AFRICAN SCHOOL OF PHYSICS

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COSMOLOGY (Theory)

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Contents:

1. INTRODUCTION

1.1 Cosmology

A study of the universe with regard to its (i) Origins, (ii) Structure, (iii) Geometry and (iv) Evolution (dynamic and thermodynamic)

1.2 Lecture Aims:

To provide an introductory theoretical interpretation to the observed characteristics of the Universe.

1.3 Characteristics

- Observations indicate the following characteristics of the Universe:

1.31 Geometry:

- At large distance scales $\gtrsim 100$ Mpc
  [$1\,\text{Mpc} \simeq 3.26 \times 10^6$ light years $\simeq 3.08 \times 10^{24}$ cm]

  the Universe is (i)
  (i) Isotropic (rotationally invariant)
  (ii) Homogeneous (translationally invariant)

1.32 Dynamics:

- The dynamics of the Universe is dominated by gravity.
  [Why? There are 4 forces in nature!]
- The dynamics is best described by General Relativity (GR)
- At scales $\ll$ smaller than the particle horizon Newtonian Mechanics provides a good approximation to cosmic dynamics.

1.33 Composition:

(a) Universe bathed in background radiation currently at 2.73°K.

(b) large matter dominance v^s anti-matter

(c) baryonic matter: 75% H, 25% He

(d) Universe dominated by dark energy ~ 75% and dark matter ~ 25%

* NB baryonic matter makes negligible contribution.

1.34 Structure:

Cosmic Microwave Background (CMB) indicates density fluctuations of 10^{-5} when Universe was ~ 1000 smaller.

Cosmology attempts to construct a theoretical picture to explain these observations.

1.4 Units and Scales

Object	Mass (gm)	Radia Size (cm)
Sun	$1 M_\odot = 2\times 10^{33}$	7×10^{10}
Galaxy	$\sim 10^{11} M_\odot \simeq 2\times 10^{44}$	$\sim 10\,KPC \sim 3\times 10^{22}$
Cluster	$\sim 10^{14} M_\odot \simeq 2\times 10^{47}$	$\sim MPC \sim 10^{25}$
Universe	$\sim \rho_c \times Horiz \simeq 8\times 10^{55}$	Horizon $\sim 3000\,MPC$

$1\,PC = 3.26$ Light Years.

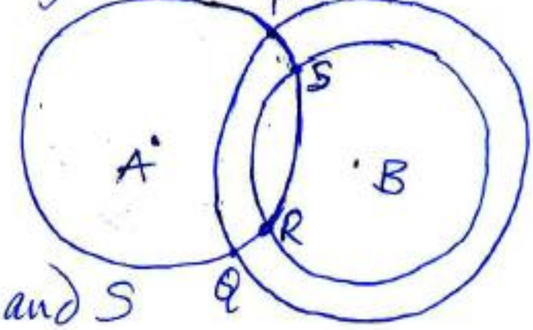
ISOTROPY AND HOMOGENEITY OF THE UNIVERSE

2.1 Defitions

- Isotropic:- Rotationally invariant.
  (looks similar in all directions)
- Homogeneous:- Translationally invariant
  (does not change (i.e. density const.) in a given direction).

- Space which looks Isotropic from 2 different points A and B is homogeneous:

Proof: Refer to diag.



- From B the points P and Q have equal density.
- From A, Points P, Q, R and S have equal density.

$\Rightarrow$ all points in the region have equal density

$\Rightarrow$ space (here) is homogeneous!

*Note: Converse not generally true

$\because$ Can have homogeneous anisotropic space.

