Introduction to Transverse Beam Dynamics

Bernhard Holzer, CERN

The Ideal World

I.) Magnetic Fields and Particle Trajectories

Luminosity Run of a typical storage ring:

LHC Storage Ring: Protons accelerated and stored for 12 hours
distance of particles travelling at about $v \approx c$
$L = 10^{10} - 10^{11}$ km
... several times Sun - Pluto and back

→ guide the particles on a well defined orbit ("design orbit")
→ focus the particles to keep each single particle trajectory
  within the vacuum chamber of the storage ring, i.e. close to the design orbit.
1.) Introduction and Basic Ideas

"... in the end and after all it should be a kind of circular machine"

\[ \text{need transverse deflecting force} \]

Lorentz force

\[ \vec{F} = q \left( \vec{E} + \vec{v} \times \vec{B} \right) \]

typical velocity in high energy machines:

\[ v = c \approx 3 \times 10^8 \text{ m/s} \]

Example:

\[
B = 1 \text{T} \quad \Rightarrow \quad F = q \times 3 \times 10^1 \frac{m}{s} \times \frac{V_s}{m} \quad \text{technical limit for el. field}
\]

\[ F = q \times 300 \frac{MV}{m} \]

equivalent el. field \[ E \]

**remember what you learned before p12 SuSh:**

In accelerator physics we ask: "What are the particles' generalized coordinates when they reach a certain point in space?"

First, we convert to a non-inertial reference frame. 
We use the 'Frenet-Serret' co-ordinate system

Particle motion is described with respect to a **reference orbit** in the non-inertial frame \((x, y, s)\). This co-ordinate system is known as **Frenet-Serret**
The ideal circular orbit

condition for circular orbit:

\[ F_L = e v B \]
\[ F_{cent} = \frac{\gamma m_e v^2}{\rho} \]
\[ \frac{\gamma m_e v^2}{\rho} = e v B \]

\[ \frac{p}{e} = B \rho \]

\[ B \rho = "\text{beam rigidity}\" \]

1.) The Magnetic Guide Field

Dipole Magnets:
define the ideal orbit
homogeneous field created
by two flat pole shoes

Define the Geometry of the Ring:

\[ \theta = \frac{l}{\rho} = \frac{B l n I}{B \rho} = \frac{\int B d l}{p/q} = 2\pi \]

Example LHC:

\[ B = 8.3 T \]
\[ p = 7000 \text{ GeV} \]
\[ 2\pi p = 17.6 \text{ km} \]
\[ \rho = 2.8 \text{ km} \]
\[ \approx 66\% \]

convenient units:

\[ B = \frac{V}{T} = \left[ \frac{V}{m^2} \right] \]
\[ p = \frac{GeV}{c} \]

field map of a storage ring dipole magnet
Example LHC:

7000 GeV Proton storage ring
dipole magnets \( N = 1232 \)
\( l = 15 \text{ m} \)
\( q = +1 \text{ e} \)

\[ \int B \, dl = N \, l \, B = \frac{2\pi \, p}{e} \]

\[ B = \frac{2\pi \, 7000 \, 10^7 \text{eV}}{1232 \, 15 \, 3 \times 10^8 \text{m/s} \, e} \approx 8.3 \text{ Tesla} \]

2.) Focusing Properties – Transverse Beam Optics

Classical Mechanics: pendulum

there is a restoring force, proportional to the elongation \( x \):

\[ F = m \, \frac{d^2 x}{dt^2} = -k \, x \]

\( \text{Ansatz} \)

\[ x(t) = A \, \cos(\omega t + \varphi) \]

\[ \dot{x} = -A \omega \, \sin(\omega t + \varphi) \]

\[ \ddot{x} = -A \omega^2 \, \cos(\omega t + \varphi) \]

\( \text{Solution} \)

\[ \omega = \sqrt{k/m}, \quad x(t) = x_0 \, \cos(\sqrt{\frac{k}{m}} t + \varphi) \]

Storage Ring: we need a Lorentz force that rises as a function of the distance \( x \) to \( \cdots \) ?

\[ F(x) = q \, v \, B(x) \]
Next, we change the independent variable from $t$ to $s$

