Transverse Beam Dynamics

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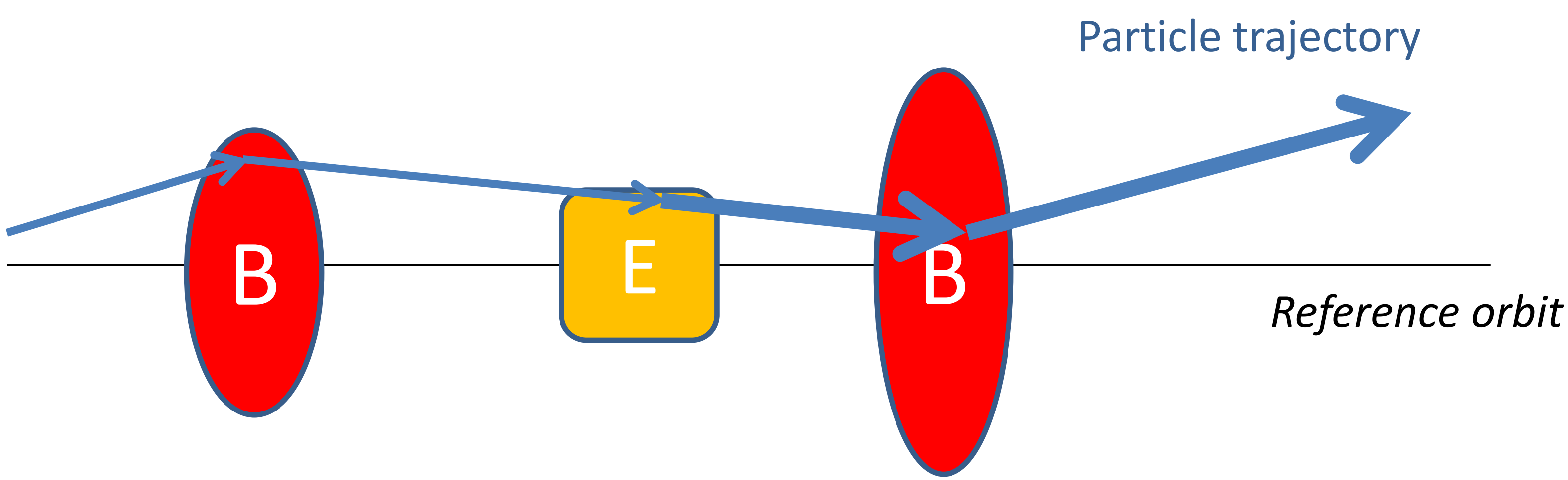
Outlook

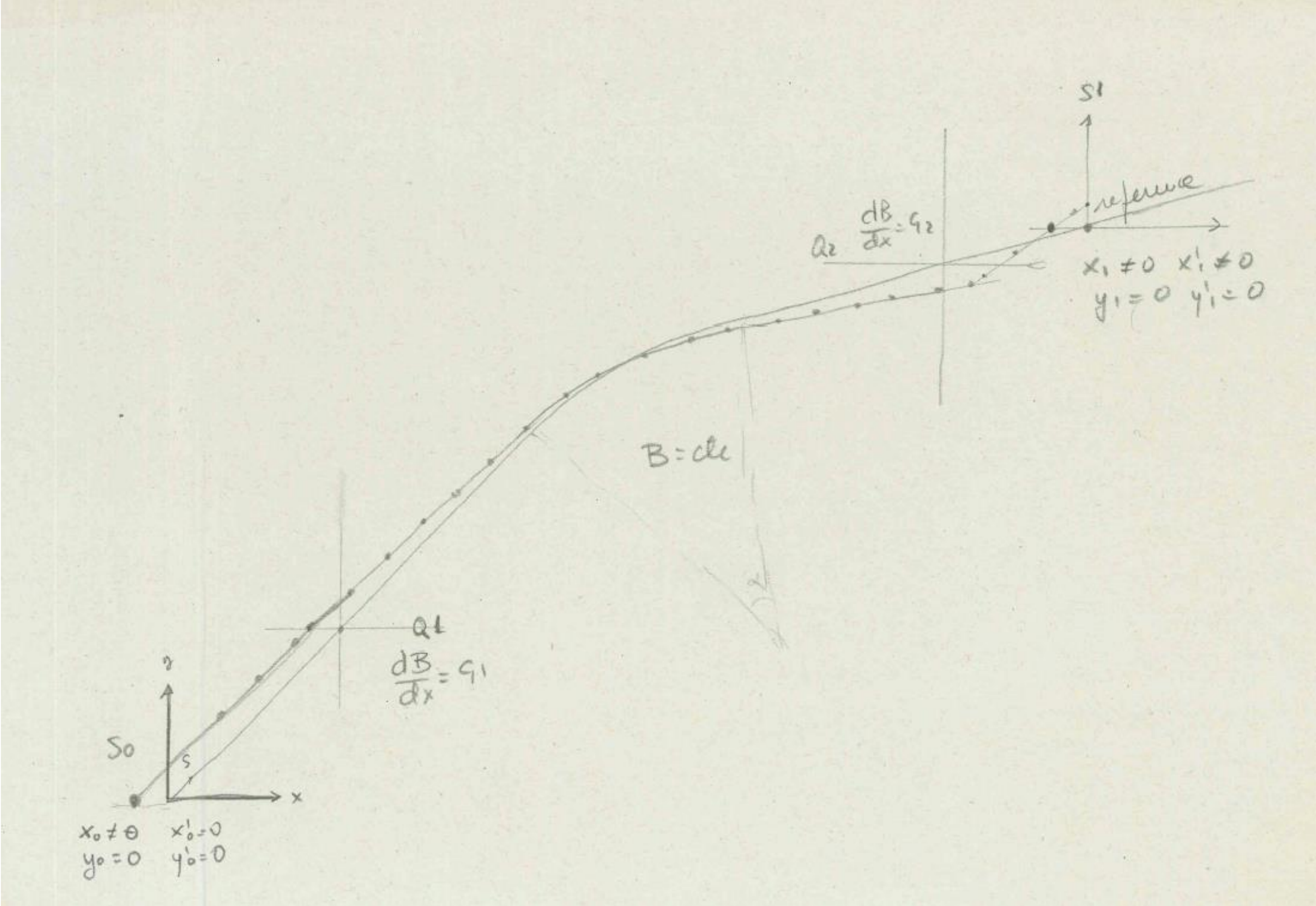
Charged particle and magnetic fields
Linear approximation
Dipoles and quadrupoles
Lorentz equation
Horizontal plane
From Lorentz equation to Hill's equation
From Hill's equation to Twiss parameter definition
Matrix representation
Storage ring – periodic solution
Stability diagram
Chromaticity
Sextupole correction

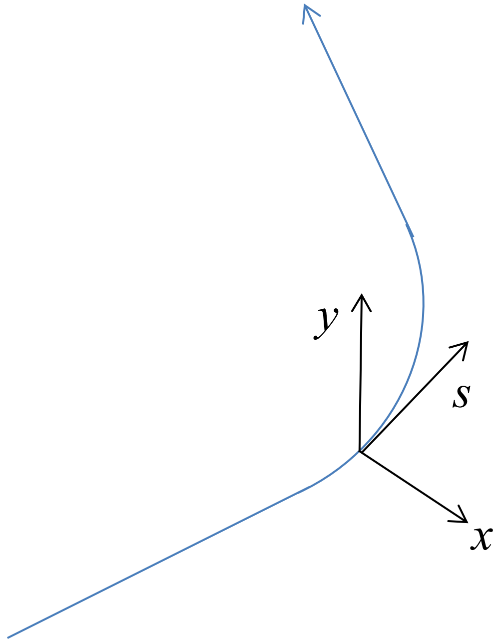
The accelerator from the particle point of view is a sequence of

Drifts – No external fields – Particles go straight
Magnetic fields – Particles are bent according to the magnetic rigidity
Electric fields – Particles go straight, gain or loose energy