Consequences of Cosmic Isotropy + Homogeneity:

$\Rightarrow$ no privilaged points in the universe

$\Rightarrow$ no privilaged observers

$\Rightarrow$ Universe looks same for all observers

$\Rightarrow$ Consequence:

— observations from one point (eg. Earth)

(a) Can represent true observable character of Universe

(b) Can be used to test Cosmology

2 THE HUBBLE LAW

2.1

— Hubble observed (1926) that at large scales

(i) galaxies recede from each other

(ii) recede with velocities v proportional to their separation distances r.

i.e.

$$\boxed{v = Hr} \quad \text{— (1)}$$

where $H = H(t)$

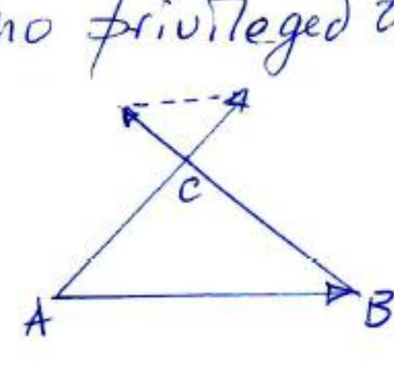
- Law ⇒ universe is dynamic and has been expanding throught its history.

⇒ Universe has age (show later)!

2.2 Hubble law is

- a consequence of Cosmic I + H
- Consistent with no privileged observer

To see this consider 2 observers A + B observing a receding point C. Then



$$\vec{v}_{CA} = H\vec{r}_{CA}$$

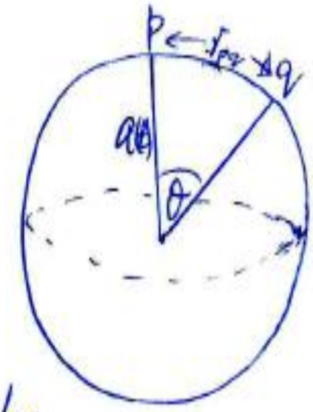
$$+ \; v_{CB} = H\vec{r}_{BA}$$

Observers must agree that Hubble law holds between them, since

$$\vec{v}_{BA} = \vec{v}_{BC} + \vec{v}_{CA} = \vec{v}_{CA} - \vec{v}_{CB} = H(\vec{r}_{CA} - \vec{r}_{CB})$$

i.e. $\vec{v}_{BA} = H\vec{r}_{BA}$

2.3 Making sense of the Hubble Law



- use analogue of an expanding 2-sphere cf. balloon with radius a(t)

- The arc length $r_{pq} = r(t)$ given by

$$r(t) = \theta a(t) \quad \text{——} \quad (2)$$

$$\Rightarrow \quad \dot{r} = \theta \dot{a} \quad \text{——} \quad (3)$$

$$2+3 \Rightarrow \quad \dot{r} = \frac{\dot{a}}{a} r$$

$$\Rightarrow \quad \boxed{v = Hr} \quad \text{——} \quad (4)$$

(4) Gives Hubble law for expanding 2-sphere if $H = \frac{\dot{a}}{a}$ —— (5)

Note: Eqs. 2–5 hold independent of expansion rate $\dot{a}$

2.4 Age of Universe Estimate

From Hubble law $v = Hr$
Can determine H for distant objects from
(i) observing recession v (from Doppler shifts)
(ii) observing dist. r (from luminosity $\sim \frac{1}{r^2}$)

use for (i) Standard Candles eg. Type 1A Supernovae
for (ii) Standard rulers (known objects of same size)

Current Estimates $\Rightarrow$ $H = 65 - 80 \text{ kms}^{-1} \text{Mpc}^{-1}$

Hubble law $\Rightarrow$ any 2 pts now separated by r, coincided in the past at

$$t_0 = \frac{v}{r} = \underbrace{\frac{1}{H_0}}_{\to (5)} \sim \underbrace{15 \text{ billion years}}_{\text{Age of Universe Estimate}}$$

Note: Estimate assumes H const. throughout past.
There are better Estimates!

3. NEWTONIAN COSMOLOGY

- Observations indicate Universe has gone through 4 Eras or Epochs:
  (i) Inflation, (ii) Radiation Era, (iii) Matter Era
  (iv) Dark Energy Era.
- Newtonian Mechanics describes matter Era well
- " is a good starting point
- rest of Era's described by General Relativity

3.1 DYNAMICAL EQUATIONS

3.11 Continuity:

- Consider an arbitrary 3-sphere, mass M, centered at an arbitrary point and expanding about it so at any time t it has a radius

$R(t) = a(t)\, x_{com}$ [$a(t) \to$ Scale factor, $x_{com} \to$ Comoving radius]

