POSIPOL 2009, Lyon

Theoretical constraints in polarised reactions

- discrete symmetries :PT+ Born, parity , chirality
- positivity
- Monte-Carlo simulations with spins

Formalism :
X. A., M. Elchikh, J-M. Richard, J. Soffer, O. Teryaev, Phys. Rep. 470 (2009)

X. Artru, IPNL

Density matrices

Spin 1/2 : $\rho=(1+\mathbf{S.}\sigma)/2$ $|\mathbf{S}| \leq 1$

S_x, S_y, S_z : expectation values of spin components

$$S_i = \langle\sigma_i\rangle = 2\,\langle s_i\rangle$$

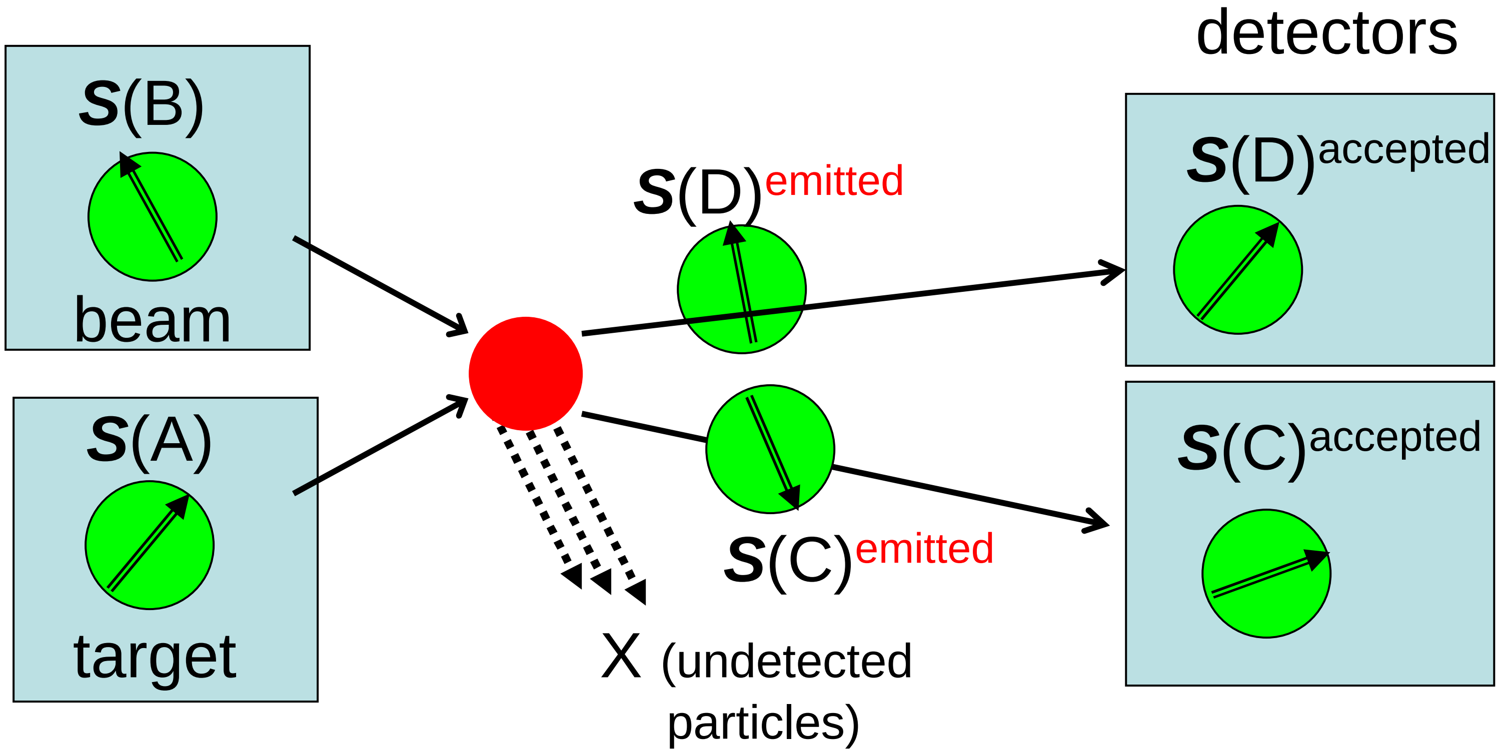
Photon : $\rho=(1+\mathbf{S.}\sigma)/2$ $|\mathbf{S}| \leq 1$

S_1, S_2, S_3, : Stokes parameters

S_3=planarity = linear polarisation (horizontal) - (vertical)
S_1=obliquity = linear polarisation (+45°) - (-45°)
S_2=helicity = circular polarisation

Polarized experiment

A+B→C+D (+X)



emitted polarization ≠ *accepted* polarization

Polarised cross section

$$d\sigma/d\Omega_{pol} = I_0 F\{ \boldsymbol{S}(A), \boldsymbol{S}(B), \boldsymbol{S}(C)^{acc}, \boldsymbol{S}(D)^{acc} \}$$

$\boldsymbol{S}(C)^{acc}$ = polarisation accepted by the detector of C.

