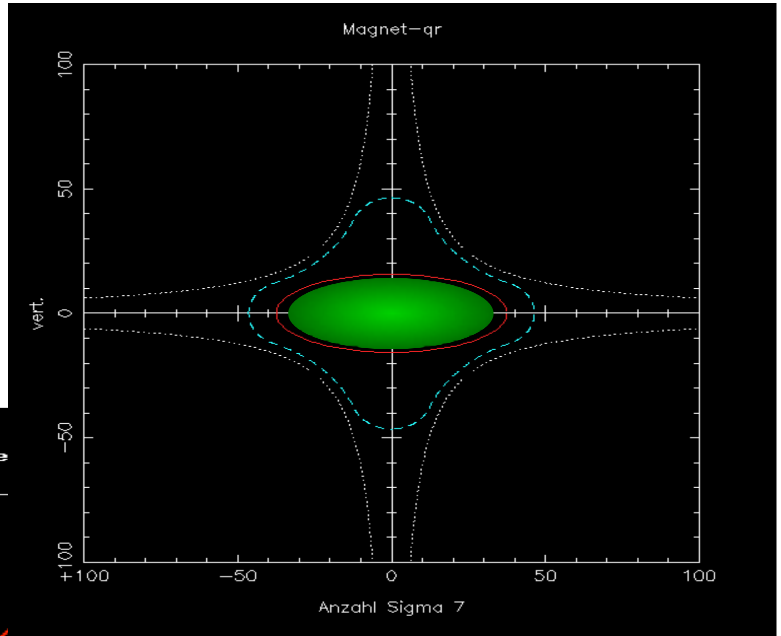
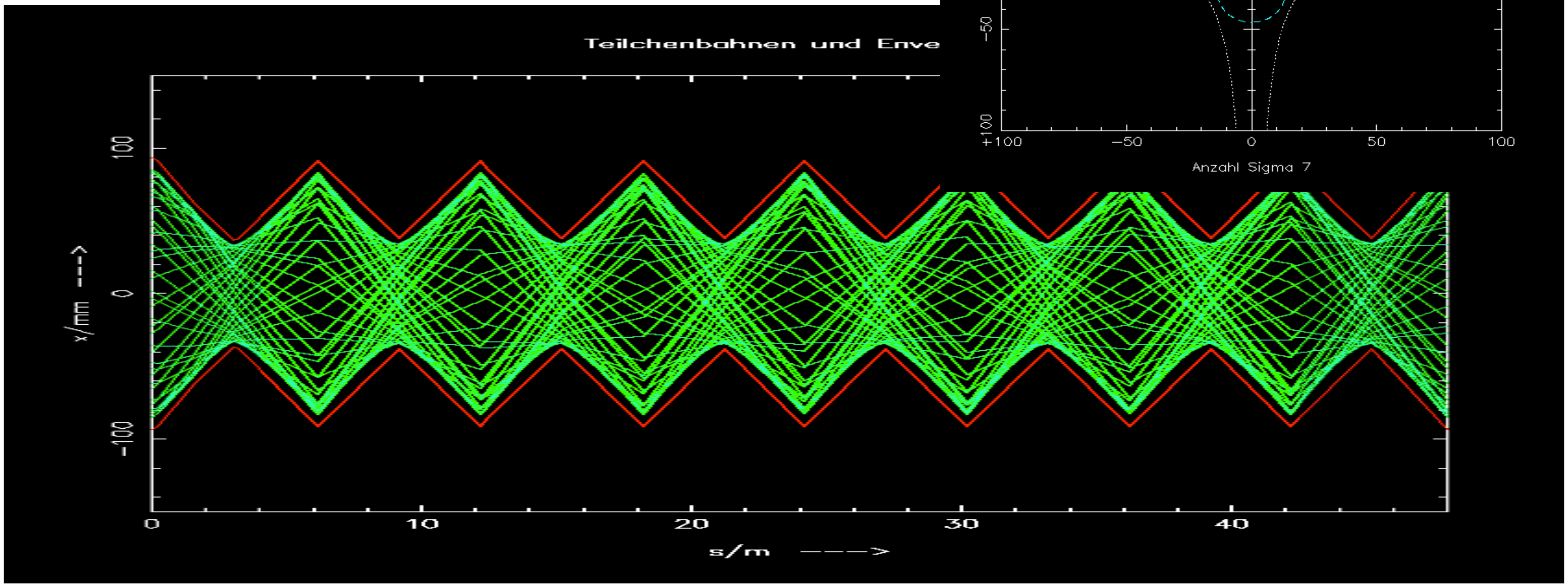
Introduction to Transverse Beam Optics

II.) Particle Trajectories, Beams & Bunch

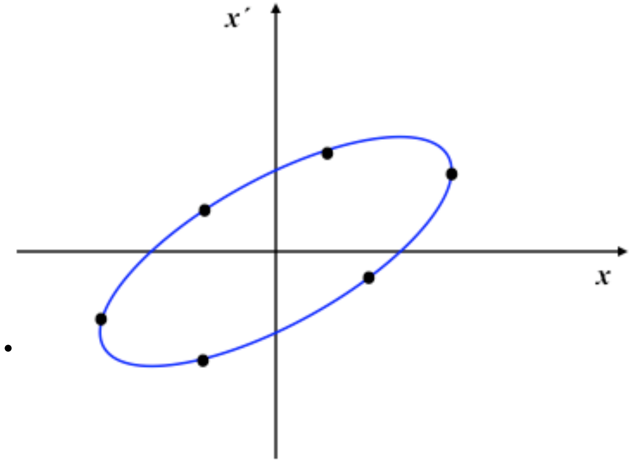
ε & β

... don't worry: it's still the "ideal world"





Phase Space Ellipse



particel trajectory: $x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s) + \phi\}$

max. Amplitude: $\boxed{\hat{x}(s) = \sqrt{\varepsilon\beta}}$ $\longrightarrow$ ***x´ at that position …***

… put $\hat{x}(s)$ ***into*** $\boldsymbol{\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\, x'^2(s)}$ ***and solve for x´***

$$\varepsilon = \gamma \cdot \varepsilon\beta + 2\alpha\sqrt{\varepsilon\beta} \cdot x' + \beta x'^2$$

$$\longrightarrow \quad \boxed{x' = -\alpha \cdot \sqrt{\varepsilon / \beta}}$$

* ***A high β-function means a large beam size and a small beam divergence.*** ***!***
… et vice versa !!!

