\Rightarrow$ 1 massive scalar q-particle ("Higgs boson")

and $\mathcal{L}_{\vec{A},\Phi} \supset |\vec{D}\varphi|^2-|D_t\varphi|^2$

$\supset(\vec{A}\cdot\vec{A}-\Phi^2)\underbrace{|\varphi|^2q^2}_{\sim m^2q^2/\lambda}$

$\Rightarrow$ 1 massive photon w/ $m_\gamma=\frac{|m|q}{\sqrt{\lambda}}>0$.

"Higgs phase" w/ no $U(1)$ symm.

What does $m_\gamma > 0$ mean? In massive electromagnetism macroscopic electric & magnetic fields are "screened", which means that they can only penetrate a distance $\sim \frac{c\hbar}{m_\gamma}$ into a region in this phase.

This is familiar in conductors (metals) where $\vec{E}$-fields are screened in their interior by charge build-up on the surface.

Here also the $\vec{B}$-field is screened, which requires a current build-up on the surface. This gives rise to persistent currents which means zero resistivity, hence a superconductor.

So Higgs phase = superconducting phase.

Still have a Coulomb to Higgs phase transition if, say, $m^2 \propto T - T_*$.

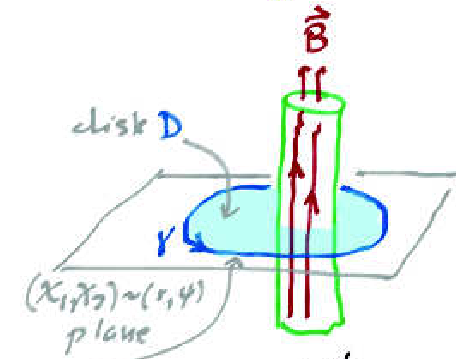
How about solitons? The infinite-energy vortex of the complex scalar theory becomes a finite energy/length magnetic flux tube.

Flux tube idea:

$$\langle \varphi(x) \rangle = \langle \varphi(r,\psi) \rangle \overset{r\to\infty}{\sim} |\varphi_0| e^{in\psi}$$

as before, but now $|\varphi_0| e^{in\psi}$ all describe <u>same</u> <u>vacuum</u>, since the $e^{in\psi}$-dependence can be 'gauged away' by $U(1)_{gauge}$ invariance. Therefore no divergent energy cost from $\vec{\nabla}\psi$ as $r\to\infty$.

This implies that the vortex must carry a quantized magnetic flux along its core:



$$\Phi_B = \int_D \vec{B}\cdot d\vec{a} = \int_D (\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \oint_\gamma \vec{A}\cdot d\vec{l}$$

On γ $\varphi \sim |\varphi_0| e^{in\psi}$.

Vacuum @ $r\to\infty$ $\Rightarrow$ φ constant $\Rightarrow$ $\vec{D}\varphi \approx 0$

$$\Rightarrow \vec{\nabla}\varphi = iq\vec{A}\varphi \Rightarrow \vec{A} = \frac{1}{iq}\frac{\vec{\nabla}\varphi}{\varphi}$$

$$\therefore \Phi_B = \frac{1}{iq}\oint_\gamma \frac{\vec{\nabla}\varphi}{\varphi}\cdot d\vec{l} = \frac{n}{q}\int_0^{2\pi}\vec{\nabla}\psi\cdot d\vec{l} = \frac{2\pi n}{q}.$$

Lessons: (1) Gauge invariance is not a symmetry.

(2) Coulomb phase with massless spin-1 (photon) & 'emergent' symmetry,

(3) can transition to a massive (gapped) Higgs phase with no symmetry,

(4) with flux tube solitons.

F. Many other examples ... for instance:

- Spatial translation & rotation symmetry (fluid phase)

  ↓ spontaneously breaks ↓

  Discrete subgroup: lattice translations & rotations

  = crystaline order (solid phase)

⇒ Momentum no longer conserved, but discrete translations imply there is still a conserved "lattice momentum". This gives rise to Brioullin zones, electronic bands, etc.

Also, the <u>rigidity</u> of solids is a direct result of spontaneously broken translation symmetry. This rigidity is the basis of all human technology.

