to the memory of Pierre Binétruy

Gravitational birefringence of light at cosmological scales

Christian Duval & Thomas Schücker

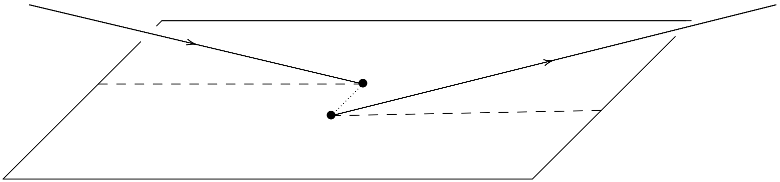

Figure: The Fedorov (1955) Imbert (1972) effect for reflection: A plane glass surface reflects an incoming, circularly polarized light beam. The dashed lines indicate the orthogonal projections of incoming and reflected light beams onto the glass surface. The dotted line (between the blobs) is the offset between incoming and reflected beams. It is of the order of the wavelength of the light beam.

O. Hosten, P. Kwiat, “Observation of the Spin Hall Effect of Light via weak measurements”, Science **319** (2008) 787–790.

K. Yu. Bliokh, A. Niv, V. Kleinert, E. Hasman, “Geometrodynamics of spinning light”, Nature Photonics **2** (2008) 748.

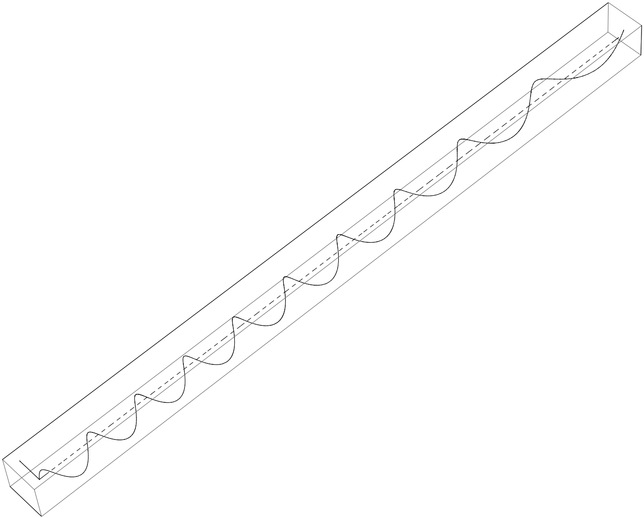

Figure: The trajectory of photons, $\mathbf{x}(t)$, in a flat Robertson-Walker universe in comoving coordinates is the helix. The dashed line is the null geodesic. The transverse spin $\mathbf{s}_e^{\perp}$ at emission time t_e is indicated by the short arrow at the left.

Express the metric in Euclidean coordinates $\mathbf{x}$ and cosmic time t:

$$\mathrm{g} = -a(t)^2 \, \|d\mathbf{x}\|^2 + dt^2$$

with scale factor $a > 0$, that we also suppose increasing, $a' > 0$.

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• Denote by $\sim$ linearization in $|\mathbf{s}_e| \, \lambda_e / (2\pi \, \hbar \, a_e)$, λ_e being the wavelength at emission. Then the instantaneous period of the helix is

$$T_{\mathrm{helix}}(t) \sim \frac{a(t)}{a_e} \, \frac{\lambda_e}{1 + q(t)} \quad \text{with} \quad q := -a \, a''(t)/a'(t)^2.$$

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- Its center at time t is located in comoving coordinates at

$$\mathbf{x}_{\mathrm{center}}(t) \sim \begin{pmatrix} x^1(t) \\ 0 \\ -\frac{\lambda_e}{2\pi \, a_e} \left(1 - a'_e x^1(t)\right) \end{pmatrix}.$$

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- The radius of the helix is

$$R_{\text{helix}}(t) \sim \frac{a(t)}{a_e} \, \frac{\lambda_e}{2\pi}.$$

Conclusions and questions

- The gravitational field of an expanding universe produces birefringence of light.
- This birefringence carries information on the acceleration of the universe.
- Can this birefringence be measured?

- Does the gravitational field of a gravitational wave also produce birefringence of light?
- If yes, what information is carried by this birefringence?
- If yes, can this birefringence be measured?

Generalizing the geodesic equation

Let $X(\tau)$ be the trajectory of a particle with spin in spacetime with
4-velocity $\dot{X} = (\dot{X}^\mu)$,
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and the *antisymmetric* spin tensor $S = (S^{\mu\nu})$.

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Their evolution is governed by the *Mathisson-Papapetrou-Dixon equations*:

$$\begin{aligned} \dot{P}^\mu &= -\frac{1}{2} R_{\alpha\beta\rho}{}^{\mu} S^{\alpha\beta} \dot{X}^\rho, \\ \dot{S}^{\mu\nu} &= P^\mu \dot{X}^\nu - P^\nu \dot{X}^\mu, \end{aligned}$$

where $R = (R_{\alpha\beta\rho}{}^{\mu})$ is the Riemann tensor of the metric g, and here (!) the overdot means covariant derivative along the wordline.