The new conjugate phase space variables are $x, p_x, y, p_y, t, -H$

And the new Hamiltonian (s-dependent) is

$$\hat{H} = -(1 + x / \rho) \left[ \frac{\left( H \phi \right)^2}{c^2} - m^2 c^2 - (p_x - eA_y)^2 - (p_y - eA_x)^2 \right]^{1/2}$$

Which is time-independent (if also $\phi, A$ are time-independent)

Expanding the Hamiltonian to second order in $p_x, p_y$

$$\hat{H} \approx -p(1 + x / \rho) + \frac{1 + x / \rho}{2 p} \left[ (p_x - eA_y)^2 + (p_y - eA_x)^2 \right] + eA_x$$

$\hat{H} - e \phi = E$ is the total particle energy

$p = \sqrt{E^2 / c^2 - m^2 c^2}$ is the total particle momentum

---

**Quadrupole Magnets:**

required: focusing forces to keep trajectories in vicinity of the ideal orbit

linear increasing Lorentz force $\Rightarrow$ linear increasing magnetic field

$$B_y = g \ x \quad B_x = g \ y$$

normalised quadrupole field:

gradient of a quadrupole magnet:

$$g = \frac{2 \mu_n I}{r^3}$$

$$k = \frac{g}{p / e}$$

simple rule:

$$k = 0.3 \frac{g(T / m)}{p(\text{GeV} / c)}$$

LHC main quadrupole magnet

$$g \approx 25 \ldots 220 \ T / m$$

what about the vertical plane:

... Maxwell

$$\nabla \times \mathbf{B} = \mu_0 \nabla \left( \frac{\phi}{c} \right) \Rightarrow \frac{\partial B_x}{\partial x} = \frac{\partial B_y}{\partial y}$$
Equation of Motion:

Consider local segment of a particle trajectory...
... and remember the old days:

\( \text{Ideal orbit: } \rho = \text{const, } \frac{d\rho}{dt} = 0 \)

\( \text{Force: } F = m\rho \left( \frac{d\theta}{dt} \right)^2 = m\rho \omega^2 \)

\( F = mv^2 / \rho \)

\( \text{radial acceleration: } \alpha_r = \frac{d^2\rho}{dt^2} - \rho \left( \frac{d\theta}{dt} \right)^2 \)

\( \text{general trajectory: } \rho \rightarrow \rho + x \)

\[ F = m \frac{d^2}{dt^2} (x + \rho) - \frac{mv^2}{x + \rho} = eB_yv \]

\( F = m \frac{d}{dt} (x + \rho) - \frac{mv^2}{x + \rho} = eB_yv \)  

1. \( \frac{d^2}{dt^2} (x + \rho) = \frac{d^2}{dt^2} x \quad \text{... as } \rho = \text{const} \)

2. remember: \( x \approx mm, \rho \approx m \quad \Rightarrow \text{develop for small } x \)

\[ \frac{1}{\rho} = \frac{1}{x + \rho} \approx \frac{1}{x} \left( 1 - \frac{x}{\rho} \right) \]

Taylor Expansion:

\[ f(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \ldots \]

\[ m \frac{d^2x}{dt^2} - \frac{mv^2}{\rho} \left( 1 - \frac{x}{\rho} \right) = eB_yv \]

\[ \ldots \Rightarrow \text{see appendix for the details} \]
Equation for the vertical motion:

\[
\frac{1}{\rho^2} = 0 \quad \text{no dipoles … in general …} \\
\rho \leftrightarrow -k \quad \text{quadrupole field changes sign}
\]

\[
y'' + ky = 0
\]

Remember p18 SuSh multipole expansion

Multipole fields

In the usual notation:

\[
B_x + iB_y = B_{ref} \sum_{n=1}^{\infty} (b_n + ib_n) \left( \frac{x + iy}{R_{ref}} \right)^{n+1}
\]

\(b_n\) are "normal multipole coefficients" (LEFT) and \(a_n\) are "skew multipole coefficients" (RIGHT) 'ref' means some reference value

n=1, dipole field  
n=2, quadrupole field  
n=3, sextupole field

Images: A. Welski, https://cds.cern.ch/record/1333874
Remarks:

* Multipoles in linear approximation
dipole & quadrupole

* Separate Function Machines:
Split the magnets and optimise them according
to their job: bending, focusing, correction, etc

Example: heavy ion storage ring TSR

* Weak Focusing:
... there seems to be a focusing even without
a quadrupole gradient „weak focusing of dipole magnets“

\[
k = 0 \quad \Rightarrow \quad x' = -\frac{1}{\rho^2} x
\]