The **emitted** polarisation of C is

$$\boldsymbol{S}(C)^{emit} = \frac{\nabla_{\boldsymbol{S}(C)}{}^{acc} F\{ \boldsymbol{S}(A), \boldsymbol{S}(B), \boldsymbol{S}(C)^{acc}, \boldsymbol{S}(D)^{acc} \}}{F\{ \boldsymbol{S}(A), \boldsymbol{S}(B), 0, 0 \}}$$

Cartesian reaction parameters

or *correlation parameters*, denoted $(\lambda\mu\ldots|\nu\tau\ldots)$

Exemple: $1/2 + 0 \rightarrow 1/2 + 0$

$$F\{\boldsymbol{S}(A),\boldsymbol{S}(C)\} = (0|0)$$
$$+ S_x(A)\,(x|0) + S_y(A)\,(y|0) + S_z(A)\,(z|0)$$
$$+ (0|x)\,S_x(C) + (0|y)\,S_y(C) + (0|y)\,S_y(C)$$
$$+ S_x(A)\,(x|x)\,S_x(C) + S_x(A)\,(x|y)\,S_y(C) + \ldots$$

$$= S_\mu(A)\,(\mu|\nu)\,S_\nu\,(C)$$

with $(0|0) = 1$.

$\boldsymbol{S}$ has been promoted to 4-vector $S_\mu = (S_0\,,\boldsymbol{S})$ with $S_0 = 1$.

Expression in terms of amplitudes

$$(\lambda\mu|\nu\tau) = \mathrm{Tr}\,\{\, M \quad \sigma_\lambda(A)\,\sigma_\mu(B) \quad M^\dagger \quad \sigma_\nu(C)\,\sigma_\mu(D) \,\}$$

Other notations : $C_{\lambda\mu\nu\tau}$

or : $\langle\, \sigma_\lambda(A)\,\sigma_\mu(B)\,\sigma_\nu(C)\,\sigma_\mu(D) \,\rangle$

Spin property of the Born approximation

If PT is conserved and M is hermitian (this is the case of Born approximation) then all PT-odd spin observable vanish.

PT - odd observables :

> σ_x , σ_y , σ_z for spin 1/2 particles

> helicity (σ_2 , in our convention, for the photon)

The linear polarisation operators σ_3 (planarity) and σ_1 (obliquity)
are PT - even.

In the Born approximation there is no single spin asymmetry for an electron or positron.
The 3-spin correlations parameters also vanish.

Chirality conservation

The helicity of an electron is approximately conserved in hard interactions, the momentum transfers are much bigger than the electron mass.

One of the numerous consequences

- For longitudinal polarisations, the e+e- cross section of a reaction (A) is of the form

$$\sigma(A) = A_1\ (1+P)(1-P') + A_2\ (1-P)(1+P').$$

The cross section ratio $\sigma(A)/\sigma(B)$ between two reactions (A) and (B) depends only on the quantity

$$(1+P_{eff}) / (1-P_{eff}) = [\ (1+P)/(1-P)\ ]\ /\ [\ (1+P')/(1-P')\ ]$$

The larger $|P_{eff}|$ is, the better is the separation between (A=signal) and (B=background). Wherefrom the advantage of having both beam polarised.

Other consequences of chirality

In a hard scattering process, a rotation of all the transverse spins by a common angle ϕ about the particle momenta does not change the cross section
(this common rotation looks like a Cardan transmission).

$\rightarrow$ Hikasa theorem :
The integration over the azimuthal angles of the particle washes out the transverse spin asymetries.

e.g. in $e^+e^- \rightarrow \mu^+ \mu^-$, one must not integrate over the azimuth of the $\mu^+ \mu^-$ axis.

Parity - 1) Classical constraints

In A+B $\rightarrow$ C+D the scattering plane (x,z) is a *symmetry plane*.

Parity conservation = invariance under *mirror reflection* Π about this plane.

σ_x and σ_z are ''Π - odd'' observables

σ_y is ''Π - even''

Classical parity rule: *all Π* ***- odd*** *observables vanish.*

Exemple : $\pi + N \rightarrow K + \Lambda$ (0+1/2 $\rightarrow$ 0+1/2)

(x|0) = (z|0) = (x|y) = 0 , etc.

Non-vanishing observables : (y|0), (x|z), etc.