Transverse dynamics





Reference system

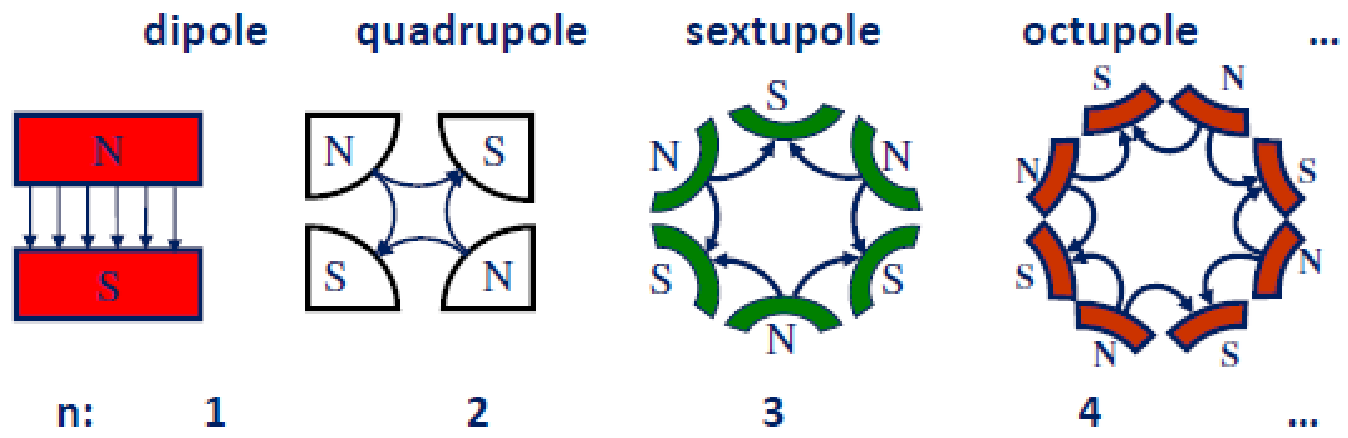
x : horizontal

y : vertical

s : longitudinal along the trajectory

Typical Magnetic Fields

2n-pole:



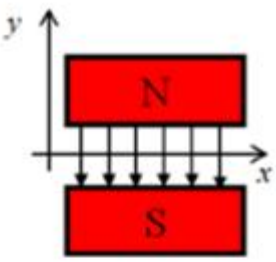
- Normal: gap appears in the horizontal plane
  - Skew: rotate around beam axis by $\pi/2n$

Properties of Typical Magnets

Dipoles: used for guiding the particle trajectories

$B_x = 0$

$B_y = B_o = constant$

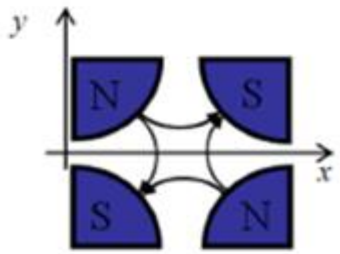


Quadrupoles: used to focus the particle trajectories

$B_x = G\,y$

$B_y = -G\,x$

$G = constant$

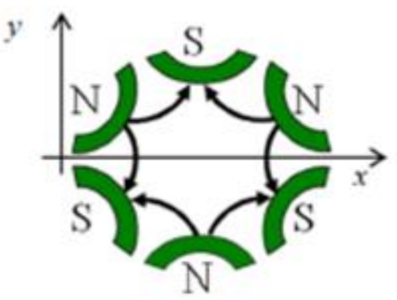


Sextupoles: used to correct chromatism and non linear terms

$B_x = 2\,S\,x\,y$

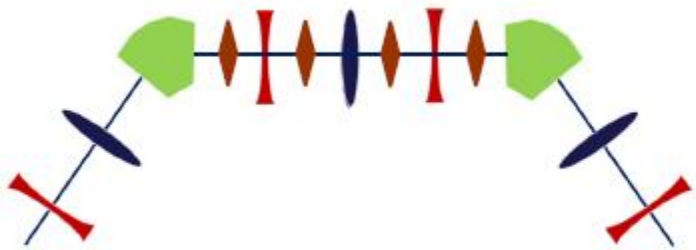
$B_y = S\,(x^2-y^2)$

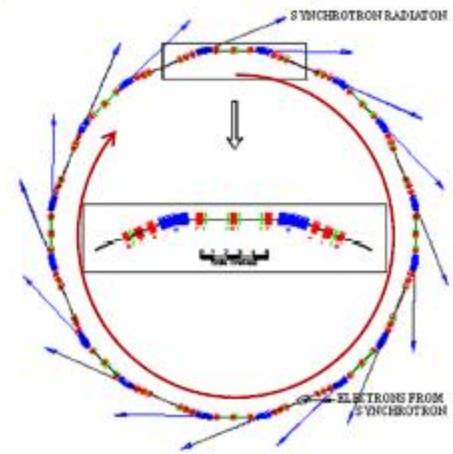
$S = constant$



Lattice in an accelerator

Lattice: Sequence of magnets interleaved with drifts (used for diagnostics, vacuum pumping, Injection, extraction, etc)





The first step in calculating a lattice is to consider only the linear components of it (quadrupoles and dipoles). Non linear effects and chromatic aberration corrections will be evaluated later.
The trajectory of the reference particle (the particle with nominal energy and initial position and divergence set to zero) along the optics is calculated.
All the other beam particles are represented in a frame moving along the reference trajectory, and where the reference particle is always in the center.
Coordinate systems used to describe the motion are usually locally Cartesian or cylindrical (typically the one that allows the easiest field representation)

Magnetic rigidity

Remember:

$$\beta \equiv \frac{v}{c} = \frac{pc}{E}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

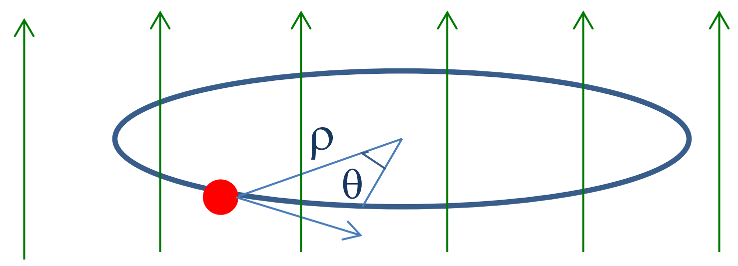
$$\text{momentum } p = \gamma m v$$

$$\text{total energy } E = \gamma m c^2$$

$$\text{kinetic energy } K = E - mc^2$$

$$E^2 = \sqrt{\left(mc^2\right)^2 + (pc)^2}$$

Constant magnetic field: B



A charged particle (charge = q) will follow a circle of radius ρ

Lorentz force: $F_L = qvB$

Centrifugal force: $F_{centr} = \frac{\gamma m_o v^2}{\rho}$

$F_L = F_{centr}$

$$\Rightarrow \quad B\rho = \frac{p}{q}$$

Equations of motion

Magnetic field representation (consider only normal terms)

$$B_y(x) = B_0 + \frac{dB_y}{dx}x + \frac{1}{2!}\frac{d^2B_y}{dx^2}x^2 + ..$$

$$B_x(y) = \frac{dB_x}{dy}y + \frac{1}{2!}\frac{d^2B_x}{dy^2}y^2 + ..$$

Concentrate in horizontal motion: only vertical fields

$$g = \frac{dB_y}{dx}, \; g' = \frac{dg}{dx}$$

Gradient and its derivative

Let's normalize to momentum

$$\frac{B_y\,(x)}{p/e} = \frac{B_0}{B_o\rho} + \frac{g\;x}{p/e} + \frac{1}{2!}\frac{eg'}{p/e} + \cdots$$

Equations of motion

Consider only linear terms:

$$\frac{B(x)}{p/e} = \frac{1}{\rho} + kx \qquad \text{where} \qquad k = \frac{g}{p/e} = \frac{1}{B\rho}\frac{dB_y}{dx}$$

In the ideal orbit, ρ = const $d\rho/dt = 0$

General trajectory: $\rho = \rho + x$, with $x \ll \rho$

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = qB_y v$$

Since $x \ll \rho$ Taylor expansion $\frac{1}{x+\rho} \approx \frac{1}{\rho}(1 - \frac{x}{\rho})$

$$\frac{d^2}{dt^2}(\rho) = 0$$

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1 - \frac{x}{\rho}) = qB_y v$$

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1 - \frac{x}{\rho}) = qv\left\{B_0 + x\frac{\partial B_y}{\partial x}\right\}$$

Equations of motion

Pass from t to s as independent variable

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{ds}\frac{ds}{dt}\right) = \frac{d}{ds}(x'v)\frac{ds}{dt}$$

$$= x''v^2 + \frac{dx}{ds}\frac{dv}{ds}v$$

Second term is zero

$$x''v^2 - \frac{v^2}{\rho}\left(1 - \frac{x}{\rho}\right) = \frac{qvB_0}{m} + \frac{qvxg}{m}$$

