Review of Relativistic Quantum Mechanics

(Lecture notes ASP2016 - Kigali)

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Abstract

These lecture notes give a short review of fundamental aspects of Relativistic Quantum Mechanics, with the aim to preparing students for lectures on the Standard Model.

Contents

1 Special Relativity

1.1 Fundamentals

A ray of light as a speed c which is independent of the inertial observer who measures it. As a consequence, the time t and space coordinates $\vec{r}$ corresponding to a massive particle must be measured differently by different inertial observers. The invariance of the speed of light implies that the infinitesimal motion of the particle is such that

$$c^2dt^2 - (d\vec{r})^2 = c^2d\tau^2 \ , \tag{1}$$

where $d\tau$ is the infinitesimal proper time measured by a clock in the particle rest frame. The velocity of the massive particle, measured in this inertial frame, is $\vec{v} = d\vec{r}/dt$, such that eq.(1) implies

$$d\tau = \frac{dt}{\gamma(v)} \equiv dt\sqrt{1 - v^2/c^2} \ . \tag{2}$$

In Newtonian Mechanics, the time t is independent of the inertial frame, such that a frame-independent definition of the particle momentum is $\vec{p} = md\vec{r}/dt$. In Relativistic Mechanics, it is rather the proper time of the particle which is independent of the frame where it is calculated, and the frame-independent definition of the particle momentum is then

$$\vec{p} \equiv m\frac{d\vec{r}}{d\tau} = \gamma(v)m\vec{v} \ . \tag{3}$$

From the identity (1) it is then easy to see that

$$[\gamma(v)mc^2]^2 = [mc^2]^2 + [pc]^2 \ , \tag{4}$$

and a Taylor expansion of the left-hand side gives

$$\gamma(v)mc^2 = mc^2 + \frac{1}{2}mv^2 + \cdots \ , \tag{5}$$

where dots represent higher orders in v^2/c^2. As a consequence, the energy of the free particle is

$$E = \gamma(v)mc^2 \ , \tag{6}$$

and contains: the rest energy mc^2, the Newtonian kinetic energy $mv^2/2$ and relativistic corrections (dots). The dispersion relation is thus

$$E^2 = m^2c^4 + p^2c^2 \ , \tag{7}$$

and is characteristic of a relativistic free motion.

1.2 Example: particle submitted to a constant and uniform force

Consider a particle of mass m, initially at rest at $x = 0$, moving along the x-axis under the influence of the constant and uniform force $\vec{f} = f\vec{e_x}$. The potential energy from which the force derives is $-fx$, and energy conservation reads

$$\gamma mc^2 - fx = \gamma(0)mc^2 - fx(0) = mc^2 \ , \tag{8}$$

such that

$$\gamma(\tau) = 1 + \frac{fx(\tau)}{mc^2} \ . \tag{9}$$

The relativistic equation of motion

$$\frac{d}{dt}\left(m\gamma\frac{dx}{dt}\right) = f \tag{10}$$

can be written in terms of the particle proper time τ as

$$m\frac{d^2x}{d\tau^2} = \gamma f \ , \tag{11}$$

where the relation (2) was used. Using the previous expression for γ, we find

$$\frac{d^2x}{d\tau^2} - \frac{f^2x}{m^2c^2} = \frac{f}{m} \ , \tag{12}$$

which, given the initial conditions, has the solution

$$x(\tau) = \frac{mc^2}{f}\left[\cosh\left(\frac{f\tau}{mc}\right) - 1\right] \ . \tag{13}$$

The gamma factor is then

$$\gamma = \cosh\left(\frac{f\tau}{mc}\right) \ , \tag{14}$$

and one can calculate the speed of the particle, measured in the laboratory frame:

$$\frac{dx}{dt} = \frac{1}{\gamma}\frac{dx}{d\tau} = c\tanh\left(\frac{f\tau}{mc}\right) \ . \tag{15}$$

As expected, the speed asymptotically goes to c when $\tau \to \infty$, and it does not diverge linearly as in the Newtonian case.