Mass spectrometer: particles are separated
according to their energy
and focused due to the \(1/\rho\)
effect of the dipole

* Hard Edge Model:

\[
\frac{1}{\rho} \quad \frac{d}{dx} \frac{1}{\rho} \quad \frac{d}{dx} \quad \frac{d}{dx} \quad \frac{d}{dx} \quad \frac{d}{dx} \quad \frac{d}{dx} \quad \frac{d}{dx}
\]

... this equation is not correct !!!

bending and focusing fields ... are functions
of the independent variable „x“

Inside a magnet we assume constant focusing
properties!

\[
\frac{1}{\rho} = \text{const} \quad k = \text{const}
\]

\[
B_{\text{eff}} = \int_{0}^{L} B \, ds
\]
4.) Solution of Trajectory Equations

remember p25 SuSh:

... c’est quoi ca ??
(engl. version: WHAT ???)

it just means that we would like to calculate the trajectories of a particle while it is traveling through our machine (or to Pluto, if you don’t mind).

Differential Equation of harmonic oscillator ... with spring constant K

Ansatz: \[ x(s) = a_1 \cos(\omega s) + a_2 \sin(\omega s) \]

general solution: linear combination of two independent solutions

\[ x'(s) = -a_1 \omega \sin(\omega s) + a_2 \omega \cos(\omega s) \]
\[ x''(s) = -a_1 \omega^2 \cos(\omega s) - a_2 \omega^2 \sin(\omega s) = -\omega^2 x(s) \]
\[ \omega = \sqrt{K} \]

general solution:
\[ x(s) = a_1 \cos(\sqrt{K} s) + a_2 \sin(\sqrt{K} s) \]
determine $a_1$, $a_2$ by boundary conditions:

$$\begin{align*}
    x(0) &= x_0, & a_1 &= x_0 \\
    x'(0) &= x'_0, & a_2 &= \frac{x'_0}{\sqrt{K}}
\end{align*}$$

**Hor. Focusing Quadrupole $K > 0$:**

$$x(x) = x'_0 \cdot \cos(\sqrt{K} x) + x''_0 \cdot \frac{1}{\sqrt{K}} \sin(\sqrt{K} x)$$

$$x'(x) = -x'_0 \cdot \sqrt{K} \cdot \sin(\sqrt{K} x) + x''_0 \cdot \cos(\sqrt{K} x)$$

For convenience expressed in matrix formalism:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s \neq 0} = M_{\text{foc}} \begin{pmatrix} x \\ x' \end{pmatrix}_{s = 0}$$

$$M_{\text{foc}} = \begin{pmatrix} \cos(\sqrt{K} x) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} x) \\ -\sqrt{K} \sin(\sqrt{K} x) & \cos(\sqrt{K} x) \end{pmatrix}$$

**Hor. defocusing quadrupole:**

$$x'' - K x = 0$$

Remember from school:

$$f(s) = \cosh(s), \quad f'(s) = \sinh(s)$$

**Ansatz:**

$$x(s) = a_1 \cdot \cosh(\omega s) + a_2 \cdot \sinh(\omega s)$$

$$M_{\text{def}} = \begin{pmatrix} \cosh(\sqrt{K} l) & \frac{1}{\sqrt{K}} \sinh(\sqrt{K} l) \\ \sqrt{K} \sinh(\sqrt{K} l) & \cosh(\sqrt{K} l) \end{pmatrix}$$

**Drift space:**

$$K = 0$$

$$M_{\text{drift}} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

"with the assumptions made, the motion in the horizontal and vertical planes are independent... the particle motion in x & y is uncoupled"
Thin Lens Approximation:

matrix of a quadrupole lens

\[
M = \begin{pmatrix}
\cos \sqrt{|k|} & \frac{1}{\sqrt{|k|}} \sin \sqrt{|k|} \\
-\frac{1}{\sqrt{|k|}} \sin \sqrt{|k|} & \cos \sqrt{|k|}
\end{pmatrix}
\]

in many practical cases we have the situation:

\[ f = \frac{1}{k l_q} \gg l_q \quad \text{... focal length of the lens is much bigger than the length of the magnet} \]

\[ l_q \to 0 \quad \text{while keeping} \quad k l_q = \text{const} \]

\[
M_x = \begin{pmatrix}
1 & 0 \\
\frac{1}{f} & 1
\end{pmatrix} \quad M_y = \begin{pmatrix}
1 & 0 \\
-\frac{1}{f} & 1
\end{pmatrix}
\]

... useful for fast (and in large machines still quite accurate) "back on the envelope calculations" ... and for the guided studies!

