Basic Maths and Physics for Accelerators

C. Biscari

Physical constants

Constant	*Symbol*	Value
Speed of light in vacuum	c	2.99792458 10^{8} m/s
Electric charge unit	e	1.60217653 10^{-19} C
Electron rest energy	$m_e c^2$	0.51099818 MeV
Proton rest energy	$m_p c^2$	938.27231 MeV
Fine structure constant	α	1/137036
Avogadro's number	A	6.021415 10^{23} 1/mol
Classical electron radius	r_c	2.8179433 10^{-15} m
Planck's constant	h	4.1356675 10^{-15} eV s
λ of 1 eV photon	$\hbar c/e$	12398.419 Å
Permittivity of vacuum	ε_o	8.85418782 10^{-12} C/(Vm)
Permeability of vacuum	μ_o	1.25663706 10^{-6} Vs/(Am)

SI measurement system

Vector calculus

Remember for example:

Gradient of a scalar function

$\varphi(x, y, z, t)$

$$\nabla\varphi = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right)$$

For a vector

$$\vec{F} = (F_1, F_2, F_3)$$

Divergence

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Curl

$$\nabla \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

A **vector potential** is a vector field whose curl is a given vector field.
A **scalar potential** is a scalar field whose gradient is a given vector field

Maxwell's Equations

Relate electric and magnetic fields generated by charge and current distributions
Put together in 1863 results known from work of Gauss, Faraday, Ampere, Biot, Savart and others

$\vec{E}$	=	electric field
$\vec{D}$	=	electric displacement
$\vec{H}$	=	magnetic field
$\vec{B}$	=	magnetic flux density
ρ	=	electric charge density
$\vec{j}$	=	current density
μ_0	=	permeability of free space, $4\pi\,10^{-7}$
ϵ_0	=	permittivity of free space, $8.854\,10^{-12}$
c	=	speed of light, $2.99792458\,10^{8}$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{\boldsymbol{B}}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{\boldsymbol{D}}}{\partial t}$$

In vacuum $\vec{D} = \epsilon_0\,\vec{E}$, $\vec{B} = \mu_0 \vec{H}$, $\epsilon_0 \mu_0 c^2 = 1$

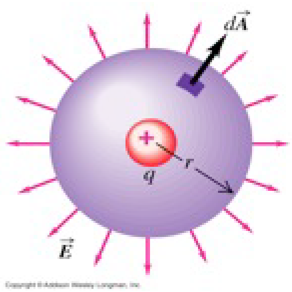
1st Maxwell equation

Equivalent to Gauss's Flux Theorem, or Coulomb's Law

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \leftrightarrow \quad \iiint_V \nabla \cdot \vec{E}\, dV = \iint_S \vec{E}\,\mathrm{d}\vec{S} = \frac{1}{\epsilon_0}\iiint_V \rho\, dV = \frac{Q}{\epsilon_0}$$

The flux of electric field out of a closed region is proportional to the total electric charge Q enclosed within the surface

Field generated by a point charge:



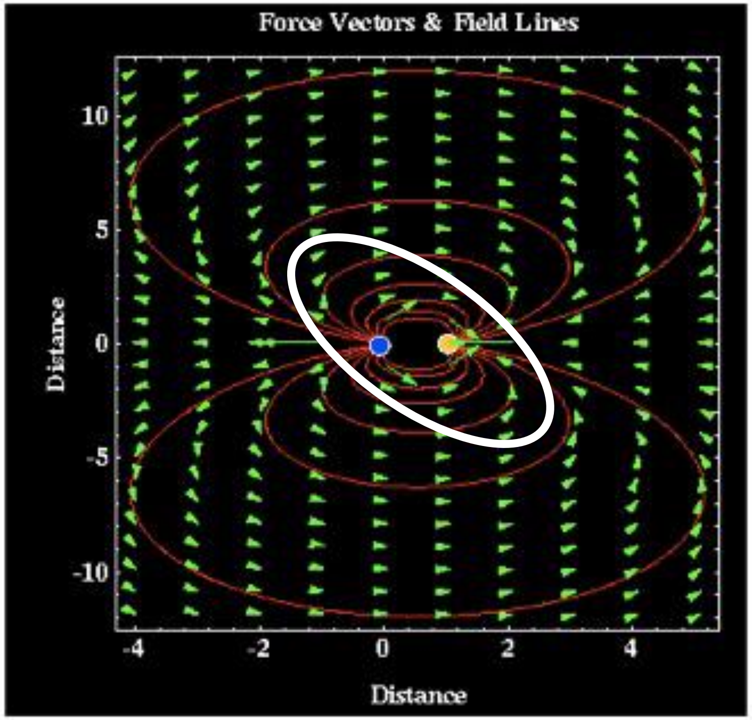
$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3} \quad \leftrightarrow \quad E_r = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}$$

$$\iint_{sphere} \vec{E}\,\mathrm{d}\vec{S} = \frac{q}{4\pi\epsilon_0}\iint_{sphere} \frac{ds}{r^2} = \frac{q}{\epsilon_0}$$

