Longitudinal beam dynamics

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- Interaction between electric fields and particles
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Path length dependence

Consider two particles with different momentum on parallel trajectories

$$p_1 = p_0 + \Delta p$$

They will have non equal velocities if they are not ultra-relativistics (protons under few GeVs, electrons under few MeVs)

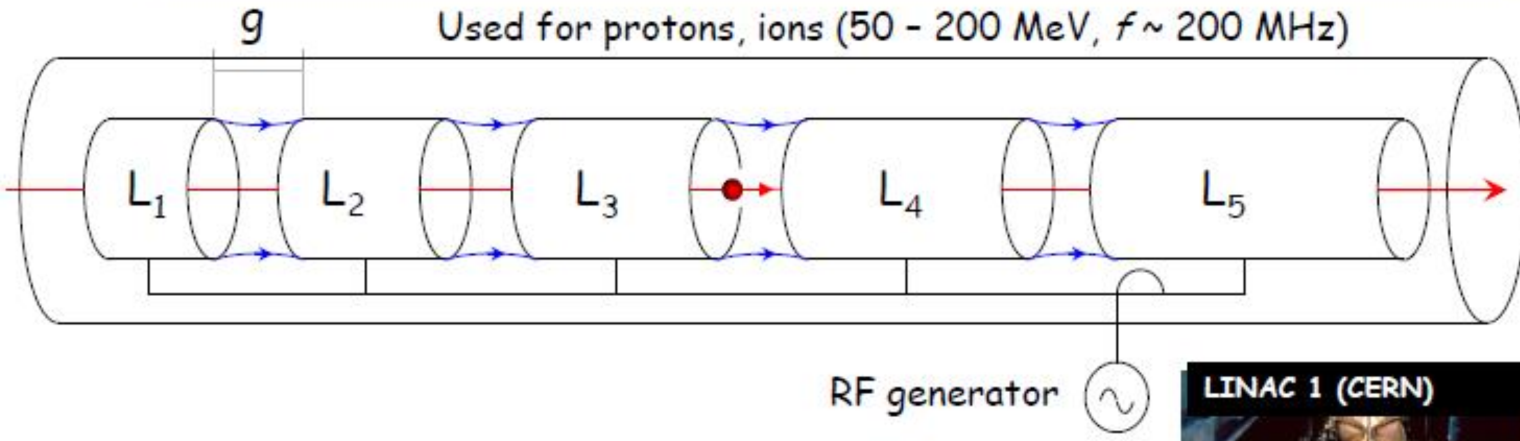
At a given instant the length travelled by the two particles will be:

$$L_0 = \beta_0 ct \qquad L_1 = (\beta_0 + \Delta\beta)ct$$

$$\frac{\Delta L}{L_0} = \frac{L_1 - L_0}{L_0} = \frac{\Delta\beta}{\beta_0}$$

RF acceleration: Alvarez Structure

Used for protons, ions (50 - 200 MeV, $f \sim 200$ MHz)



Synchronism condition $(g \ll L)$

$$L = v_s T_{RF} = \beta_s \lambda_{RF}$$

$$\omega_{RF} = 2\pi \frac{v_s}{L}$$

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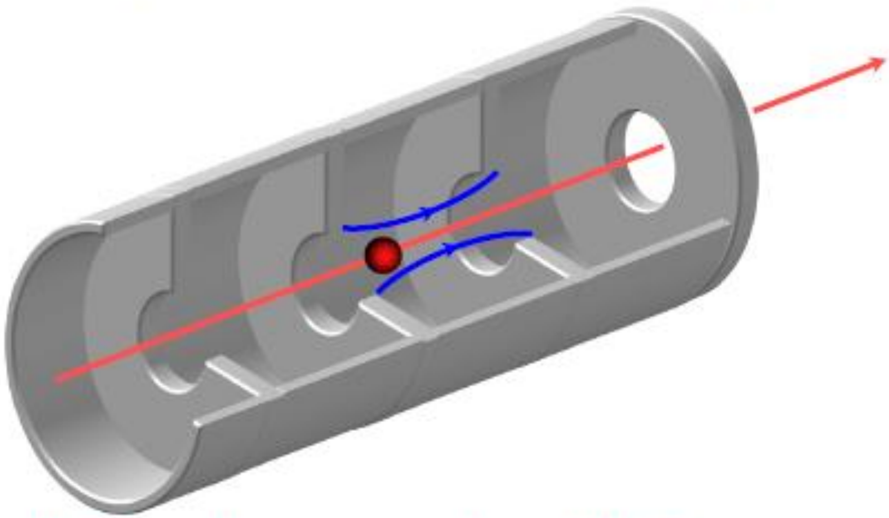
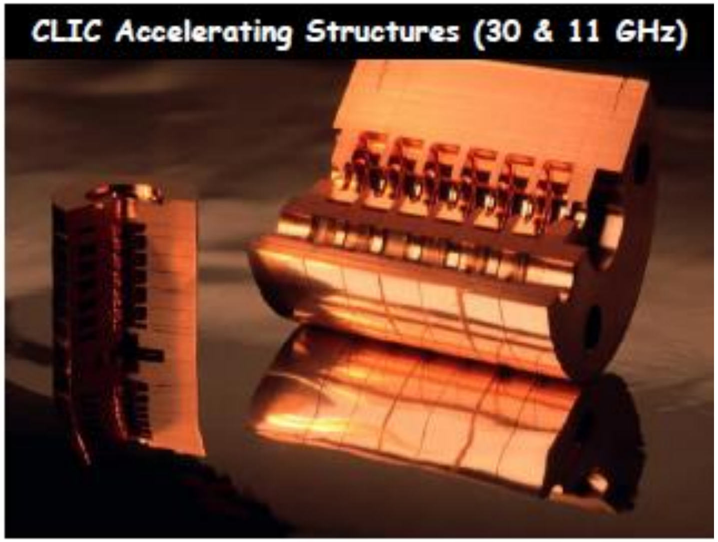
Disc loaded traveling wave structures