1.3 Covariant notations

In what follows we set $c = 1$.
The aim of covariant notations is to provide compact expressions for relativistic quantities, by extending the usual 3-dimensional space vector notations to 4-dimensional spacetime. We define the 4-vector position

$$x^\mu \equiv (t, \vec{r}) \ , \tag{16}$$

where the Greek index μ runs from 0 to 3. The infinitesimal change in proper time (2) can then be written

$$d\tau^2 = dx^\mu dx^\nu \eta_{\mu\nu} \ , \tag{17}$$

where a repeated index implies the summation over its values, and the matrix $\eta_{\mu\nu}$ is the Minkowski metric

$$\eta_{\mu\nu} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{18}$$

One also defines $x_\mu \equiv \eta_{\mu\nu} x^\nu = (t, -\vec{r})$, such that $d\tau^2 = dx^\mu dx_\mu$. Note that one can also define the opposite convention $\eta_{\mu\nu}$ =diag$(-1, 1, 1, 1)$.
The derivative of a quantity with respect to x^μ gives a lower index and vice versa:

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} \quad \text{and} \quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu} \ . \tag{19}$$

The 4-momentum is

$$p^\mu \equiv m \frac{dx^\mu}{d\tau} = (E, \vec{p}) \ , \tag{20}$$

and the dispersion relation (7) reads then $p^\mu p_\mu = m^2$. Finally, one defines $\eta_{\mu\nu}$ as the elements of the inverse metric $\eta^{\mu\nu}$, which have the same values as shown in the expression (18), since the metric is diagonal, with unit matrix elements. The trace of Minkowski metric (18) is then

$$\eta_{\mu\nu}\eta^{\mu\nu} = 4 \ . \tag{21}$$

1.4 Classical Electrodynamics

We know that the electromagnetic field can be expressed in terms of the potentials $V, \vec{A}$ as

$$\begin{aligned} \vec{B} &= \vec{\nabla} \times \vec{A} \\ \vec{E} &= -\vec{\nabla} V - \partial_t \vec{A} \ , \end{aligned} \tag{22}$$

and these potentials are not unique: the physical fields $\vec{E}$ and $\vec{B}$ are invariant under the gauge transformation

$$\begin{aligned} \vec{A} &\rightarrow \vec{A} + \vec{\nabla}\Lambda \\ V &\rightarrow V - \partial_t \Lambda \ , \end{aligned} \tag{23}$$

where Λ is any differentiable function of spacetime coordinates. One then defines the 4-vector potential

$$A^\mu \equiv (V, \vec{A}) \ , \tag{24}$$

and the gauge transformation (23) can be written

$$A^\mu \ \rightarrow \ A^\mu - \partial^\mu \Lambda \ . \tag{25}$$

The field strength tensor is

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} , \tag{26}$$

from which the Maxwell equations read

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= j^\nu \\ \partial^\rho F^{\mu\nu} + \partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} &= 0 \ , \end{aligned} \tag{27}$$

where the 4-current $j^\nu = (\rho, \vec{j})$ is made from the charge density ρ and the current density $\vec{j}$, and the second identity is a consequence of the definition of $F^{\mu\nu}$.

Electric charge conservation can be derived from the continuity equation as follows. Given the antisymmetric nature of $F^{\mu\nu}$, one necessarily has

$\partial_\mu \partial_\nu F^{\mu\nu} = 0$, which, together with the equation of motion (27), leads to current conservation $\partial_\nu j^\nu = 0$. In terms of the charge and current densities, this takes the form of the continuity equation

$$\partial_t \rho + \vec{\nabla} \cdot \vec{j} = 0 \ . \tag{28}$$

The latter can be integrated over a 3-dimensional volume $\mathcal{V}$ to give

$$\frac{dQ}{dt} + \int_{\mathcal{V}} \vec{\nabla} \cdot \vec{j}\ d^3x = 0 \ , \tag{29}$$

where Q is the total charge, and Gauss theorem leads then to

$$\frac{dQ}{dt} + \oint_{\partial\mathcal{V}} \vec{j} \cdot \vec{n}\ d^2a = 0 \ , \tag{30}$$

where ∂V is the closed boundary of $\mathcal{V}$, with unit normal vector $\vec{n}$ at each point. The latter relation is valid whatever the volume $\mathcal{V}$, which can be taken to infinity. The flux of $\vec{j}$ across its boundary $\partial\mathcal{V}$ goes then to 0, since the sources are localised in space, and the total electric charge satisfies $dQ/dt = 0$, which shows charge conservation.