* ***In the middle of a quadrupole β = maximum,***
α = zero

$$\left.\begin{array}{l} \beta = \text{maximum} \\ \alpha = \text{zero} \end{array}\right\} \quad x' = 0$$

… and the ellipse is flat

Phase Space Ellipse

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\, x'^2(s)$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

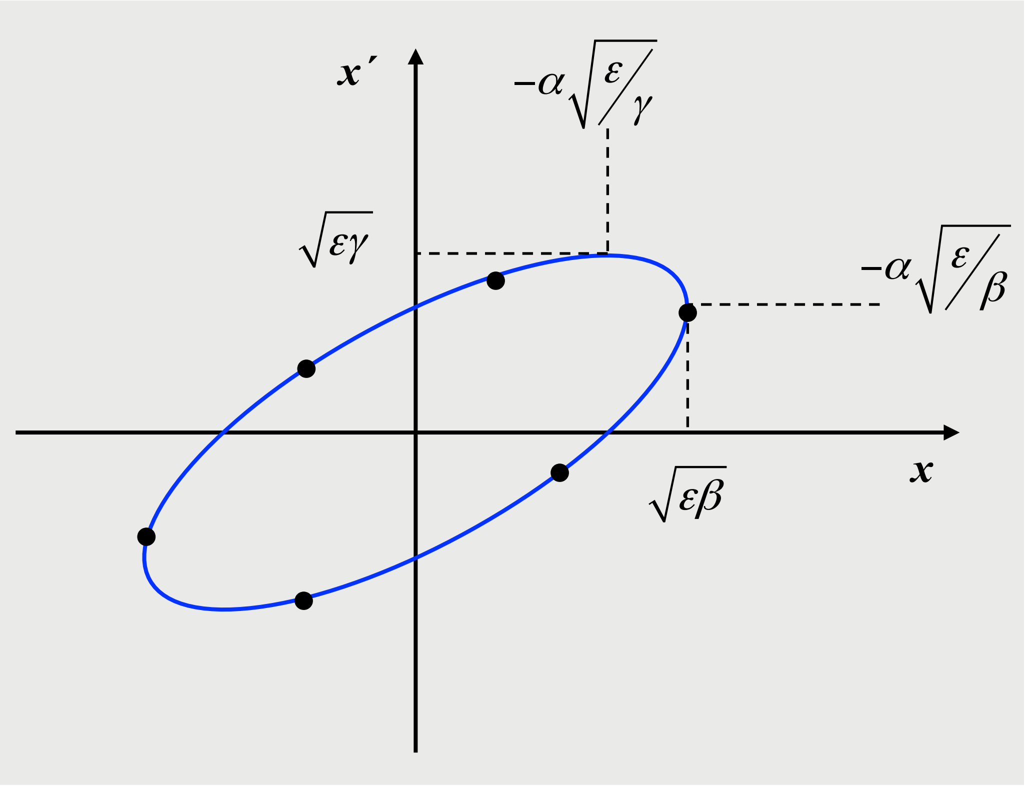
$$\longrightarrow \quad \varepsilon = \frac{x^2}{\beta} + \frac{\alpha^2 x^2}{\beta} + 2\alpha \cdot xx' + \beta \cdot x'^2$$

… solve for x′ $\quad x'_{1,2} = \dfrac{-\alpha \cdot x \pm \sqrt{\varepsilon\beta - x^2}}{\beta}$

… and determine $\hat{x}'$ ***via:*** $\quad \dfrac{dx'}{dx} = 0$

$$\longrightarrow \quad \boxed{\hat{x}' = \sqrt{\varepsilon\gamma}}$$

$$\longrightarrow \quad \boxed{\hat{x} = \pm\alpha\sqrt{\varepsilon/\gamma}}$$



shape and orientation of the phase space ellipse depend on the Twiss parameters β α γ

Remember:

Beam Emittance and Phase Space Ellipse:

equation of motion: $$x''(s) - k(s)\,x(s) = 0$$

general solution of Hills equation: $$x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\cos(\psi(s) + \varphi)$$

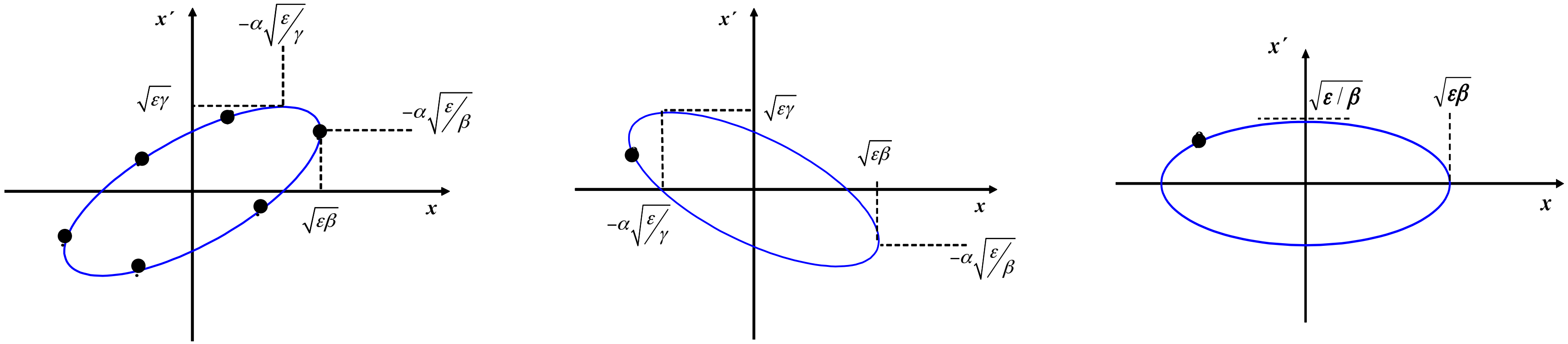
beam size: $$\sigma = \sqrt{\varepsilon\beta} \approx \text{"mm"}$$