-When particles gets ultra-relativistic (v~c) the drift tubes become very long unless the operating frequency is increased. Late 40's the development of radar led to high power transmitters (klystrons) at very high frequencies (3 GHz).

-Next came the idea of suppressing the drift tubes using traveling waves. However to get a continuous acceleration the phase velocity of the wave needs to be adjusted to the particle velocity.





solution: slow wave guide with irises ==> iris loaded structure

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Energy gain in Linacs

The kinetic energy gain for a particle with charge e that moves in an electric field along a path L is given by:

$$\Delta W = e\int_0^L \vec{E}\,(\vec{r},t)d\vec{s} = eV$$

V is the voltage gain for the particle, depends on the particle trajectory and includes the contribution of all electric fields (rf fields, space charge, interaction with vacuum chamber,..)

Assuming a sinusoidal electric field:

$$E_z = E_0 \sin(\omega_{RF} t + \phi_s)$$

The synchronous particle passes at the middle of the gap g, at time t = 0:

The energy gain is

$$\Delta W = eE_0 \int_{-\frac{g}{2}}^{\frac{g}{2}} \sin(\omega_{RF} t + \phi_s)\ dz = eE_0 \int_{-\frac{g}{2}}^{\frac{g}{2}} (\sin\omega_{RF} t \cos\phi_s + \cos\omega_{RF} t \sin\phi_s) dz$$