2^{nd} Maxwell equation

Gauss Law for Magnetism

$$\nabla \cdot \vec{B} = 0 \qquad \iint \vec{B} d\vec{S} = 0$$



The net magnetic flux out of any closed surface is zero.
Surround a magnetic dipole with a closed surface.
The magnetic flux directed inward towards the south pole will equal the flux outward from the north pole.
If there were a magnetic monopole source, this would give a non-zero integral.

The magnetic field can be derived from a vector potential ***A*** defined by

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}.$$

3rd Maxwell equation

Relates electric field and non constant magnetic field
Equivalent to Faraday's law of Induction

$$\nabla \times \vec{E} = -\frac{\partial \vec{\boldsymbol{B}}}{\partial t}$$

$$\iint_S \nabla \times \vec{E} \cdot d\vec{S} = -\iint_S \frac{\partial \vec{\boldsymbol{B}}}{\partial t} d\vec{S}$$

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\iint_S \vec{B} \cdot d\vec{S} = -\frac{d\Phi}{dt}$$

Faraday's Law is the basis for electric generators, inductors and transformers.

The electromotive force round a circuit
is proportional to the rate of change of flux of
magnetic field through the circuit.

4^{th} Maxwell equation

Relates magnetic field and not constant electric field. Equivalent to Faraday's law of Induction

$$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{\boldsymbol{D}}}{\partial t} \leftrightarrow \nabla \times \left(\frac{1}{\mu}\vec{B}\right) = \mu_o \vec{j} + \frac{1}{c^2}\frac{\partial\left(\epsilon\vec{E}\right)}{\partial t}$$

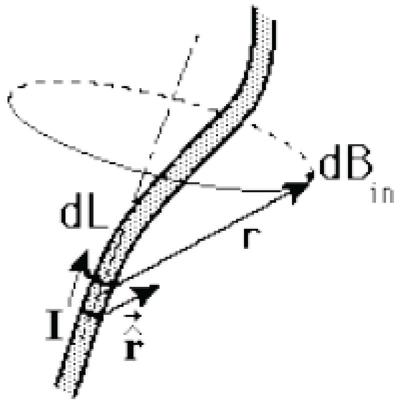
From Ampere's (Circuital) Law : $\nabla \times \vec{B} = \mu_0 \vec{j}$

$$\oint_C \vec{B} \cdot d\vec{l} = \iint_S \nabla \times \vec{B} \cdot d\vec{S} = \mu_0 \iint_S \vec{j} \cdot d\vec{S} = \mu_0 I$$

Satisfied by the field for a steady line current (Biot-Savart Law, 1820):

$$\vec{B} = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l} \times \vec{r}}{r^3}$$

Which for a straight line current $\vec{B} = \frac{\mu_0 I}{2\pi r}$

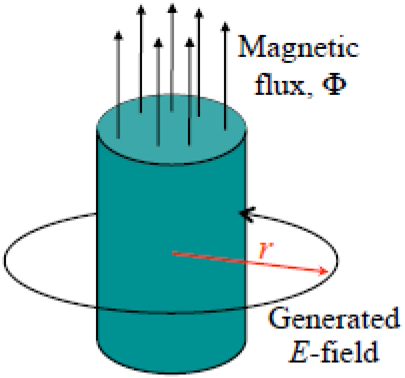


Example **The Betatron**

Particles accelerated by the rotational electric field generated by a time-varying magnetic field

$$\oint \vec{E} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \iint \vec{B} \cdot \mathrm{d}\vec{S}$$

$$\Longrightarrow \quad 2\pi r E_\theta = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