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$

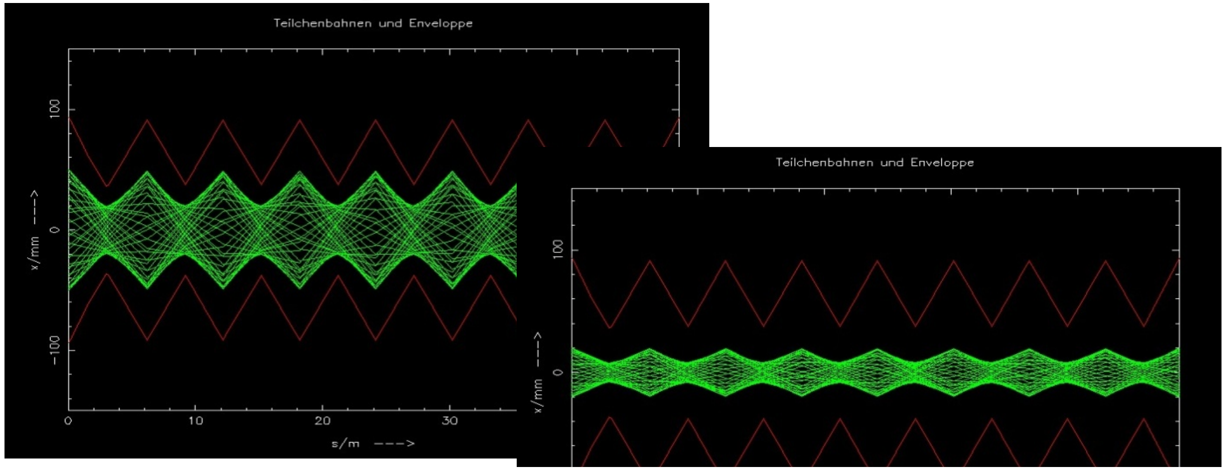
$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

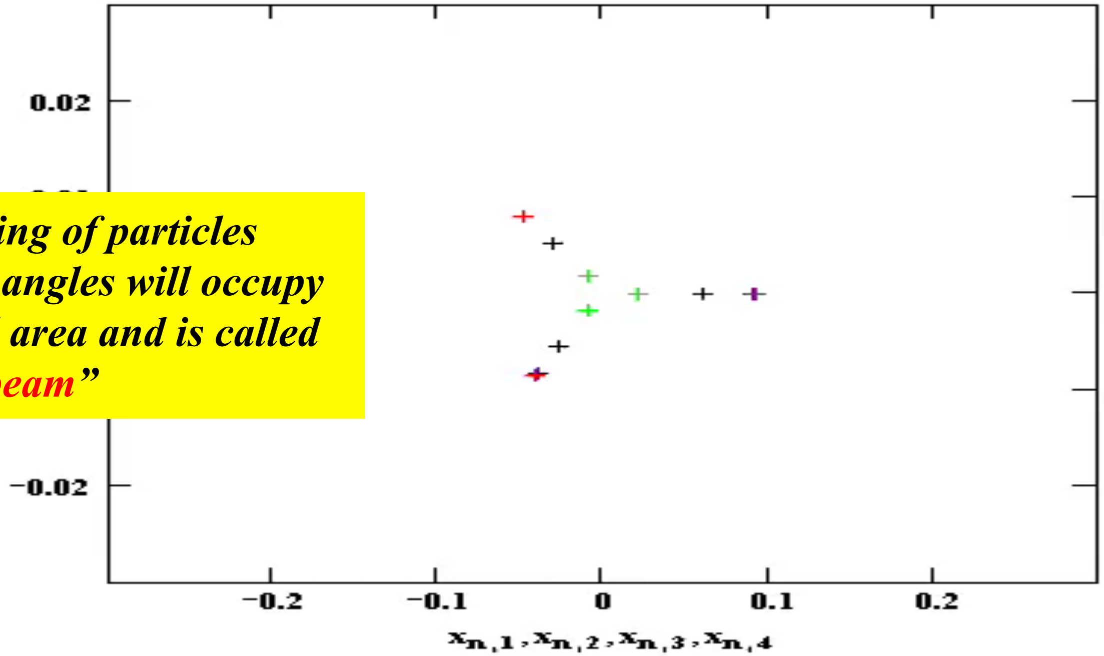
$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

** ε is a constant of the motion … it is independent of „s“*
** parametric representation of an ellipse in the x x ‘ space*
** shape and orientation of ellipse are given by α, β, γ*



Emittance of the Particle Ensemble:

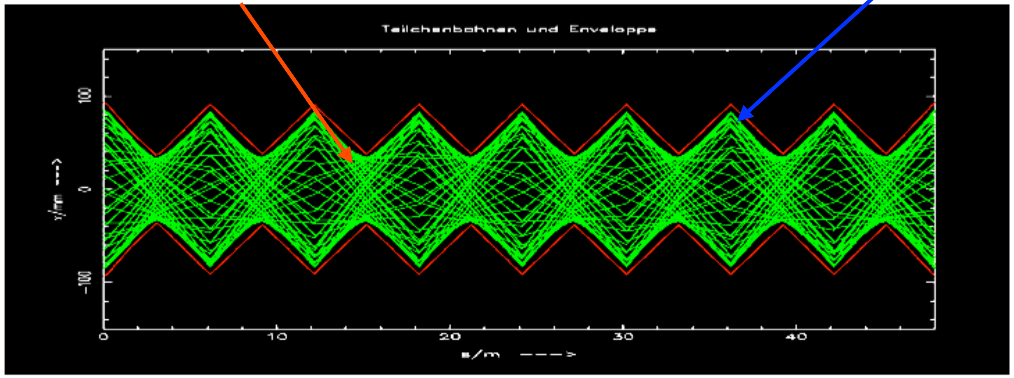




A particle ensemble consisting of particles with small amplitudes and angles will occupy in phase space only a small area and is called a "cold" or "high quality beam"

Emittance of the Particle Ensemble:

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cdot\cos(\Psi(s)+\phi) \qquad \hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



single particle trajectories, $N \approx 10^{11}$ per bunch

Gauß Particle Distribution:

$$\rho(x) = \frac{N\cdot e}{\sqrt{2\pi}\sigma_x}\cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}}$$

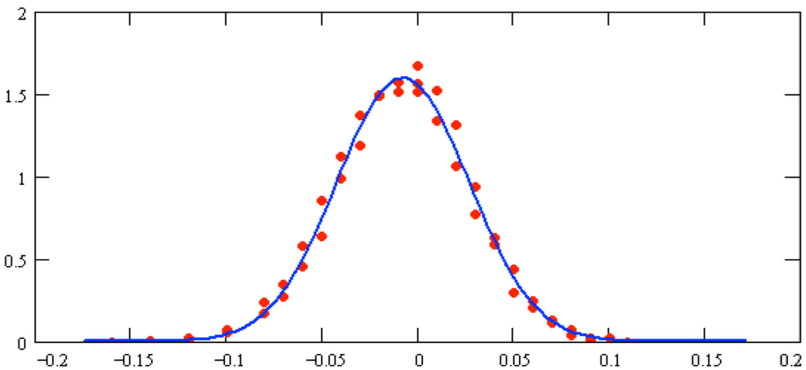
particle at distance 1 σ from centre
↔ 68.3 % of all beam particles