1.5 Action for the Electromagnetic field

We show here that the action from which the equation of motion (27) arises is

$$S_{EM} = \int d^4x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - j_\mu A^\mu \right) \ . \tag{31}$$

For this, we note that

$$\frac{\delta}{\delta A_\nu(x)} \int d^4y\ j_\rho(y) A^\rho(y) = \int d^4y\ j_\rho(y) \delta^{(4)}(x-y) \eta^{\rho\nu} = j^\nu(x) \ , \tag{32}$$

and

$$\begin{aligned} \frac{\delta}{\delta A_\nu(x)} \int d^4y\ \partial^\sigma A^\rho(y) \partial_\rho A_\sigma(y) &= 2 \int d^4y\ \partial_\sigma A_\rho(y) \partial^\rho \delta^{(4)}(x-y) \eta^{\nu\sigma} \\ &= -2 \partial^\nu \partial^\rho A_\rho \ . \end{aligned} \tag{33}$$

as well as

$$\frac{\delta}{\delta A_\nu(x)} \int d^4y\ \partial^\sigma A^\rho(y) \partial_\sigma A_\rho(y) = 2 \int d^4y\ \partial_\sigma A_\rho(y) \partial^\sigma \delta^{(4)}(x-y) \eta^{\nu\rho}$$
$$= -2\Box A^\nu(x) \ , \tag{34}$$

where

$$\Box \equiv \partial^\sigma \partial_\sigma = \frac{\partial^2}{\partial t^2} - \nabla^2 \ . \tag{35}$$

Note that the surface terms arising from the integration by parts vanish, because fields are assumed to decrease quickly enough at infinity. The equation of motion (27) is then obtained from the variational principle

$$0 = \frac{\delta S_{EM}}{\delta A_\nu(x)} = \Box A^\nu - \partial^\nu \partial^\rho A_\rho - j^\nu$$
$$= \partial_\mu F^{\mu\nu} - j^\nu \ . \tag{36}$$

2 Quantum Mechanics

2.1 Heisenberg uncertainties

Particle-wave duality can be expressed by the equivalence of the two monochromatic plane waves representations

$$\exp(i\vec{k}\cdot\vec{x} - i\omega t) = \exp\left(i\frac{\vec{p}\cdot\vec{x}}{\hbar} - i\frac{Et}{\hbar} \right) \ . \tag{37}$$

and a general signal can be represented as a wave packet, sum of different plane waves with different weights. For simplicity we consider a wave packet $\psi(x,t)$ propagating in one dimension, with the relativistic dispersion relation $E^2 = m^2 + p^2$, and the Gaussian weight

$$A \exp\left(-\frac{(p-p_0)^2}{\Delta p^2} \right) \ . \tag{38}$$

This weight is centered on the momentum p_0 and has the width Δp. In the limit where $m << p_0$, the energy is $E \simeq |p|$ and the wave packet is

$$\psi(t,x) = A \int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp\left(\frac{ipx}{\hbar} - \frac{iEt}{\hbar} - \frac{(p-p_0)^2}{\Delta p^2} \right) \tag{39}$$
$$\simeq Ae^{ip_0(x-t)/\hbar} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp\left(\frac{i}{\hbar} p(x-t) - \frac{p^2}{\Delta p^2} \right) \ , \tag{40}$$

where the mistake made from the replacement $|p| \to p$ is negligible if the main contribution of the integrand occurs for p around $p_0 > 0$. The Gaussian integral gives then

$$\psi(x,t) \propto \exp\left(\frac{i}{\hbar}p_0(x-t) - \frac{(x-t)^2}{\Delta x^2}\right) , \tag{41}$$

where $\Delta x \equiv 2\hbar/\Delta p$. The wave packet features two characteristics:
• The wave packet propagates with the momentum p_0;
• The wave packet has a Gaussian shape, centered on the event $x = t$, with width satisfying $\Delta x \Delta p = 2\hbar$. Given the dispersion relation $E = |p|$, the width in energy and the width in time are also related by $\Delta t \Delta E = 2\hbar$.

More generally, the Fourier transform relating a wave/particle to its spectrum, is such that the widths satisfy the Heisenberg uncertainties

$$\Delta x \Delta p \simeq \Delta t \Delta E \simeq \hbar . \tag{42}$$

2.2 States and operators

The static wave function

$$\psi(x) = \int \frac{dp}{2\pi} f(p) e^{ipx/\hbar} \tag{43}$$

can be seen as a vector living in an infinite-dimensional vector space (Hilbert space), with coordinates $f(p)$ in the basis provided by the exponentials $\exp(ipx/\hbar)$. The scalar product is defined as

$$< \psi_1|\psi_2 > \equiv \int dx \psi_1^\star \psi_2 = \int \frac{dp}{2\pi} f_1^\star(p) f_2(p) , \tag{44}$$

and $|\psi|^2 =< \psi|\psi >$ is the probability density to have the system in the state ψ. Given that

$$- i\hbar \frac{\partial}{\partial x}\, e^{ipx/\hbar} = p e^{ipx/\hbar} , \tag{45}$$