Divide by v^2

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = \frac{B_0}{p/q} + \frac{xg}{p/q}$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = -\frac{1}{\rho} + kx$$

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = 0$$

For vertical plane

$$\frac{1}{\rho^2} = 0, k = -k$$

$$y'' + ky = 0$$

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = 0$$

$$y'' + ky = 0$$

Solution of equations of motion

Putting

Horizontal : $K = \frac{1}{\rho^2} - k$

Vertical : $K = k$

If K is constant we get the differential equation of harmonic oscillator with spring constant K

$$x'' + Kx = 0$$

$$x(s) = a_1 \cos(\omega s) + a_2 \sin(\omega s)$$

$$\omega = \sqrt{K}$$

At $s = 0 \;\Rightarrow\; x = x_0,\; x' = x'_0$

(where now *x* can represent both *x* or *y)*

$a_1 = x_0 \quad a_2 = \frac{x'_0}{\sqrt{K}}$,

$$x(s) = x_0 \cos\left(\sqrt{K}s\right) + x'_0 \frac{1}{\sqrt{K}} \sin\left(\sqrt{K}s\right)$$

$$x'(s) = -x_0 \sqrt{K} \sin\left(\sqrt{K}s\right) + x'_0 \cos\left(\sqrt{K}s\right)$$

Matrix formalism

We can write the equations in matrix formalism: coordinates at point s_1 can be obtained knowing the coordinates at s_0

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} = M \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

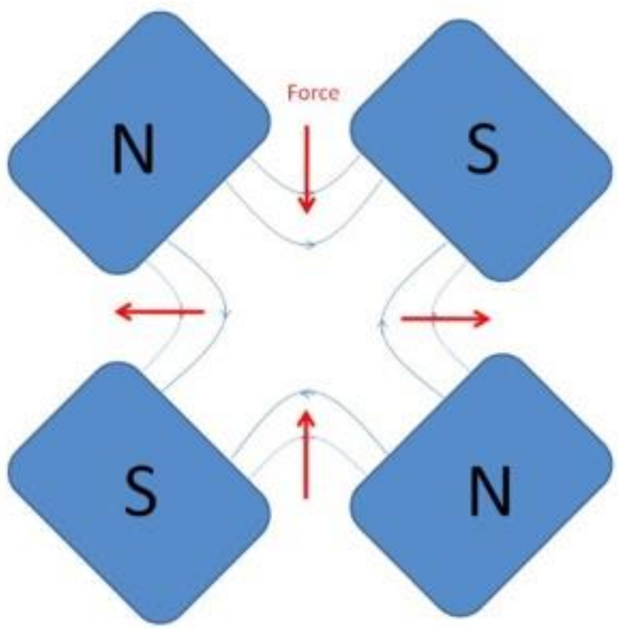
$$M = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix}$$

Example: Drift
Length: L
$K = 0$

$$M_{Drift} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$$x_1 = x_0 + Lx'_0$$
$$x'_1 = x'_0$$

Matrix formalism



Focusing quadrupole:
Length L , $K > 0$

$$M = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix}$$

$$x_1 = x_0 cos\sqrt{K}L + \frac{x'_0}{\sqrt{K}} sin\sqrt{K}L$$
$$x'_1 = -x'_0\sqrt{K}sin\sqrt{K}L + x'_0 cos\sqrt{K}L$$

Defocusing quadrupole:
Length L , $K < 0$

$$M = \begin{pmatrix} \cosh\left(\sqrt{|K|}L\right) & \frac{1}{\sqrt{|K|}}\sinh\left(\sqrt{|K|}L\right) \\ \sqrt{|K|}\sinh\left(\sqrt{|K|}L\right) & \cosh\left(\sqrt{|K|}L\right) \end{pmatrix}$$

Thin lens approximation

In parallel with optics

$$M = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix}$$

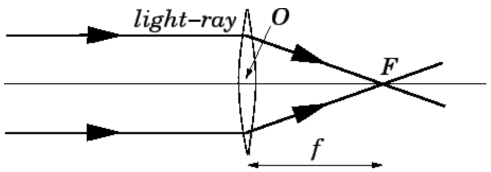
L=> 0 KL constant

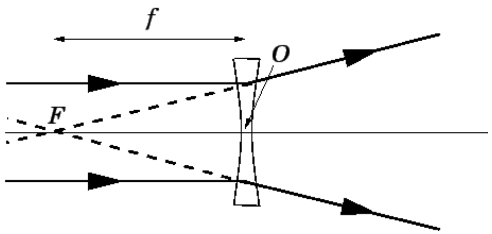
$\frac{\sin(\sqrt{K}L)}{\sqrt{K}L}$ -> 1

$$M = \begin{pmatrix} 1 & 0 \\ -KL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \pm\frac{1}{f} & 1 \end{pmatrix}$$

$$f = \frac{1}{KL} \gg L$$

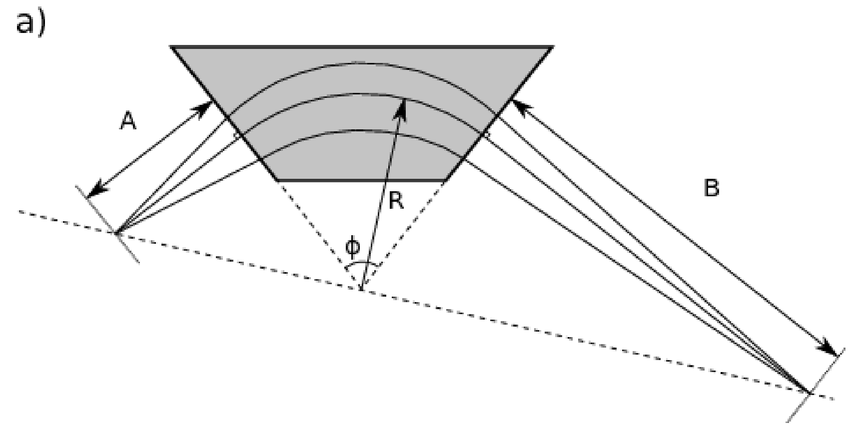
f positive or negative depending on quad





Dipole

Sector magnet:
Nominal particle trajectory is perpendicular to dipole entrance
Horizontal plane: $K = {}^{1}/_{\rho^2} - k$
Vertical plane: $K = k$



If $k = 0, L = \rho\theta$

$$M_H = \begin{pmatrix} cos\theta & \rho sin\vartheta \\ -\frac{1}{\rho} sin\theta & cos\theta \end{pmatrix}$$

$$M_V = \begin{pmatrix} 1 & \rho\vartheta \\ 0 & 1 \end{pmatrix}$$

Magnet with field index: $k \neq 0$

Exercise – write the matrix

System of lattice elements: Drifts (M_D), quads (M_Q), bendings (or dipoles) (M_B)

Starting with $\begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$ The final position and divergence of the particle will be $\begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}$

$$\begin{pmatrix} x_1 \\ x'_1 \end{pmatrix} = M_{Dn} \cdot M_{Qn} \cdot M_{Dn-1} \cdots \cdot M_{B1} \cdot M_{D2} \cdot M_{Q1} \cdot M_{D1} \cdot \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

Or simpler

$$\begin{pmatrix} x_1 \\ x'_1 \end{pmatrix} = M(s_1, s_0) \cdot \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

The mathematical representation of an accelerator lattice is a sequence of matrices

Twiss parameters – Betatron tune

$x'' + Kx = 0$ If K = constant /harmonic oscillator

$x'' + K(s)x = 0$ If K varies with s: Hill's equation

The solution of the Hill equation is given by:

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}cos(\varphi(s) + \varphi_0)$$