=0

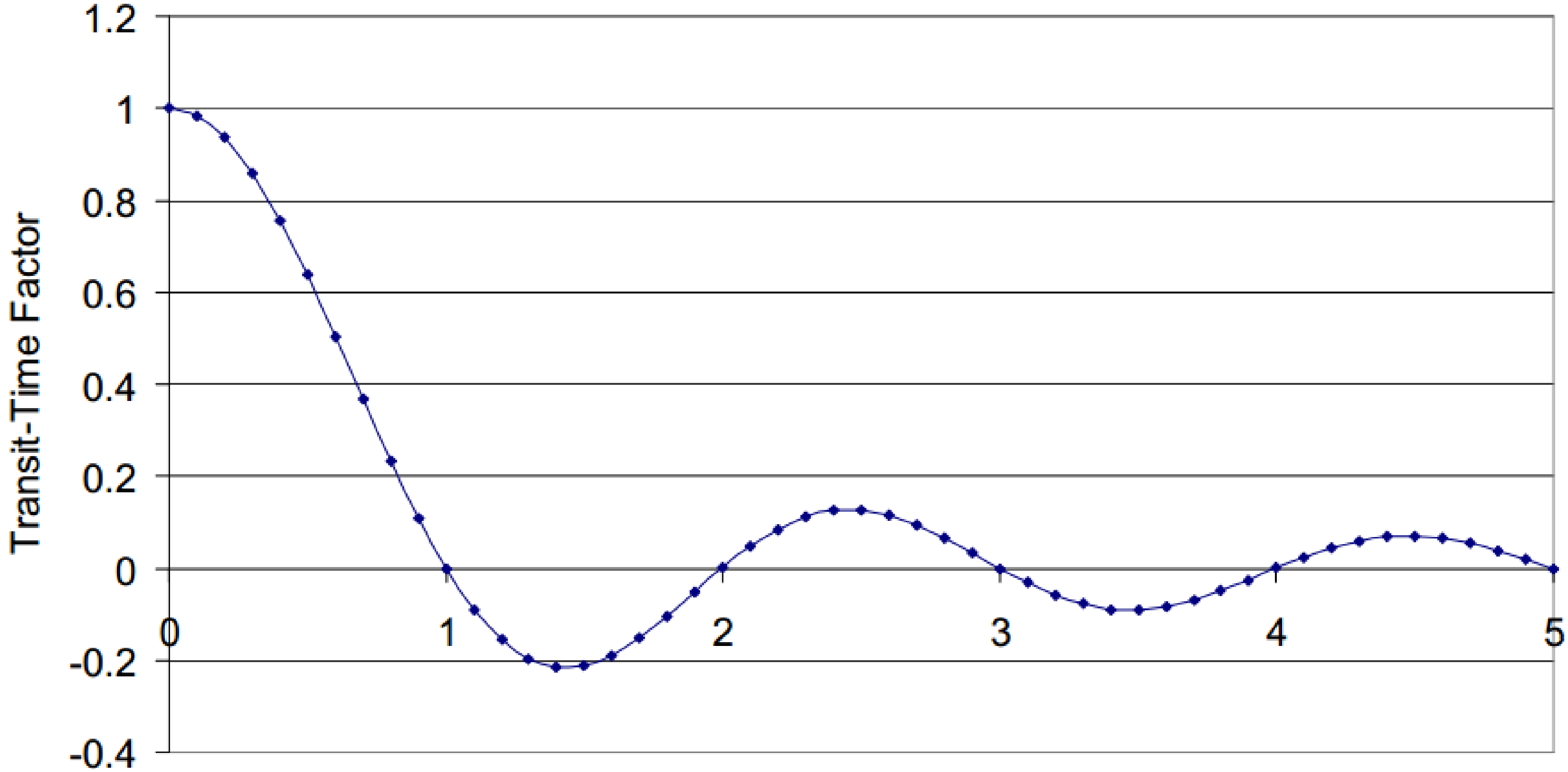
Considering that

$$\omega_{RF} t = \omega_{RF}\frac{z}{v} = \frac{2\pi c}{\lambda}\frac{z}{\beta c} = \frac{2\pi}{\beta\lambda} z \Rightarrow \Delta W = eE_0 \sin\phi_s \frac{\beta\lambda}{2\pi}\left[\sin\frac{2\pi}{\beta\lambda} z\right]_{-\frac{g}{2}}^{\frac{g}{2}} = eE_0 \frac{\sin(\pi g/\beta\lambda)}{\pi g/\beta\lambda} g \sin\phi_s$$

$$\Delta W = eVT \sin\phi_s$$

Where T is the transit time

$$T = \frac{\sin(\pi g/\beta\lambda)}{\pi g/\beta\lambda}$$



$g/\beta\lambda$

For efficient acceleration by RF fields, we need to properly match the gap length g to the distance that the particle travels in one RF wavelength, βλ

In general

$$T = \frac{\int_{-g/2}^{g/2} E(0,z)\cos(\omega_{RF} t)\,dz}{\int_{-g/2}^{g/2} E(0,z)\,dz}$$

T is the transit-time factor: a factor that takes into account the time variation of the field during particle transit through the gap

The cavities are spaced $L = \beta_s \lambda$

β_s beta after cavity, λ wavelength

The fast particle arrives at $t_a < t_s$ and gains energy $\Delta W_a < \Delta W_s$

The slow particle arrives at $t_b > t_s$ and gains energy $\Delta W_b > \Delta W_s$

The synchronous particle arrives at the synchronous phase

a,b oscillate in phase (time) about the synchronous particle =>

synchrotron oscillation

Synchrotron motion

- Energy and phase are related through the rf acceleration. The nominal particle is the one which is in phase with the rf and has the nominal energy. The variation of phase and energy with respect to the nominal ones represents the synchrotron motion

Energy phase equations in Linacs

Rate of energy gain for the synchronous particle:

$$\frac{dW_s}{ds} = \frac{dp_s}{ds} = eE_o sin\phi_s$$

The non synchronous particles with 'reduced' coordinates, w and φ, gains

$$w = W - W_s$$

$$\varphi = \phi - \phi_s \text{ (small)}$$

$$\frac{dw}{ds} = eE_o[sin(\phi_s + \varphi) - sin\phi_s] \approx eE_o cos\phi_s \cdot \varphi$$

And its phase changes by

$$\frac{d\varphi}{ds} = \omega_{RF}\left[\frac{dt}{dz} - \left(\frac{dt}{dz}\right)_s\right] = \omega_{RF}\left(\frac{1}{v} - \frac{1}{v_s}\right) \cong -\frac{\omega_{RF}}{{v_s}^2}(v - v_s)$$

Since

$$v - v_s = c(\beta - \beta_s) \cong \frac{c}{2\beta_s}\left(\beta^2 - {\beta^2}_s\right) \cong \frac{w}{m_o v_s {\gamma_s}^3}$$

Energy-phase oscillations

$$\frac{d\varphi}{ds} = -\frac{\omega_{RF}}{m_o {v_s}^3 {\gamma_s}^3} w$$

Combining the two first order equations into a second order one:

$$\frac{d^2\varphi}{ds^2} + {\Omega_s}^2 \varphi = 0$$

With

$${\Omega_s}^2 = \frac{eE_o \omega_{RF} \cos\phi_S}{m_o {v_s}^3 {\gamma_s}^3}$$

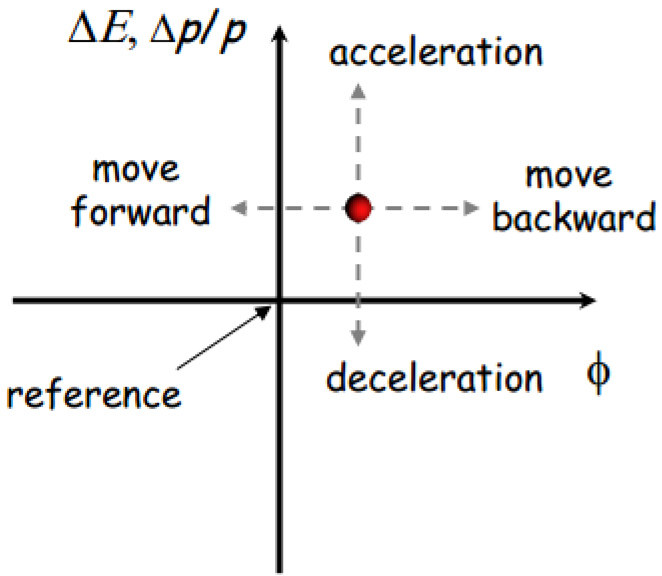
Stable oscillations occur if

${\Omega_s}^2 > 0$ and real -> $\cos\phi_S > 0$

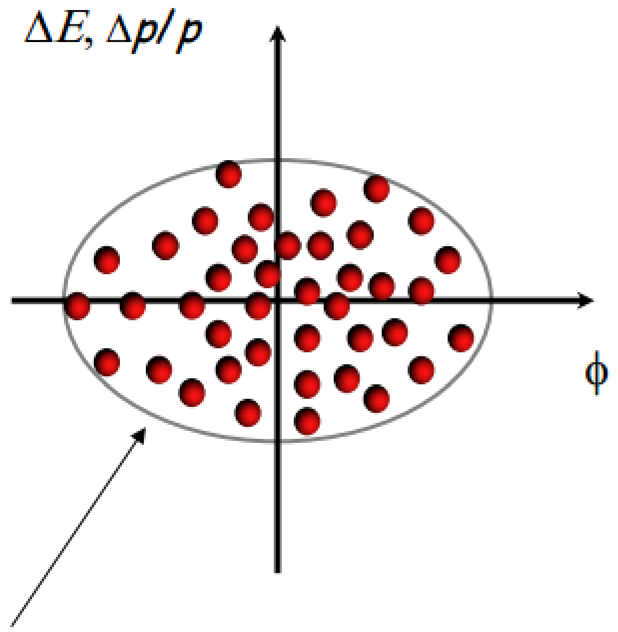
And since for acceleration $\sin\phi_S > 0$

$$0 < \phi_S < \frac{\pi}{2}$$

Longitudinal phase space, longitudinal emittance



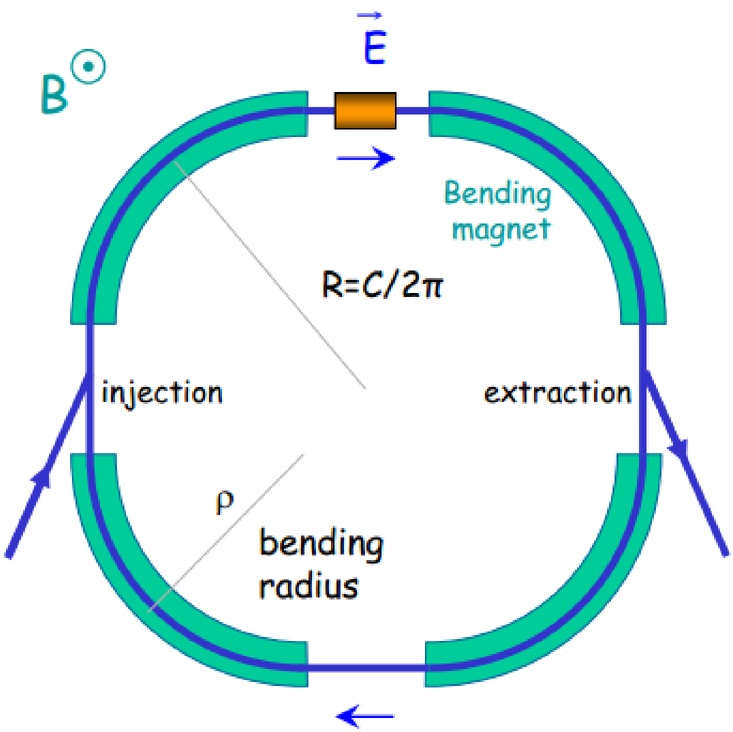
The particle trajectory in the phase space ($\Delta p/p$, ϕ) describes its longitudinal motion.



Emittance: phase space area including all the particles

Synchrotrons – storage rings

The synchrotron is a synchronous accelerator since there is a synchronous RF phase for which the energy gain fits the increase of the magnetic field at each turn. That implies the following operating conditions:



$e\hat{V}\sin\phi$ ⟶ Energy gain per turn

$\phi = \phi_s = cte$ ⟶ Synchronous particle

$\omega_{RF} = h\omega_r$ ⟶ RF synchronism (h - harmonic number)

$\rho = cte \quad R = cte$ ⟶ Constant orbit

$B\rho = P/e \Rightarrow B$ ⟶ Variable magnetic field

ALBA RF

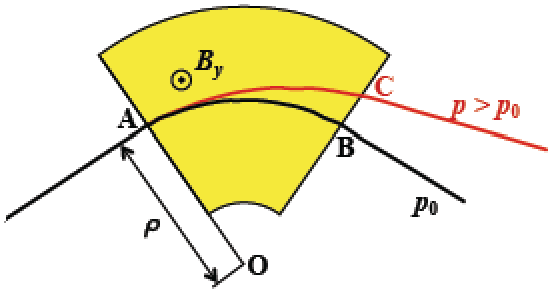
6 RF cavities (DAMPY, designed by a EU collaboration)

- 500 MHz
- Normal conducting
- nose cone
- HOM damped
- Designed for 500 kW beam power (400 mA)

Path length dependence on energy with dipoles – Storage rings

Since $p = \beta\gamma m_0 c$ =>

$$\frac{\Delta p}{p_0} = \gamma^2 \frac{\Delta\beta}{\beta} \qquad \frac{\Delta L}{L_0} = \frac{1}{\gamma^2}\frac{\Delta p}{p_0}$$



$$\rho = \frac{p}{eB_y} = \frac{\beta\gamma m_0 c}{eB_y}$$

$$\frac{L - L_0}{L_0} \propto \frac{p - p_0}{p_0}$$

$$\frac{\Delta L}{L_0} = \alpha \frac{\Delta p}{p_0}$$

Alfa constant

For γ>>1

$$\frac{\Delta L}{L_0} = \alpha \frac{\Delta p}{p_0} \cong \alpha \frac{\Delta E}{E_0}$$

