# UNITARITY-BASED TECHNIQUES FOR HIGHER-ORDER QCD
# (PART I)

## PIERPAOLO MASTROLIA

HERE, TODAY

- Theoretical Challenges in Perturbation Theory

- Unitarity-based Strategies
  - Semi-numerical Techniques
  - Analytic Techniques

- OutLook

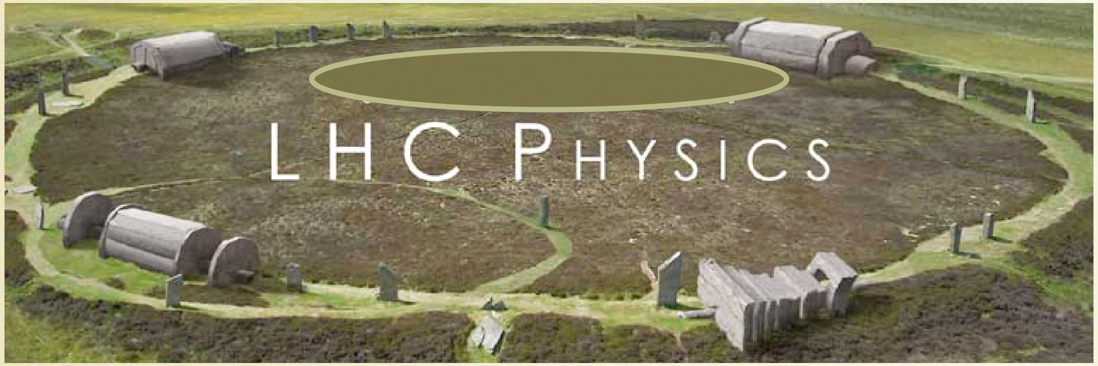
# WHY NLO ?

- Less Sensitivity to unphysical input scales (renormalization & factorization)
  - first predictive normalization of observables at NLO
  - more accurate estimates of backgrounds to new-physics
  - confidence on cross-sections for precision measurements
- More realistic process modeling
  - initial state radiation
  - jet clustering
  - richer virtuality
- Crossing path with other techniques
  - matching with resummed calculations
  - NLO parton showers

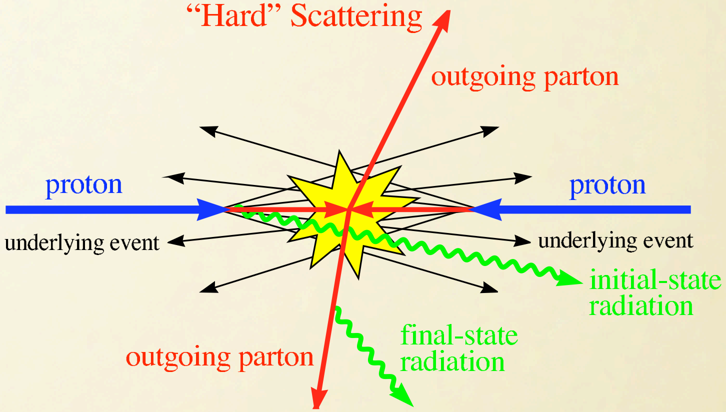
# Where NLO ?

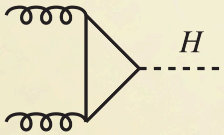
- Front-line in Theoretical Particle Physics

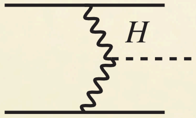
@ LHC Phenomenology

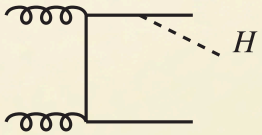


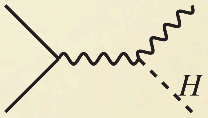
p-p collision @ XY TeV c.m.e.