$\exp(ipx/\hbar)$ is an eigenvector of the space-derivative operator $-i\hbar\partial_x$, with eigenvalue p. The space derivative is then identified with the momentum operator $\hat{p}$. The position operator $\hat{x}$ acts as a multiplication by x and, given the identity $\partial_x(x\psi) = \psi + x\partial_x\psi$ for any function $\psi(x)$, we have

$$[\hat{p}, \hat{x}] \equiv \hat{p}\hat{x} - \hat{x}\hat{p} = -i\hbar\mathbf{1} , \tag{46}$$

where $\mathbf{1}$ is the identity operator.

2.3 Schrödinger equation

We consider here a wave function $\psi(\vec{x},t)$ depending on space and time in 3 space dimensions. Since

$$i\hbar\frac{\partial}{\partial t}\, e^{-iEt/\hbar} = Ee^{-iEt/\hbar}\ , \tag{47}$$

$\exp(-iEt/\hbar)$ is eigenvector of the time-derivative operator $i\hbar\partial_t$ with eigenvalue E. The time derivative is then identified with the Hamiltonian operator $\hat{H}$, and the time evolution of a state ψ is given by the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi\ , \tag{48}$$

where the specific form of $\hat{H}$ depends on the system studied.
Quantisation is based on the remarks (45) and (47), by replacing physical quantities by operators

$$E \to i\hbar\frac{\partial}{\partial t} \quad \text{and} \quad \vec{p} \to \hat{\vec{p}} = -i\hbar\vec{\nabla}\ , \tag{49}$$

which act on the wave function of the system, in the Hilbert space of states. For a free particle of mass m, the Newtonian dispersion relation $E = p^2/(2m)$ thus leads to

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2\nabla^2}{2m}\psi\ . \tag{50}$$

In the situation where the system interacts with a source with potential $V(\vec{x})$, the Schrödinger equation is

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2\nabla^2}{2m}\psi + V(\vec{x})\psi\ . \tag{51}$$

If one then looks for stationary solutions of the form

$$\psi(\vec{x},t) = e^{-iEt/\hbar}\chi(\vec{x})\ , \tag{52}$$

the Schrödinger equation (48) consists in solving the eigenvector problem

$$\left(-\frac{\hbar^2\nabla^2}{2m} + V(\vec{x})\right)\chi(\vec{x}) = E\chi(\vec{x})\ , \tag{53}$$

and quantisation of energy levels arises from the boundary conditions.

2.4 Conserved current

A continuity equation of the form (28) can be obtained from the free Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = \frac{-\hbar}{2m}\nabla^2\psi \ . \tag{54}$$

Multiplying both sides by the complex conjugate $\psi^\star$ and taking the complex conjugate of the whole equation leads to

$$\begin{aligned} i\frac{\partial\psi}{\partial t}\psi^\star &= \frac{-\hbar}{2m}\psi^\star\nabla^2\psi \\ -i\frac{\partial\psi^\star}{\partial t}\psi &= \frac{-\hbar}{2m}\psi\nabla^2\psi^\star \ , \end{aligned} \tag{55}$$

and the difference of these two equation reads

$$\frac{\partial(\psi\psi^\star)}{\partial t} = \frac{i\hbar}{2m}(\psi^\star\nabla^2\psi - \psi\nabla^2\psi^\star) = -\frac{i\hbar}{2m}\vec{\nabla}\cdot(\psi\vec{\nabla}\psi^\star - \psi^\star\vec{\nabla}\psi) \ . \tag{56}$$

The latter identity can be written as eq.(28), where

$$\rho = |\psi|^2 \tag{57}$$

$$\vec{j} = \frac{i\hbar}{2m}\left(\psi\vec{\nabla}\psi^\star - \psi^\star\vec{\nabla}\psi\right) \ , \tag{58}$$

and the continuity equation obtained in Quantum Mechanics corresponds to the conservation of probabilities.