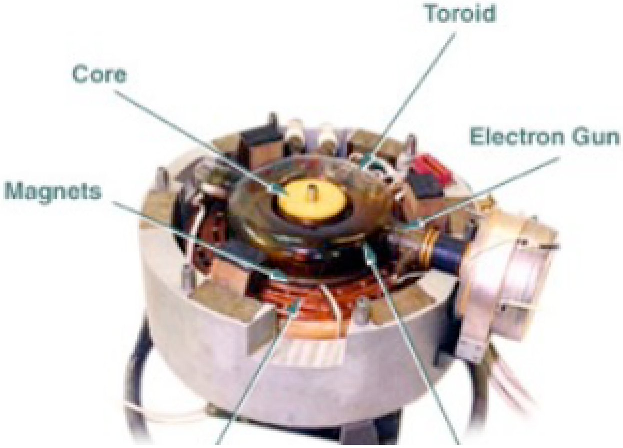


For circular motion at a constant radius:

$$-\frac{mv^2}{r} = evB \quad \Longrightarrow \quad B = -\frac{p}{er}$$

$$\Longrightarrow \frac{\partial}{\partial t} B(r,t) = -\frac{1}{er}\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{E}{r} = \frac{1}{2\pi r^2}\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

$$\Longrightarrow B(r,t) = \frac{1}{2}\frac{1}{\pi r^2}\iint B\,\mathrm{d}S$$

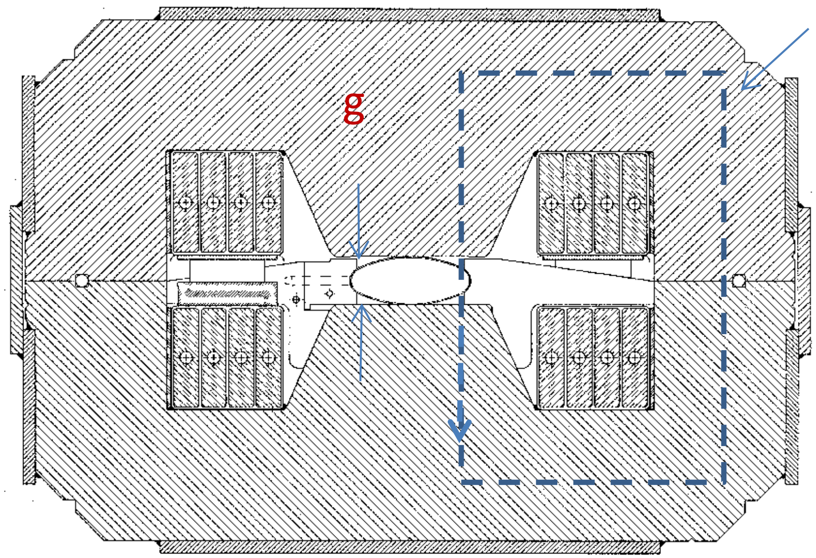


B-field on orbit needs to be one half the average *B* over the circle. This imposes a limit on the energy that can be achieved. Nevertheless the constant radius principle is attractive for high energy circular accelerators.

16

Example: Field in a permeable dipole

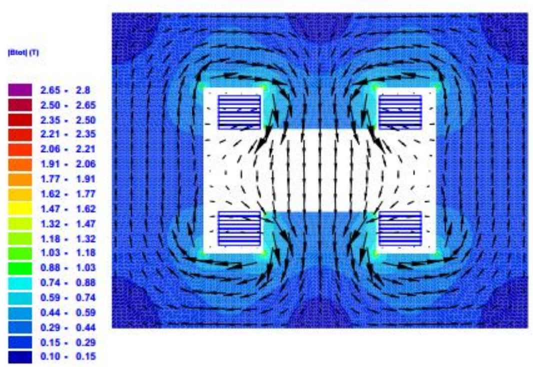
- Cross section of dipole magnet



$$\oint_C \vec{H}\bullet d\vec{l} = \frac{1}{\mu_{steel}} \int_{\text{path in steel}} \vec{B}\bullet d\vec{l} + \frac{B_{gap}g}{\mu_0}$$

$$\approx \frac{B_{gap}g}{\mu_0} = I_{enclosed}$$

$\mu_{steel} \gg \mu_{gap}$



$$\Rightarrow B_{gap} \approx \frac{\mu_0 N_{turns} I}{g}$$

Relativity

For the most part, we will use SI units, except

- *Energy: eV (keV, MeV, etc) [1 eV = 1.6x10^{-19} J]*
- *Mass: eV/c^2 [proton = 1.67x10^{-27} kg = 938 MeV/c^2]*
- *Momentum: eV/c [proton @ β=.9 = 1.94 GeV/c]*

$$\beta \equiv \frac{v}{c} = \frac{pc}{E}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