Signals:

- Decays: $H \to VV$ $(V=\gamma,W,Z)$
- $PP \to H+0,1,2$ jets (Gluon Fusion)
- $PP \to H+2$ jets (Weak Boson Fusion)
- $PP \to H+t\bar{t}$
- $PP \to H+W,Z$

Backgrounds:

- $PP \to t\bar{t}+0,1,2$ jets
- $PP \to VV+0,1,2$ jets
- $PP \to V+0,1,2,3$ jets
- $PP \to VVV+0,1,2,3$ jets

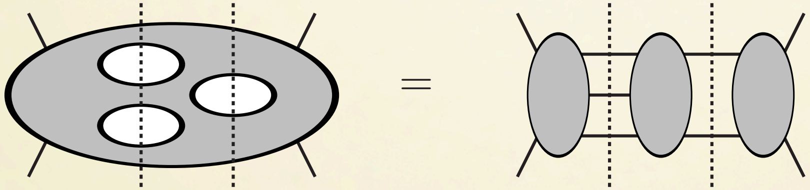
# WHERE NLO ?

Front-line in Theoretical Particle Physics

@ LHC Phenomenology

@ QFT Stucture

- ElectroWeak Symmetry Breaking: Higgs mechanism
- Beyond the Standard Model (SuSy, Dark Matter, . . . )
- Unveiling the *Iterative Structure* of Scattering Amplitudes in gauge-Theory



Anastasiou, Bern, Dixon, Kosower
Bern, Dixon, Smirnov;
Bern, Czakon,Dixon, Kosower;
Beisar, Eden, Staudacher;
Drummond, Korchemsky, Sokatchev;
Brandhuber, Heslop, Travaglini;
Alday, Maldacena;
Roiban, Spradlin, Volovich;
....

# WHERE NLO ?

Front-line in Theoretical Particle Physics

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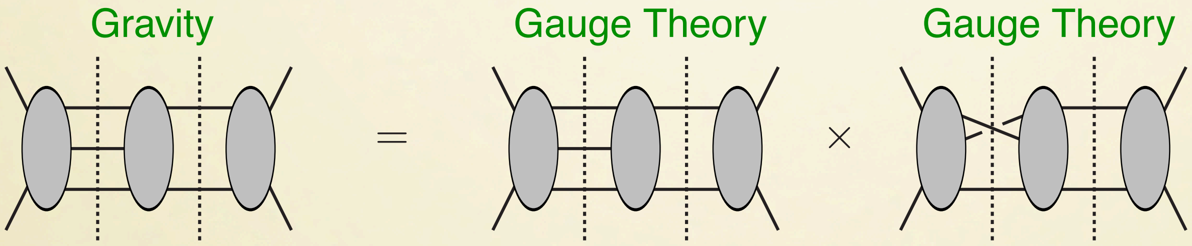
@ QFT Stucture

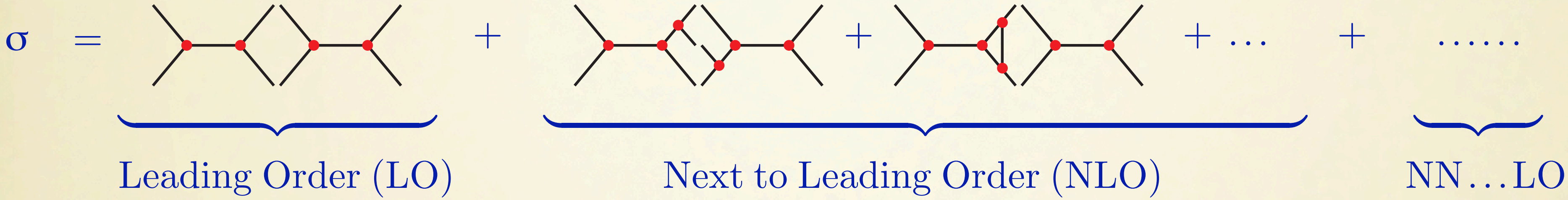
- ElectroWeak Symmetry Breaking: Higgs mechanism
- Beyond the Standard Model (SuSy, Dark Matter, . . . )
- Unveiling the Iterative Structure of Scattering Amplitudes in gauge-Theory
- Exploring the *Finiteness of Supergravity*