ε and φ_0 integration constants

Inserting 1 in the equation of motion it can be shown that the phase advance is related to β by

$$\varphi(s) = \int_0^s \frac{ds}{\beta(s)}$$

In storage rings (length of circumference = L) beta is periodic

$$\beta(s + L) = \beta(s)$$

One complete turn: phase advance in one turn: ***Betatron Tune***

$$Q_{x,y} = \frac{1}{2\pi}\oint \frac{ds}{\beta_{x,y}(s)}$$

More on next slides

Solution of Hill's equation

$$x = \sqrt{\varepsilon\beta}\cos\phi$$

With $\phi = \varphi(s) + \varphi_0$ and β depending on s

$$\alpha = -\frac{1}{2}\frac{d\beta}{ds}$$

$$\gamma = \frac{1+\alpha^2}{\beta}$$

with

$$x' = \sqrt{\varepsilon}\frac{d\sqrt{\beta}}{ds}\cos\phi - \sqrt{\varepsilon\beta}\frac{d\phi}{ds}\sin\phi$$

$$x' = -\alpha\sqrt{\frac{\varepsilon}{\beta}}\cos\phi - \sqrt{\varepsilon\beta}\frac{d\phi}{ds}\sin\phi$$

$$x'' = \sqrt{\varepsilon}\frac{d^2\sqrt{\beta}}{ds^2}\cos\phi - \sqrt{\varepsilon}\frac{d\sqrt{\beta}}{ds}\frac{d\phi}{ds}\sin\phi - \sqrt{\varepsilon}\frac{d\sqrt{\beta}}{ds}\frac{d\phi}{ds}\sin\phi - \sqrt{\varepsilon\beta}\frac{d^2\phi}{ds^2}\sin\phi - \sqrt{\varepsilon\beta}\left(\frac{d\phi}{ds}\right)^2\cos\phi$$

Substituting in the 2nd order equation, and equating to zero terms multipliying sin and cos we get:

$$-2\sqrt{\varepsilon}\frac{d\sqrt{\beta}}{ds}\frac{d\phi}{ds} - \sqrt{\varepsilon\beta}\frac{d^2\phi}{ds^2} = 0$$

Dividing by $\sqrt{\varepsilon}$ and differentiating $\sqrt{\beta}$:

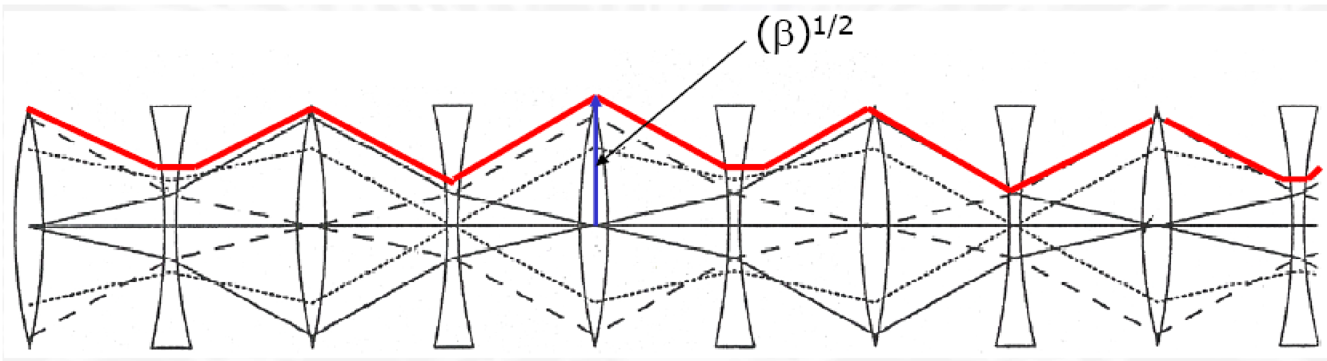
$$\frac{d\beta}{ds}\frac{d\phi}{ds} + \beta\frac{d^2\phi}{ds^2} = 0$$

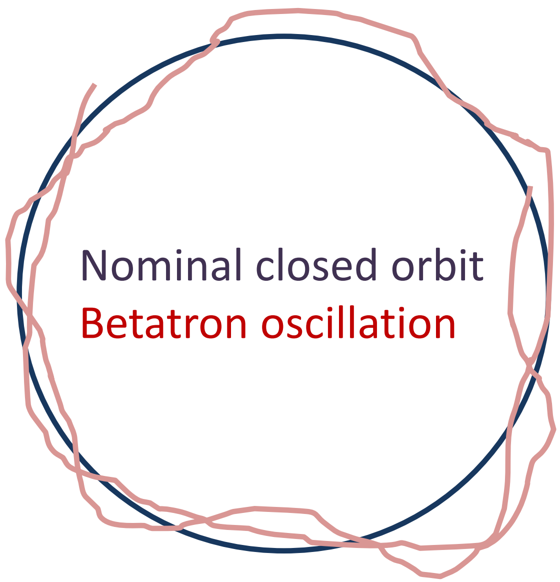
$$(\beta\phi'\,)' = 0$$

$$\beta\phi' \quad = cte = 1$$

$$\phi' = \frac{1}{\beta} \qquad\qquad \varphi(s) = \int_0^s \frac{ds}{\beta(s)}$$

Betatron oscillations





Particles oscillate around the closed orbit, a number of times which is given by the betatron tune.
The square of the β function by the emittance represents the envelope of the betatron oscillations

Twiss parameters

Amplitude of an oscillation

$$\hat{x}(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}$$

$\beta(s)$represents the envelope of all particle trajectories at a given position s in a storage ring

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}cos(\varphi(s) + \varphi_0)$$

$$x'(s) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\{\alpha(s)\cos(\varphi(s) + \varphi_0) + sin(\varphi(s) + \varphi_0)\}$$

$$\alpha(s) = -\frac{1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1 + \alpha^2(s)}{\beta(s)}$$

α, β and γ are the Twiss parameters

Emittance

Inserting in x' eq.

$$\varepsilon = \gamma(s)x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s){x'}^2(s)$$

ε is a constant of motion, not depending on *s*.
Parametric representation of an ellipse in *x,x'* phase space defined by alfa, beta, gamma: **Courant-Snyder invariant** emittance ε
For a single particle, different positions in the storage ring and different turns:

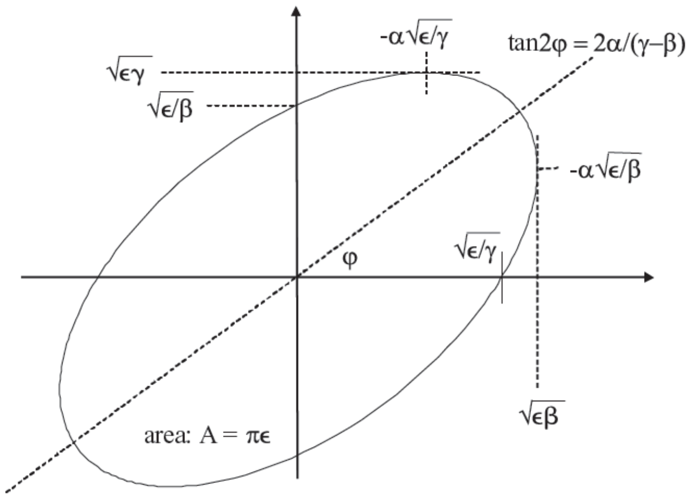
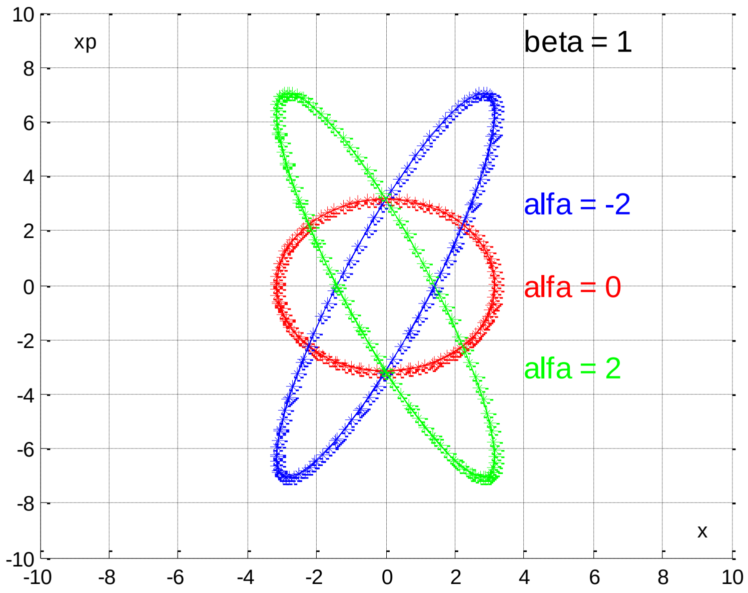