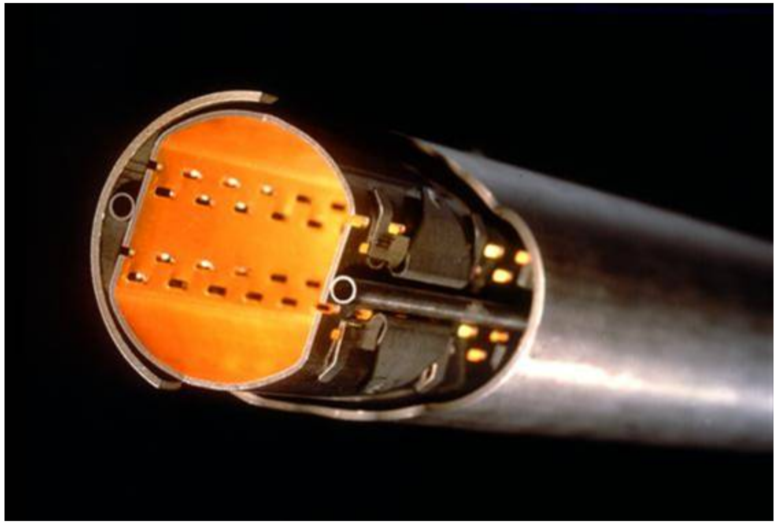
LHC:

$$\beta = 180\,m$$

$$\varepsilon = 5*10^{-10}\,m\,rad$$

$$\sigma = \sqrt{\varepsilon*\beta} = \sqrt{5*10^{-10}m*180m} = 0.3\,mm$$





aperture requirements: $r_0 = 12 * \sigma$

9.) Transfer Matrix M

… yes we had the topic already

general solution of Hill´s equation

$$\begin{cases} x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\} \\ x'(s) = \dfrac{-\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left[\alpha(s)\cos\{\psi(s)+\phi\} + \sin\{\psi(s)+\phi\}\right] \end{cases}$$

remember the trigonometrical gymnastics: sin(a + b) = … etc

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta_s}\left(\cos\psi_s\cos\phi - \sin\psi_s\sin\phi\right)$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta_s}}\left[\alpha_s\cos\psi_s\cos\phi - \alpha_s\sin\psi_s\sin\phi + \sin\psi_s\cos\phi + \cos\psi_s\sin\phi\right]$$

starting at point $s(0) = s_0$, where we put $\Psi(0) = 0$

$$\left.\begin{aligned} \cos\phi &= \frac{x_0}{\sqrt{\varepsilon\beta_0}} \quad , \\ \sin\phi &= -\frac{1}{\sqrt{\varepsilon}}\left(x'_0\sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right) \end{aligned}\right\}$$

inserting above …

$$x(s) = \sqrt{\frac{\beta_s}{\beta_0}}\left\{\cos\psi_s + \alpha_0 \sin\psi_s\right\} x_0 + \left\{\sqrt{\beta_s\beta_0}\sin\psi_s\right\} x_0'$$

$$x'(s) = \frac{1}{\sqrt{\beta_s\beta_0}}\left\{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s\right\} x_0 + \sqrt{\frac{\beta_0}{\beta_s}}\left\{\cos\psi_s - \alpha_s \sin\psi_s\right\} x_0'$$

which can be expressed ... for convenience ... in matrix form

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$

* ***we can calculate the single particle trajectories between two locations in the ring,***
if we know the α β γ at these positions.

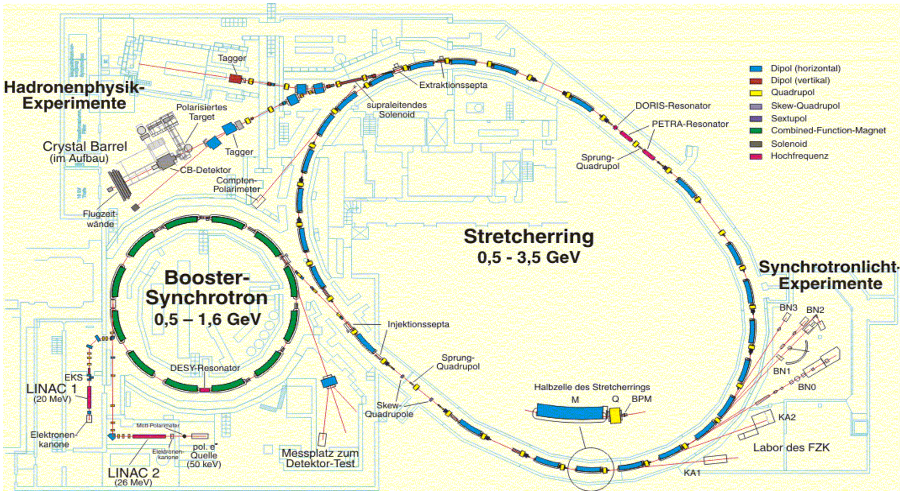
* ***and nothing but the α β γ at these positions.***

* ... ***!***

* *Äquivalenz der Matrizen*

10.) Periodic Lattices

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s \beta_0}\,\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$



ELSA Electron Storage Ring

„This rather formidable looking matrix simplifies considerably if we consider one complete revolution …"

$$\boldsymbol{M(s)} = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

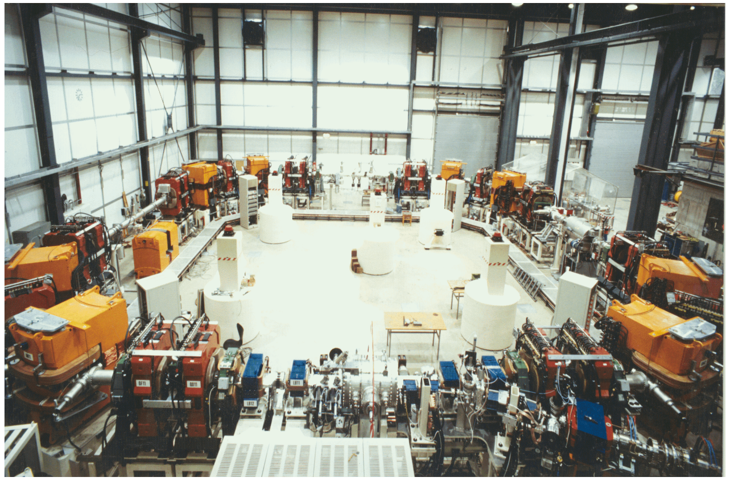
$$\psi_{turn} = \int_s^{s+L} \frac{ds}{\beta(s)}$$

ψ_{turn} = phase advance per period

***Tune:* Phase advance per turn in units of 2π**

$$\boldsymbol{Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}}$$

Stability Criterion:



Question: ***what will happen, if we do not make too many mistakes and your particle performs one complete turn ?***

Matrix for 1 turn:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_f = M_{turn} * \begin{pmatrix} x \\ x' \end{pmatrix}_i$$

$$M = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix} = \cos\psi \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{1}} + \sin\psi \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{\mathbf{J}}$$

Matrix for N turns:

$$M^N = (1 \cdot \cos\psi + J \cdot \sin\psi)^N = 1 \cdot \cos N\psi + J \cdot \sin N\psi$$

The motion for N turns remains bounded, if the elements of M^N remain bounded

$$\psi = real \quad \leftrightarrow \quad |\cos\psi| \le 1 \quad \leftrightarrow \quad Tr(M) \le 2$$

stability criterion …. proof for the disbelieving collegues !!