Combining the two planes:

Clear enough (hopefully ...?): a quadrupole magnet that is focussing o-in one plane acts as defocusing lens in the other plane ... et vice versa.

\[
M_{\text{hor foc. quadrupole lens}} = \begin{pmatrix}
\cos(\sqrt{|k|} k_x) & \frac{1}{\sqrt{|k|}} \sin(\sqrt{|k|} k_x) \\
-\sqrt{|k|} \sin(\sqrt{|k|} k_x) & \cos(\sqrt{|k|} k_x)
\end{pmatrix}
\]

matrix of the same magnet in the vert. plane:

\[
M_{\text{defoc}} = \begin{pmatrix}
\cosh(\sqrt{|k|} k_y) & \frac{1}{\sqrt{|k|}} \sinh(\sqrt{|k|} k_y) \\
\sqrt{|k|} \sinh(\sqrt{|k|} k_y) & \cosh(\sqrt{|k|} k_y)
\end{pmatrix}
\]

\[
\begin{pmatrix}
x' \\
y'
\end{pmatrix} = \begin{pmatrix}
\cos(\sqrt{|k|} k_x) & \frac{1}{\sqrt{|k|}} \sin(\sqrt{|k|} k_x) & 0 & 0 \\
-\sqrt{|k|} \sin(\sqrt{|k|} k_x) & \cos(\sqrt{|k|} k_x) & 0 & 0 \\
0 & 0 & \cosh(\sqrt{|k|} k_y) & \frac{1}{\sqrt{|k|}} \sinh(\sqrt{|k|} k_y) \\
0 & 0 & \sqrt{|k|} \sinh(\sqrt{|k|} k_y) & \cosh(\sqrt{|k|} k_y)
\end{pmatrix} \begin{pmatrix}
x \\
x' \\
y \\
y'
\end{pmatrix}
\]

! with the assumptions made, the motion in the horizontal and vertical planes are independent ... the particle motion in x & y is uncoupled!
Transformation through a system of lattice elements
combine the single element solutions by multiplication of the matrices

\[ M_{\text{total}} = M_{QF} \cdot M_{D} \cdot M_{QD} \cdot M_{\text{band}} \cdot M_{D} \ldots \]

\[
\begin{pmatrix}
  x \\
  x'
\end{pmatrix}_{x2} = M(S_{x},S_{x}) \cdot 
\begin{pmatrix}
  x \\
  x'
\end{pmatrix}_{x1}
\]

in each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator \( x \).

5.) Orbit & Tune:

Tune: number of oscillations per turn

64.31
59.32

Relevant for beam stability: non integer part

LHC revolution frequency: 11.3 kHz \( 0.31 \times 11.3 = 3.5 \text{kHz} \).
**LHC Operation: Beam Commissioning**

*First turn steering "by sector:"
- One beam at the time
- Beam through 1 sector (1/8 ring), correct trajectory, open collimator and move on.

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**LHC Operation: the First Beam**

Beam 1 on OTR screen
1st and 2nd turn
**Question:** what will happen, if the particle performs a second turn?

... or a third one or ... 10th turns

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**Astronomer Hill:**

differential equation for motions with periodic focusing properties

„Hill’s equation“

\[
\ddot{x} + k(s)x(s) = 0
\]

Example: particle motion with periodic coefficient

restoring force \( \neq \text{const} \),
\( k(s) = \text{depending on the position } s \),
\( k(s+L) = k(s) \), periodic function

we expect a kind of quasi harmonic oscillation: amplitude & phase will depend on the position \( s \) in the ring.
6.) The Beta Function

General solution of Hill’s equation:

\[(i) \quad x(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi)\]

\(\varepsilon, \phi = \) integration constants determined by initial conditions

\(\beta(s)\) periodic function given by focusing properties of the lattice \(\leftrightarrow\) quadrupoles

\(\beta(s + L) = \beta(s)\)

Inserting (i) into the equation of motion …

\[\Psi(s) = \int_0^s \frac{ds}{\beta(s)}\]

\(\Psi(s) = \) “phase advance” of the oscillation between point “0” and “s” in the lattice.

