THE (REFINED) TOPOLOGICAL VERTEX

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OUTLINE

- Topological A-model & Topological Vertex
- Refined Vertex *'at work'* Local $\mathbb{P}^1$
- Link invariants

TOPOLOGICAL A-MODEL

The genus g topological string amplitudes appear in the effective action of the corresponding 4d theory

$$\int d^4x (\partial_i \partial_j F_0(t_i)) F_i^+ \wedge F_j^+, \qquad g = 0$$

$$\int d^4x \, F_g(t_i) \, R_+^2 \, F_+^{2g-2}, \qquad g > 0$$

genus g amp. — self dual Riemann tensor (contr.) — self dual graviphoton

Organize the amplitudes into a generating function

$$F(t_i) = \log Z = \sum_g \lambda^{2g-2} F_g(t_i), \qquad \lambda = \langle F_+ \rangle$$

TOPOLOGICAL A-MODEL

According to the Gopakumar&Vafa formalism the topological string amplitudes can be recast as

$$F = \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{k=1}^{\infty} \sum_{j_L} (-1)^{2j_L} N_\beta^{j_L} e^{-kT_\beta} \left(\frac{q^{-2j_L k} + \ldots + q^{+2j_L k}}{k(q^{k/2} - q^{-k/2})^2} \right), q = e^{ig_s}$$

There exist a more *refined* version of this generating function

$$F = \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{k=1}^{\infty} \sum_{j_L, j_R} e^{-kT_\beta} \frac{(-1)^{2j_L+2j_R} N_\beta^{(j_L,j_R)} \left((t\,q)^{-kj_L} + \ldots + (t\,q)^{+kj_L}\right) \left(\left(\frac{t}{q}\right)^{-kj_R} + \ldots + \left(\frac{t}{q}\right)^{+kj_R}\right)}{k(t^{k/2} - t^{-k/2})(q^{k/2} - q^{-k/2})}$$

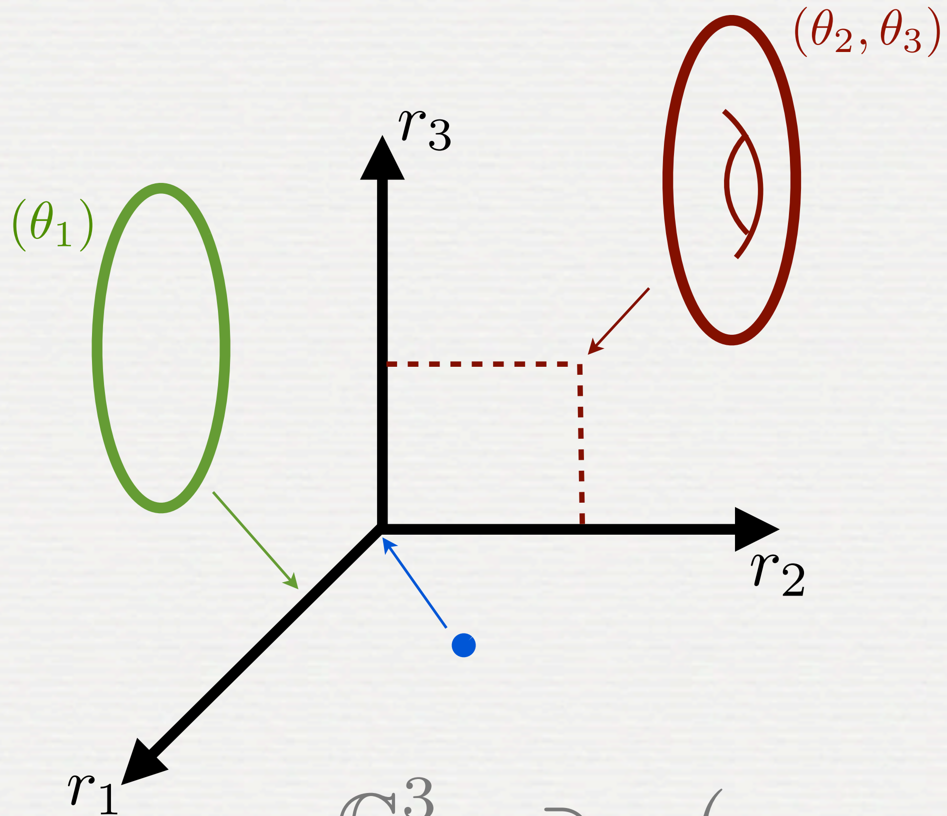
such that $N_\beta^{j_L} = \sum_{j_R} N_\beta^{(j_L,j_R)} (-1)^{2j_R} (2j_R + 1)$.