Particles with higher energies do longer paths in the sector magnet

Momentum compaction in a storage ring:

$$\frac{dl}{ds} = 1 + \frac{x(s)}{\rho(s)}$$

Momentum compaction (in storage rings)

Let's call $\delta = \frac{\Delta p}{p_0}$

$$x(s) = \delta D(s)$$

The difference in trajectory is clearly linked to the dispersion function
Integrating along the circumference C

$$\oint_L dl = L + \Delta L = L + \delta \oint_L \frac{D(s)}{\rho(s)} ds$$

$$\frac{\Delta L}{L_0} = \alpha_C \delta$$

$$\alpha_C = \frac{1}{L_o} \int_0^{L_0} \frac{D}{\rho} ds$$

is the momentum compaction. It measures the relative change in circumference per unit relative momentum offset. If α_C is small the different trajectories are 'packed'

Transition Energy

Relationship with time difference:
Two particles with different momentum

$$\Delta t = t - t_0 = \frac{s + \alpha_C \delta s}{v} - \frac{s}{v_0} \approx \frac{s}{v_0}\left(\frac{v_0}{v} - 1 + \alpha_C \delta\right)$$

$$\frac{\Delta t}{t_0} \approx \frac{v_0}{v} - 1 + \alpha_C \delta$$

For small δ

$$v \approx v_0 \left(1 + \frac{\delta}{\gamma^2}\right)$$

$$\frac{\Delta t}{t_0} \approx \left(\alpha_c - \frac{1}{\gamma^2}\right)\delta = \eta_C \delta$$

η_C = slip factor

the energy for which the slip factor is zero is the transition energy:

$$\frac{1}{\gamma_t^2} = \alpha_C$$

During acceleration, passing through the transition energy changes the sign of the dependence on momentum dispersion of particle revolution frequency

If v = c

$$\frac{\Delta t}{t_0} \approx \frac{\Delta L}{L_0}$$

If the slip factor is zero the ring is isochronous: all particles have the same revolution frequency

1^{st} energy-phase equation

Harmonic number h:

$$f_{RF} = hf_r$$

f_r is the revolution frequency

$$\Delta\phi = -h\Delta\theta \text{ with } \theta = \int \omega_r dt$$

h is the maximum n. of bunches that can be stored in a storage ring

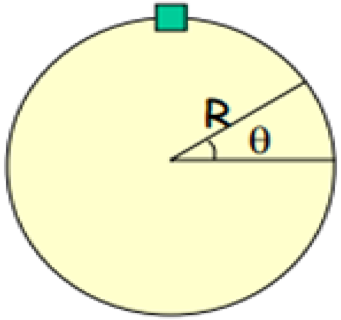
For a given particle with respect to the reference one

$$\Delta\omega_r = \frac{d}{dt}(\Delta\theta) = -\frac{1}{h}\frac{d}{dt}(\Delta\phi) = -\frac{1}{h}\frac{d\phi}{dt}$$

Since

$$\eta = \frac{p_s}{\omega_{rs}}\left(\frac{d\omega_r}{dp}\right)_s \text{ and } E^2 = E_o{}^2 + p^2c^2$$

$$\Delta E = v_s\Delta p = \omega_{rs}R_s\Delta p, \quad R_S = \frac{C}{2\pi} = \frac{v_S}{\omega_{rs}}$$



Then

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_sR_s}{h\eta\omega_{rs}}\frac{d(\Delta\phi)}{dt} = -\frac{p_sR_s}{h\eta\omega_{rs}}\dot{\phi}$$

2nd energy-phase equation

The particle gains energy with the rate

$$\frac{dE}{dt} = e\hat{V}\sin\phi\,\frac{\omega_r}{2\pi}$$

The rate of change with respect to the reference particle is

$$2\pi\Delta\left(\frac{\dot{E}}{\omega_r}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

$$T_r = \frac{2\pi}{\omega_r} \text{ => revolution time}$$

$$\Delta\left(\dot{E}T_r\right) \cong \dot{E}\Delta T_r + T_{rs}\Delta\dot{E} = \Delta E T_r + T_{rs}\Delta\dot{E} = \frac{d}{dt}(T_{rs}\Delta E)$$

And therefore

$$2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}(\sin\phi - sin\phi_s)$$

Equations of longitudinal motion

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}}\frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}}\dot{\phi}$$

$$2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

- Deriving and combining:

$$\frac{d}{dt}\left(\frac{p_s R_s}{h\eta\omega_{rs}}\frac{d\phi}{dt}\right) + \frac{e\hat{V}}{2\pi}(\sin\phi - \sin\phi_s) = 0$$

Small amplitude oscillations

For slowly varying R_s, p_s, w_s and eta:

$$\ddot{\phi} + \frac{{\Omega_s}^2}{\cos\phi_s}(\sin\phi - \sin\phi_s) = 0$$

With

$${\Omega_s}^2 = \frac{h\eta\omega_{rs}e\hat{V}\cos\phi_s}{2\pi p_s R_s}$$

Frequency of the oscillation

For small variations from the reference particle:

$$\sin\phi - \sin\phi_s = \sin(\phi_s + \Delta\phi) - \sin\phi_s \cong \varphi\cos\phi_s$$