$$\text{momentum } p = \gamma m v$$

$$\text{total energy } E = \gamma m c^2$$

$$\text{kinetic energy } K = E - mc^2$$

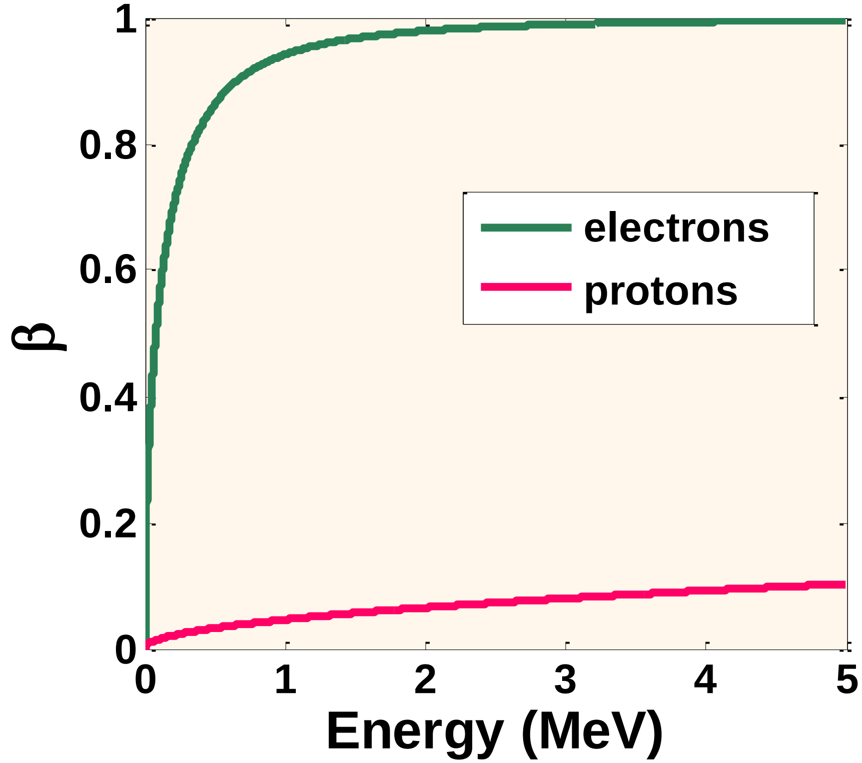
$$E^2 = \sqrt{\left(mc^2\right)^2 + \left(pc\right)^2}$$

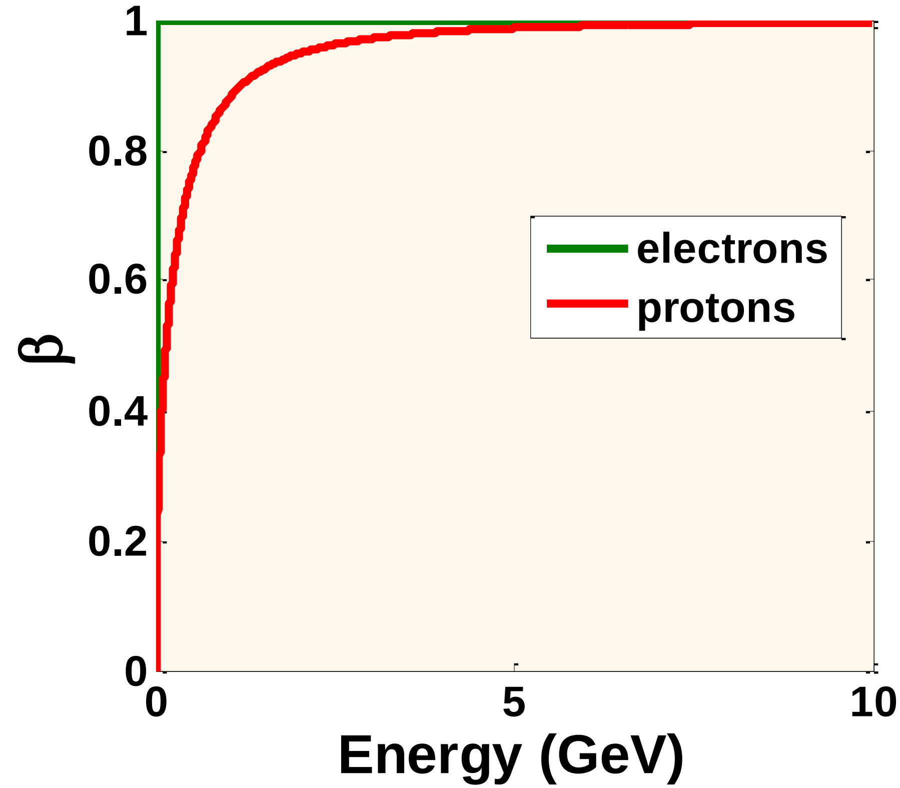
Incremental relationships between energy, velocity and momentum