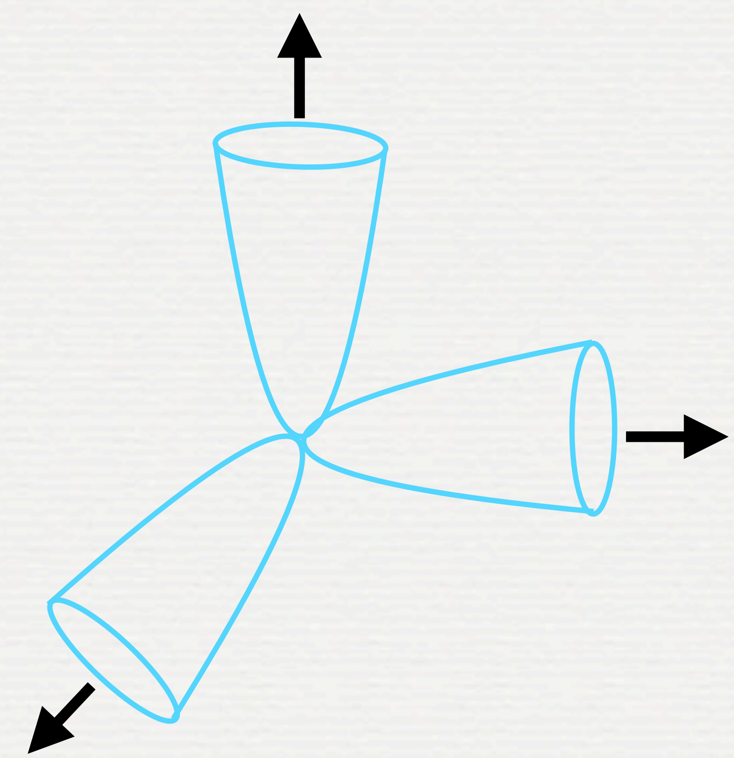
$N_\beta^{(j_L,j_R)}$ is the degeneracy of particles with spin (j_L, j_R).

TOPOLOGICAL VERTEX

The topological vertex completely solves the problem of computing topological string amplitudes on toric 3-folds.