2.5 Bra and Ket

The interpretation of Quantum Mechanics in the context of linear algebra allows a more abstract formalism, independently of the representation in terms of functions of space and time. A state is represented by the bra $|\psi\rangle$, which is decomposed on a basis $|k\rangle$ as

$$|\psi\rangle = \sum_k a_k\,|k\rangle \ , \tag{59}$$

and the Hermitian conjugate of the bra is the ket $\langle k| \equiv |k\rangle^\dagger$. The basis is orthonormal if

$$\langle k|\, l\rangle = \delta_{kl} \ , \tag{60}$$

and the projector operator on a specific vector basis $|k\rangle$ is $P_k = |k\rangle\langle k|$. The sum over all the projectors must be equal to the identity since, for any state of the form (59), we have

$$\sum_k |k\rangle\langle k|\,|\psi\rangle = \sum_{k,l} a_l\,|k\rangle\langle k|\,|l\rangle = \sum_{k,l} a_l\delta_{kl}\,|k\rangle = \sum_k a_k\,|k\rangle = |\psi\rangle \ , \tag{61}$$

such that

$$\sum_k |k\rangle\langle k| = \mathbf{1} \ . \tag{62}$$

2.6 Fermi's Golden Rule

A Hamiltonian $\hat{H}^0$ has eigenstates $|n\rangle$ with eigenvalues E_n, such that $\hat{H}^0|n\rangle = E_n|n\rangle$. We assume a time-independent perturbation $\epsilon\hat{H}^1$ to the Hamiltonian, with $\epsilon << 1$, which induces transitions between the different unperturbed states $|n\rangle$. The aim is to calculate the transition probability from an initial state $|i\rangle$, with energy E_i, to a final state $|f\rangle$, with energy E_f (we first assume the energy levels to be discrete). The Hamiltonian eigenstates form an orthonormal basis for the Hilbert space

$$\langle n|\,m\rangle = \delta_{nm} \ . \tag{63}$$

The time-dependent state of the system $|\psi(t)\rangle$ is decomposed on the basis of unperturbed states

$$|\psi(t)\rangle = \sum_n a_n(t)\,|n\rangle \ , \tag{64}$$

and satisfies the Schrödinger equation

$$\begin{aligned} i\hbar\partial_t\,|\psi(t)\rangle &= (\hat{H}^0 + \epsilon\hat{H}^1)\,|\psi(t)\rangle & (65)\\ &= \sum_n a_n(t)(\hat{H}^0 + \epsilon\hat{H}^1)\,|n\rangle \\ &= \sum_n a_n(t)(E_n + \epsilon\hat{H}^1)\,|n\rangle \ . & (66)\end{aligned}$$

We project this equation on the state $|m\rangle$ to obtain (the only time dependence is in $a_n(t)$)

$$i\hbar\sum_n a_n'(t)\,\langle m|\,n\rangle = \sum_n a_n(t)\left(E_n\,\langle m|\,n\rangle + \epsilon\,\langle m|\,\hat{H}^1\,|n\rangle\right) \ , \tag{67}$$

where a prime denotes a time derivative. Taking into account the condition (63), we obtain then

$$i\hbar a_m'(t) = E_m a_m(t) + \epsilon \sum_n H^1_{mn} a_n(t) \ , \tag{68}$$

where $H^1_{nm} \equiv \langle m| \hat{H}^1 |n\rangle$ is the perturbation matrix element for the transition form $|n\rangle$ to $|m\rangle$.

Zeroth order solution
Neglecting the perturbation, we obtain from eq.(68)

$$a_m(t) = a_m(0) \exp\left(-\frac{i}{\hbar} E_m t\right) + \mathcal{O}(\epsilon) \ , \tag{69}$$

which is expected for the stationary states of the unperturbed Hamiltonian $\hat{H}^0$.

First order solution
Taking into account the 0^{th} order solution (69), the equation (68) can be written

$$i\hbar a_m'(t) = E_m a_m(t) + \epsilon \sum_n H^1_{mn} a_n(0) e^{-iE_n t/\hbar} + \mathcal{O}(\epsilon^2) \ . \tag{70}$$

The initial state is such that $a_n(0) = \delta_{in}$, such that, when ignoring terms of order ϵ^2, the coefficient $a_f(t)$ satisfies

$$i\hbar a_f'(t) = E_f a_f(t) + \epsilon H^1_{fi} \ e^{-iE_i t/\hbar} \ , \tag{71}$$

which can also be written

$$i\hbar \partial_t \left(a_f(t) e^{iE_f t/\hbar}\right) = \epsilon H^1_{fi} \ e^{i\omega_{fi} t} \ , \tag{72}$$

with $\hbar\omega_{fi} \equiv E_f - E_i$. For $f \neq i$, the initial condition is $a_f(0) = 0$, and the previous equation is solved as

$$i\hbar a_f(t) = \frac{\epsilon H^1_{fi}}{i\omega_{fi}} \left(e^{-iE_i t/\hbar} - e^{-iE_f t/\hbar}\right) \tag{73}$$

$$= 2\epsilon H^1_{fi} \ \frac{\sin(\omega_{fi} t/2)}{\omega_{fi}} \ e^{-i(E_f + E_i)t/2\hbar} \ . \tag{74}$$