	$\frac{\Delta\beta}{\beta}$	$\frac{\Delta p}{p}$	$\frac{\Delta T}{T}$	$\frac{\Delta E}{E} = \frac{\Delta\gamma}{\gamma}$
$\frac{\Delta\beta}{\beta} =$	$\frac{\Delta\beta}{\beta}$	$\frac{1}{\gamma^2}\frac{\Delta p}{p}$ $\frac{\Delta p}{p} - \frac{\Delta\gamma}{\gamma}$	$\frac{1}{\gamma(\gamma+1)}\frac{\Delta T}{T}$	$\frac{1}{\beta^2\gamma^2}\frac{\Delta\gamma}{\gamma}$ $\frac{1}{\gamma^2-1}\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta p}{p} =$	$\gamma^2\frac{\Delta\beta}{\beta}$	$\frac{\Delta p}{p}$	$\frac{\gamma}{\gamma+1}\frac{\Delta T}{T}$	$\frac{1}{\beta^2}\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta T}{T} =$	$\gamma(\gamma+1)\frac{\Delta\beta}{\beta}$	$\left(1+\frac{1}{\gamma}\right)\frac{\Delta p}{p}$	$\frac{\Delta T}{T}$	$\frac{\gamma}{\gamma-1}\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta E}{E} =$	$(\beta\gamma)^2\frac{\Delta\beta}{\beta}$	$\beta^2\frac{\Delta p}{p}$	$\left(1-\frac{1}{\gamma}\right)\frac{\Delta T}{T}$	$\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta\gamma}{\gamma} =$	$(\gamma^2-1)\frac{\Delta\beta}{\beta}$	$\frac{\Delta p}{p} - \frac{\Delta\beta}{\beta}$		

Particle velocity as a function of kinetic energy

$$\beta = \frac{v}{c} \qquad \beta = 1 \longrightarrow \text{Particle at light velocity } c$$





Electrons are ultrarelativistic at few MeV

Protons at few GeV (mass ~2000 times electron mass)

Electrodynamic Potentials

We can write the electric and magnetic fields in terms of Vector and Scalar potentials

$$\vec{B} = \vec{\nabla} \times \vec{A}(\vec{r}, t)$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{F} = e\left(\vec{E} + \vec{v} \times \vec{B}\right) = \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt}; \quad \text{for } v << c$$

$$= \frac{d\vec{p}}{dt}; \quad \text{relativistically correct}$$

Lorentz's Force

Particle dynamics are governed by the Lorentz force law

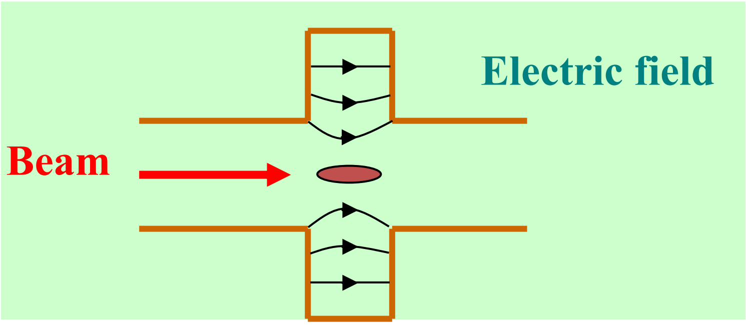
$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