- Rotational symmetry of atomic spins (paramagnetic phase)

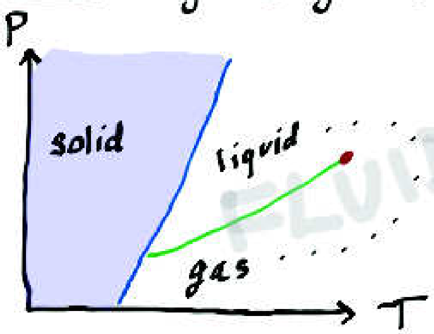
  ↓ spontaneously breaks ↓

  Ferromagnetic order (like the $\mathbb{Z}_2$ Ising case)

- Combine above ⇒ Anti-ferromagnetic orders.

- Also, many other choices exist for low energy effective degrees of freedom (other than scalars): fermions (spinors), vectors, ...

- Note: Symmetry-breaking does not explain all phase transitions. E.g.,



Liquid-gas transition is between the same phase (fluid).

- In particle physics, we assume Poincaré symmetry, which is space & time translations and Lorentz transformation (rotations and boosts) as symmetries of the microscopic description.
  And we (usually) assume that this symmetry is unbroken in the vacuum. We then examine various choices of low energy degrees of freedom and "internal" symmetries (analogs of the $\mathbb{Z}_2$ and $U(1)$ symmetries described above).
  - Poincaré symm. means more constrained effective actions than in CM physics. But microscopic "origins" are unknown, so less constrained...

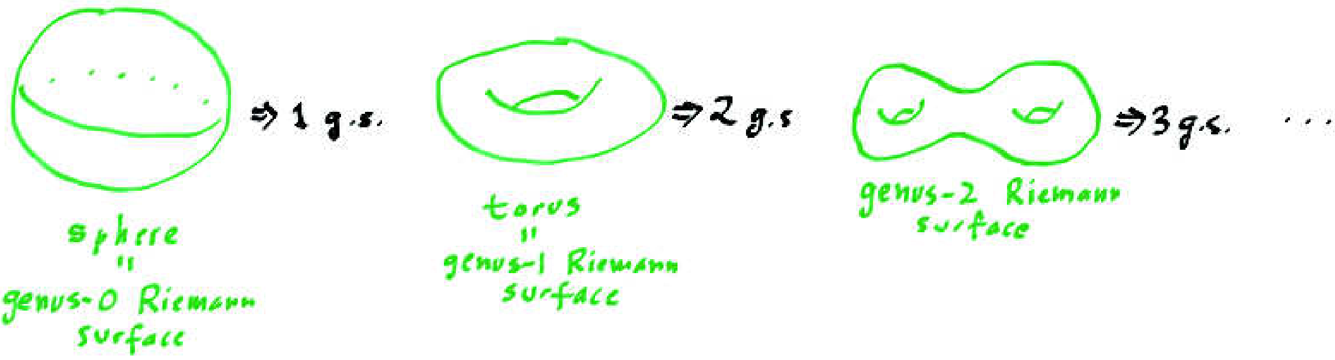
III. Other kinds or order

So far, I have been telling a story that has been well-understood since the 1970s. More modern developments, especially since 1990, have focussed on kinds order (phases) which do not depend on broken symmetries.

The simplest of these are typically "gapped" phases, so may seem boring ("trivial"), but it can happen that non-local quantum effects can persist in the low-energy effective theory. Examples of this were understood theoretically in models of "anyons" in 2+1-dimensions, and first seen experimentally in the fractional quantum Hall effect.

These phases are described by low-energy effective theories with no propagating degrees of freedom. This because at energies below the gap, i.e. below the energy of the least energetic mode, there is simply no modes left to excite. So what is there left to say?

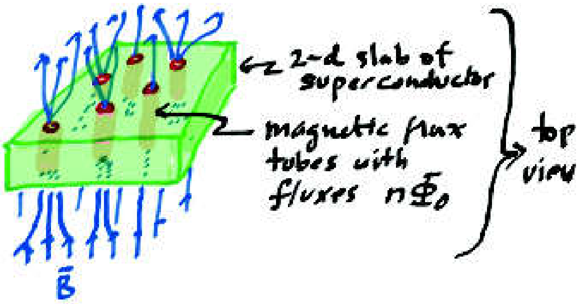
Find a subtle non-local behavior: the number of vacua depends on the topology of the space it lives in. For example, in such a system in 2 space dimensions, on a 2-d plane it will have a single ground state. But if you put it on other geometries you find diffent numbers of ground states:

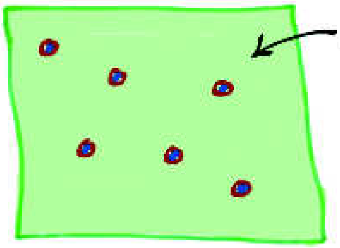


So this gives a way of getting multiple ground states without symmetry-breaking, called topological order.