The transition probability is finally

$$P_{fi} \equiv |\langle f|\ \psi(t)\rangle|^2 = |a_f(t)|^2 = 4(\epsilon H^1_{fi})^2\ \frac{\sin^2(\omega_{fi}t/2)}{(\hbar\omega_{fi})^2}\ . \tag{75}$$

Transition to the continuum
In the situation where E_f is in a continuum of energies (scattering state), the result (75) is replaced by

$$P_{fi} \to P = 4\int_{-\infty}^{\infty} dE_f\ \rho(E_f)(\epsilon H^1_{fi})^2\ \frac{\sin^2(\omega t/2)}{(\hbar\omega)^2}\ , \tag{76}$$

where $\rho(E_f)$ denotes the density of energy states. In realistic situations, ρ is a smooth function of the energy, and the integrand is thus sharply peaked around the value $\omega = 0$ (or $E_f = E_i$), such that

$$\begin{aligned} P &\simeq \frac{2t}{\hbar}\rho(E_i)(\epsilon H^1_{fi})^2\int_{-\infty}^{\infty} dx\ \frac{\sin^2(x)}{x^2} \\ &= \frac{2\pi t}{\hbar}\rho(E_i)(\epsilon H^1_{fi})^2\ . \end{aligned} \tag{77}$$

The transition rate is therefore constant, which is Fermi's Golden Rule:

$$\frac{dP}{dt} = \frac{2\pi}{\hbar}\rho(E_i)(\epsilon H^1_{fi})^2 \tag{78}$$

3 Relativistic Quantum Mechanics

In this section we set $c = \hbar = 1$

3.1 Klein-Gordon equation

The Schrödinger equation is obtained by quantisation of the non-relativistic dispersion relation $E = p^2/(2m)$, based on the replacements (49). Making the same replacements in the relativistic dispersion relation $E^2 = m^2 + p^2$ leads to the Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi(x) = 0\ , \tag{79}$$

where ϕ is a real scalar field and x is a collective notation for the coordinates x^μ. By definition, a scalar field is invariant under a change of inertial frame: $\phi'(x') = \phi(x)$ and

$$\left(\frac{\partial^2}{\partial t'^2} - \nabla'^2 + m^2\right)\phi'(x') = 0 \ , \tag{80}$$

where primes denote another inertial frame. The action for the real scalar field, for which the variational principle leads to the equation of motion (79), is

$$S_{KG} = \frac{1}{2}\int d^4x \left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) \ . \tag{81}$$

Indeed, the functional derivative of the action leads to

$$\begin{aligned}\frac{\delta S_{KG}}{\delta\phi(x)} &= \int d^4y \left(\partial_\mu\phi\partial^\mu\delta^{(4)}(x-y) - m^2\phi\delta^{(4)}(x-y)\right) \\ &= -\int d^4y \left(\partial^\mu\partial_\mu\phi + m^2\phi\right)\delta^{(4)}(x-y) \\ &= -\left(\Box + m^2\right)\phi(x) \ ,\end{aligned} \tag{82}$$

where surface terms are omitted. The action for a free complex scalar field is

$$\tilde{S}_{KG} = \int d^4x \left(\partial_\mu\phi\partial^\mu\phi^\star - m^2\phi\phi^\star\right) \ , \tag{83}$$

and the equation of motion for ϕ is obtained as

$$\frac{\delta\tilde{S}_{KG}}{\delta\phi^\star} = 0 \ . \tag{84}$$

3.2 Dirac equation

The relativistic dispersion relation (7) has two solutions: $E = \pm\sqrt{m^2+p^2}$, that Dirac interpreted as the energies of the particle and the antiparticle. Dirac then looked for the equation of motion which corresponds to the "square root" dispersion relation, and thus which involves differential operators $\mathcal{D}_\pm$ such that

$$\mathcal{D}_+\mathcal{D}_- = \Box + m^2 \ . \tag{85}$$

The simplest solution is

$$\mathcal{D}_\pm = \pm i\gamma^\mu\partial_\mu - m \ , \tag{86}$$

where the 4-vector γ^μ must satisfies $\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$, in order to obtain

$$\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = \frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\partial_\mu\partial_\nu = \partial_\mu\partial^\mu = \Box \; . \tag{87}$$