— the density evolves as

$$\rho(t) = \frac{M}{\frac{4}{3}\pi R(t)^3}$$

∵ M is conserved at some known time $t = t_0$ (e.g. now)

$$\rho_0 = \frac{M}{4\frac{\pi}{3} R_0^3}$$

Thus

$$\frac{\rho(t)}{\rho_0} = \frac{R_0^3}{R(t)^3} = \left(\frac{a_0}{a(t)}\right)^3$$

or

$$\boxed{\rho(t) = \left(\frac{a_0}{a(t)}\right)^3 \rho_0} \quad \text{— 3.11}$$

The density evolves as

$$\dot{\rho} = \frac{d\rho}{dt} = -3\left(\frac{\dot{a}}{a}\right)\left(\frac{a_0}{a}\right)^3 \rho_0$$

or $$\boxed{\dot{\rho} = -3H\rho(t)} \quad \text{— 3.12}$$

∵ H and ρ are purely functions of t we can rewrite Eq. 3.. as

$$\boxed{\dot{\rho} + \nabla\cdot(\rho\vec{v}) = 0} \quad \text{— 3.13}$$

3.3 is a particular case of the continuity Equation $\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$ — 3.14
for which $\rho(\vec{r}, t) = \rho(t)$

3.2 Cosmic Deceleration

— Recall the Hubble law predicts an expanding universe
But Universe contains matter which gravitates to
to slow down Expansion.

— We model this by considering a test particle mass m on surface of previous sphere, $R(t) = a(t)\chi_{com}$ and mass M

— An attractive force F exists given by

$$= -\frac{GmM}{R^2}$$

$$\Rightarrow m\ddot{R} = \frac{GmM}{R^2} \longrightarrow 3.21$$

$$\Rightarrow \ddot{R} = \frac{4\pi G}{3}\underbrace{\left(\frac{M}{\frac{4\pi}{3}R^3}\right)}_{\rho} R = \frac{4\pi}{3} G\rho(t) R$$

use $R = a\chi_{com} \Rightarrow$

$$\boxed{\ddot{a} = -\frac{4\pi}{3} G\rho a(t)} \longrightarrow 3.22$$

SUMMARY Eqs. 3.12 + 3.22 i.e.

i.e. $\boxed{\dot{\rho} + 3H\rho = 0}$ ← dynamics of matter

+ $\boxed{\ddot{a} + \frac{4\pi}{3} G\rho a = 0}$ ← dynamics of scale factor

sufficiently describe cosmic dynamics in Newtonian Cosmology.

3.3 Solutions

Recall from Eq. 3.11 $\rho(t) = \left(\frac{a_0}{a(t)}\right)^3 \rho_0$

subst. into Eq. 3.22 $\Rightarrow$

$$\ddot{a}(t) + \frac{4\pi}{3} G \rho_0 \frac{a_0^3}{a(t)^2} = 0 \longrightarrow 3.31$$

Can integrate Eq. 3.31 to find

$$\frac{1}{2}\dot{a}^2(t) - \left(\frac{4\pi G}{3} a_0^3 \rho_0\right)\frac{1}{a(t)} = E$$

or

$$\boxed{\frac{1}{2}\dot{a}^2 + V(a) = E} \longrightarrow 3.32$$

where E is an integration const. and $V(a) = \left(\frac{4\pi G}{3} a_0^3\right)\frac{1}{a}$

- Eq. 3.32 resembles Eq. of motion of a particle with energy E moving in gravitational pot'l of a planet.
- Depending on sign of E, particle may (i) fall back, (ii) stay in orbit or (iii) escape
- Similarly gravity retards cosmic expansion and the nature of expansion depends on the sign of E.

$\Rightarrow$

— In Newtonian cosmology, scale factor $a(t)$ has no absolute meaning.
— Only sign of E is physically meaningful.

Recall from 3.31 $\ddot{a}(t) = \frac{4\pi}{3} G \rho_0 \frac{a_0^3}{a(t)^2}$
divide through by a^2 $\Rightarrow$

$$\boxed{H^2 - \frac{2E}{a^2} = \frac{8\pi}{3} G \rho} \longrightarrow 3.33$$

Universe can not be static !!!