$$\vec{F} = m\frac{d\vec{v}}{dt} \quad \text{for } v << c$$

$$\vec{F} = \frac{d\vec{p}}{dt} \quad \text{for any } v$$

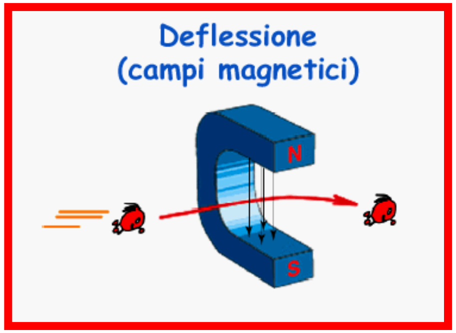
Acceleration

$\vec{E} = \text{electric field}$



Bending and focusing

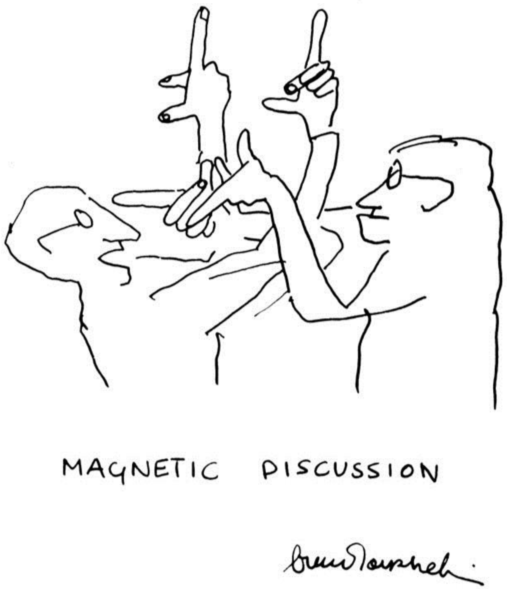
$\vec{B} = \text{magnetic field}$



Particle in a magnetic field

The magnetic force always acts at right angles to the charge motion, the magnetic force can do no work on the charge. The B-field cannot speed up or slow down a moving charge; it can only change the direction in which the charge is moving.

The **component of velocity** of the charged particle that is **parallel** to the magnetic field is **unaffected**, i.e. the charge moves at a constant speed along the direction of the magnetic field.
Negatively charged particles circulate in the opposite direction as positively charged particles.
The direction can be found using the right hand rule applied to the perpendicular component of the velocity



Motion in constant electric field

$$\frac{d(m_o \gamma \vec{v})}{dt} = \mathrm{q}\,\vec{E}$$

$$\gamma \vec{v} = \frac{q\vec{E}}{m_o} t$$

$$\gamma^2 = 1 + \left(\frac{\gamma \vec{v}}{c}\right)^2 \qquad \gamma = \sqrt{1 + \left(\frac{q\vec{E}t}{m_o c}\right)^2}$$

If the electric field is constant:

$$\vec{E} = (E, 0,0)$$

It can be demonstrated that

$$m_o c^2 (\gamma - 1) = qE(x - x_o)$$

Uniform acceleration in straight line

Motion in a constant magnetic field

- A charged particle moves in circular arcs of radius ρ with angular velocity ω:

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(\gamma m\vec{v}\right) = \gamma m\frac{d\vec{v}}{dt} = q\vec{v}\times\vec{B}$$

$$\vec{v} = \vec{\omega}\times\vec{\rho} \quad \Rightarrow \quad q\vec{v}\times\vec{B} = \gamma m\vec{\omega}\times\frac{d\vec{\rho}}{dt} = \gamma m\vec{\omega}\times\vec{v}$$

$$qvB = \frac{\gamma m v^2}{\rho}$$

$$\omega = \frac{v}{\rho} = \frac{qB}{\gamma m}$$

Example: Cyclotron (1930's)

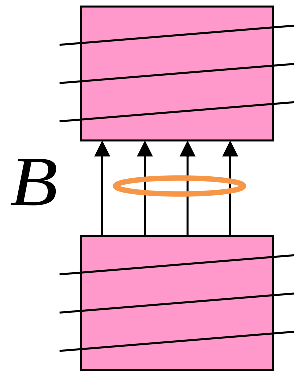
- A charged particle in a uniform magnetic field will follow a circular path of radius

$$\rho = \frac{mv}{qB} \quad (v << c)$$

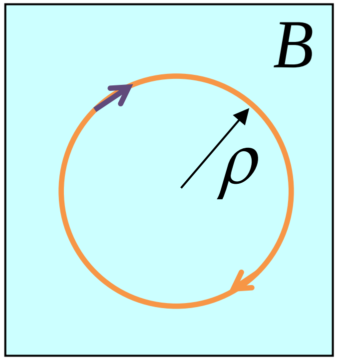
$$f = \frac{v}{2\pi\rho} = \frac{qB}{2\pi m} \text{ (constant!!)}$$