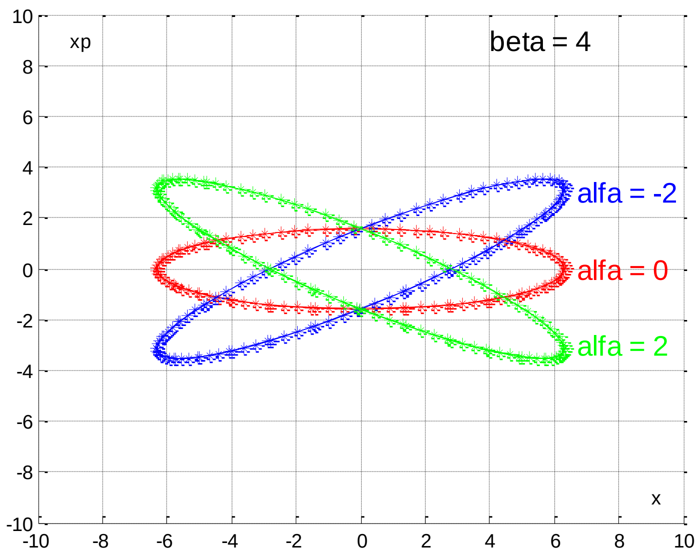
Fig. 5.2. Phase space ellipse

Drawing particle position in different points of a storage ring

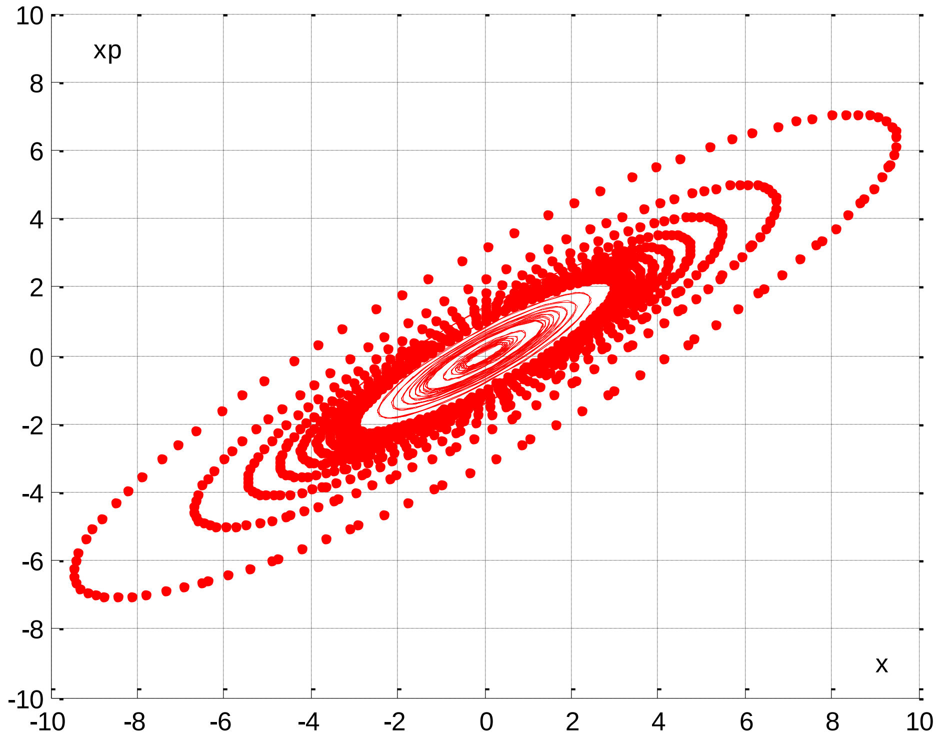
```
alfa = 0;
beta = 1;
gamma = (1+alfa*alfa)/beta;
eps = 10;
 for i =1 : npoints
thdeg (i) = i*13;
teta(i) = thdeg(i)*rad;
x(i) = sqrt(eps*beta)*cos(teta(i));
xp(i) = -sqrt(eps/beta)*( alfa*cos(teta(i)) + sin(teta(i)) );
 hold on
 end
grid on
plot(x,xp,'-r*')
```

Same area, same emittance,
Different orientation





Liouville's theorem



Each particle, in absence of non conservative forces, has a constant invariant.
Under the influence of conservative forces the particle density in phase space is constant.
Magnetic fields of dipoles and quadrupoles are conservative:
In a beam the phase space is maintained constant

Beam size $x_{max} = \sqrt{\beta(s)\varepsilon} \quad x'_{max} = \sqrt{\gamma(s)\varepsilon}$

Evolution of particle trajectory related to evolution of Twiss Parameters

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}cos(\varphi(s) + \varphi_0)$$

$$x'(s) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\{\alpha(s)\cos(\varphi(s) + \varphi_0) + sin(\varphi(s) + \varphi_0)\}$$

Developing *sin(a+b)*

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}(cos\varphi(s)cos\varphi_0 - sin\varphi(s)sin\varphi_0)$$

$$x'(s) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\{\alpha(s)\cos\varphi(s)cos\varphi_0 - \alpha(s)\sin\varphi(s)sin\varphi_0 + \sin\varphi(s)cos\varphi_0 + \cos\varphi(s)sin\varphi_0\}$$

Starting at $s(0) = 0 \quad \varphi(0) = 0 \quad cos\varphi_0 = \frac{x_0}{\sqrt{\varepsilon\beta_0}},\ sin\varphi_0 = -\frac{1}{\sqrt{\varepsilon}}\left(x'_0\sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right)$

Substituting above

$$x(s) = \sqrt{\frac{\beta(s)}{\beta_0}}[\cos\varphi(s) + \alpha_0 sin\varphi(s)]x_0 + \left[\sqrt{\beta(s)\beta_0}\,sin\varphi(s)\right]x'_0$$

$$x'(s) = \frac{1}{\sqrt{\beta(s)\beta_0}}[(\alpha_0 - \alpha(s))\cos\varphi(s) - (1 + \alpha_0\alpha(s))sin\varphi(s)]x_0 + \sqrt{\frac{\beta(s)}{\beta_0}}[\mathrm{cos}\varphi(\mathrm{s}) - \alpha(s)\,sin\varphi(s)]x'_0$$

Particle coordinates evolution as a function of Twiss parameters

These equations can be written under matrix formalism

$\beta(s) \rightarrow \beta_s \quad \alpha(s) \rightarrow \alpha_s \quad \varphi(s) \rightarrow \varphi_s$

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}[\cos\varphi_s + \alpha_0 sin\varphi_s] & \sqrt{\beta_s\beta_0}\, sin\varphi_s \\ \frac{1}{\sqrt{\beta_s\beta_0}}[(\alpha_0 - \alpha_s)\cos\varphi_s - (1 + \alpha_0\alpha_s) sin\varphi_s] & \sqrt{\frac{\beta_s}{\beta_0}}[\cos\varphi_s - \alpha_s\, sin\varphi_s]x'_0 \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

Knowing Twiss parameters in two points and phase advance in between it is possible to define the particle position in the second point from the first one with no need of knowing the elements in between

One turn matrix

For one revolution:

$$\beta(s) = \beta_0 \quad \alpha(s) \to \alpha_0 \quad \varphi(s) \to 2\pi Q_{x,y}$$

One turn matrix

$$M_T(s) = \begin{pmatrix} \cos 2\pi Q + \alpha_o \sin 2\pi Q & \beta_o \sin 2\pi Q \\ -\gamma_o \sin 2\pi Q & \cos 2\pi Q - \alpha_o \sin 2\pi Q \end{pmatrix} = \cos 2\pi Q \cdot 1 + \sin 2\pi Q \cdot J$$

With $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad and \quad J = \begin{pmatrix} \alpha & \beta \\ -\gamma & \alpha \end{pmatrix}$

It can be easily demonstrated that

$$|M_T| = 1$$

Knowing the one turn matrix at a certain location and the total phase advance, β and α are defined at that location

Exercise: write the expression of β and α as a function of one-turn matrix terms

N-turns Matrix

$$M^N = (1 \cdot cos2\pi Q + J \cdot sin2\pi Q)^N = 1 \cdot cos2\pi NQ + J \cdot sin2\pi NQ$$

Motion bounded (stable) if the elements of M^N are bounded:

$$|cos2\pi Q| \leq 1$$

Or

$$Tr(M) \leq 2$$

↑

Stability condition

Transformation of Twiss Parameters

Start from two positions, s_o and s

$$\begin{pmatrix} x \\ x' \end{pmatrix} = M \cdot \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

Or

$$\varepsilon = \beta_s x'^2 + 2\alpha_s x x' + \gamma_s x^2$$

$$\varepsilon = \beta_0 x'^2_0 + 2\alpha_0 x_0 x'_0 + \gamma_0 x_0^2$$

$$\begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = M^{-1} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}$$

$$x_0 = S'x - Sx'$$

$$x'_0 = -C'x + Cx'$$

Inserting in ε and rearranging the terms on x and x'

$$\beta(s) = C^2\beta_0 - 2SC\alpha_0 + S^2\gamma_0$$

$$\alpha(s) = -CC'\beta_0 + (SC' + S'C)\alpha_0 - SS'\gamma_0$$

$$\gamma(s) = C'^2\beta_0 - 2S'C'\alpha_0 + S'^2\gamma_0$$

Transformation of Twiss Parameters

And can be written in matrix notation: given the Twiss parameters α, β, γ at any point in the lattice we can transform them and calculate their values at any other point in the ring if we know the transfer matrix.