3.4 Critical Density

— In rocket motion, one knows Earth's pot'l and finds v to determine sign of E
— In cosmology one knows $H = \frac{\dot{a}}{a}$ but needs ρ to determine sign of E.

- From Eq. 3.33, the Equilibrium state $E = 0$, is given a critical density

$$\boxed{\rho_c = \frac{3}{8\pi G} H^2} \longrightarrow 3.41$$

Note: $\because H = H(t) \Rightarrow \rho_c = \rho_c(t)$
In most literature ρ_c refers usually to $(\rho_c)_{now}$

3.5 Cosmological Parameter

Recall from Eq. 3.33

$$H^2 - \frac{2E}{a^2} = \frac{8\pi}{3} G \rho$$

use Eq. 3.41 i.e. $\rho_c = \frac{3}{8\pi G} H^2$

$$\Rightarrow \frac{8\pi G}{3}\rho_{cr} - \frac{2E}{a^2} = \frac{8\pi G}{3}\rho$$

$$\Rightarrow \frac{2E}{a^2} = \frac{8\pi G}{3}(\rho_{cr} - \rho)$$

or

$$\boxed{E = \frac{4\pi G}{3} a^2 \rho_{cr}\left[1 - \Omega(t)\right]} \longrightarrow 3.51$$

where

$$\boxed{\Omega(t) = \frac{\rho(t)}{\rho_{cr}(t)}} \longrightarrow 3.52$$

is the cosmological parameter.

— Note: Knowledge of $\Omega(t)$ in 3.52 can be used to determine the sign of E in 3.51

— * Assuming E fixed, its sign does not change and determining Ω_{now} determines sign of E for all time.

$\Rightarrow$ Why Ω is important!

3.6 Curvature

Recall in Newtonian Cosmology, dynamics of Universe governed by

$$H^2 - \frac{2E}{a^2} = \frac{8\pi}{3}\rho \quad \text{or} \quad E = \frac{4\pi G}{3} a^2 \rho_{cr}\left[1 - \Omega(t)\right]$$

The sign of E and hence the value of Ω are related to the Curvature k of the Universe (see GR later) so that

$$H^2 + \frac{k}{a^2} = \frac{8\pi}{3}\rho$$

	E-Sign	Ω	k
open	>0	<1	$+1$
Flat	0	1	0
Closed	<0	>1	-1

3.7 Flat Universe with dust

3.71 Scale factor Evolution

- Can now determine dynamics of flat ($E=0$) Universe by finding $a = a(t)$
- Result can be used as standard gauge for other solutions.

Recall from 3.32, $\frac{1}{2}\dot{a}^2(t) - \left(\frac{4}{3}\pi G a_0^3 \rho_0\right)\frac{1}{a(t)} = E$

For $E=0$, $\dot{a}^2 = \frac{\alpha}{a}$ where $\alpha = \frac{8}{3}\pi G \rho_0 a_0^3$
$= \text{const.}$

$\Rightarrow \dot{a}^2 a = \alpha$

To integrate consider

$$\left[\frac{d}{dt} a^{\frac{3}{2}}\right]^2 = \left(\frac{3}{2}\dot{a}a^{\frac{1}{2}}\right)^2 = \frac{9}{4}\dot{a}^2 a$$

$$\Rightarrow a\dot{a}^2 = \frac{4}{9}\left[\frac{d}{dt} a^{3/2}\right]^2 = \alpha$$

$$\Rightarrow a^{\frac{3}{2}} = \left(\tfrac{9}{4}\alpha\right)^{\frac{1}{2}} t$$

i.e.

$$\boxed{a(t) = \left(\tfrac{9}{4}\alpha\right)^{\frac{1}{3}} t^{\frac{2}{3}}} \longrightarrow 3.71$$

$$\boxed{\text{i.e. } a(t) \sim t^{\frac{2}{3}}}$$

Eq. 3.71 gives Evolution of dust or matter dominated Universe

3.72 Age of the Universe

Recall $H = \frac{\dot{a}}{a}$

For flat Universe, use result $a(t) \sim t^{2/3}$

$$\Rightarrow H = \frac{d}{dt}t^{2/3} \Big/ t^{2/3}$$

or

$$\boxed{H = \frac{2}{3}\frac{1}{t}} \longrightarrow 3.72a$$

Eq. 3.72 $\Rightarrow$ if universe is flat ($E=0$) and dust dominated, its current age is given by

$$\boxed{t = \frac{3}{2}H_0} \longrightarrow 3.7$$

Cf value $t = \frac{1}{H_0}$ previously obtained without gravity considerations.