Parity - 2) quantum constraints

Parity conservation for *amplitudes* reads

$$M = \Pi \, M \, \Pi^{-1}$$

Using $(\mu|\nu) = \mathrm{Tr}\,\{ M \,\sigma_\mu\, M^\dagger\, \sigma_\nu \}$, one obtains new constraints :

for $\pi + N \rightarrow K + \Lambda$,

$$(y|y) = (0|0) \; ,$$
$$(0|y) = (y|0) \; ,$$

which could not be guessed from classical arguments.

Positivity - 1) classical constraints

F{$\boldsymbol{S}$(A),$\boldsymbol{S}$(B),$\boldsymbol{S}$(C),…. } is positive *for any set of polarization vectors* $\boldsymbol{S}$(A),$\boldsymbol{S}$(B),$\boldsymbol{S}$(C), etc.

(subject to the conditions $|\boldsymbol{S}| \leq 1$)

Exemple: $1/2 + 1/2 \rightarrow X$ (isotropic case)

$$F\{ \boldsymbol{S}(A),\boldsymbol{S}(B) \} = 1 + \mathbf{c}\ \boldsymbol{S}(A).\boldsymbol{S}(B)$$

$F \geq 0$ for

$$-1 \leq \mathbf{c} \leq +1.$$

Positivity - 2) ***quantum*** constraints

Same exemple $1/2 + 1/2 \to X$

The ***cross section matrix*** is defined as

$$R = (\mu\nu|)\, \sigma_\mu(A) \otimes \sigma_\nu(B) \qquad (\sigma_0 = I)$$

Quantum positivity constraint:

R is ***semi-positive*** (like a density matrix)

$$\langle \Psi_{A+B} | R | \Psi_{A+B} \rangle \geq 0$$

Classical versus quantum positivity constraint

Using only ***separable*** states ($\Psi_{A+B} = \Psi_A \otimes \Psi_B$), one obtains nothing but the ***classical*** positivity constraint **$-1 \leq c \leq 1$**

Using separable and ***entangled*** states ($\Psi_{A+B} \neq \Psi_A \otimes \Psi_B$) one obtains the more severe ***quantum positivity*** *constraint* :

$$-1 \leq c \leq 1/3$$

The constraint $c \leq +1/3$ is required for Ψ_{A+B} = singlet state.

Usefull lesson:

" fully opposite spins (c=-1) are allowed,

fully identical spins (c=+1) are forbiden "

A+B entangled states are not easy to prepare,
but not impossible.
Exemple:
$e^+e^- \rightarrow \gamma\gamma$, when e^+e^- form a *para-positronium.*

Quantum positivity constraints.

2) case $1/2 + 0 \rightarrow 1/2 + X$

Polarized cross section

$\sim F\{ \boldsymbol{S}(A), \boldsymbol{S}(B) \} = (\mu|\nu)\ S_\mu(A)\ S_\nu(B)$

$\Downarrow$

cross section matrix (again ***semi-positive***)

$R_{A\text{-}B}\{ \sigma(A), \sigma(B) \} = (\mu|\nu)\ \sigma_\mu(A) \otimes \sigma^t_\nu(B)$

Note the *transposition* in $\sigma^t(B)$, related to the *crossing* from the $1/2 + 1/2 \rightarrow X$ case.

Second exemple of quantum positivity constraint : $1/2 \to 1/2 + X$ (crossed reaction of the preceding exemple)

The cross section matrix is

$R_{A\text{-}C} = (\mu|\nu)\ \sigma_\mu(A) \otimes \sigma^{t}_\nu(C)$

Let us assume

$F\{\ \boldsymbol{S}(A), \boldsymbol{S}(C)\ \} = 1 + \mathbf{d}\ \boldsymbol{S}(A).\boldsymbol{S}(C)$, that is to say $(\mu|\nu) = \delta_{\mu\nu}$

Classical positivity : $-1 \leq \mathbf{d} \leq 1$

Quantum positivity : $1/3 \leq \mathbf{d} \leq 1$

"full spin transmission (d=+1) is allowed"

"full spin reversal (d=-1) is forbiden"