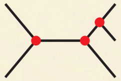
Bern, Dixon, Kosower, Perlestein, Rozowski, Roiban;
Bern, Bjerrum-Borh, Dunbar, Forde, Ita, Perkins, Risager;
Chalmers; Green, Vanhove, Russo;
Badger, Bjerrum-Borh, Vanhove,
Bern, Carrasco, Johanson;
Arkani-Hamed, Cachazo, Kaplan;
....

# NLO Building Blocks

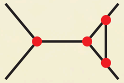
- Partonic Cross-Section



☑ tree-graphs with (n+1)-partons
  - soft/collinear divergences



☐ virtual-graphs with n-partons
  - divergences from loop-integration



$$\rightarrow \quad I^{\mu\nu\rho\ldots} = \int d^D\ell\, \frac{\ell^\mu\ell^\nu\ell^\rho\ldots}{D_1 D_2\ldots}$$

☑ extracting IR-singularities from both and combining them
  - phase-space slicing, subtractions, dipoles, antennas

# OBJECTIVES

## ✱ Challenges

- ▷ Accurate description of *multi-particle* final states, involving $t\,\bar{t}, W, Z, \text{jets}$, @ NLO
- ▷ Automation
  - multi-process tool
  - numerical stability and speed
  - improve the factorial growth
- ▷ Combined treatment of *Real* & *Virtual* contributions

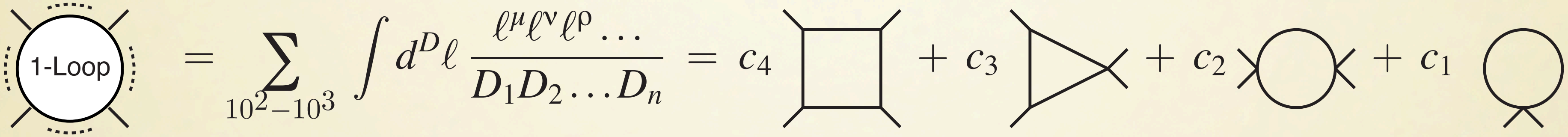
# PROCESS-INDEPENDENT STRATEGY

## ✱ Properties of the S-Matrix

- a general mathematical property: **Analyticity** of Scattering-Amplitudes
  - ▷ *Scattering Amplitudes are determined by their poles and branch-cuts*
- a general physical property: **Unitarity** of Scattering-Amplitudes
  - ▷ *The residues at poles and branch-points are products of simpler amplitudes, with lower number of particles and/or less loops*

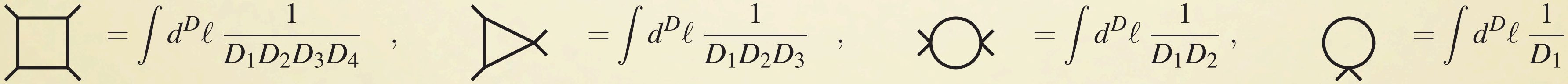
# ONE-LOOP SCATTERING AMPLITUDES

- $n$-particle Scattering: $1+2 \rightarrow 3+4+\ldots+n$
- Reduction to a Scalar-Integral Basis Passarino-Veltman



$$= \sum_{10^2-10^3} \int d^D\ell \, \frac{\ell^\mu \ell^\nu \ell^\rho \ldots}{D_1 D_2 \ldots D_n} = c_4 \; + c_3 \; + c_2 \; + c_1$$