We have to add an equation of state:

$$SP = 0,$$

which implies $P^2 = P_\mu P^\mu = \mathrm{const}$ & $\mathrm{Tr}(S^2) = -S_{\mu\nu}S^{\mu\nu} = \mathrm{const}$.
For photons we set

$$P^2 = 0 \qquad \& \qquad -\frac{1}{2}\mathrm{Tr}(S^2) = s^2\,,$$

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Then the resulting equations of motion read [Souriau-Saturnini'76]

$$\begin{aligned} \dot{X} &= P + \frac{2}{R(S)(S)}\, S \cdot R(S) \cdot P\,, \\ \dot{P} &= -s\, \frac{\sqrt{-\det(R(S))}}{R(S)(S)}\, P\,, \\ \dot{S} &= P \wedge \dot{X}\,, \end{aligned}$$

where $R(S)(S) := R_{\mu\nu\rho\sigma}S^{\mu\nu}S^{\rho\sigma}$ must not vanish!

Flat Robertson Walker metrics and $3+1$ decomposition

- The 4-momentum of the photon is decomposed as

$$P = \begin{pmatrix} \frac{1}{a}\mathbf{p} \\ \|\mathbf{p}\| \end{pmatrix}, \qquad \mathbf{p} \in \mathbb{R}^3 \setminus \{0\},$$

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$$S^{\cdot}{}_{\cdot} = \begin{pmatrix} j(\mathbf{s}) & -\dfrac{(\mathbf{s}\times\mathbf{p})}{a||\mathbf{p}||} \\ -\dfrac{a\,(\mathbf{s}\times\mathbf{p})^T}{||\mathbf{p}||} & 0 \end{pmatrix}, \quad \mathbf{s} \in \mathbb{R}^3, \quad j(\mathbf{s})\cdot := \mathbf{s}\times\cdot\,.$$

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$$\begin{aligned}
\frac{d\mathbf{x}}{dt} &= \frac{1}{a}\left[-q\frac{\mathbf{p}}{\|\mathbf{p}\|} + (1+q)\,\frac{\mathbf{s}}{s}\right], \\
\frac{d\mathbf{p}}{dt} &= -\frac{a'}{a}\left[-q\,\mathbf{p} + \|\mathbf{p}\|\,(1+q)\,\frac{\mathbf{s}}{s}\right], \\
\frac{d\mathbf{s}}{dt} &= (1+q)\,\frac{\mathbf{s}}{s}\times\mathbf{p} - \frac{a'}{a}\,\mathbf{s} + \frac{a'}{a}\left[\frac{\|\mathbf{s}\|^2}{s}\,(1+q) - s\,q\right]\frac{\mathbf{p}}{\|\mathbf{p}\|}\,.
\end{aligned}$$

Conservation laws: Noetherian quantities

From invariance under translations and rotations we get:

$$\mathcal{P} = -a\mathbf{p} + a'\mathbf{s} \times \frac{\mathbf{p}}{||\mathbf{p}||} = \text{const},$$

$$\mathcal{L} = \mathbf{x} \times \mathcal{P} + \mathbf{s} = \text{const}.$$

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These 7 constants of motion imply 2 more, “the scalar spin” and “the transverse spin”:

$$s = \frac{\mathbf{s}\cdot\mathbf{p}}{||\mathbf{p}||} = \mathrm{const}, \qquad \mathcal{S} = a'\,||\mathbf{s}^{\perp}|| = \mathrm{const},$$

with $\mathbf{s}^{\perp} := \mathbf{s} - s\,\mathbf{p}/||\mathbf{p}||$.

Numerical integration in flat ΛCDM

$$a(t) \;=\; a_0 \left(\frac{\cosh[\,\sqrt{3\Lambda}\,t] - 1}{\cosh[\,\sqrt{3\Lambda}\,t_0] - 1} \right)^{1/3}$$

Our strategy:

- Express spin $\mathbf{s}(t)$ and momentum $\mathbf{p}(t)$ in terms of conserved quantities $\mathcal{P}$, $\mathcal{L}$, $\mathcal{E}$, s and $\mathcal{S}$; plug those into the equation for the velocity $d\mathbf{x}/dt$.
- We remain with 3 equations in 3 unknowns $\mathbf{x}(t)$:

$$\frac{d\mathbf{x}}{dt} = A(t)\,\mathbf{x} + B(t).$$

- Integrate the latter using the Runge-Kutta algorithm with initial conditions at emission time t_e:

$$\mathbf{x}_e = 0, \qquad \mathbf{p}_e = \begin{pmatrix} \|\mathbf{p}_e\| \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{s}_e = \begin{pmatrix} s \\ s_e^{\perp} \\ 0 \end{pmatrix}.$$