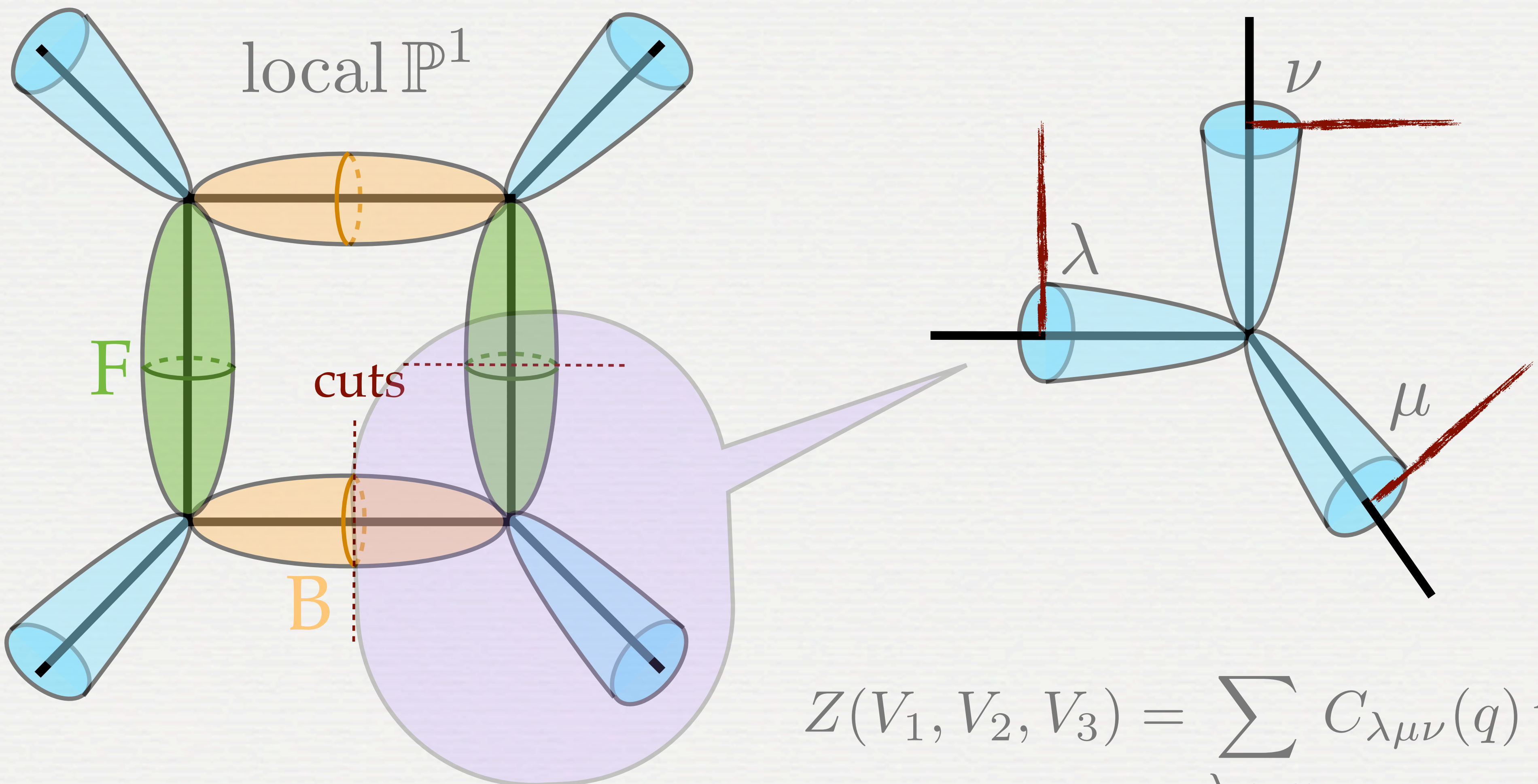
Example Geometry: $\mathbb{C}^3$





$$\begin{aligned} \mathbb{C}^3 \quad &\ni \quad (z_1, z_2, z_3) \\ &= \quad (r_1 e^{\theta_1}, r_2 e^{i\theta_2}, r_3 e^{i\theta_3}) \end{aligned}$$

TOPOLOGICAL VERTEX



$$Z(V_1, V_2, V_3) = \sum_{\lambda,\mu,\nu} C_{\lambda\mu\nu}(q)\, \mathrm{tr}_\lambda V_1\, \mathrm{tr}_\mu V_2\, \mathrm{tr}_\nu V_3$$

1) Divide the geometry
2) Compute the amplitudes for individual vertices
3) Glue the vertices to get the full amplitude

$$V_i = \mathrm{P}\exp\left[\oint_{\partial D_i} A\right]$$

TOPOLOGICAL VERTEX

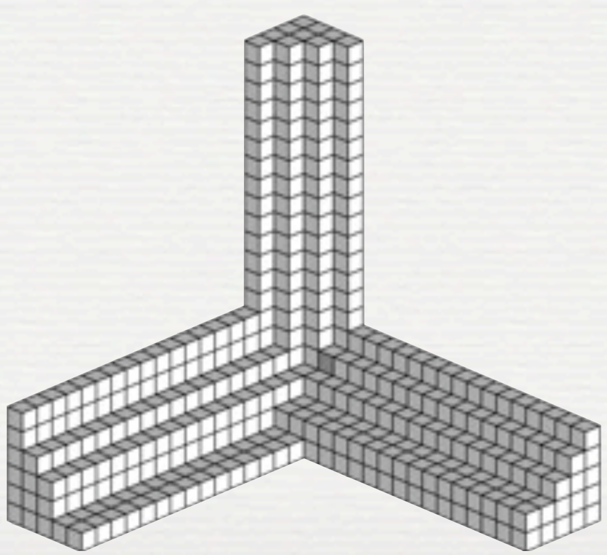
The *usual* topological vertex has the form:

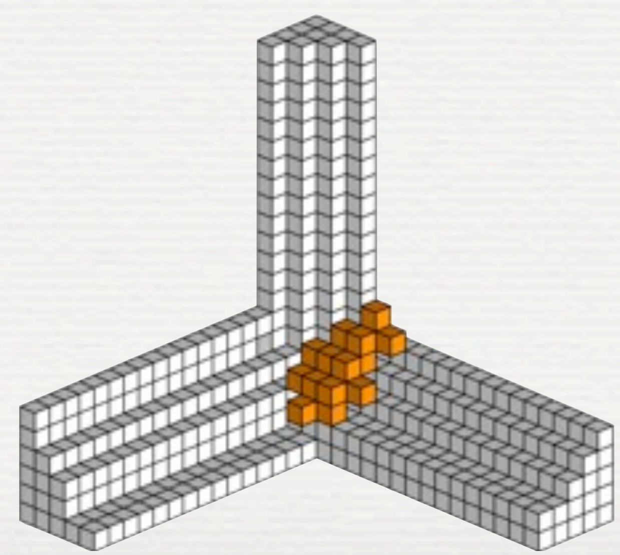
$$C_{\lambda\mu\nu}(q) = q^{\frac{\kappa(\mu)}{2}} s_{\nu^t}(q^{-\rho}) \sum_{\eta} s_{\lambda^t/\eta}(q^{-\nu-\rho}) s_{\mu/\eta}(q^{-\nu^t-\rho})$$

The *refined* topological vertex has a similar form:

$$C_{\lambda\mu\nu}(t,q) = \left(\frac{q}{t}\right)^{\frac{\|\mu\|^2+\|\nu\|^2}{2}} t^{\frac{\kappa(\mu)}{2}} P_{\nu^t}(t^{-\rho};q,t) \sum_{\eta} \left(\frac{q}{t}\right)^{\frac{|\eta|+|\lambda|-|\mu|}{2}} s_{\lambda^t/\eta}(q^{-\nu}t^{-\rho}) s_{\mu/\eta}(t^{-\nu^t}q^{-\rho})$$

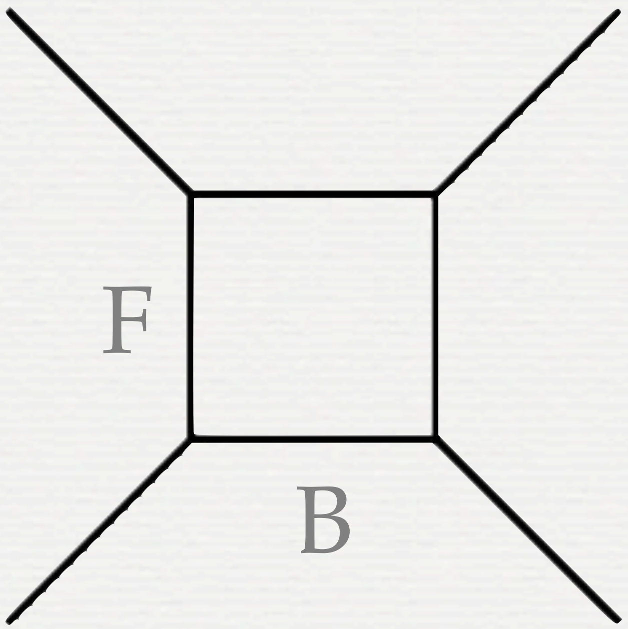
Both topological vertices have crystal melting interpretation





