Qualitative behavior of string solutions with light-like endpoints in AdS/CFT

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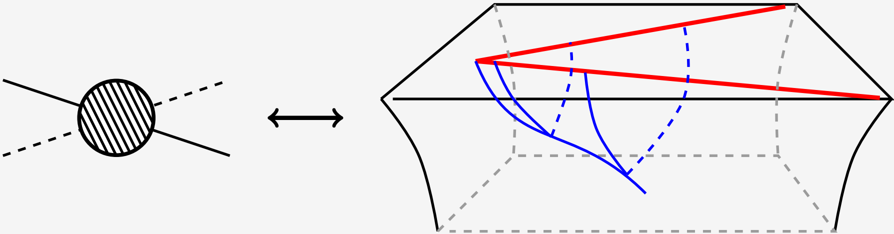
Gauge/Gravity for solving Quantum Field Theory

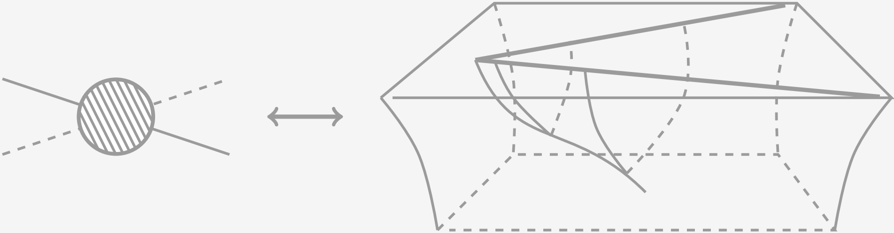
We can perform strong coupling QFT calculation calculations through weakly coupled gravity (classical string theory).



Gauge Theory **Gravity Theory**

Figure: Some Gauge theories and Gravity theories are conjectured to be the same theory under a field redefinition.

Gauge/Gravity for solving Quantum Field Theory



$N = 4$ SYM $\qquad\qquad$ $AdS_5 \times S_5$

We don't know the dual theory to Quantum Chromodynamics, but we do know the dual of a QCD-like theory, $N = 4$ SYM which is $AdS_5 \times S_5$.

$$ds^2_{AdS} = \frac{1}{u^2}\left(\eta_{\mu\nu} dx^\mu dx^\nu + du^2\right) \tag{1}$$

The Wilson Loop

Wilson Loop measures the potential energy between two points.

$$W = \frac{1}{N} \langle 0 | \mathrm{tr} \{ P \mathrm{exp} \left(i \oint_C dx \cdot A(x) \right) \} | 0 \rangle \qquad (2)$$

Here C specifies the path of two back-to-back particles along the lightcone.

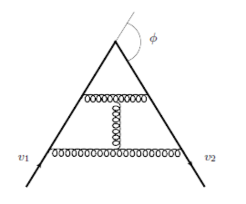


Figure: Visual representation of of the Wilson loop.

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From the Gauge/Gravity correspondence we say that:

$$\lim_{\lambda \to \infty} W \sim e^{-S_{cl}} \tag{4}$$

Where S_{cl} is the area of the worldsheet tracing out the string solution in the gravitational theory, and the boundary conditions of the end points of the string match the countour C.

Equation of Motion for General Geometry

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- We expect this background to be Asymptotically AdS so we impose

$$f(u) = \frac{1}{u^2}e^{-\int^u g(u')} \text{ with } \lim_{u\to 0} f(u) \to \frac{1}{u^2} \tag{6}$$

where $g(u)$ is a real valued function.

The Wilson Loop at Strong Coupling

The Equation of Motion for general $g(u)$:

$$\boxed{s\frac{\ddot{u}}{1-\dot{u}^2} = -\dot{u} + s\left(\frac{2}{u} + g(u)\right)} \tag{7}$$

with $s = \sqrt{t^2 - \vec{x}^2}$.

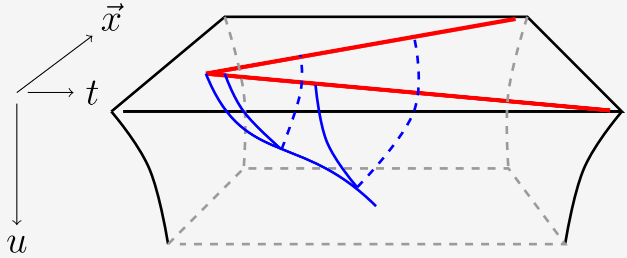

Figure: String in AdS geometry distorted by $g(u)$.

Solution for $g(u) = 0$.

In this case

$$s\frac{\ddot{u}}{1-\dot{u}^2} = -\dot{u} + s\frac{2}{u} \tag{8}$$

we see that $u(s) = \sqrt{2}s$ will solve the equation of motion.