This seems contrived: when do we ever find physical systems on genus-g Riemann surfaces? And, if we did, so what? What dynamics could change the spatial topology of a system?

Reconsider the Higgs phase: no symmetry, completely gapped ... boring except we saw that it has very heavy flux tubes. Put a slab of material in this phase transverse to a magnetic field:





2-d gapped phase on space with topology of plane with 6 "punctures".

Each punctures is a heavy fluxon w/ flux $n_i \Phi_0$ with $(n_1, n_2, \dots, n_6)$ integers.

You can move the punctures (fluxons) around without disturbing the state of the surrounding superconductor state. Can do 2 kinds of things:

1)  exchange fluxons

2)  "fuse" 2 fluxons $\rightarrow$ 1 fluxon (& reverse: split $1 \rightarrow 2$).

In a superconductor, each fluxon is labelled by an integer "n_i" measuring the amount of magnetic flux, $n_i \Phi_0$, threading the flux tube. Thus under fusion:

$$n_1 \circ n_2 = (n_1 + n_2) \quad \text{(i.e. } \vec{B} \text{ fluxes simply add).}$$

$\uparrow$ "fuse"

Though not obvious, these operations are closely related to the topological multiplicity of ground states given earlier:

Ask: are we back at the same state we started from? If not, then system must possess an extra ground state for each handle.

Unfortunately, nothing interesting happens for Higgs phase fluxons.

But we can imagine generalizing this to allow the most general properties for fusing fluxons:

- Label different "charges" of fluxons by letters $a, b, c, \ldots$
- Possible results of fusing a with b is given by a fusion rule

$$a \circ b = \sum_c N_{ab}^c \cdot c \;, \quad N_{ab}^c = N_{ba}^c \text{ are non-negative integers.}$$

- N_{ab}^c counts possible outcomes of fusing a & b:

  $N_{ab}^c = 0 \Rightarrow$ can't get c from $a \circ b$

  $N_{ab}^c > 0 \Rightarrow$ counts number of "channels" for getting c from $a \circ b$ (# of distinct amplitudes that can interfere)

(Fusion rule do not encode actual value of probability of getting c from $a \circ b$, just whether or not that probability can be non-zero. The actual value depends on the microscopic details of just how a and b are brought together.)

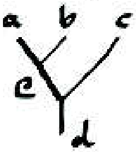
Call the Hilbert space of (ground) states of theory in the presence of an 'a' and 'b' fluxon with total flux 'c' by $\mathcal{H}^c_{ab}$.
Then $N^c_{ab} = \dim(\mathcal{H}^c_{ab})$.

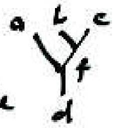
If there is a fusion rule for which there can be more than one outcome on the r.h.s., we call the fluxons "non-abelian anyons".

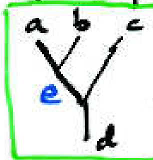
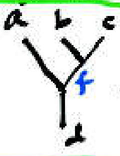
For superconductors, label fluxons by their quantized (integer) fluxes, n_i, and get fusion rule: $n_1 \circ n_2 = n_3$ w/ $n_3 = n_2 + n_1$. So always just one possibility on r.h.s., so does <u>not</u> give topological order. To get topological order, will need to look at other systems.

For simplicity, let's assume all N_{ab}^{c} = either 0 or 1 from now on.
So $\dim(\mathcal{H}_{ab}^{c}) = 0$ or 1. Denote a basis state of $\mathcal{H}_{ab}^{c}$ when $N_{ab}^{c}=1$ by the vertex . So, such a vertex only exists if $N_{ab}^{c}=1$.