REFINED VERTEX *'AT WORK'*

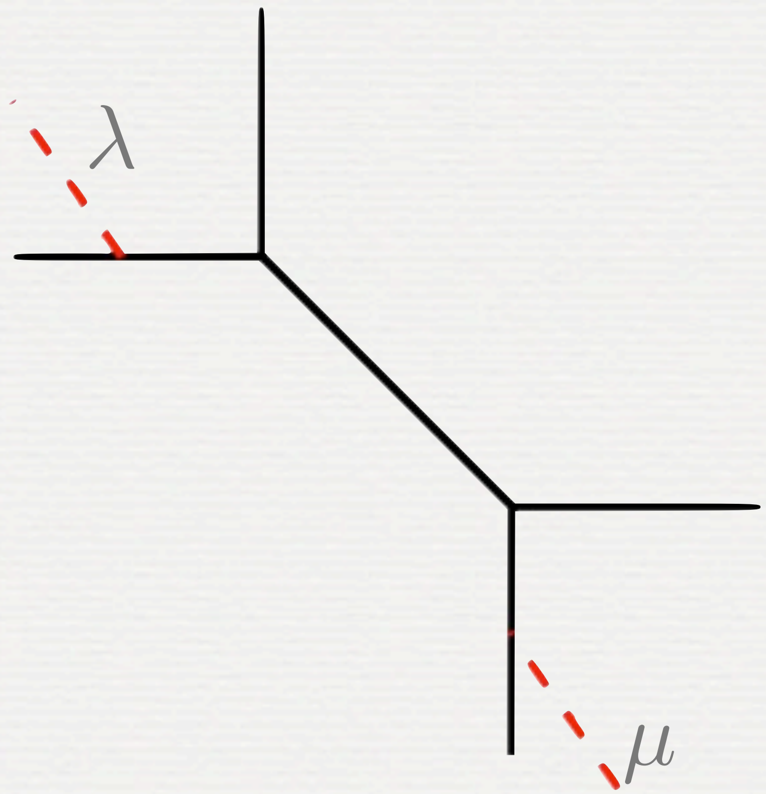
Local $\mathbb{P}^1$



$C_{n,m}$	$\sum_{j_L,j_R} N_C^{(j_L,j_R)}(j_L,j_R)$
$B+mF,\ m\geq 0$	$(0,m+\frac{1}{2})$
$2B+2F$	$(\frac{1}{2},4)\oplus(0,\frac{7}{2})\oplus(0,\frac{5}{2})$
$2B+3F$	$(1,\frac{11}{2})\oplus(\frac{1}{2},5)\oplus(\frac{1}{2},4)\oplus 2(0,\frac{9}{2})\oplus(0,\frac{7}{2})\oplus(0,\frac{5}{2})$
$3B+3F$	$(2,\frac{15}{2})\oplus(\frac{3}{2},7)\oplus(\frac{3}{2},6)\oplus 3(1,\frac{13}{2})\oplus 2(1,\frac{11}{2})\oplus(1,\frac{9}{2})$ $\oplus(\frac{1}{2},7)\oplus 3(\frac{1}{2},6)\oplus 3(\frac{1}{2},5)\oplus 2(\frac{1}{2},4)\oplus(\frac{1}{2},3)\oplus 4(0,\frac{11}{2})$ $\oplus 3(0,\frac{9}{2})\oplus 3(0,\frac{7}{2})\oplus(0,\frac{5}{2})\oplus(0,\frac{3}{2})$
$3B+4F$	$(3,\frac{19}{2})\oplus(\frac{5}{2},9)\oplus(\frac{5}{2},8)\oplus 3(2,\frac{17}{2})\oplus(\frac{3}{2},9)\oplus 2(2,\frac{15}{2})$ $\oplus 4(\frac{3}{2},8)\oplus(1,\frac{17}{2})\oplus(2,\frac{13}{2})\oplus 4(\frac{3}{2},7)\oplus 7(1,\frac{15}{2})\oplus 2(\frac{1}{2},8)$ $\oplus(0,\frac{17}{2})\oplus 2(\frac{3}{2},6)\oplus 6(1,\frac{13}{2})\oplus 7(\frac{1}{2},7)\oplus(0,\frac{15}{2})\oplus(\frac{3}{2},5)$ $\oplus 5(1,\frac{11}{2})\oplus 8(\frac{1}{2},6)\oplus 7(0,\frac{13}{2})\oplus 2(1,\frac{9}{2})\oplus 6(\frac{1}{2},5)$ $\oplus 6(0,\frac{11}{2})\oplus(1,\frac{7}{2})\oplus 4(\frac{1}{2},4)\oplus 7(0,\frac{9}{2})\oplus 2(\frac{1}{2},3)$ $\oplus 4(0,\frac{7}{2})\oplus(\frac{1}{2},2)\oplus 3(0,\frac{5}{2})\oplus(0,\frac{3}{2})\oplus(0,\frac{1}{2})$