Since γ^μ do not commute, they must be matrices, such that the wave function ψ for which the equations of motion are $D_\pm\psi = 0$ must have several components. In 4-dimensional space time, a massive fermion has 4 components and represents a 1/2-spin particle. It contains 4 degrees of freedom: 2 spin states for the particle and 2 spin states for the antiparticle. The Dirac equation is then

$$(i\not{\partial} - m)\psi = 0 \quad \text{where } \not{\partial} \equiv \gamma^\mu\partial_\mu \; . \tag{88}$$

3.3 Properties of the gamma matrices

We list here few fundamental properties of the gamma matrices, which are necessary to calculate a Feynman graph involving fermions.

$$\begin{aligned} \{\gamma^\mu, \gamma^\nu\} &= 2\eta^{\mu\nu}\, \mathbf{1} \\ \gamma^{\mu\dagger} &= \gamma^0\gamma^\mu\gamma^0 \\ \mathrm{tr}\,(\gamma^\mu) &= 0 \\ \mathrm{tr}\,(\gamma^\mu\gamma^\nu) &= 4\eta^{\mu\nu} \;, \end{aligned} \tag{89}$$

where $\mathbf{1}$ is the unit matrix with respect to Dirac indices.
Also, one defines the matrix $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$, which anticommutes with all the other ones: $\{\gamma^5, \gamma^\mu\} = 0$, and which allows to define the projectors on helicity states (projection of spin on momentum)

$$\begin{aligned} \psi_R &\equiv P_R\psi \;, \quad \text{with} \quad P_R \equiv \frac{1}{2}(\mathbf{1} + \gamma^5)\psi \\ \psi_L &\equiv P_L\psi \;, \quad \text{with} \quad P_L \equiv \frac{1}{2}(\mathbf{1} - \gamma^5)\psi \; . \end{aligned} \tag{90}$$

Since P_R, P_L are projectors, they satisfy $P_R + P_L = \mathbf{1}$ and $P_R P_L = 0$.

3.4 Conserved current and fermionic action

The Dirac equation (88) has Hermitian conjugate

$$i\partial_\mu\psi^\dagger\gamma^0\gamma^\mu\gamma^0 + m\psi^\dagger = 0 \; . \tag{91}$$

Given that $(\gamma^0)^2 = \mathbf{1}$, a multiplication by γ^0 on the right gives

$$i\partial_\mu \overline{\psi}\gamma^\mu + m\overline{\psi} = 0 \ , \tag{92}$$

where $\overline{\psi} \equiv \psi^\dagger\gamma^0$. Multiplying eq.(88) by $\overline{\psi}$ on the left, and multiplying eq.(92) by ψ on the right leads to

$$\begin{aligned} \overline{\psi} i\partial_\mu \gamma^\mu \psi &= m\overline{\psi}\psi \\ i\partial_\mu \overline{\psi}\gamma^\mu \psi &= -m\overline{\psi}\psi \ , \end{aligned} \tag{93}$$

such that

$$\partial_\mu(\overline{\psi}\gamma^\mu\psi) = 0 \ . \tag{94}$$

This is the continuity equation (28) for the conservation of the density of probability 4-current $j^\mu = (\rho, \vec{j})$, with

$$\begin{aligned} \rho &= \overline{\psi}\gamma^0\psi = \psi^\dagger\psi \\ \vec{j} &= \overline{\psi}\vec{\gamma}\psi \ . \end{aligned} \tag{95}$$

The Dirac equation (88) can be obtained from the variational principle $\delta S_D/\delta\overline{\psi} = 0$, with

$$S_D = \int d^4x \ \overline{\psi}(i\not{\partial} - m)\psi \ , \tag{96}$$

and the equation of motion (92) can be obtained from the variational principle $\delta S_D/\delta\psi = 0$:

$$\begin{aligned} \frac{\delta S_D}{\delta\psi(x)} &= \int d^4y (-)\overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\delta^{(4)}(x-y) \\ &= \int d^4y \ \left(i\partial_\mu\overline{\psi}\gamma^\mu + m\overline{\psi}\right)\delta^{(4)}(x-y) \\ &= i\partial_\mu\overline{\psi}\gamma^\mu + m\overline{\psi} \ , \end{aligned} \tag{97}$$

where the sign $(-)$ arises from the anticommutation of ψ and $\overline{\psi}$.