Say $N_{ab}^{e}=1$ and $N_{ec}^{d}=1$. Then we can fuse $a \circ b \to e$ and $e \circ c \to d$ to get a state in $\mathcal{H}_{abc}^{d}$, the space of states with total flux "d" in the presence of "a", "b", & "c" fluxons. We denote this state by:



But we could have fused $b \circ c$ first, i.e. done $a \circ (b \circ c)$ instead of $(a \circ b) \circ c$. This is denoted by . So there must be a way of expressing the  state in terms of the  basis states:

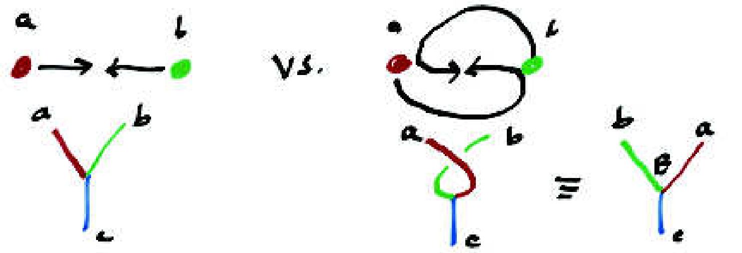
 $= \sum_f$  $\cdot \left(F^{d}_{abc}\right)^{f}_{e}$

set of complex numbers: "fusion tensor"

This is reminiscent of group theory: fusion ~ group multiplication, and fusion tensor relation ~ associativity.

Also, there is a natural identity fluxon "0" = state of absence of any fluxon, with fusion rule $0 \circ a = a$ for all a.

For group multiplication can have $a \cdot b \neq b \cdot a$. How about for fusion? At the level of the fusion rules, $a \circ b = b \circ a$, since there is no concept of the order of fusing 2 fluxons. But we can ask what is the relation between the ${}^{a}Y^{b}_{c}$ state and the state you get upon first interchanging a & b along a simple counter-clockwise path before fusing them:



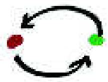
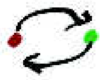
My notation: "B" for "braid"

In our simplified case with all $N^{c}_{ab} \in \{0,1\}$, the possible relation is very simple:

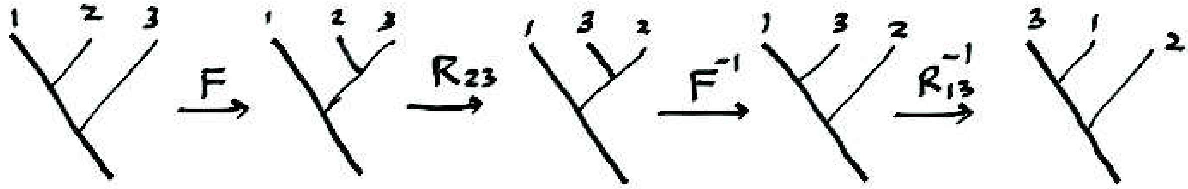
$$\boxed{{}^{a}\overset{B}{Y}{}^{b}_{c} = R^{c}_{ab}\; {}^{b}Y^{a}_{c}}$$

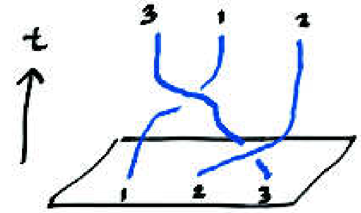
"Braid relations"

some complex phases ≡ "Braiding tensor"

So the braid relation represents the "statistics" that result from exchanging a & b along a path in plane:  $\sim R$.
Can also exchange along the reverse path:  $\sim R^{-1}$.

This is special to 2 spatial dimensions: "braid statistics".
Arbitrarily-complicated braidings of fluxons can be described using successive braid (R) and fusion (F) operations & their inverses.
E.g.