We can write $\dot{\varphi} = \dot{\phi}$, $\ddot{\varphi} = \ddot{\phi}$

And therefore we obtain again the harmonic oscillator equation:

$$\ddot{\varphi} + {\Omega_s}^2\varphi = 0$$

Stable if ${\Omega_s}^2 > 0$ and Ω_s is real

For $\gamma < \gamma_{tr}$ $\eta > 0$ $0 < \phi_s < \pi/2$

$\gamma > \gamma_{tr}$ $\eta < 0$ $\pi/2 < \phi_s < \pi$

Synchrotron tune

The oscillation frequency defines the synchrotron tune Q_s (in analogy with betatron tune)
Number of oscillations in one turn

$$Q_s = \sqrt{-\frac{h\eta e\hat{V}\cos\phi_s}{2\pi p_s R_s \omega_{rs}}} = \sqrt{-\frac{h\eta e\hat{V}\cos\phi_s}{2\pi E_s {\beta_s}^2}}$$

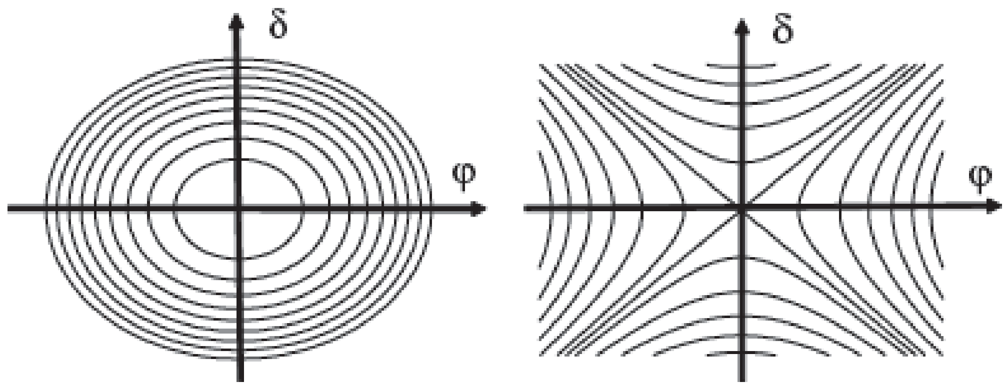


Fig. 6.4. Synchrotron oscillations in phase space for stable motion ($\Omega^2 > 0$) (left) and for unstable motion ($\Omega^2 < 0$) (right)

Longitudinal emittance

The longitudinal Twiss parameter β_L can be defined, analogously to transverse plane

$$\beta_L = \frac{|\eta| h \lambda_{RF}}{2\pi {\beta_s}^2 E_s c Q_s} = \frac{\lambda_{RF}}{c\beta_s}\sqrt{-\frac{\eta h}{2\pi e \hat{V} E_s \cos\phi_s}}$$

And we can write the equations of motion in matrix formalism:

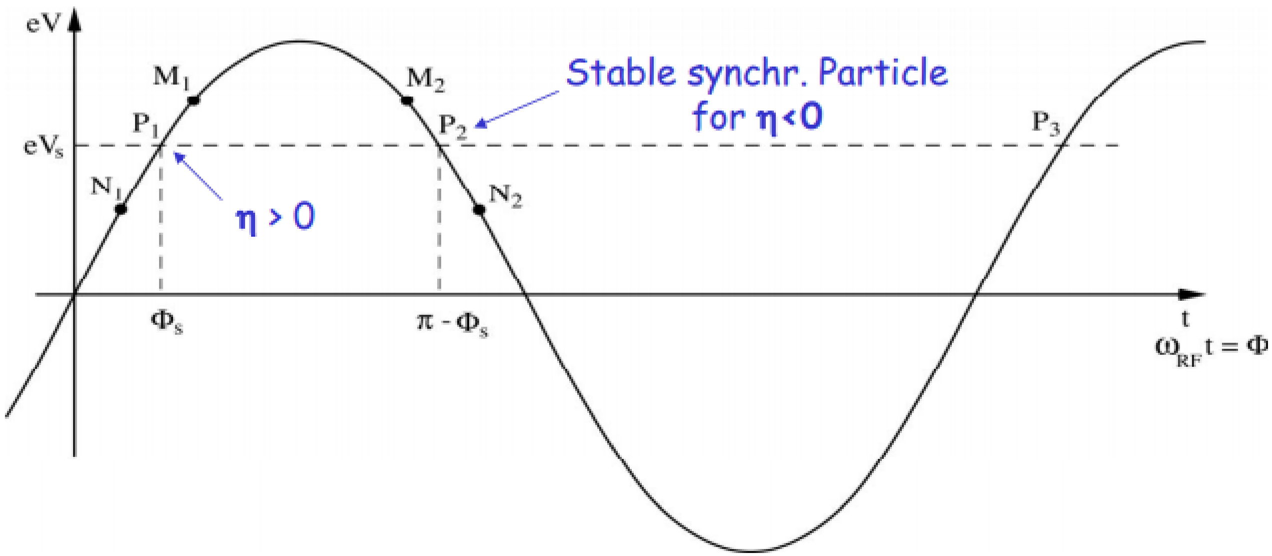
$$\begin{pmatrix}\Delta t\\ \Delta E\end{pmatrix}_n = \begin{pmatrix}\cos 2\pi n Q_s & \beta_L \sin 2\pi n Q_s\\ -\frac{1}{\beta_L}\sin 2\pi n Q_s & \cos 2\pi n Q_s\end{pmatrix}\begin{pmatrix}\Delta t\\ \Delta E\end{pmatrix}_o$$