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$$u(s) = \sqrt{2}s \implies \begin{cases} \ddot{u} = 0 \\ -\dot{u} = -\sqrt{2} \\ s\frac{2}{u} = \sqrt{2} \end{cases} \tag{9}$$

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So we choose our initial conditions to be that $u \to \sqrt{2}s$ when $s \to 0$ for general $g(u)$ to always match the AdS solution.

Qualitative Behaviour for General $g(u)$

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$$\left.\begin{aligned} \ddot{u} &= -\operatorname{csch}(v)\dot{v} \\ 1-\dot{u}^2 &= -\operatorname{csch}(v) \end{aligned}\right\} \implies \frac{\ddot{u}}{1-\dot{u}^2} = \dot{v} \tag{11}$$

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This gives us two first order equations:

$$\begin{cases} s\dot{v} = -\dot{u} + s\left(\frac{2}{u} + g(u)\right) \\ \dot{u} = \coth v \end{cases} \tag{12}$$

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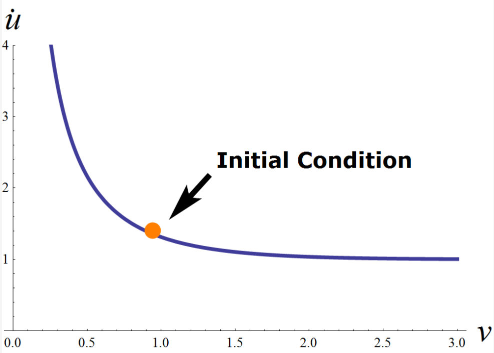

Figure: Path constraining $\dot{u} = \coth v$. Here $\dot{u}(0) = \sqrt{2}$.

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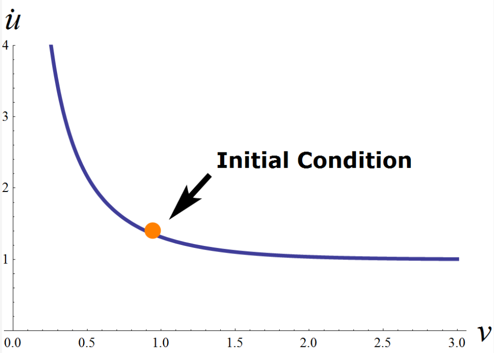

Figure: Path constraining $\dot{u} = \coth v$. Here $\dot{u}(0) = \sqrt{2}$.

$$\boxed{1 < \dot{u} < \infty} \tag{14}$$

The case $g(u) > 0$

Substituting the initial condition $\lim_{s \to 0} u = \sqrt{2}s$ into the EOM

$$s\dot{v} = -\dot{u} + s\left(\frac{2}{u} + g(u)\right) \implies \dot{v} \sim g(u). \qquad (15)$$

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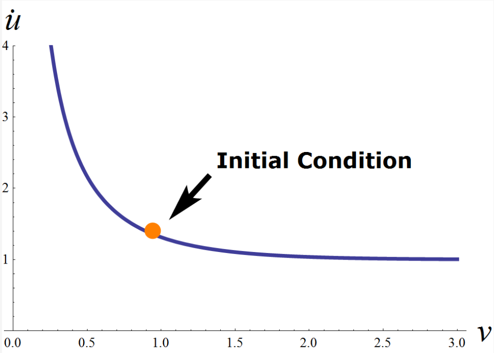

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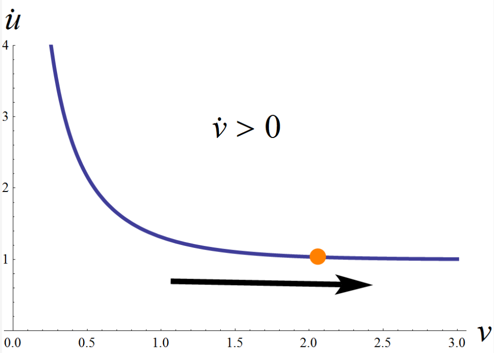

Figure: For $g(u) > 0$, $u(s)$ receives an initial kick into $\dot{u} \in (1, \sqrt{2})$.

The case $g(u) > 0$

Consider the region $1 < \dot{u} < \sqrt{2}$. Because $u(0) = 0$ we can conclude that $u(s)$ is bounded by the functions s and $\sqrt{2}s$.