3.73 Density Evolution (dilution) law for flat Universe

Recall for $E=0$, $\rho = \rho_c = \frac{3}{8\pi G}H^2$
(Eq. 3.41)

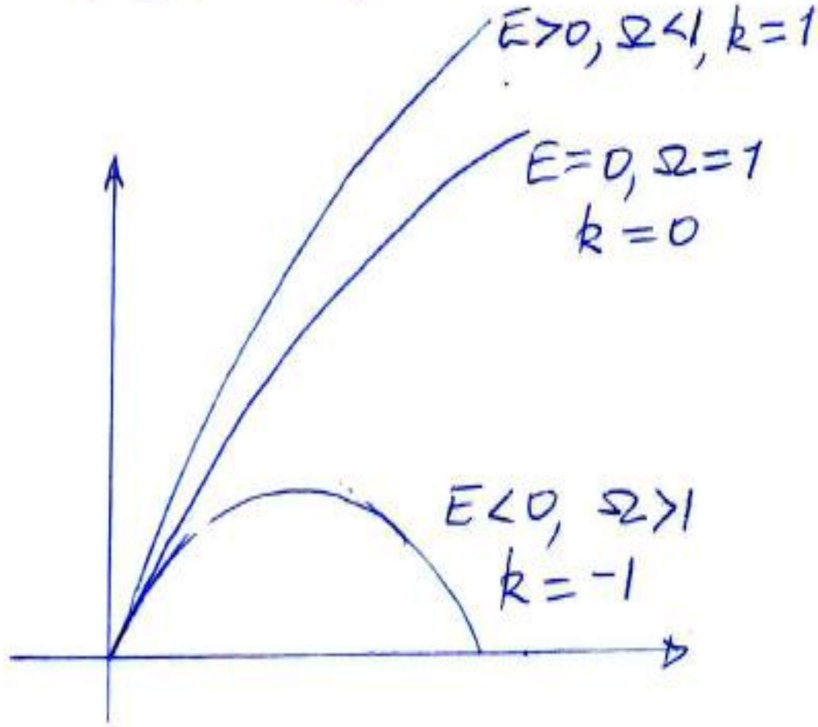
use Eq. 3.72a $\Rightarrow$ $\frac{2}{3}\left(\frac{1}{t}\right) = \frac{8\pi}{3}G\rho_{cr}$

i.e.

$$\boxed{\rho_{cr}(t) = \left(\frac{1}{6\pi G}\right)\frac{1}{t^2}} \longrightarrow 3.73$$

Eq. 3.73 gives time evolution of the critical density ρ_{cr}

3.74 Graphical Solutions of Universe.



Sketches summarize solutions of dynamics of the Universe in Newtonian model.

(1) $E>0$, Universe open, Expands for ever

(2) $E=0$, Universe flat (critically open) Expands for ever

(3) $E<0$, Universe closed, Expands to a maximum and recolapses.

4. RELATIVISTIC COSMOLOGY

- The Universe is dynamical.
- Its dynamics and its geometry depend on its energy content.
- Dynamics of the universe different during the various epochs.
- Newtonian Cosmology only described matter era dynamics.
- Need GR to describe entire dynamics.

4.1 ELEMENTS OF GENERAL RELATIVITY

- The content of Einsteins GR is that matter fields (of any form) modify the geometry of spacetime through gravity.
- The GR theory is summarized in a set of 16 differential Equations

$$G_{\mu\nu} = T_{\mu\nu} \longrightarrow 4.1$$

- Eq. 4.1 is a tensor (2nd rank) Equation

The lhs represents the curvatures of spacetime

$\Longrightarrow$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu}$$

- Both the Ricci tensor $R_{\mu\nu}$ and the Ricci scala R are functions of the metric $g_{\mu\nu}$
- The metric tensor $g_{\mu\nu}$ which encodes the gravity information, is the solution to Eq. 4.1

- The rhs $T_{\mu\nu}$ encodes information of the energy fieds that generate gravity.