An invariant of the motion is

$$\frac{1}{\beta_L}(\Delta t_n)^2 + \beta_L(\Delta E_n)^2 = \varepsilon_L = longitudinal\ emittance$$

Phase Stability in a Synchrotron

From the definition of η it is clear that below transition an increase in energy is followed by a higher revolution frequency (increase in velocity dominates) while the reverse occurs above transition (v ≈ c and longer path) where the momentum compaction (generally > 0) dominates.



Large Amplitude Oscillations

For large phase or energy deviations the 2nd order differential equation is non-linear:

$$\ddot{\phi} + \frac{{\Omega_s}^2}{\cos\phi_s}(\sin\phi - \sin\phi_s) = 0$$

Multypling by $\dot{\phi}$ and integrating gives an invariant of the motion I:

$$\frac{\dot{\phi}^2}{2} - \frac{{\Omega_s}^2}{\cos\phi_s}(\cos\phi + \phi\sin\phi_s) = I$$

Which for small amplitudes:

$$\dot{\phi}^2 + {\Omega_s}^2(\Delta\phi)^2 = 2I$$

And similar for the second variable ΔE

When ϕ reaches π-Φ_s, the force goes to zero and beyond it becomes non restoring.

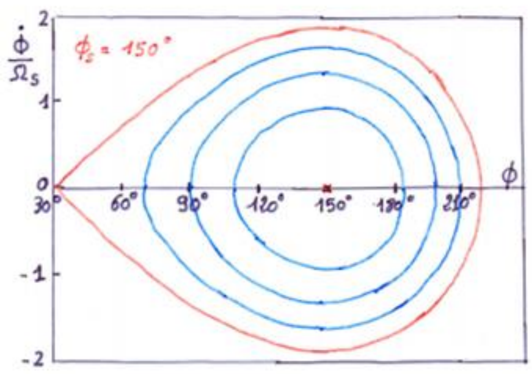
π-Φ_s is an extreme amplitude for a stable motion which in the phase space ($\dot{\phi}^2/\Omega_s$, $\Delta\Phi$) is shown as closed trajectories

The separatrix is the limit between the stable and the unstable oscillations

$$\frac{\dot{\phi}^2}{2} - \frac{{\Omega_s}^2}{\cos\phi_s}(\cos\phi + \phi\sin\phi_s) = -\frac{{\Omega_s}^2}{\cos\phi_s}(\cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s)$$

Where is zero:

$$\cos\phi_m + \phi_m\sin\phi_s = \cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s$$



Energy acceptance

$\dot{\phi}$ is maximum when $\ddot{\phi} = 0$, corresponding to $\phi = \phi_s$

Introducing this into the separatrix equation gives:

$$\dot{\phi}_{max}{}^2 = 2\Omega_s{}^2[2 + (2\phi_s - \pi)\tan\phi_s]$$

Which means energy acceptance:

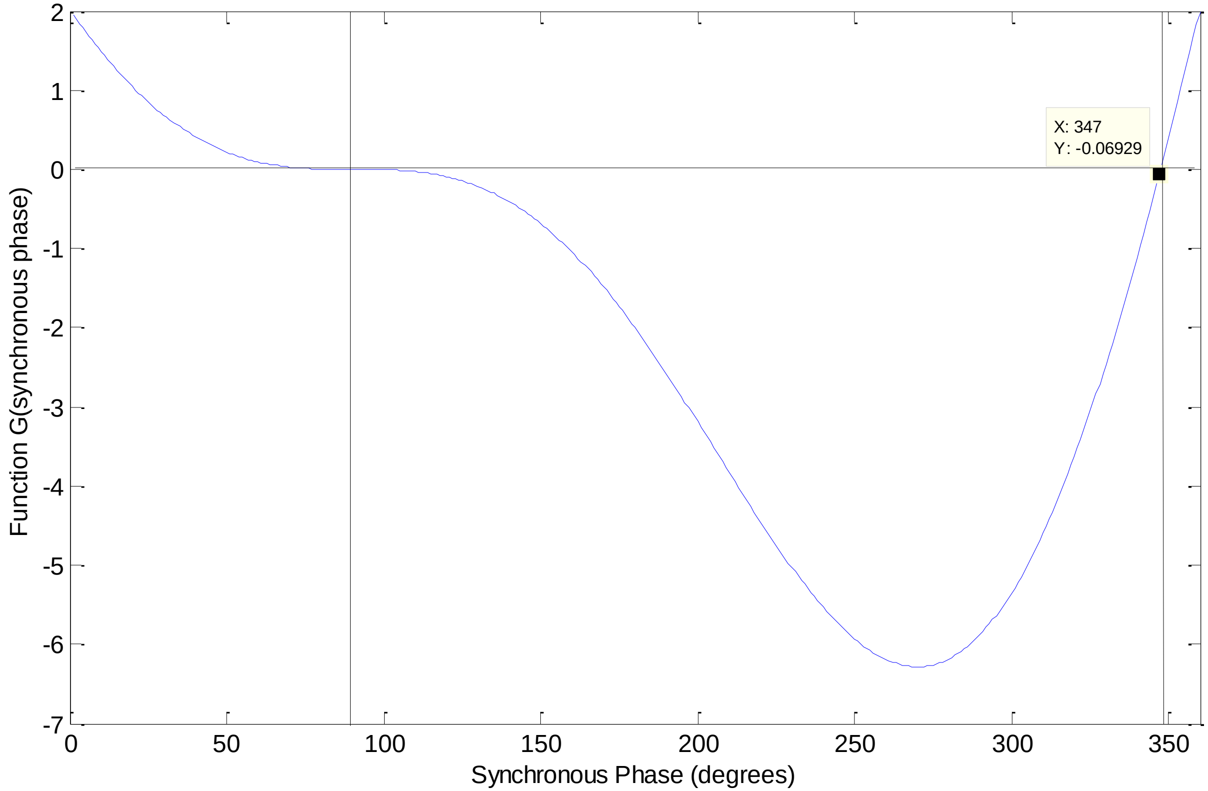
$$\left(\frac{\Delta E}{E_s}\right)_{max} = \pm\beta\sqrt{-\frac{e\hat{V}}{\pi h\eta E_s}G(\phi_s)}$$