Matrix for 1 turn:

$$M = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix} = \cos\psi \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\boldsymbol{I}} + \sin\psi \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{\boldsymbol{J}}$$

Matrix for 2 turns:

$$\boldsymbol{M}^2 = (\boldsymbol{I}\cos\psi_1 + \boldsymbol{J}\sin\psi_1)(\boldsymbol{I}\cos\psi_2 + \boldsymbol{J}\sin\psi_2)$$

$$= \boldsymbol{I}^2 \cos\psi_1 \cos\psi_2 + \boldsymbol{IJ}\cos\psi_1 \sin\psi_2 + \boldsymbol{JI}\sin\psi_1 \cos\psi_2 + \boldsymbol{J}^2 \sin\psi_1 \sin\psi_2$$

now …

$$\boldsymbol{I}^2 = \boldsymbol{I}$$

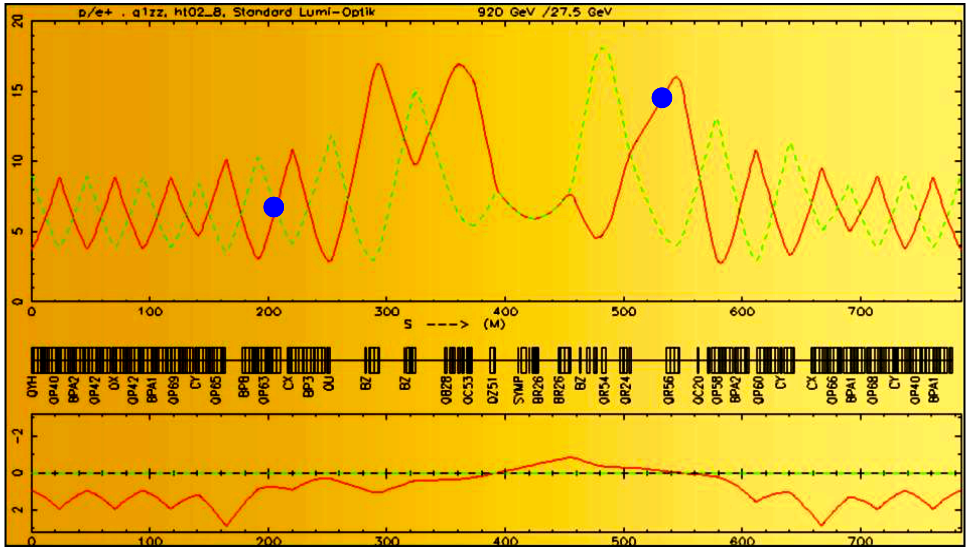
$$\left.\begin{aligned} \boldsymbol{I}\,\boldsymbol{J} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \\ \boldsymbol{J}\,\boldsymbol{I} &= \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \end{aligned}\right\} \quad \boldsymbol{I}\,\boldsymbol{J} = \boldsymbol{J}\,\boldsymbol{I}$$

$$\boldsymbol{J}^2 = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} * \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha^2 - \gamma\beta & \alpha\beta - \beta\alpha \\ -\gamma\alpha + \alpha\gamma & \alpha^2 - \gamma\beta \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\boldsymbol{I}$$

$$\boldsymbol{M}^2 = \boldsymbol{I}\cos(\psi_1 + \psi_2) + \boldsymbol{J}\sin(\psi_1 + \psi_2)$$

$$\boldsymbol{M}^2 = \boldsymbol{I}\cos(2\psi) + \boldsymbol{J}\sin(2\psi)$$

11.) Transformation of α, β, γ



Betafunction in a storage ring

consider two positions in the storage ring: s_0 , s

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

since ε = const (Liouville):

$$\varepsilon = \beta_s x'^2 + 2\alpha_s x x' + \gamma_s x^2$$

$$\varepsilon = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0^2$$

... remember W = CS´-SC´ = 1

$$\begin{pmatrix} x \\ x' \end{pmatrix}_0 = M^{-1} * \begin{pmatrix} x \\ x' \end{pmatrix}_s$$

$$M^{-1} = \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix}$$

$$\rightarrow \quad x_0 = S'x - Sx'$$

$$x_0' = -C'x + Cx'$$

… inserting into ε

$$\varepsilon = \beta_0 (Cx' - C'x)^2 + 2\alpha_0 (S'x - Sx')(Cx' - C'x) + \gamma_0 (S'x - Sx')^2$$

sort via x, x´and compare the coefficients to get ....

$$\beta(s) = C^2\beta_0 - 2SC\alpha_0 + S^2\gamma_0$$

$$\alpha(s) = -CC'\beta_0 + (SC' + S'C)\alpha_0 - SS'\gamma_0$$

$$\gamma(s) = C'^2\beta_0 - 2S'C'\alpha_0 + S'^2\gamma_0$$

in matrix notation:

$$\begin{pmatrix}\beta \\ \alpha \\ \gamma\end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + CS' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix}\beta_0 \\ \alpha_0 \\ \gamma_0\end{pmatrix}$$

!

1.) this expression is important

2.) given the twiss parameters α, β, γ at any point in the lattice we can transform them and calculate their values at any other point in the ring.

3.) the transfer matrix is given by the focusing properties of the lattice elements, the elements of M are just those that we used to calculate single particle trajectories.