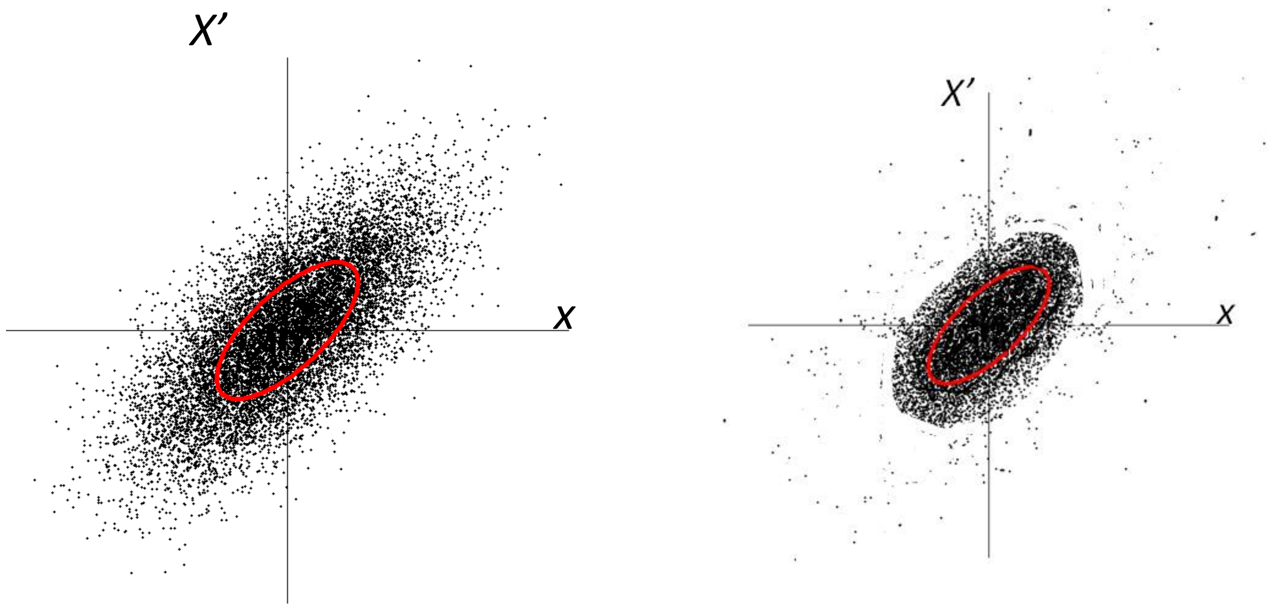
$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} C^2 & -2CS & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta_o \\ \alpha_o \\ \gamma_o \end{pmatrix}$$

rms Emittance

The emittance is the area of the phase space occupied by all particles in a beam
Each particle has its own ‘invariant emittance’
rms emittance represents the beam characteristics, and is defined as:

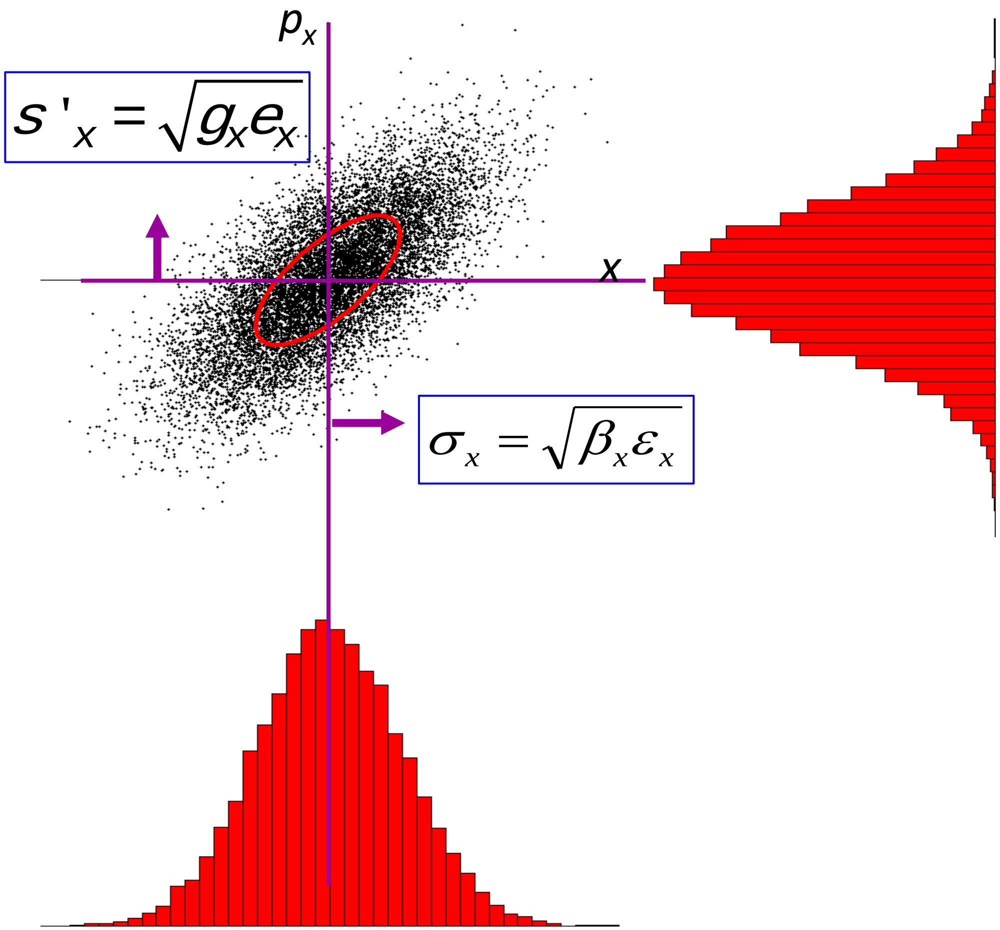
$$\varepsilon_x{}^{rms} = \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle x\, x'\rangle^2}$$

Rms values behave the same for all distributions in linear systems
Most usual beam distributions are gaussian