For one complete revolution: number of oscillations per turn “Tune”

\[Q_y = \frac{1}{2\pi} \int \frac{ds}{\beta(s)}\]

The Beta Function

Amplitude of a particle trajectory:

\[x(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi)\]

Maximum size of a particle amplitude

\[\hat{x}(s) = \sqrt{\varepsilon} \beta(s)\]

\(\beta\) determines the beam size

(... the envelope of all particle trajectories at a given position “s” in the storage ring.

It reflects the periodicity of the magnet structure.)
7.) Beam Emittance and Phase Space Ellipse

general solution of Hill equation

\[
\begin{align*}
(1) & \quad x(s) = \sqrt{\epsilon} \sqrt{\beta(s)} \cos(\psi(s) + \phi) \\
(2) & \quad x'(s) = -\frac{\sqrt{\epsilon}}{\sqrt{\beta(s)}} \left[ \alpha(s) \cos(\psi(s) + \phi) + \sin(\psi(s) + \phi) \right]
\end{align*}
\]

from (1) we get

\[
\cos(\psi(s) + \phi) = \frac{x(s)}{\sqrt{\epsilon} \sqrt{\beta(s)}}
\]

\[
\alpha(s) = \frac{1}{2} \beta'(s) \\
\gamma(s) = \frac{1 + \alpha(s)^2}{\beta(s)}
\]

Insert into (2) and solve for \( \epsilon \)

\[
\epsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s) x'^2(s)
\]

* \( \epsilon \) is a constant of the motion ... it is independent of "s"

* parametric representation of an ellipse in the \( x, x' \) space

* shape and orientation of ellipse are given by \( \alpha, \beta, \gamma \)

---

Beam Emittance and Phase Space Ellipse

\[
\epsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s) x'^2(s)
\]

\[
\text{Liouville: in reasonable storage rings area in phase space is constant.} \quad A = \pi \epsilon = \text{const}
\]

\( \epsilon \) beam emittance = wobblycity of the particle ensemble, intrinsic beam parameter, cannot be changed by the foc. properties.

Scientifically speaking: area covered in transverse \( x, x' \) phase space ... and it is constant !!!
**Particle Tracking in a Storage Ring**

*Calculate x, x’ for each linear accelerator element according to matrix formalism*

*plot x, x’ as a function of s***

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... and now the ellipse:

*note for each turn x, x’ at a given position s1* and plot in the phase space diagram
Résumé:

beam rigidity: \( B \cdot \rho = \frac{p}{q} \)

bending strength of a dipole: \( \frac{1}{\rho} \text{[m}^{-1}] = \frac{0.2998 \cdot B_a(T)}{p(GeV/c)} \)

focusing strength of a quadrupole: \( k \text{[m}^{-2}] = \frac{0.2998 \cdot g}{p(GeV/c)} \)

focal length of a quadrupole: \( f = \frac{1}{k' \cdot l_0} \)

equation of motion: \( x'' + Kx = \frac{1}{\rho} \frac{\Delta p}{p} \)

matrix of a foc. quadrupole: \( x_{x_2} = M \cdot x_{x_1} \)

\[
M = \begin{pmatrix}
\cos \sqrt{K} & \frac{1}{\sqrt{K}} \sin \sqrt{K} \\
-\sqrt{K} \sin \sqrt{K} & \cos \sqrt{K}
\end{pmatrix}
\]

\[
M = \begin{pmatrix}
1 & 0 \\
\frac{1}{f} & 1
\end{pmatrix}
\]

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3.) The equation of motion:

**Linear approximation:**

* ideal particle → design orbit

* any other particle → coordinates x, y small quantities

x, y << \( \rho \)

→ magnetic guide field: only linear terms in x & y of B

have to be taken into account

**Taylor Expansion of the B field:**

\[
B_y(x) = B_{y0} + \frac{dB_y}{dx} x + \frac{1}{2!} \frac{d^2 B_y}{dx^2} x^2 + \frac{1}{3!} \frac{d^3 B_y}{dx^3} x^3 + \ldots
\]

\[
B(x) = \frac{B_x}{p/e} + \frac{g x}{p/e} + \frac{1}{2!} \frac{eg'}{p/e} x^2 + \frac{1}{3!} \frac{eg''}{p/e} x^3 + \ldots
\]