3.5 Towards Quantum Field Theory

The Klein-Gordon and Dirac equations describe free fields, with a constant number of particles, and which can thus be described by Quantum

Mechanics. If one introduces interactions, then it is possible to create or annihilate particles, because of the equivalence between mass and energy. One then needs a formalism which allows for an infinite number of particles, which is QFT, and the space of all the possible states is the Fock space.

As a toy model for the Standard Model, let us consider the following fields: *(i)* an Abelian vector A_μ with field strength $F_{\mu\nu}$; *(ii)* a fermion ψ coupled to A_μ, with charge e; *(iii)* a complex scalar ϕ coupled to A_μ, with charge g; *(iv)* a real scalar φ coupled to the fermion with Yukawa coupling y. The Lagrangian reads

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\left(i\partial\!\!\!/ - eA\!\!\!/ - m\right)\psi \\ &\quad +(\partial_\mu + igA_\mu)\phi(\partial^\mu - igA^\mu)\phi^\star \\ &\quad +\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi,\phi) - y\varphi\overline{\psi}\psi\ .\end{aligned} \tag{98}$$

where $V(\varphi,\phi)$ is the potential, including the scalar mass terms and interactions. This Lagrangian is invariant under the simultaneous set of gauge transformations

$$\begin{aligned} A_\mu &\rightarrow A_\mu - \partial_\mu\Lambda \\ \psi &\rightarrow e^{ie\Lambda}\psi \qquad \text{and} \qquad \overline{\psi} \rightarrow e^{-ie\Lambda}\overline{\psi} \\ \phi &\rightarrow e^{ig\Lambda}\phi \qquad \text{and} \qquad \phi^\star \rightarrow e^{-ig\Lambda}\phi^\star \\ \varphi &\rightarrow \varphi\ . \end{aligned} \tag{99}$$

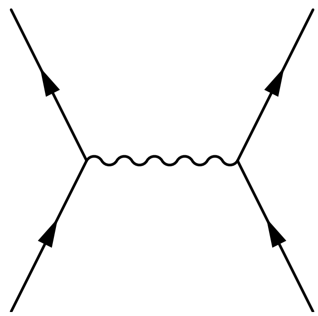
Gauge invariance is important to respect, since it implies the conservation of electric charge for example, when A_μ is the electromagnetic field. We note that gauge invariance is automatically respected if one performs the following minimal substitution in the free Lagrangian

$$\begin{aligned} \partial_\mu &\rightarrow \partial_\mu - ieA_\mu \qquad \text{for the fermion field} \\ \partial_\mu &\rightarrow \partial_\mu - igA_\mu \qquad \text{for the complex scalar field}\ , \end{aligned} \tag{100}$$

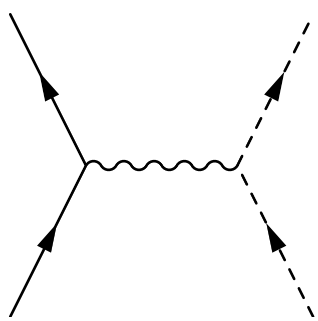
in order to include interactions.

One can see that the Lagrangian (98) does not couple the real scalar φ to the photon A_μ, or the complex scalar ϕ to the fermion ψ. But all theses fields are actually indirectly coupled, through the exchange of a gauge field of through a fermion loop, as can be read from the different interactions in the Lagrangian:

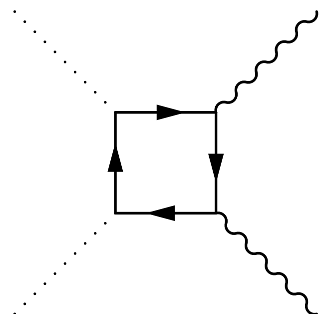
- Fermion/fermion scattering
  A full line represents the fermion propagator, the arrow is the current density, the wavy line is the photon propagator, the 3-leg vertices arise from the cubic coupling $\overline{\psi}\not{A}\psi$. Because of the two vertices, each proportional to e, this process is proportional to e^2.



- Fermion/charged scalar scattering
  A dashed line represents the charged scalar propagator, the 3-leg vertex photon-scalar arises from the derivative cubic interaction $\phi A_\mu \partial^\mu \phi^\star$. This scattering process is proportional to eg



- Neutral scalar/photon scattering
  A doted line represents the real scalar (neutral). This one-loop graph corresponds to a quantum correction, proportional to $y^2 e^2$.



- Neutral scalar annihilation into a photon
  This one-loop quantum correction is proportional to $y^2 e$