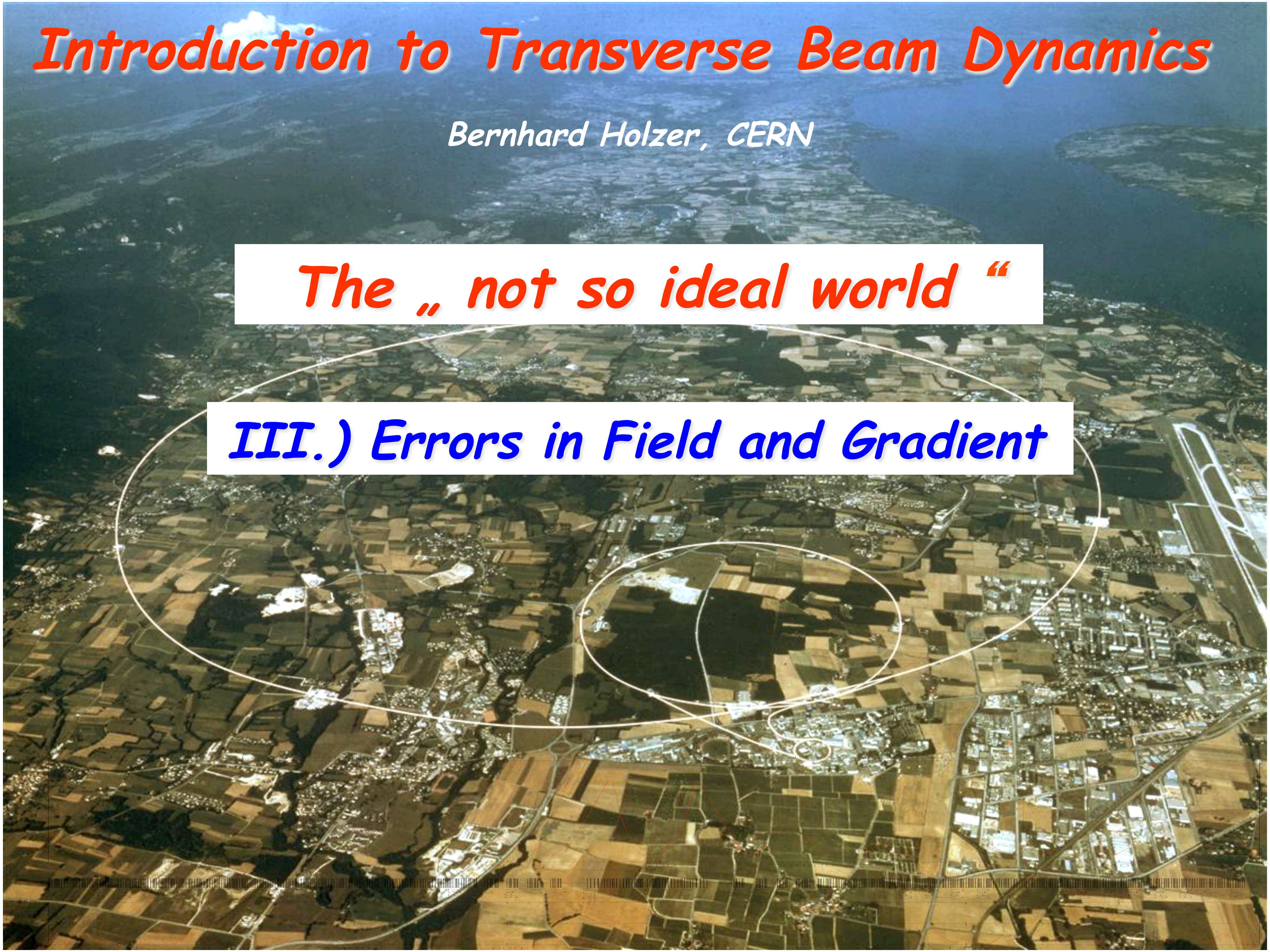
4.) go back to point 1.)