Normalise to momentum

\( p/e = B\rho \)

---

**The Equation of Motion:**

\[
\frac{B(x)}{p/e} = \frac{1}{\rho} + k x + \frac{1}{2!} \mu x^2 + \frac{1}{3!} \lambda x^3 + \ldots
\]

only terms linear in x, y taken into account

dipole fields

quadrupole fields

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**Separate Function Machines:**

Split the magnets and optimise them according to their job:

bending, focusing etc

---

Example:

heavy ion storage ring TSR

* more tools used dipole and quadrupole beams
Equation of Motion:

Consider local segment of a particle trajectory ...

... and remember the old days: (Goldstein page 27)

radial acceleration:

\[ a_r = \frac{d^2 \rho}{dt^2} - \rho \left( \frac{d\theta}{dt} \right)^2 \]

Ideal orbit: \( \rho = \text{const} \), \( \frac{d\rho}{dt} = 0 \)

Force: \( F = m \rho \left( \frac{d\theta}{dt} \right)^2 = m \rho \omega^2 \)

\( F = m \frac{v^2}{\rho} \)

general trajectory: \( \rho \rightarrow \rho + x \)

\[ F = m \frac{d^2}{dt^2} (x + \rho) - \frac{mv^2}{x + \rho} = eB_y v \]

\( F = m \frac{d^2}{dt^2} (x + \rho) - \frac{mv^2}{x + \rho} = eB_y v \)

1. \[ \frac{d^2}{dt^2} (x + \rho) = \frac{d^2}{dt^2} x \] \( \ldots \) as \( \rho = \text{const} \)

2. remember: \( x \approx m \), \( \rho \approx m \ldots \) develop for small \( x \)

\[ \frac{1}{x + \rho} = \frac{1}{\rho} \left( 1 - \frac{x}{\rho} \right) \]

Taylor Expansion

\[ f(x) = f(x_0) + \frac{(x - x_0)}{1!} f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \]

\[ m \frac{d^2 x}{dt^2} - \frac{mv^2}{\rho} (1 - \frac{x}{\rho}) = eB_y y \]
guide field in linear approx.

\[ B_y = B_0 + x \frac{\partial B_y}{\partial x} \]

\[ m \frac{d^2 x}{dt^2} - \frac{m v^2}{\rho} (1 - \frac{x}{\rho}) = e v \left( B_0 + x \frac{\partial B_y}{\partial x} \right) \]

\[ \frac{d^2 x}{dt^2} - \frac{v^2}{\rho} (1 - \frac{x}{\rho}) = \frac{e v B_0}{m} + \frac{e v x g}{m} \]

independent variable: \( t \rightarrow s \)

\[ \frac{dx}{dt} = \frac{ds}{dt} \]

\[ \frac{d^2 x}{dt^2} = \frac{d}{dt} \left( \frac{dx}{ds} \right) \frac{ds}{dt} = \frac{d}{ds} \left( \frac{dx}{ds} \right) \frac{ds}{dt} \]

\[ \frac{d^2 x}{dt^2} = x^* v^2 + \frac{dx}{ds} \frac{dv}{ds} v \]

\[ x^* v^2 = \frac{v^2}{\rho} (1 - \frac{x}{\rho}) = \frac{e v B_0}{m} + \frac{e v x g}{m} \]

\[ x^* \frac{1}{\rho} \left( 1 - \frac{x}{\rho} \right) = \frac{e B_0}{mv} + \frac{e x g}{mv} \]

\[ x^* \frac{1}{\rho} + \frac{x}{\rho} \frac{1}{p^2} = \frac{B_0}{p / e} + \frac{x g}{p / e} \]

\[ x^* \left( 1 - \frac{1}{\rho} \right) k = \frac{1}{\rho} + k x \]

\[ x^* + x \left( \frac{1}{\rho^2} - k \right) = 0 \]

\[ \frac{1}{\rho} = 0 \quad \text{no dipoles in general ...} \]

\[ k \leftrightarrow -k \quad \text{quadrupole field changes sign} \]

\[ y^* + k y = 0 \]

Equation for the vertical motion:

\[ \frac{B_0}{p / e} = \frac{1}{\rho} \]

\[ \frac{g}{p / e} = k \]

\( y^* \)