- Known: Master Integrals

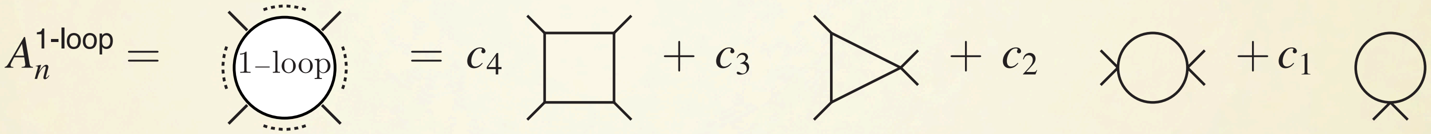


$$= \int d^D\ell \, \frac{1}{D_1 D_2 D_3 D_4} \,, \quad = \int d^D\ell \, \frac{1}{D_1 D_2 D_3} \,, \quad = \int d^D\ell \, \frac{1}{D_1 D_2} \,, \quad = \int d^D\ell \, \frac{1}{D_1}$$

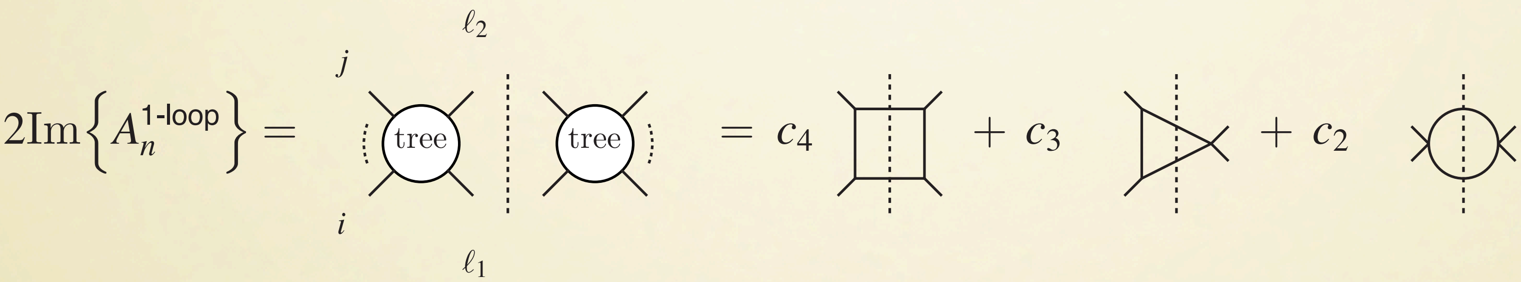
- Unknowns: $c_i$ are rational functions of external kinematic invariants

# Unitarity & Cutting Rules

- Optical Theorem from Unitarity $S \equiv 1 + iT \;:\; S^\dagger S = 1 \;\Rightarrow\; 2\mathrm{Im}T = -i(T - T^\dagger) = T^\dagger T$
- One-loop Amplitude:



- Discontinuity of Feynman Amplitudes Cutkosky-Veltman; Bern, Dixon, Dunbar & Kosower



on − shell condition : $\dfrac{1}{(\ell_i^2 - m_i^2 + i0)} \rightarrow \delta\left(\ell_i^2 - m_i^2\right) \qquad (i = 1, 2)$

Method ▷ Matching the cuts of any amplitudes onto the cuts of Master Integrals

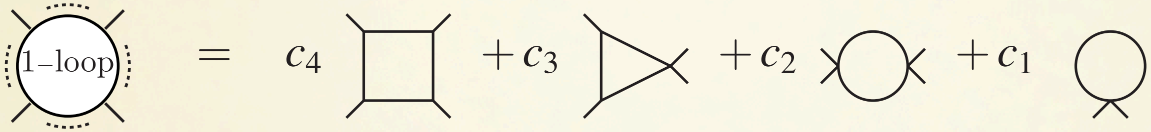
Advantage 1 ▷ iterative construction: one-loop amplitudes by sewing tree-level amplitudes

Advantage 2 ▷ simplified input: tree-amplitudes *vs* Feynman graphs
tree-amplitudes are gauge-invariant on-shell quantities,
corresponding to sums of off-shell Feynman diagrams.