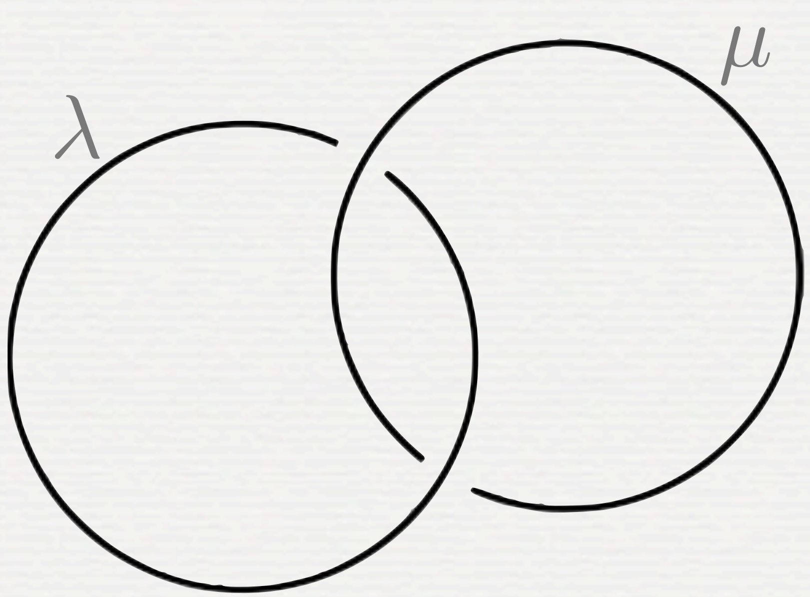
$$
\begin{aligned}
Z_{inst}(Q_b,Q_f,t,q) &= \sum_{\nu_1,\nu_2} Q_b^{|\nu_1|+|\nu_2|}\, q^{||\nu_2^t||^2}\, t^{||\nu_1^t||^2}\, \widetilde{Z}_{\nu_1}(t,q)\widetilde{Z}_{\nu_2^t}(t,q)\,\widetilde{Z}_{\nu_2}(q,t)\widetilde{Z}_{\nu_1^t}(q,t) \\
&\times \prod_{i,j=1}^{\infty} \frac{(1-Q_f\, t^{i-1}q^j)(1-Q_f\, q^{i-1}t^j)}{(1-Q_f\, t^{i-1-\nu_{2,j}}q^{j-\nu_{1,i}})(1-Q_f\, q^{i-1-\nu_{1,j}}t^{j-\nu_{2,i}})}
\end{aligned}
$$

LINK INVARIANTS



$$Z(V_1, V_2) = \sum_{\lambda,\mu} Z_{\lambda\mu}(t,q)\,\mathrm{tr}_\lambda V_1\,\mathrm{tr}_\mu V_2$$

$Z_{\lambda\mu}(t,q)$ can be computed using the refined topological vertex



We not only found perfect agreement with the known results for the refined link invariants, but also *predict* these invariant for various representations !

SUMMARY

- Topological vertex completely solves the problem of finding the string amplitudes for toric 3-folds
- Through crystal interpretation it offers a new approach for the space-time geometry
- The refined topological vertex suggests the existence of a refined Chern-Simons theory and is evidence for two-parameter (g_s, g_s') extension of the topological string theory and its duality web