$$\Omega_s = 2\pi f = \frac{qB}{m}$$

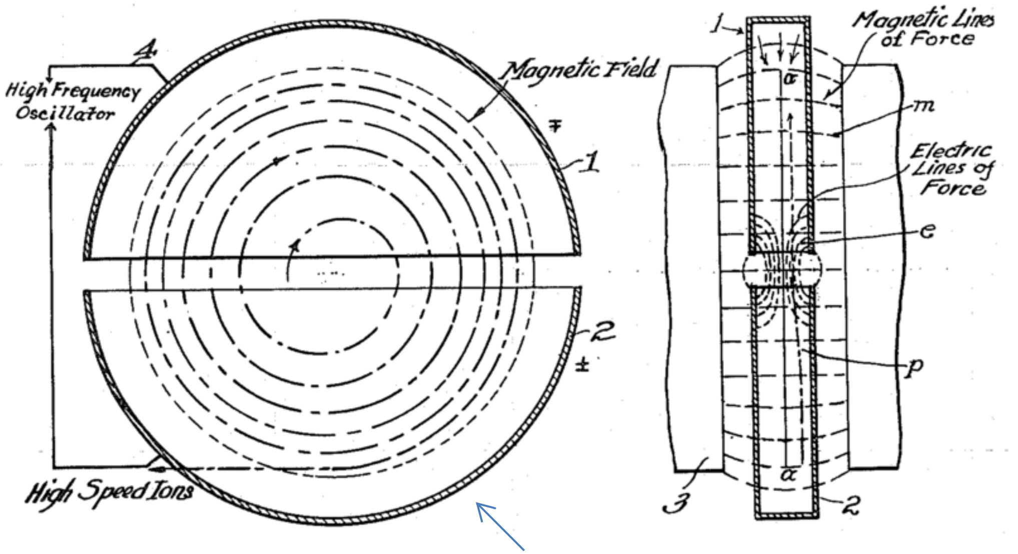
side view



top view



"Cyclotron Frequency"



Accelerating "DEES"

For a proton:

$$f_C = 15.2 \times B[T] \text{ MHz}$$

Rigidity
Relation between radius and momentum

$$B\rho = \frac{p}{q}$$

How hard (or easy) is a particle to deflect?

- Often expressed in [T-m] (easy to calculate B)
- Be careful when q≠e!!

$$B\rho[Tm] \approx 3.33 \frac{p[\frac{GeV}{c}]}{q[e]}$$

Electrons and protons (now in MeV)

$$E_o = 0.511\ MeV \quad or\ 938.27\ MeV$$
$$E_{tot} = E_{kin} + E_o$$
$$\gamma = \frac{E_{tot}}{E_o}, \beta = \sqrt{1 - \frac{1}{\gamma^2}}$$
$$p = \beta E_{tot}$$

$$B\rho = 3.33\ 10^{-3} p$$

Energia [MeV]	*Rigidità Bρ [Tm]*	
	p	*e⁻*
1	0,14	0,005
10	0,44	0,035
100	1,45	0,34
1.000	5,66	3,34
10.000	36,35	33,36
100.000	336	336
1.000.000	3335	3335

Ions

Kinetic energy per nucleon: E_{kin}

Total charge = Qe

$$E_e = 0.511\ MeV \qquad E_p = 938.27\ MeV \qquad E_n = 939.57 MeV$$

$$N_e, Z, N \Rightarrow A = Atomic\ mass\ (Z + N)$$

$$E_o = ZE_p + NE_p + N_e E_e - A * 0.8$$

$$E_{tot} = A\ E_{kin} + E_o$$

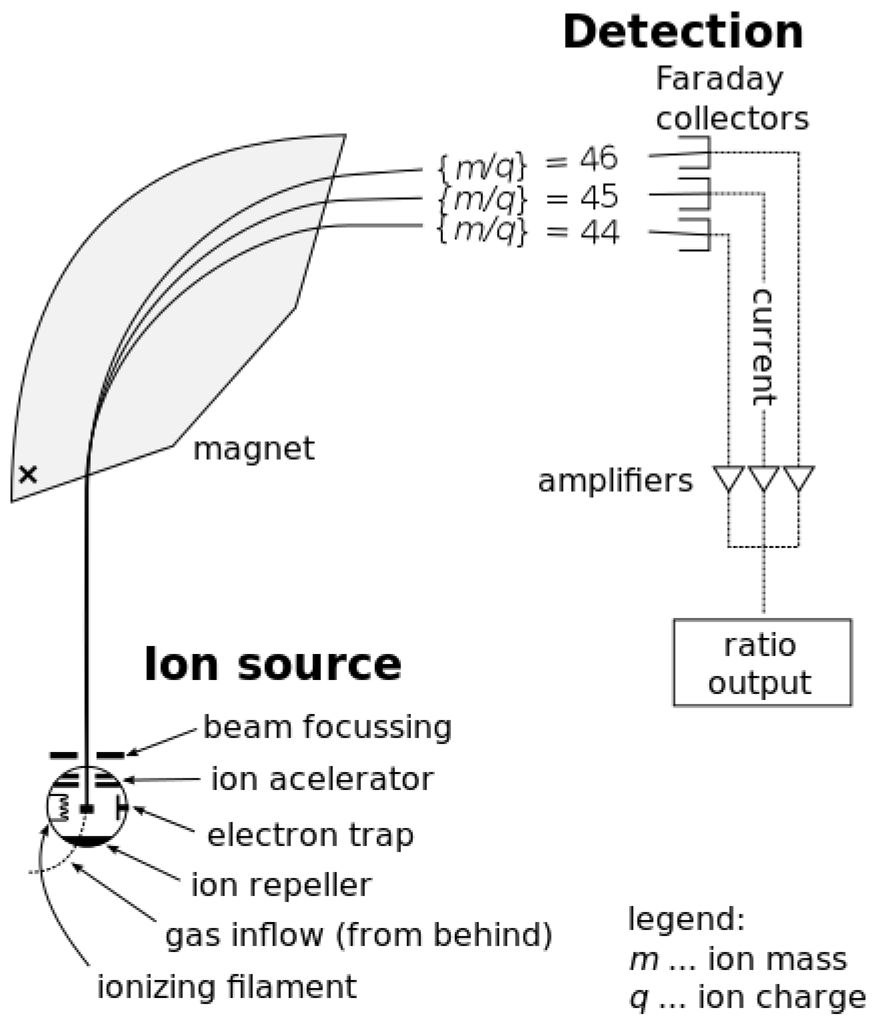
$$\gamma = \frac{E_{tot}}{E_o}, \beta = \sqrt{1 - \frac{1}{\gamma^2}}$$