# THE STRATEGY: GENERALISED UNITARITY

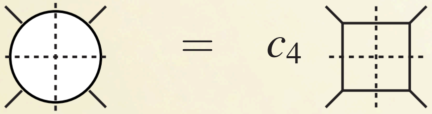
- One-loop Amplitude:

$A_n^{\text{1-loop}} =$

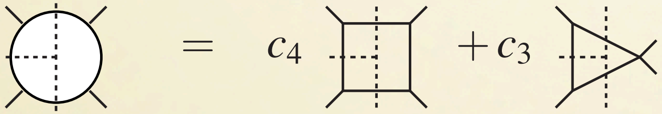


- Multiple-cuts as *optical filters*

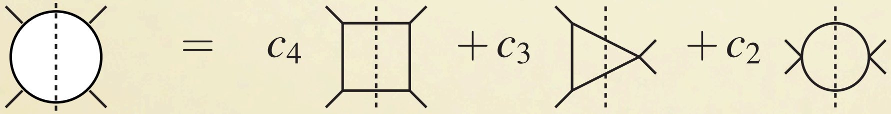
Replacing the original amplitude with simpler integrals fulfilling the same algebraic decomposition



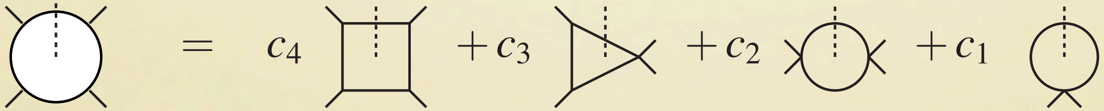
Britto, Cachazo, Feng



Bern, Dixon, Dunbar, Kosower
P.M.
Forde
Bjerrum-Bohr, Dunbar, Perkins



Bern, Dixon, Dunbar, Kosower
Brandhuber, McNamara, Spence, Travaglini
Britto, Buchbinder, Cachazo, Feng, $\oplus$ P.M.
Anastasiou, Britto, Feng, Kunszt, P.M.
Forde; Badger



Glover, Williams
Britto, Feng

The more you cut, the simpler it gets, the more you loose

# CUT-CONDITIONS

- Loop momentum decomposition [Isotropic Tetrads]

$$q^2 = p^2 = \varepsilon^{\pm 2} = 0 = \varepsilon^{\pm} \cdot p = \varepsilon^{\pm} \cdot q \, , \qquad \boxed{\ell_\mu = x_1 \, p_\mu + x_2 \, q_\mu + x_3 \, \varepsilon^+_\mu + x_4 \, \varepsilon^-_\mu}$$

Pittau, de l'Aguila
Ossola, Papadopoulos, Pittau
Forde

$$g_{\mu\nu} = \frac{1}{2p \cdot q} \left( p_\mu q_\nu + q_\mu p_\nu - \varepsilon^+_\mu \varepsilon^-_\nu - \varepsilon^-_\mu \varepsilon^+_\nu \right)$$

- under Multiple On-shellness Conditions :

- the loop-momentum becomes complex ;
- some of its components (if not all) are frozen;
- the left over free components are *integration*-variable

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## What do we do with the unfrozen variables?

- Reduction Algorithms
  - ☑ Semi-numeric Reduction of the Integrands
  - ☑ Analytic Integration of the Phase-Space Integrals

# Semi-numeric Reduction

# SEMI-NUMERIC REDUCTION

Ossola, Papadopoulos, Pittau

Ellis, Giele, Kunszt

Giele, Kunszt, Melnikov

- `OPP`-decomposition

$$A_m = \int d^4q \frac{N(q)}{D_0 \ldots D_{m-1}}$$

$$\begin{aligned}
N(q) \quad = \quad & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0i_1i_2i_3) + \tilde{d}(q;i_0i_1i_2i_3)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i \\
+ \quad & \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0i_1i_2) + \tilde{c}(q;i_0i_1i_2)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
+ \quad & \sum_{i_0<i_1}^{m-1} \left[b(i_0i_1) + \tilde{b}(q;i_0i_1)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i \\
+ \quad & \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q;i_0)\right] \prod_{i\neq i_0}^{m-1} D_i
\end{aligned}$$