Introduction to Transverse Beam Dynamics

Bernhard Holzer, CERN

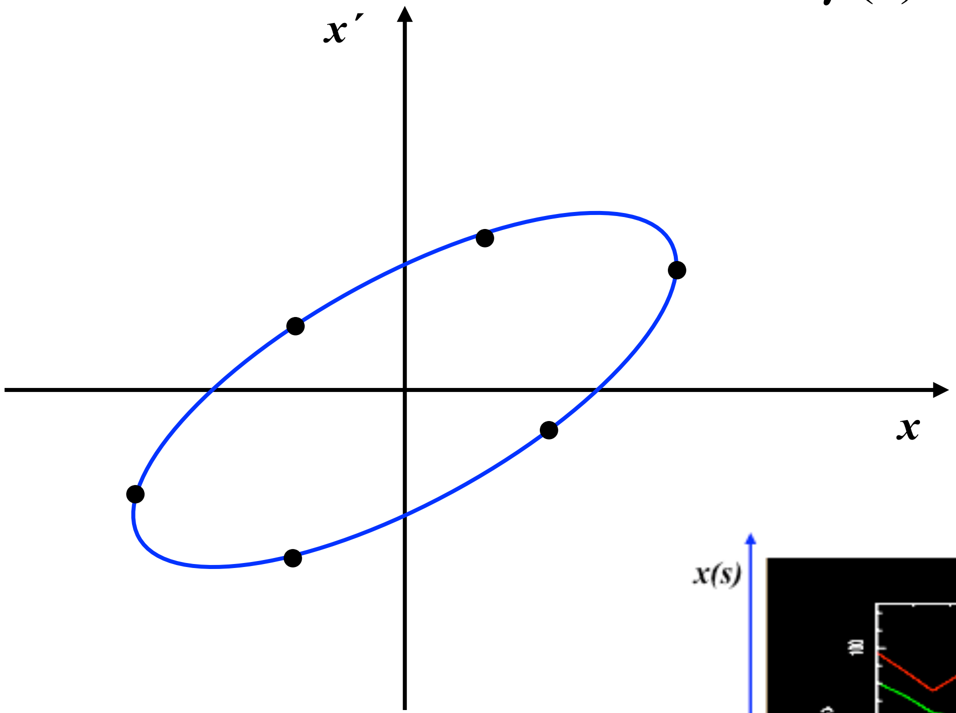
The „ not so ideal world “

III.) Errors in Field and Gradient



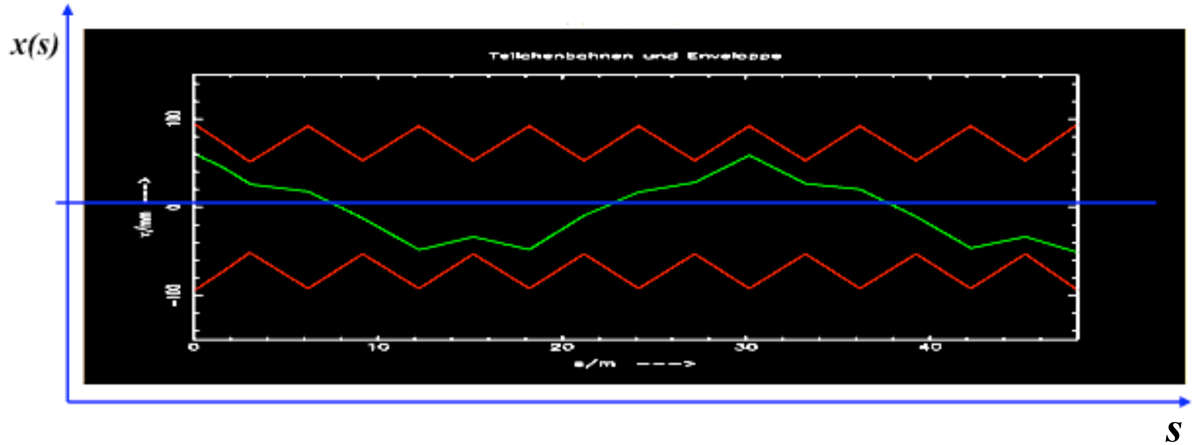
Remember: Beam Emittance and Phase Space Ellipse

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\, x'^2(s)$$



Liouville: ***in reasonable storage rings***
area in phase space is constant.

$A = \pi * \varepsilon = const$

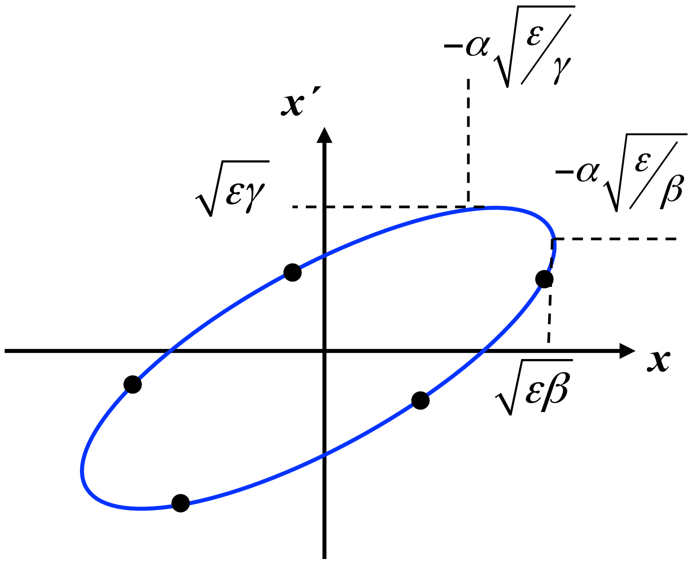


ε beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter,
cannot be changed by the foc. properties.

Scientifiquely spoken: area covered in transverse x, x´ phase space … and it is constant !!!

13.) Liouville during Acceleration

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$



***Beam Emittance** corresponds to the area covered in the*
x, x´ Phase Space Ellipse

***Liouville:** Area in phase space is constant.*

But so sorry ... $\varepsilon \neq$ const !

Classical Mechanics:

phase space = diagram of the two canonical variables
position** & **momentum
x p_x

According to Hamiltonian mechanics:

phase space diagram relates the variables q and p

Liouvilles Theorem:

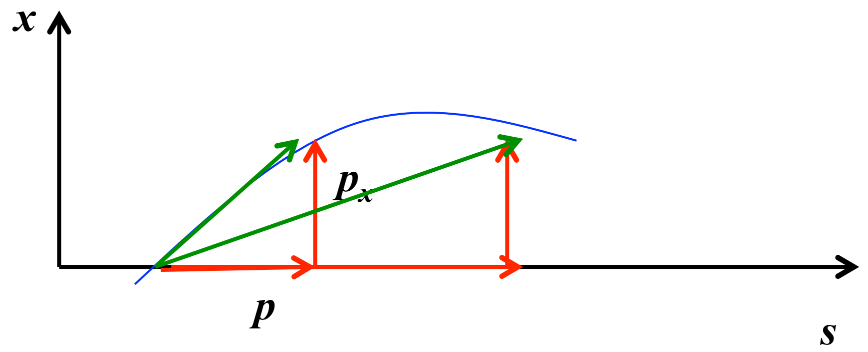
$$\int p\, dq = const$$

$$\int p_x\, dx = const$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad ; \quad \beta_x = \frac{\dot{x}}{c}$$

for convenience (i.e. because we are lazy bones) we use in accelerator theory:

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{\beta_x}{\beta} = \frac{p_x}{p}$$



$$\underbrace{\int x'\, dx}_{\varepsilon} = \frac{\int p_x\, dx}{p} \propto \frac{const}{m_0 c \cdot \gamma\beta}$$

$$\boxed{\Rightarrow \quad \varepsilon = \int x' dx \propto \frac{1}{\beta\gamma}}$$

the beam emittance shrinks during acceleration $\varepsilon \sim 1/\gamma$

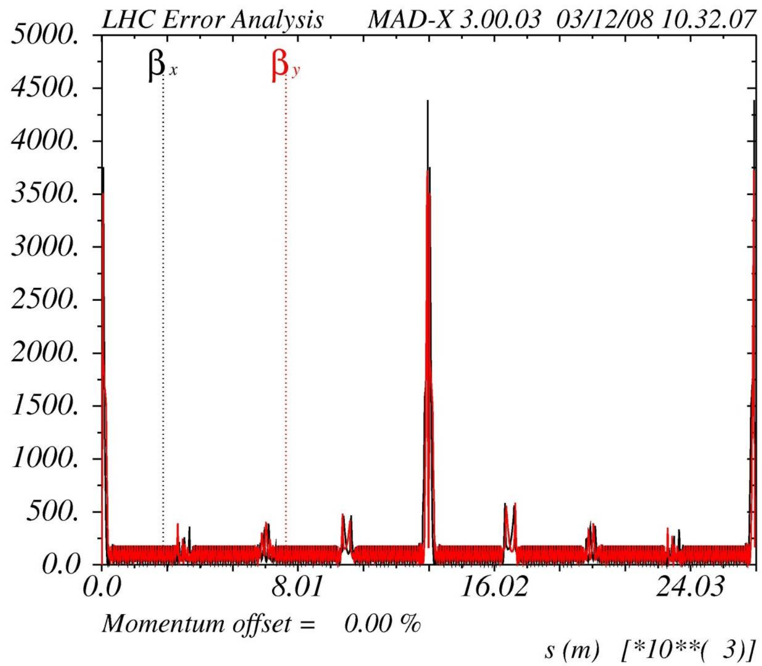
Nota bene:

1.) A proton machine … or an electron linac … needs the highest aperture at injection energy !!!
as soon as we start to accelerate the beam size shrinks as $\gamma^{-1/2}$ in both planes.