$$p = \beta E_{tot}$$

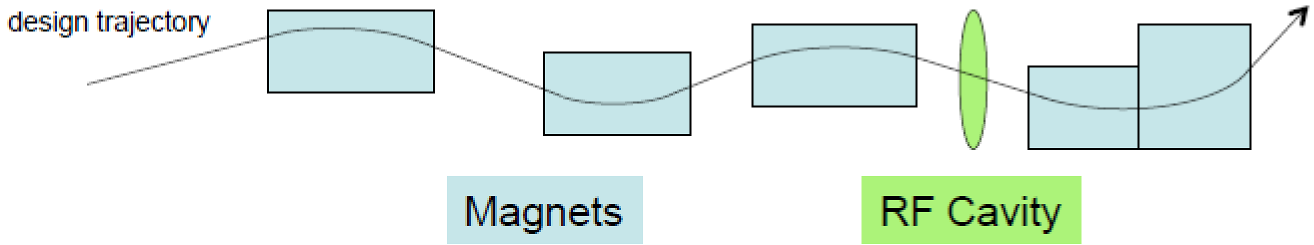
$$B\rho = 3.33\ 10^{-3} p/\mathrm{Q}$$

Example: particle spectrometer

Identify particle momentum by measuring bending angle from a calibrated magnetic field B



Simplified Particle Motion



Design trajectory

- Particle motion will be expanded around a **design trajectory** or **orbit**
- This orbit can be over linacs, transfer lines, rings

Separation of fields: Lorentz force

- Magnetic fields from static or slowly-changing magnets transverse to design trajectory
- Electric fields from high-frequency RF cavities in direction of design trajectory
- Relativistic charged particle velocities

Slide from T. Satogata / January 2015 USPAS Accelerator Physics

Others

1r curs	2n curs

1r curs

- Mecànica i Relativitat
- Electricitat i Magnetisme
- Química per a Físics
- Ones i Òptica
- Càlcul I
- Càlcul II
- Àlgebra I
- Àlgebra II
- Temes de Ciència Actual
- Iniciació a la **Física** Experimental

2n curs

- Estructura de la Matèria i Termodinàmica
- Mètodes Numèrics I
- Mecànica Clàssica
- Electromagnetisme
- Laboratori de Mecànica
- Laboratori d'Electromagnetisme
- Equacions Diferencials
- Càlcul de Vàries Variables
- Complements de Matemàtiques

Specially:

Calcul
Matrix formalism
Differential equations
Electromagnetism
Special relativity
Waves and Optics

References

- Particle Accelerator Physics – Helmut Wiedemann, Third Edition, Springer
- Cern Accelerator School (CAS)

(http://cas.web.cern.ch/cas/) , including links to other schools

- U.S. Particle Accelerator School (USPAS) (http://uspas.fnal.gov/)

Exercises

- Find the velocity in terms of *c* of an electron at 3 GeV (ALBA), of a proton at 250 MeV (CNAO) and at 6.5 TeV (LHC)

- Derive the expression of the relationship between momentum and energy change

$$\frac{dp}{p} = \frac{1}{\beta^2}\frac{dE}{E}$$