- $\tilde{d}, \tilde{c}, \tilde{b}, \tilde{a}$ ($q$-dependent) vanish upon integration (Lorentz inv.)
- $d, c, b, a$ ($q$-independent) $\equiv$ wanted coefficients

- Solution
  - ☑ Fitting d-, c-, b-, a-coefficients by numerical evaluating N(q) at different q-values (suitably chosen among the solutions of multiple-cut conditions)

# POLYNOMIAL FITTING

- OPP-reduction Ossola, Papadopoulos, Pittau (2006)

From the knowledge of the multi-variate polynomial-structure of the Integrand, all $n$-point coefficients can be determined by fitting a system of polynomial equations:

$$P_m(x) = c_0 + c_1 x + c_2 x^2 + \dots\ c_m x^m$$

▷ Problem: $P_m(x)$ is given; determine $c_i$.

▷ Solution:

  ▷ step 1: sample $P_m(x)$ at $(m+1)$ random-points, $P_{m,k} \equiv P_m(x_k)$, $(k = 0, \dots, m)$ :

$$\begin{pmatrix} P_{m,0} \\ P_{m,1} \\ \vdots \\ P_{m,m} \end{pmatrix} = \begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^m \\ 1 & x_1 & x_1^2 & \dots & x_1^m \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_m & x_m^2 & \dots & x_m^m \end{pmatrix} \times \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_m \end{pmatrix}$$

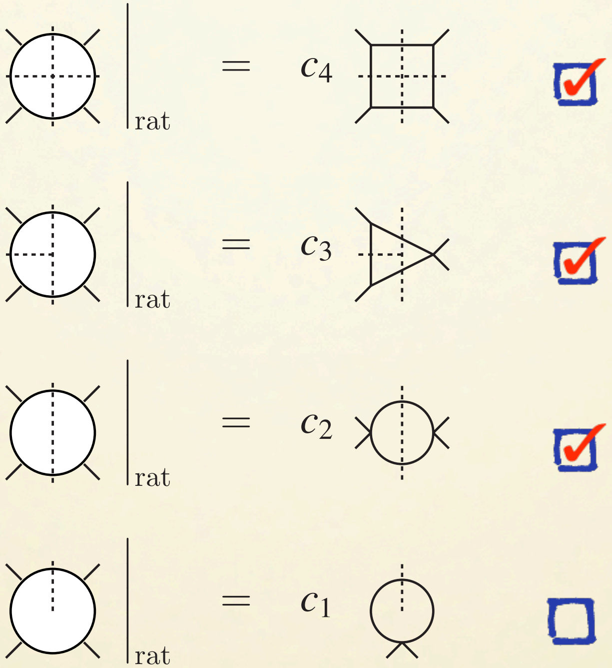
  ▷ step 2: find $c_i$ by system inversion.

- Improved Reduction with DFT Ossola, Papadopoulos, Pittau, & P.M.
  Blackhat

# ANALYTIC INTEGRATION

# MULTIPLE-CUTS PROJECTION

- Any $n$-particle cut, after integration, might contain a *Rational Term* and a *Logarithmic Term*
- Extract the $n$-point coefficient from the *Rational Term* of the $n$-particle cut.