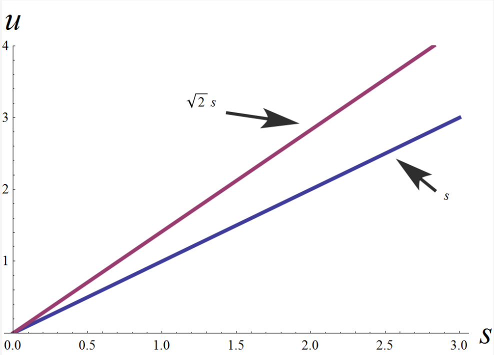

Figure: Functions constraining $u(s)$ in region $\dot{u} \in (1, \sqrt{2})$

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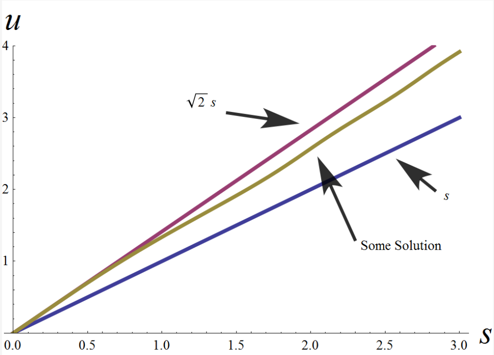

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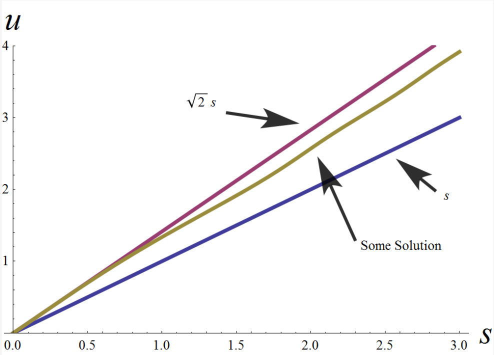

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$$\boxed{s < u(s) < \sqrt{2}s \implies \sqrt{2} < \frac{2s}{u} < 2} \tag{17}$$

The case $g(u) > 0$

Using that $\dot{u} \in (1, \sqrt{2})$, $s\frac{2}{u} \in (\sqrt{2}, 2)$, and that $g(u) > 0$ we have that

$$\dot{v} = -\dot{u} + s\left(\frac{2}{u} + g(u)\right) > -\dot{u} + s\left(\frac{2}{u}\right) > 0. \qquad (18)$$

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$$\dot{v} = -\dot{u} + s\left(\frac{2}{u} + g(u)\right) > -\dot{u} + s\left(\frac{2}{u}\right) > 0. \qquad (18)$$

So v will become more and more positive, and $\dot{u} \to 1$.

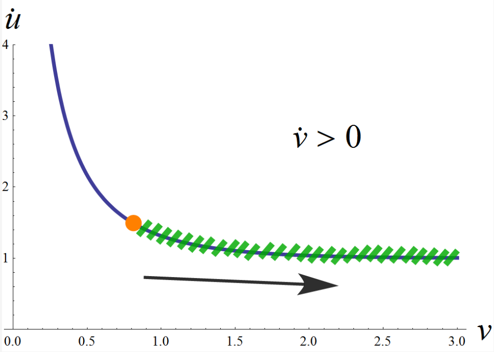
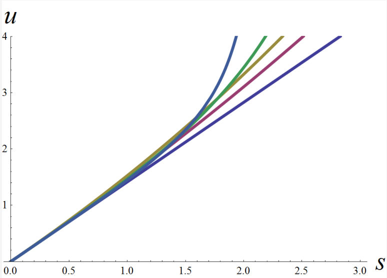


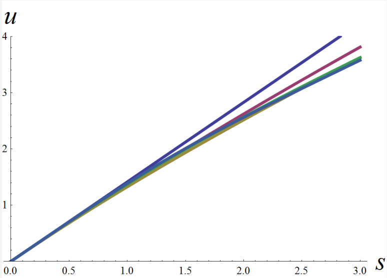
Figure: Path constraining $\dot{u} = \coth v$ for all real v.

Qualitative Solution

The trajectory of $u(s)$ can be determined by the sign of $g(u)$.



(a) Evolution for $g(u) < 0$



(b) Evolution for $g(u) > 0$

Figure: Trajectories for $u(s)$ with various test functions $g(u)$.

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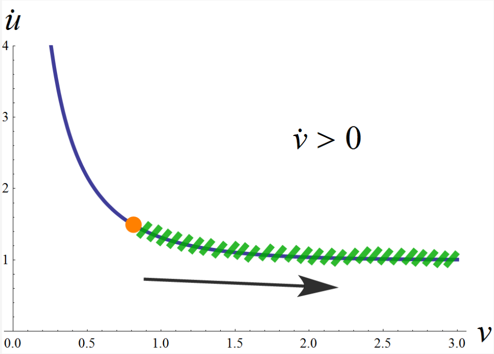
- Qualitatively different solutions can not have a positive definite $g(u)$.
- Similar analysis could be used to calculate Wilson loops of other shapes: Lines, circles, rectangles, etc.

Physical Implications



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