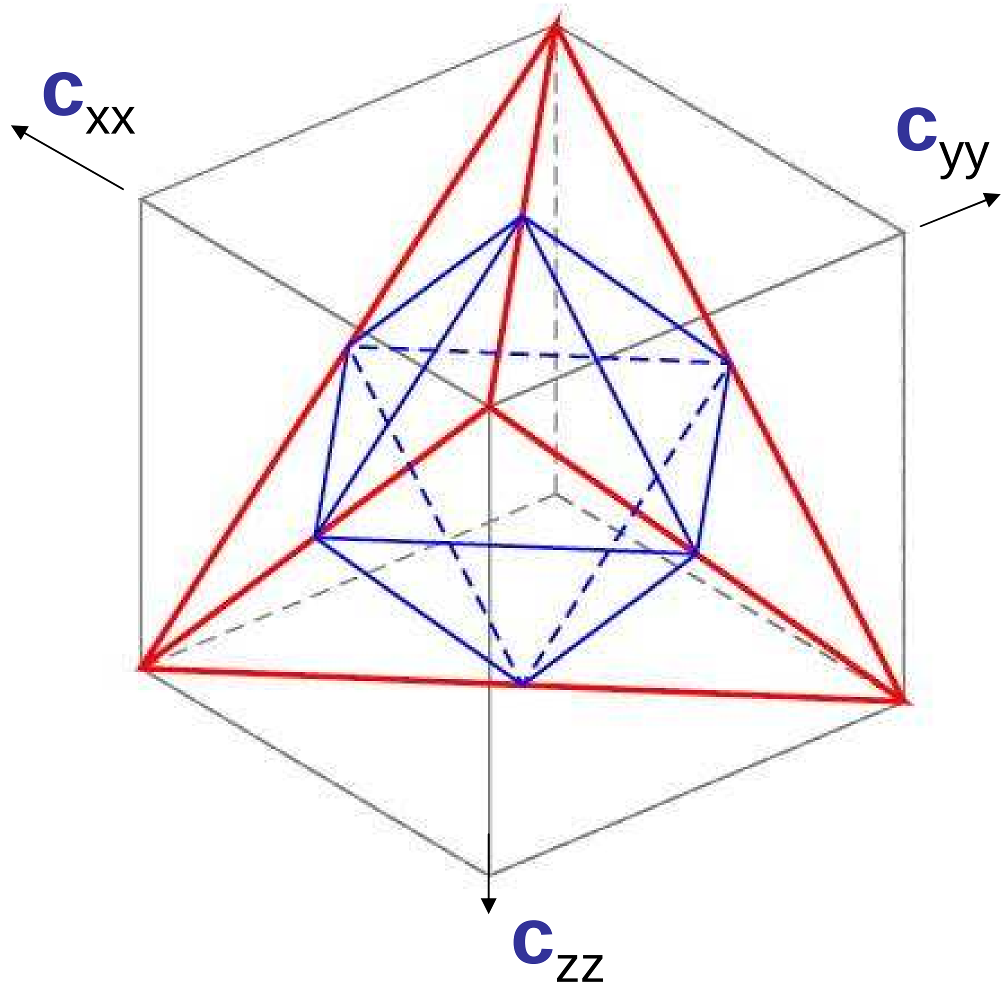
1/2 + 1/2 → X again, but non isotropic

There are 15 cartesian parameters C_{ij} .
Let us focus on the diagonal ones.

(cube)
classical positivity

(tetraedron)
quantum positivity

(octaedron)
R_{A+B} separable



Non-separable cross section matrices and Monte-Carlo

The easiest way to introduce spin in Monte-Carlo simulations of showers is to assign a polarisation vector to each particle. By doing this, one neglects *entanglements* effects. Strictly speaking, this is permitted only if the cross section matrix of each elementary process is *separable* (e.g. in the octahedron of the last figure).

Otherwise a more elaborate method invented by Collins and Knowles should be used.

However this method is not necessary if one is only interested in single particle spectrum, not in correlations (e.g. in momentum or angle) between two particles.

Cross section matrix from amplitudes

$$\langle a' b' c' d' |R| a b c d \rangle = \langle c d |M| a b\rangle \langle a'b' | M^{\dagger}| c'd'\rangle$$

(a, b, c, d = helicities, transversities or other spin variables)

This form is manifestly semi-positive.
Therefore, if one expresses the polarised cross section in terms of *amplitudes*, all positivity constraints are automatically satisfied - even if the amplitudes are approximate.

If, instead, one makes approximations at the level of *cross sections*, then one may violate some positivity constraints.

This may be the case in Olesen & Maximon method.

Olesen & Maximon approximation (Ph.Rev.114)

Amplitudes for Bremsstrahlung or pair production :
B = Born term unscreened
b = Born term screened
C = exact Coulomb solution, unscreened
c = exact solution with screened Coulomb potential

Olesen & Maximon receipe :
The screening corrections are the same for Born and all-order solutions. But they apply it to *cross sections*

$$|c|^2 - |C|^2 = |b|^2 - |B|^2 \qquad (1)$$

Question : why not apply it to *amplitudes* ?

$$c - C = b - B \qquad (2)$$

Receipe (1) is dangerous for positivity….

Conclusions

- constraints from discrete symmetries (parity, time reversal, charge conjugation, identical particles) and from positivity can be useful checks. Some of them are classical, some other are quantum ones.
- Chirality has also interesting consequences
- Non-separable cross section matrices makes the Monte-Carlo simulations with spins rather tricky,unless one decides not to predict correlations.