- Closer look at the Integrand Structure

Numerator and denominator of the $n$-particle cut-integrand are mutivariate-polynomials in $(4-n)$ complex-variables:

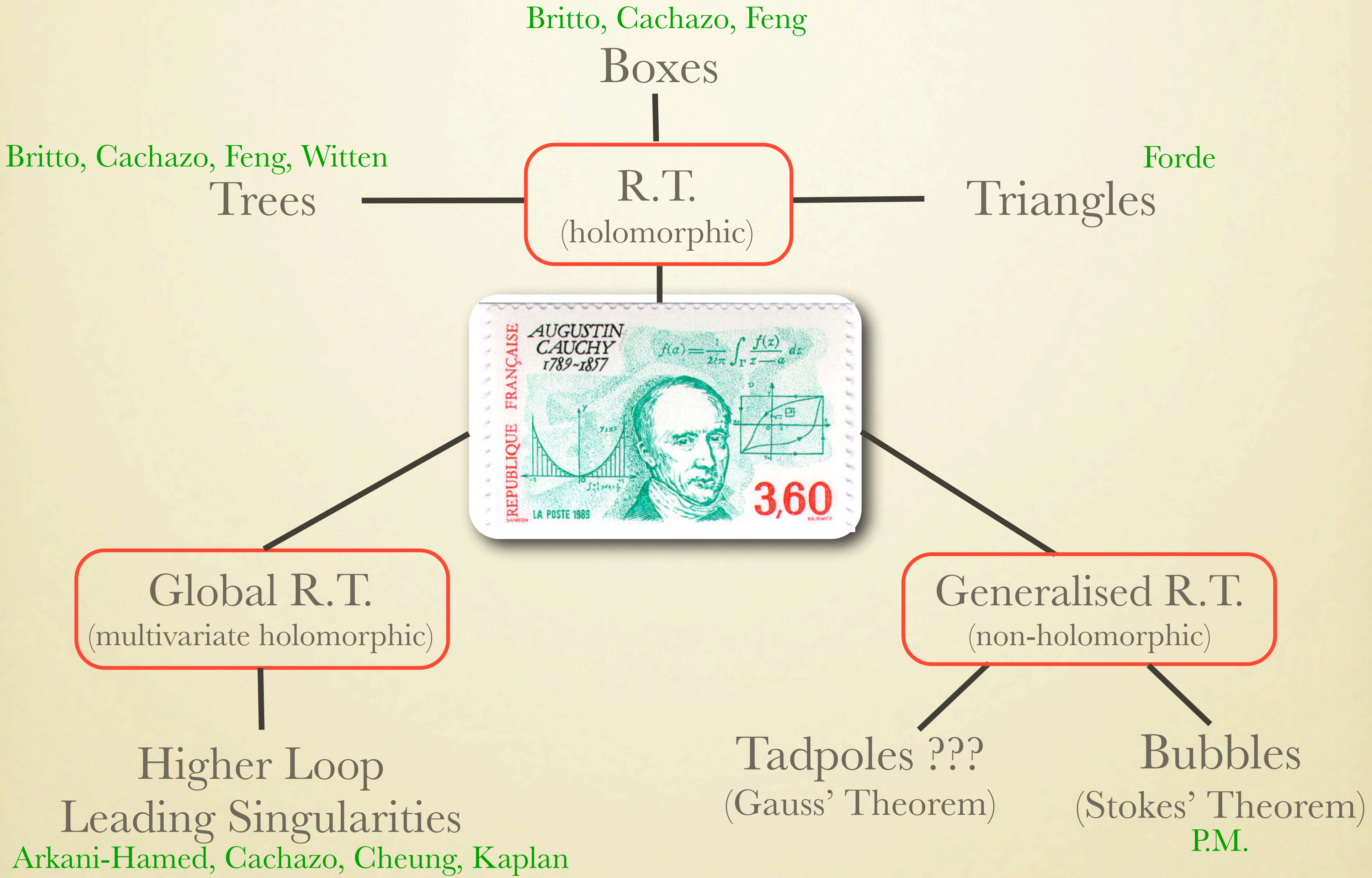
$$\mathrm{Cut}_n = \oint dx_1 \ldots dx_{4-n} \, \frac{P(x_1, \ldots, x_{4-n})}{Q(x_1, \ldots, x_{4-n})}$$

- Singularity Classification

Master Integrals characterized by the *location of the poles*.