Explicit expression for the velocity

Let us define

$$F(t) := \frac{q(t)/a(t)}{\mathcal{E}\left[1 + \frac{s^2}{\mathcal{E}^2} a'(t)^2 + \frac{\mathcal{S}^2}{\mathcal{E}^2}\right]} .$$

Then the velocity equation reads:

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \left\{ F \left[-\frac{a'}{\mathcal{E}} \, j(\mathcal{P})^2 + \frac{a'^2}{\mathcal{E}^2} (\mathcal{L} \cdot \mathcal{P}) \, j(\mathcal{P}) \right] + \frac{1}{a\,s} \, (1+q) \, j(\mathcal{P}) \right\} \mathbf{x} \\ & \quad + F \left[\mathcal{P} + \frac{a'}{\mathcal{E}} \mathcal{L} \times \mathcal{P} + \frac{a'^2}{\mathcal{E}^2} (\mathcal{L} \cdot \mathcal{P}) \, \mathcal{L} \right] + \frac{1}{a\,s} \, (1+q) \, \mathcal{L} \, . \end{aligned}$$

• Special solutions: straight lines

$$\mathbf{s}_e^\perp = 0 \quad \Rightarrow \quad \tilde{\mathbf{x}}(t) = \frac{\mathbf{p}_e}{\|\mathbf{p}_e\|} \int_{t_e}^{t} \frac{d\tau}{a(\tau)}, \quad \mathbf{p}(t) = \frac{a_e}{a(t)}\mathbf{p}_e, \quad \mathbf{s}(t) = s\,\frac{\mathbf{p}_e}{\|\mathbf{p}_e\|}$$

These are the null geodesics (spin is “enslaved”).

[1] Astro-units such that: $c = 1\,\mathrm{am}/\mathrm{as}$, $\hbar = 1\,\mathrm{ag}\,\mathrm{am}^2/\mathrm{as}$ and $H_0 = 1/\mathrm{as}$.

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• “Precessing” solutions:
Realistic initial conditions $s_e^\perp = \hbar$ (Quantum Mechanics) and e.g. $\lambda_{\mathrm{Ly}_\alpha} = 8.72 \cdot 10^{-34}\,\mathrm{am}$,[1] $z = 2.4$. Then with $\Lambda = 3 \cdot 0.685/\mathrm{am}^2$ and $t_0 = 0.951\,\mathrm{as}$ the time of emission is $t_e = 0.188\,\mathrm{as}$.

For a more modest $\lambda = 1.2 \cdot 10^{-2}\,\mathrm{am}$, Runge & Kutta readily tell us:

- ⋆ $R(S)(S) > 0$.
- ⋆ The longitudinal offset of the trajectory from its companion null geodesic is

$$|x^1(t) - \tilde{x}^1(t)| = \mathcal{O}(\epsilon^2)\,, \quad \epsilon := s_e^\perp/\mathcal{E}\,.$$

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Perturbative solutions

We return to *generic*, flat RW spacetimes and *linearize* the equations of motion w.r.t. the small dimensionless parameters

$$\eta := \frac{s}{\mathcal{E}} \qquad \& \qquad \epsilon := \frac{s_e^{\perp}}{\mathcal{E}}.$$

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Put $(x^1, x^2, x^3) = (\tilde{x}^1, \epsilon y^2, \epsilon y^3) + \mathcal{O}(\epsilon^2)$ and linearize $d\mathbf{x}/dt$:

$$\frac{dx^1}{dt} \sim \frac{1}{a} + a_e' \, \frac{1+q}{a} \, y^2 \frac{\epsilon^2}{\eta} \, ,$$

$$\frac{dy^2}{dt} \sim \frac{1+q}{a} \left[1 - a_e' x^1 + y^3\right] \frac{1}{\eta} \, ,$$

$$\frac{dy^3}{dt} \sim - \; \frac{1+q}{a} \, y^2 \frac{1}{\eta}.$$

Birefringence

- Recall that $(x^1(t), 0, 0)$ is (up to second order terms) the null geodesic; with the change of time coordinate

$$t \mapsto \theta(t) \sim \frac{1}{|\eta|}\left[x^1(t) + \frac{1}{a'(t)} - \frac{1}{a'_e}\right]$$

the transverse trajectory is now governed by the equations

$$\frac{dy^2}{d\theta} \sim \mathrm{sign}(\eta)\,(y^3 + 1 - a'_e x^1), \qquad \frac{dy^3}{d\theta} \sim -\mathrm{sign}(\eta)\, y^2.$$

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- With the previous initial conditions and setting $\epsilon = |\eta|$, we obtain:

$$y^2(t) \sim \mathrm{sign}(\eta)\sin\theta(t) \qquad \& \qquad y^3(t) \sim \cos\theta(t) - 1 + a'_e x^1(t).$$

The trajectory is therefore a Left/Right helix depending on the helicity $\mathrm{sign}(\eta) = \mathrm{sign}(s)$ of the photon, i.e. birefringence of light.