Emittance and beam dimensions

- The emittance is the area of the phase space occupied by the particles
  With the emittance and the Twiss parameters in a point of the accelerator, the beam dimensions are obtained : $\sigma_{x,y}$ e $\sigma'_{x,y}$



$$\varepsilon_x = \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

Ellipse area= $\pi\varepsilon_x$

$$\langle x^2\rangle = \beta_x\varepsilon_x$$

$$\langle x'^2\rangle = \gamma_x\varepsilon_x$$

$$\langle xx'\rangle = -\alpha_x\varepsilon_x$$

$$\beta_x\gamma_x - \alpha_x^2 = 1$$

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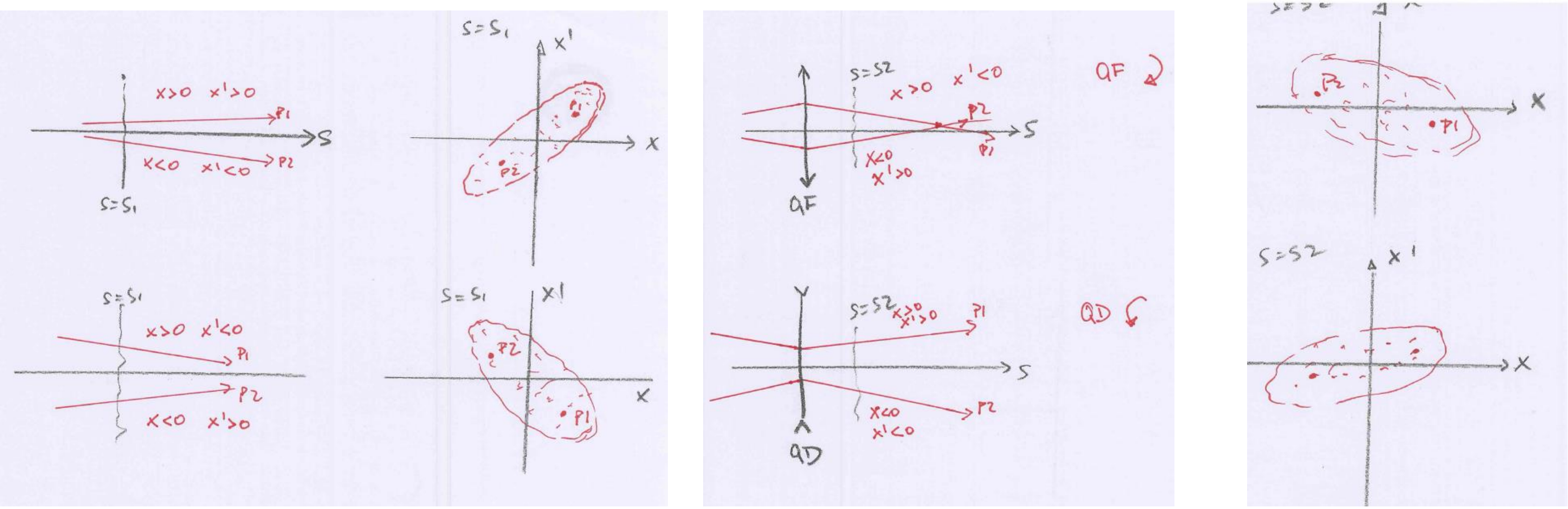
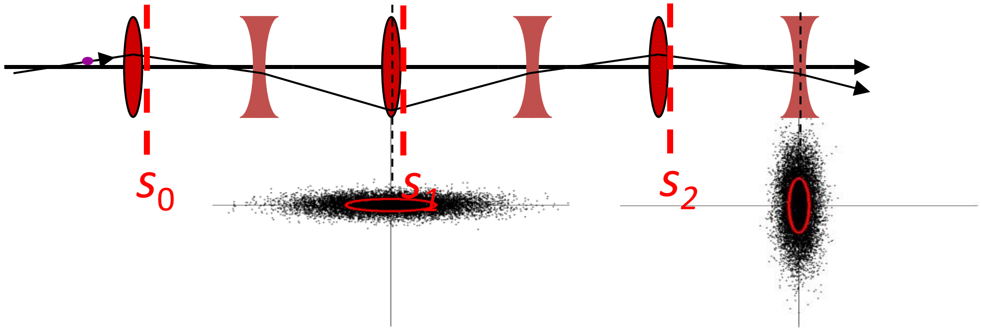
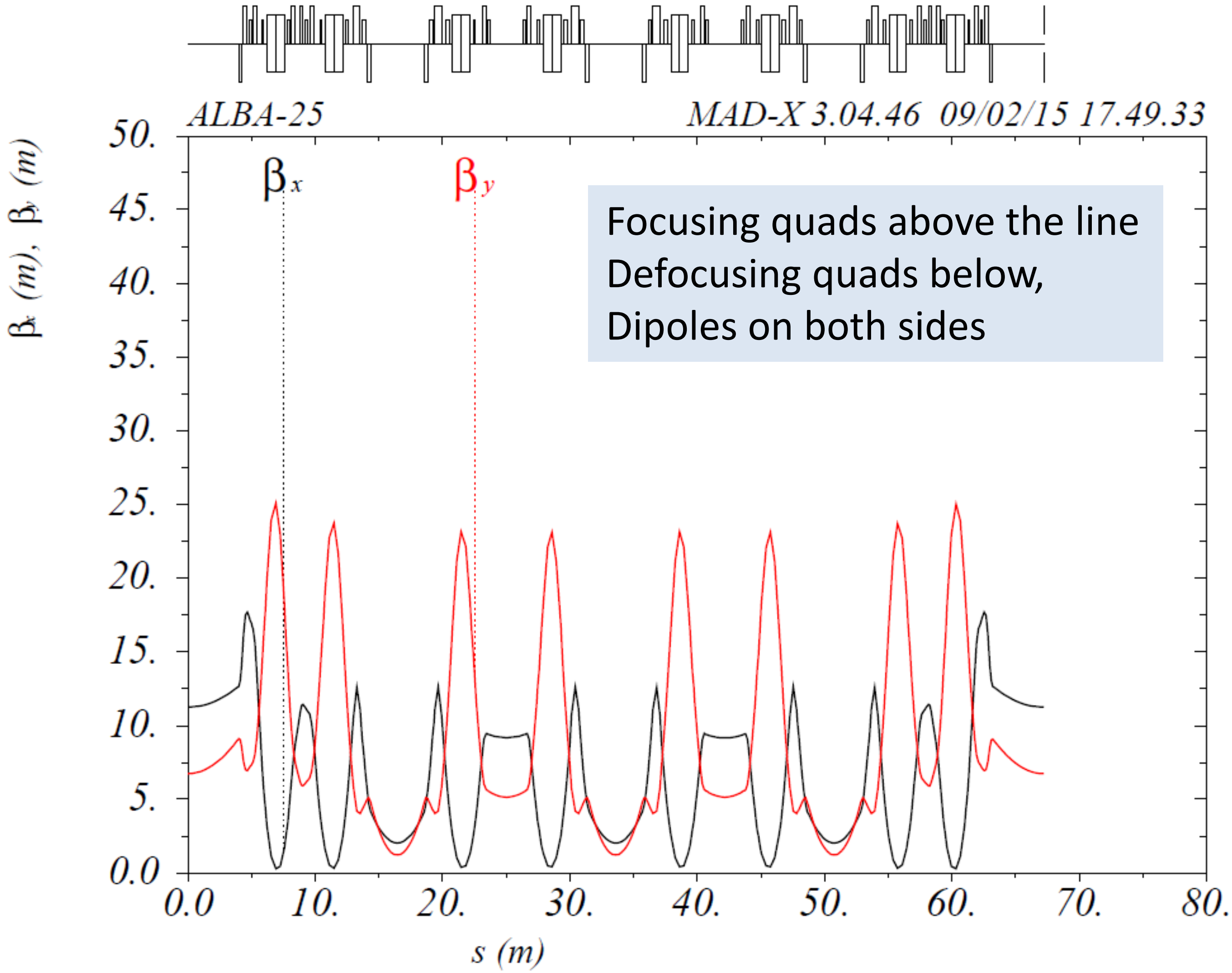
Effect of a quad on the phase space

The phase space orientation indicates if the beam trajectories are focused or defocused (at rms values):





Example of betatron functions in a storage ring - ALBA



Focusing quads above the line
Defocusing quads below,
Dipoles on both sides

Emittance in linacs and transfer lines

Adiabatic damping:

The Courant-Snyder invariant emittance ε decreases if we accelerate the particle. This is called "adiabatic damping".