☑ Integration ~ Series Expansion

# CAUCHY'S RESIDUE THEOREM @ WORK



Britto, Cachazo, Feng
Boxes

Britto, Cachazo, Feng, Witten
Trees

R.T.
(holomorphic)

Forde
Triangles

Global R.T.
(multivariate holomorphic)

Generalised R.T.
(non-holomorphic)

Higher Loop
Leading Singularities
Arkani-Hamed, Cachazo, Cheung, Kaplan

Tadpoles ???
(Gauss' Theorem)

Bubbles
(Stokes' Theorem)
P.M.

# UNITARITY-BASED RESULTS

- Numerical Results (virtual amplitudes)
  - $\gamma\gamma \to \gamma\gamma\gamma\gamma$ Ossola, Papadopoulos, Pittau ’07
  - $gg \to gggggggggggggggggg$ Giele, Zanderighi ’08
  - $pp \to t\bar{t}b\bar{b}$ van Hameren, Papadopoulos, Pittau ’09
  - $pp \to b\bar{b}b\bar{b}$ van Hameren, Papadopoulos, Pittau ’09
  - $pp \to VV + 2$ jets van Hameren, Papadopoulos, Pittau ’09
  - $pp \to VVb\bar{b}$ van Hameren, Papadopoulos, Pittau ’09
  - $e^+e^- \to e^+e^-\gamma$ Actis, Ossola, P.M. ’09
  - $e^+e^- \to \mu^+\mu^-\gamma$ Actis, Ossola, P.M. ’09

- Numerical Results @ NLO
  - $pp \to VVV$ Binoth, Ossola, Papadopoulos, Pittau ’08
  - $pp \to W + 3$ jets Berger, Bern, Dixon, Forde, Febres-Cordero, Ita, Kosower, Gleisberg ’08-’09

    Ellis, Giele, Kunszt, Melnikov, Zanderighi ’08-’09 [wait for Darren]
  - $pp \to Z + 3$ jets Berger, Bern, Dixon, Forde, Febres-Cordero, Ita, Kosower, Gleisberg (prelim.)
  - $pp \to t\bar{t}b\bar{b}$ Bevilacqua, Czakon, Papadopoulos, Pittau, Worek ’09 [wait for Stefano]

- Analytic calculations (virtual amplitudes)
  - $gg \to gggg$ Bern, Dixon, Dunbar, Kosower ’96; ... (we are here) ...; Xiao, Yang, Zhu ’08
  - $\gamma\gamma \to \gamma\gamma\gamma\gamma$ Binoth, Gehrmann, Heinrich, P.M. ’07
  - $pp \to H + 2$ jets Badger, Berger, Campbell, Del Duca, Dixon, Ellis, Glover, Risager, Sofianatos, Williams, P.M., ’06 - ’09

# OutLook

- Amplitudes from their singularity structure, accessed through complex momenta
  - ☑ Analyticity: amplitudes determined by their poles and branch-point
  - ☑ Unitarity: residues formed by products of simpler, on-shell, amplitudes

- NLO
  - ☐ Numerical Unitarity in Production Mood: Automation
    - Virtual: Blackhat, Rocket, CutTools, Helac-OneLoop, ...
    - Real: MadDipole, MadFKS, ... (aren't we going too Mad?)
  - ☐ Analytic Unitarity: R&D
    - Integration by Series Expansion (Newton's legacy)
    - Exploring the Complex Structure of Scattering Amplitudes
    - Principles of Algebraic Geometry

- N...NLO
  - ☐ Extending Unitarity-based methods @ Higher-Loop
  - ☐ Wilson Loops and String/Gauge Theory Correspondence
  - ☐ IR-structure to all-order