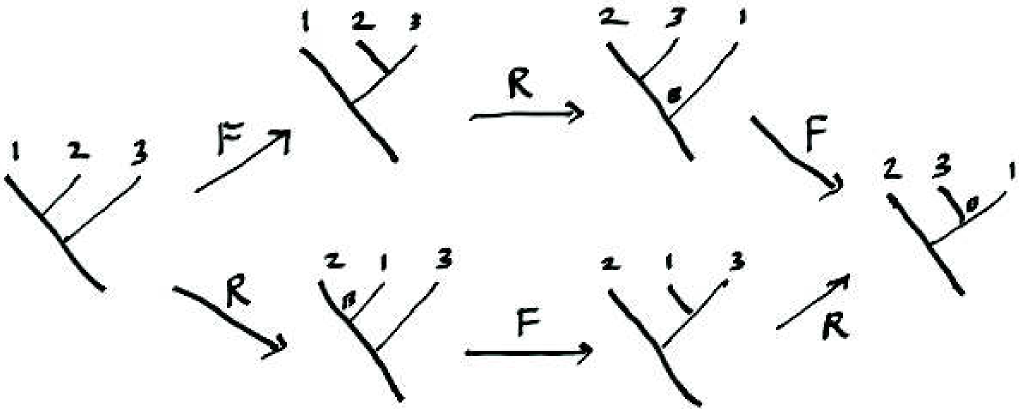


$\Leftrightarrow$ 

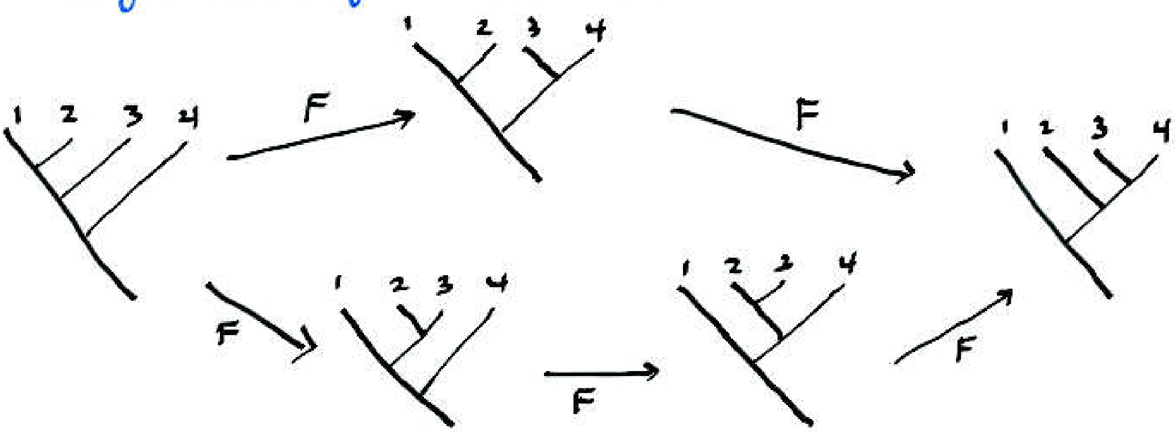
follow path of fluxons as a function of time.

But there are conditions F & R must satisfy, since there are different ways of fusing and braiding 3 & 4 fluxons to get the same final result...

Hexagon identity: "FRF = RFR"



Pentagon identity: "FFF = FF"



It turns out that these are the only independent identities R & F satisfy: all others follow from them. (In mathematics this is known as the "MacLane coherence theorem" in category theory; in physics they are the "Moore-Seiberg equations" in conformal field theory.)

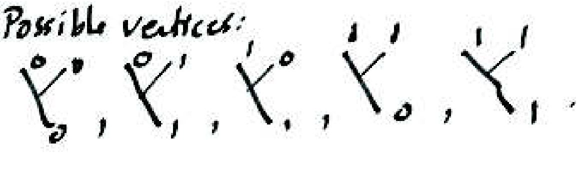
Upshot: Topological order (at least in 2-d) is related to an algebraic structure which is some kind of generalization of the theory of group representations.

Simplest non-abelian anyon (Yang-Lee model, or Fibonacci anyon)

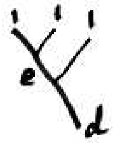
Just 2 fluxon labels: "0" = trivial (= no fluxon)
"1" = something (fluxon)

and fusion rules:
$$\begin{cases} 0 \cdot 0 = 0 \\ 0 \cdot 1 = 0 \\ 1 \cdot 1 = 0 + 1 \end{cases} \Rightarrow$$

Possible vertices:



Solve the pentagon identity. Find the only non-trivial fusion tensors are

 $= \sum_f$  $\cdot \left(F^{d}_{111}\right)^{e}_{f}$ with (though of as a 2×2 matrix on the (e,f) indices)

$$\left(F^{0}_{111}\right)^{e}_{f} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \left(F^{1}_{111}\right)^{e}_{f} = \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & -\tau \end{pmatrix} \quad \text{with } \tau \equiv \tfrac{1}{2}(\sqrt{5}-1).$$

Now solve the hexagon identity to find the only nontrivial braid tensors

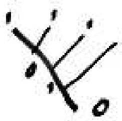
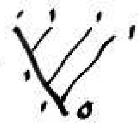
 $= R^{c}_{11}$  with $R^{0}_{11} = e^{4\pi i/5}$ and $R^{1}_{11} = -e^{2\pi i/5}$.