Specifically:

$$T_{\mu\nu} = \begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix}$$

Energy density

Energy flux

Shear stress (viscosity

Momentum density

Momentum flux

Pressure

- Generally $g_{\mu\nu}$ is solution to the 16 Eqs.
- Symmetry (spacetime) can reduce the Equations
- Eg. (i) Isotropic spacetime $\rightarrow$ 4 Eqs.
  (ii) I + H, universe " $\rightarrow$ 2 Eqs.

4.2 FRIEDMAN-ROBERTSON-WALKER (FRW) GEOMETRY

[GEOMETRY OF THE UNIVERSE]

Recall:

Universe is isotropic + homogeneous

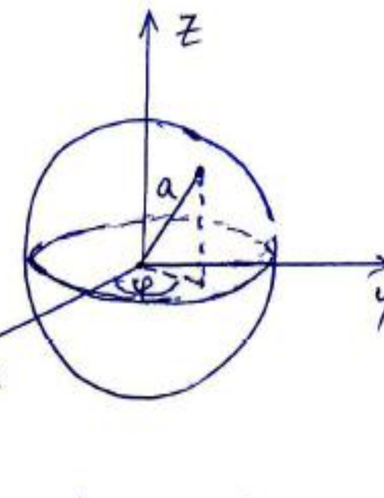
$\Rightarrow$ Universe can be represented as time-ordered sequence of 3D space-like hyperfaces

$\Rightarrow$ i.e. surfaces of constant time.

— Symmetry considerations allow only 3 possible I+H space types: spherical, flat + hyperbolic

4.21 Robertson-Walker Metric (Construction)

— An isotropic, homogeneous space can be imbedded in a higher dimensional space.



— Consider (e.g.) imbedding of a 2-sphere, radius a in 3D Euclidean space given by

$$x^2+y^2+z^2=a^2 \longrightarrow 4.21$$

On the 2-surface: $2x\,dx + 2y\,dy + 2z\,dz = 0$

Then $$dz = -\frac{[x\,dx + y\,dy]}{z=\sqrt{a^2-x^2-y^2}} \quad \text{—— } 4.21$$

Also infinitesimal displacement in 3D Euclidean space is
$$dl^2 = dx^2 + dy^2 + dz^2 \quad \text{— } 4.22$$

- On the surface (use 4.21 in 4.22)

$$dl^2 = dx^2 + dy^2 + \ldots$$

Set $x = r\cos\varphi$, $y = r\sin\varphi$ and use to find

$$dl^2 = \left[\frac{1}{1-\frac{r^2}{a^2}}\right] dr^2 + r^2 d\varphi^2 \quad \text{— } 4.22$$

- Can generalize to 3-sphere in Euclidean space (use $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$
$z = r\cos\theta$

—

Find $dl^2 = a^2\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + \sin^2\theta\, d\varphi^2\right]$

— 4.23

- The universe can be represented by a coo-rdinate system $(x^0, x^1, x^2, x^3) = (t, r, \theta, \varphi)$.
- The infinitesimal line element $ds^2 = dt^2 - dl^2$ is
$$\boxed{ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2 d\theta^2 + \sin^2\theta\, d\varphi^2\right]}$$
— 4.24

- Eq. 4.24 is the RW metric element
- Describes spacetime of Universe

4.3 THE FRIEDMAN EQUATIONS

- Recall the dynamics of the universe at any era depends on the energy terms in $T_{\mu\nu}$ that dominate.

- The dynamics is determined by solving
$$G_{\mu\nu} = T_{\mu\nu}$$
for the metric $g_{\mu\nu}$

- Here we motivate the approach through construction of the relevant differential equations using thermodynamic arguments to modify the non relativistic equations.