$$G(\phi_s) = 2\cos\phi_s + (2\phi_s - \pi)sin\phi_s$$

This rf acceptance depends on ϕ_s and is important for the beam capture at injection and the stored beam lifetime.

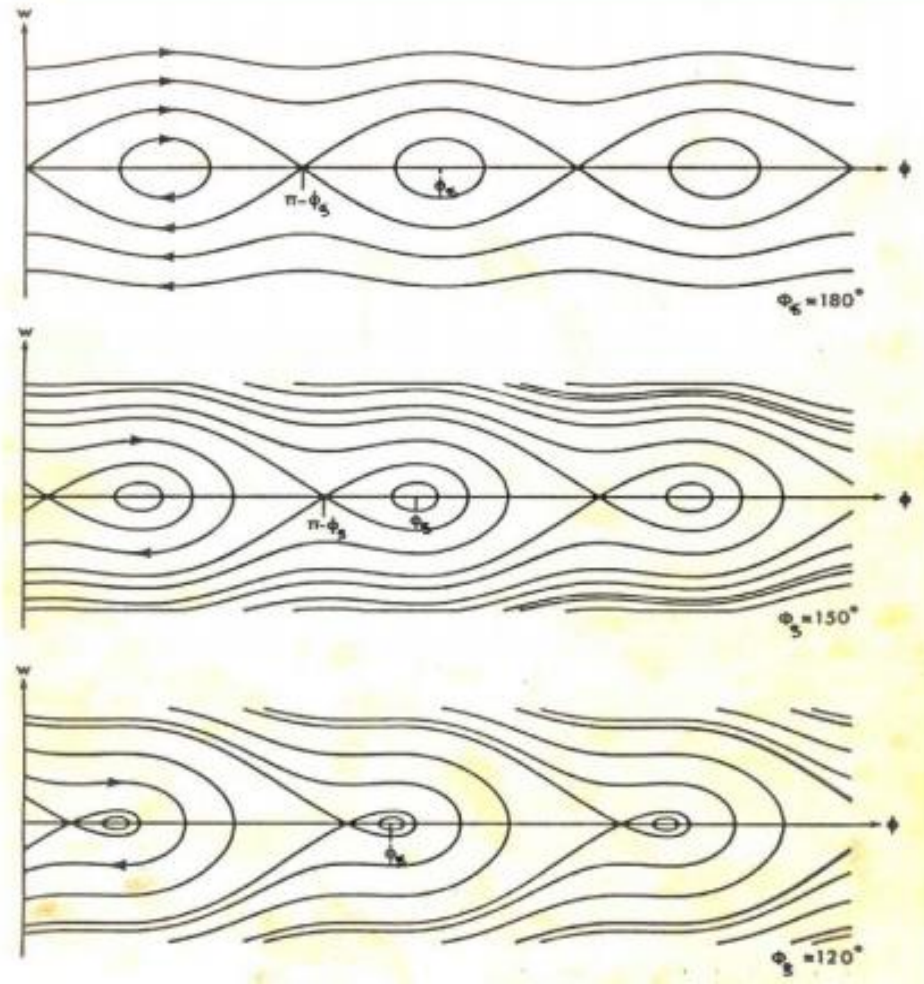
The higher the voltage the larger the energy acceptance

Energy acceptance versus synchronous phase



$$G(\phi_s) = 2\cos\phi_s + (2\phi_s - \pi) sin\phi_s$$

RF Acceptance versus Synchronous Phase



As the synchronous phase gets closer to 90° the area of stable motion (closed trajectories) gets smaller. These areas are often called "BUCKET".

The number of circulating buckets is equal to "h".

The phase extension of the bucket is maximum for ϕ_s =180° (or 0°) which correspond to no acceleration . The RF acceptance increases with the RF voltage.

Exercise

- $C = 628$ m
- $\rho = 70$ m
- $h = 84$
- $\alpha_c = 0.027$
- $\eta = \alpha_c - 1/\gamma^2$
- E = ? GeV
- f_{rf} ?