Call the Hilbert space of N "1" fluxons in a net "0" fluxon charge state $\mathcal{H}_N$, and let's compute $\dim(\mathcal{H}_N) \equiv d_N$.

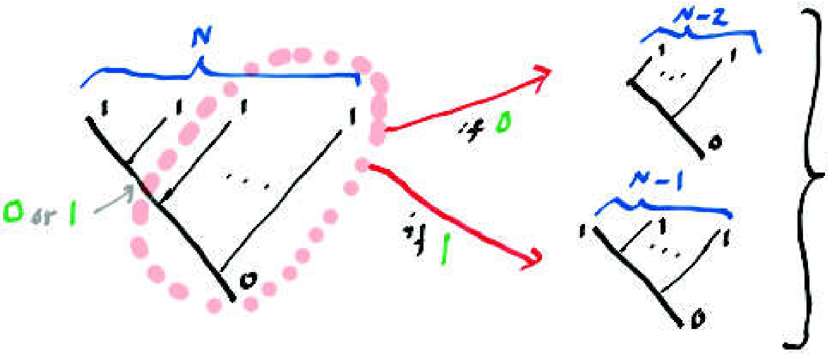
<u>N=1</u>  ← doesn't exist ⇒ $d_0 = 0$.

<u>N=2</u>  ⇒ $d_1 = 1$.

<u>N=3</u>  ← only way ⇒ $d_3 = 1$

<u>N=4</u>  &  ⇒ $d_4 = 2$ …

What is the pattern?



$\Rightarrow d_N = d_{N-1} + d_{N-2}$

$\Rightarrow d_N$ are the Fibonacci numbers.

d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	....	d_N	...
0	1	1	2	3	5	8	13	...	$c \cdot \phi^N$	...

with $\phi \equiv \frac{1}{2}(\sqrt{5} - 1) \approx 1.618$ "Golden mean."

In 'usual' QM systems, if put together system from N identical parts each with Hilbert space $\mathcal{H}_1$, then $\mathcal{H}_N = \underbrace{\mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_1}_{N}$,

implying $\dim \mathcal{H}_N = (\dim \mathcal{H}_1)^N \Rightarrow d_N = d_1^N$.

For non-abelian anyons, we have just seen $d_N \neq d_1^N$. It does obey $d_N \sim \phi^N$ for large N, but ϕ is not an integer!

This is a signature of non-locality, the "topological" nature of the ground state.

Some concluding questions:

- Is there an effective field theory descriptions of non-abelian anyons?
  - → Yes: e.g., $U(1)$ Chern-Simons gauge theory in $(2+1)$-dimensions at level $k \geq 2$.
  - → But: not known if can or how to write effective theory for all possible solutions of the hexagon and pentagon identities.

- Are topological orders in $d>2$ also all associated to a similar fusion algebra structure?
  - → Not known.

- Are there physical realizations of non-abelian anyons?
  - → Yes: certain fractional quantum Hall systems.

- Does topological order exhaust the possible kinds of quantum orders?
  - → No: there is "symmetry-protected topological order" (e.g. top. insulators, integer qu. Hall effect, Chern-Simons at $k=1$, ...)
  - → Not known whether there are others ...

Suggestions for further reading

- Renormalization, effective theories, phases:

  J. Cardy "Scaling and renormalization in statistical physics" (1996, Cambridge)

- Topological and other quantum orders:

  D. Tong "The quantum Hall effect", arXiv: 1606.06687

  J. McGreevy "TASI lectures on quantum matter", arXiv: 1606.08953

  E. Witten "Three lectures on topological phases of matter," arXiv: 1510.07698

On an "application" of non-abelian anyons to quantum computing:

J. Preskill "Chapter 9: Topological quantum computation", at www.theory.caltech.edu/people/preskill/ph229/