- Recall in Newtonian Cosmology the dynamics is described by
$$\dot{\rho}(t) + 3H\rho(t) = 0 \quad \text{——} \quad 3.12$$
and
$$\ddot{a} + \frac{4\pi}{3} G \rho(t)\, a(t) = 0 \quad \text{——} \quad 3.22$$

- Consider now that the universe is a dynamic system. Thus the work done in expansion (at pressure p) occurs at expense of energy change $dE \; d(\rho V)$

i.e. $dE = -p\,dV \Rightarrow d(\rho V) = -p\,dV \longrightarrow 3.23$

$\because V \propto a^3$

$\Rightarrow a^3 d\rho + 3\rho a^2 da = -3pa^2 da$

$$d\rho = -3(\rho + 3p)\frac{da}{a}$$

i.e. $$\boxed{\dot{\rho} = -3(\rho + p)H} \quad \text{———} \quad 4.25$$

- Eq. 3.24 is the new conservation Eq.
- It takes into account contribution due to pressure
- It is the GR Energy conservation Eq. ($T^{\mu}_{0;\mu}$)

- Further consideration of the gravitational effects of pressure (diagonal elements) modifies the acceleration (Eq. 3.22) to

$$\boxed{\ddot{a} + \frac{4}{3}\pi G(\rho + 3p)a = 0} \quad \text{———} \quad 4.26$$

Eq. 4.26 is the 1st Friedman Equation

Integration: multiply through 4.26 by $\dot{a}$ and subst. for p from 4.25 $\Rightarrow$

$$\dot{a}\ddot{a} = \frac{4}{3}\pi G\left[\dot{\rho}a^2 + 2\rho\dot{a}a\right]$$

which integrates to yield

$$\boxed{H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}} \longrightarrow 4.27$$

where $k = \{-1, 0, 1\}$ is curvature of universe.

- Eq. 4.27 is the (more familiar) 2^{nd} Friedman Eq.
- Resembles the motion Eq. 3.33 in Newtonian Cosmology

$$H^2 - \frac{2E}{a^2} = \frac{8\pi}{3} G\rho$$

provided we set $k = 2E$.

Differences:

(i) Eq. 4.27 applies an arbitrary Eq. of state (Power of GR!)

(ii) $k = \{-1, 0, 1\}$ represent spacetime curvature. + for $k = \pm$, $a(t)$ is radius of curvature!

- The geometry of universe is determined by the Cosmological parameter $\Omega = \frac{\rho}{\rho_{cr}}$

Thus

$$\Omega \begin{cases} > 1 \Rightarrow k = +1 & \text{Closed Universe} \\ = 1 \Rightarrow k = 0 & \text{flat Universe} \\ < 1 \Rightarrow k = -1 & \text{Open Universe} \end{cases}$$

— The Friedman Eq.

$$H^2 = \frac{8}{3}\pi G \rho - \frac{k}{a^2} \quad — 4.27$$

Along with either

$$\ddot{a} + \tfrac{4}{3}\pi G(\rho + 3p)a, = 0 \quad — 4.26$$

or $$\dot{\rho} + 3(\rho + P)H = 0, \quad — 4.25$$

Completly determine the dynamics of an isotropic and homogeneous universe, by determining $a(t)$ and $\rho(t)$.

— These equations must be supplemented by an equation of state

$$p = p(\rho) \quad — 4.28$$

4.4 SUMMARY OF RESULTS

- The FRW geometry describes the universe to be isotropic and homogeneous with the following dynamical solutions, depending on the dominating fields.

Content	Eqn. of State	Dilution Law	Time Evolution
(Pressureless) Matter	$P=0$	$\rho \sim a^{-3}$	$a \sim t^{2/3}$
Radiation	$=\frac{1}{3}\rho$	$\rho \sim a^{-4}$	$a \sim t^{1/2}$
Curvature		$H^2 = \frac{-k}{a^2}$	$a \sim t$
Dark Energy	$P=-\rho$	$\rho = \frac{\Lambda}{8\pi G}$	$a \sim e^{Ht}$
Generic	$P=w\rho$	$\rho \sim a^{-3(1+w)}$	

— 4.29

The geometry is described by the RW line element,

$$\boxed{ds^2 = dt^2 - a^2(t)\left[\frac{1}{1-kr^2}dr^2 + d\Omega^2\right]} \quad \rightarrow 4.30$$