Particle with momentum p0:

$$p_0^2 = p_{s0}^2 + p_{x0}^2 + p_{y0}^2$$

The slope of the trajectory is

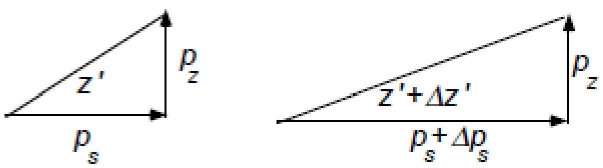
$$z'' = \frac{p_z}{p_s} \quad (z = x \text{ or } y)$$

Accelerating the particle: p_s increases but p_z does not change

$$z' + \Delta z' = \frac{p_z}{p_s + \Delta p_s} = \frac{p_z}{p_s\left(1 + \frac{\Delta p_s}{p_s}\right)} \approx z'\left(1 - \frac{\Delta p_s}{p_s}\right)$$

And therefore

$$\Delta z' = -z' \frac{\Delta p}{p}$$



In the phase space (z,z'):

For each particle i: $z_i = \sqrt{\varepsilon\beta}\cos(\varphi + \varphi_0)$ $z'_i = -\sqrt{\frac{\varepsilon}{\beta}}\sin(\varphi + \varphi_0)$

The emittance of the particle is:

$$\varepsilon = \beta z'^2_i + \gamma z^2_i$$

The change in ε applying $\Delta z'$

$$\Delta\varepsilon = 2\beta z'_i \Delta z'_i = -2\beta z'^2_i \frac{\Delta p}{p} = -2\varepsilon \frac{\Delta p}{p} sin^2(\varphi + \varphi_0)$$

Averaging over all particles:

$$\langle\Delta\varepsilon\rangle = -\varepsilon\frac{\Delta p}{p} \;=>\; \frac{d\varepsilon}{\varepsilon} = -\frac{dp}{p}$$

$$\varepsilon(p) = \varepsilon_0 \frac{p_0}{p}$$

The emittance decreases in linacs as the beam is accelerated.
Normalized emittance is defined as the invariant part:

$$\varepsilon_n = \varepsilon\beta\gamma$$

Where now β and γ are not the Twiss parameters, but: $\beta = \frac{v}{c}$ and $\gamma = \sqrt{\frac{1}{1-\beta^2}}$

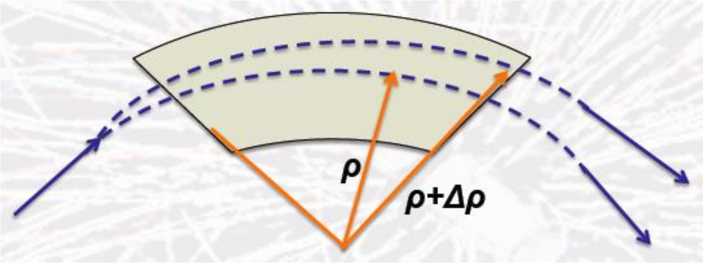
In absence of other phenomena, the normalized emittance does not change during the acceleration

Off-Momentum Particles

Magnets are chromatic elements:

Dipoles: $\rho = \frac{p}{eB} \rightarrow \frac{\Delta\rho}{\rho_0} = \frac{\Delta p}{p_0} = \frac{\Delta\theta}{\theta_0}$

Quads: $K = \frac{G}{B\rho} = \frac{Ge}{p}$



$$\Delta\theta = -\theta_0 \frac{\Delta p}{p_0}$$

$$\Delta K = -K_0 \frac{\Delta p}{p_0}$$

$\Delta p/p \ll 1$

Off-momentum particles are not oscillating around design orbit, but around a different closed orbit (chromatic closed orbit). The displacement between the design and chromatic orbits is regulated by the dispersion function *D*(*s*).

For particles with energy deviation the Hill's equation has an extra term and is not homogeneous:

$$x'' + K(s)x = \frac{1}{\rho}\frac{\Delta p}{p_0}$$

The solution is the sum of the solution of the homogenous equation + a term of dispersion:

$$x = x_{Hom} + D(s)\frac{\Delta p}{p_0}$$

Dispersion equation

The dispersion equation is:

$$D''(s) + K(s)D(s) = \frac{1}{\rho}$$

The dispersion in a storage ring is a closed orbit.

Example: **Dipole**

The solution of the dispersion equation is given by the homogeneous equation solution plus a particular one of the non homogenous:

$$D(s) = D_0 \cos\left(\frac{s}{\rho}\right) + D'_0 \rho \sin\left(\frac{s}{\rho}\right) + \rho\left[1 - \cos\left(\frac{s}{\rho}\right)\right]$$

$$D'(s) = -\frac{D_0}{\rho} \sin\left(\frac{s}{\rho}\right) + D'_0 \cos\left(\frac{s}{\rho}\right) + \sin\left(\frac{s}{\rho}\right)$$

Or in matrix formalism, representing the particle coordinates including the energy deviation:

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix} = M_{3x3} \begin{pmatrix} x_0 \\ x'_0 \\ \Delta p/p \end{pmatrix}$$

$$M_{3x3} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_{sector} = \begin{pmatrix} cos\theta & \rho sin\theta & \rho(1 - cos\theta) \\ -\frac{1}{\rho} sin\theta & cos\theta & sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

Also quadrupole focusing properties depend on particle energy:
The focusing strength is

$$K = \frac{1}{B\rho}\frac{dB}{dx}$$

Their implication on the dispersion function is of second order

$$k_{\Delta p} = -\frac{e}{p}G = -\frac{e}{p_0+\Delta p}G \approx -\frac{e}{p_0}\left(1-\frac{\Delta p}{p_0}\right)G = k_0 - \Delta k \Rightarrow \text{error}: \Delta k = \frac{\Delta p}{p_0}k_0$$

But they produce chromaticity

Particles with different momentum have different betatron tunes
(The higher the energy the lower the tunes)
Chromatism is the spread in tune divided by the spread in momentum

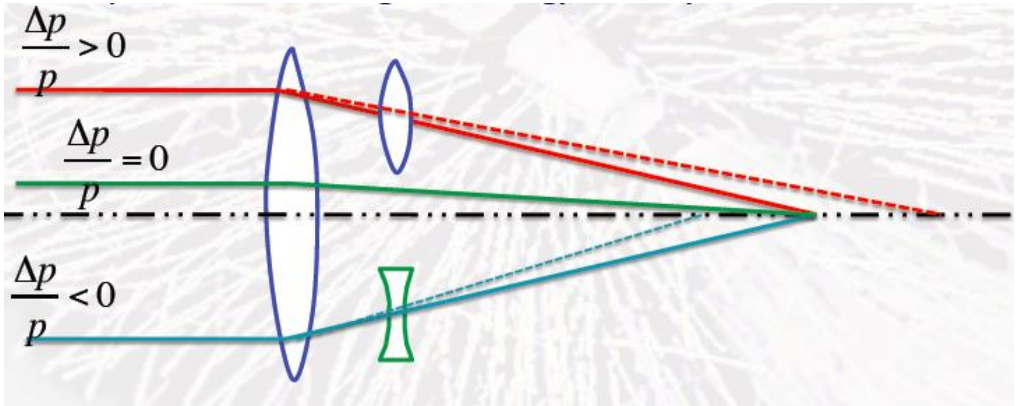
$$\xi_{x,y} = \frac{\Delta Q_{x,y}}{\Delta p / p} \quad = \quad - \frac{1}{4\pi} \sum_i k_i L_i \beta_{x,i}$$

And is given by the sum of the contributions of all elements in the ring

Chromaticity correction

- The natural chromaticity produced by quadrupoles is always negative. β_x has maxima at focusing quads (QF, $k_x > 0$) and minima at defocusing ones (QD). Horizontal chromaticity is dominated by the focusing quads, and β_x at their positions.
- Chromaticity must be corrected to avoid large tune spread (driving resonances or collective effects like head-tail instability)

Sextupoles placed where D is not zero, act on particles which are not on the nominal orbit, but on
$x = x_0 + D\, \Delta p/p$
The highest the energy deviation, the largest x, the strongest the focusing(defocusing) force



Sextupoles

- The field is quadratic :

$$B_x = m\,xy$$

$$B_y = \frac{1}{2} m \left(x^2 - y^2\right)$$

- Normalized gradient:

$$m = \frac{1}{B\rho} \frac{d^2 B_y}{dx^2} \quad [m^{-3}]$$

- Kick:

$$\Delta x' = -m x^2, \; \Delta y' = 2mxy$$

- Change in tune due to sextupole:

$$\Delta Q_x = \frac{1}{4\pi} \beta_{x,s0} m L_s x \qquad con \qquad x = D \frac{\Delta p}{p}$$

$$\frac{\Delta Q_x}{\Delta p / p} = \frac{1}{4\pi} \beta_{x,s0} m L_s D$$

Chromaticity correction

- Sextupole position are chosen so that the produced chromaticity counteracts the natural one created by quadrupoles
- Mainly where the dispersion is high and β_x e β_y are well separated.
- Total chromaticity is then:

$$\xi_x = \frac{\Delta Q_x}{\Delta p / p} = -\frac{1}{4\pi}\sum_i \beta_{x,i} k_i L_q + \frac{1}{4\pi}\sum_i \beta_{x,i} m L_s D$$

Quadrupoles **Sextupoles**