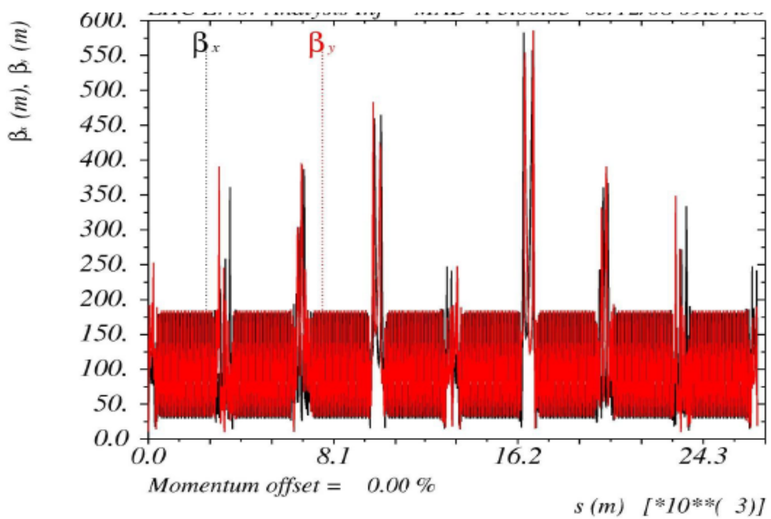
$$\sigma = \sqrt{\varepsilon\beta}$$

2.) At lowest energy the machine will have the major aperture problems,
→ here we have to minimise $\hat{\beta}$

3.) we need different beam optics a
A Mini Beta concept will only be



LHC mini beta optics at 7000 GeV



LHC injection optics at 450 GeV

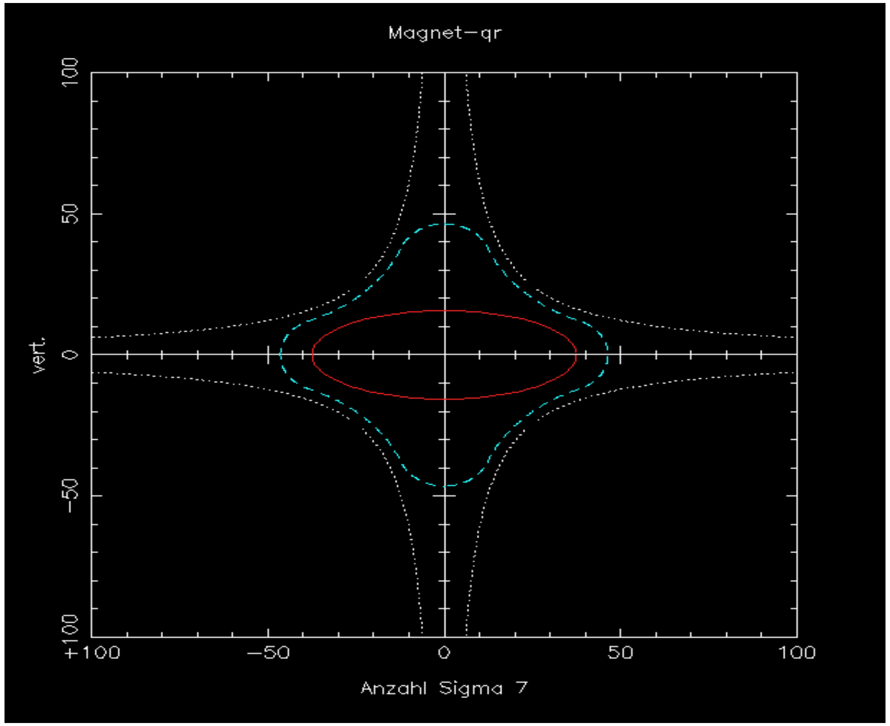
Example: HERA proton ring

injection energy: 40 GeV $\gamma = 43$

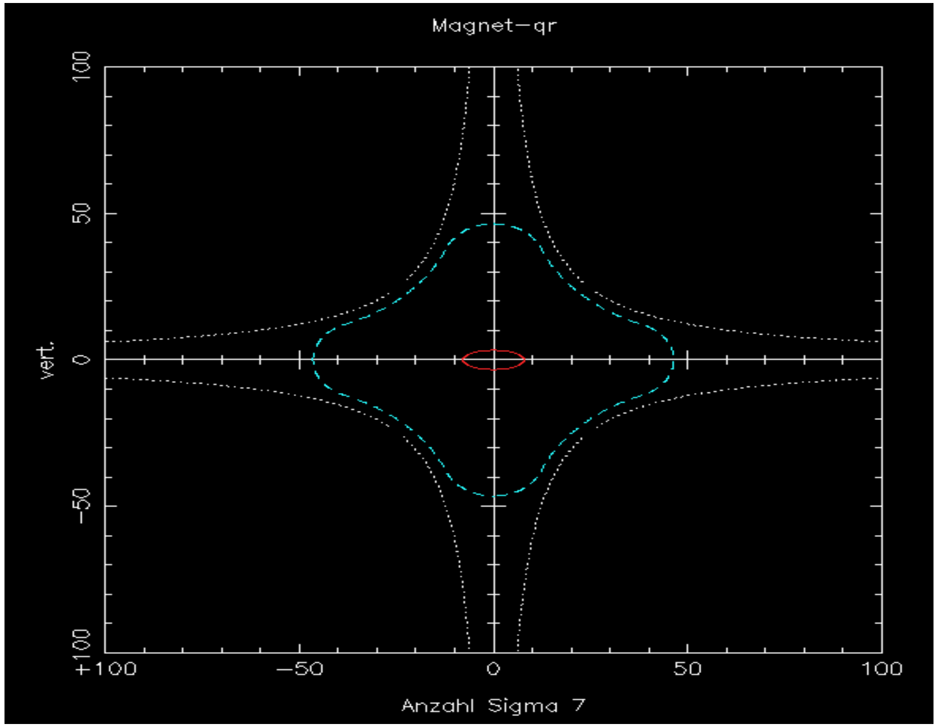
flat top energy: 920 GeV $\gamma = 980$

emittance $\varepsilon\ (40GeV) = 1.2 * 10^{-7}$

$\varepsilon\ (920GeV) = 5.1 * 10^{-9}$



7 σ beam envelope at E = 40 GeV



… and at E = 920 GeV

The „ not so ideal world "

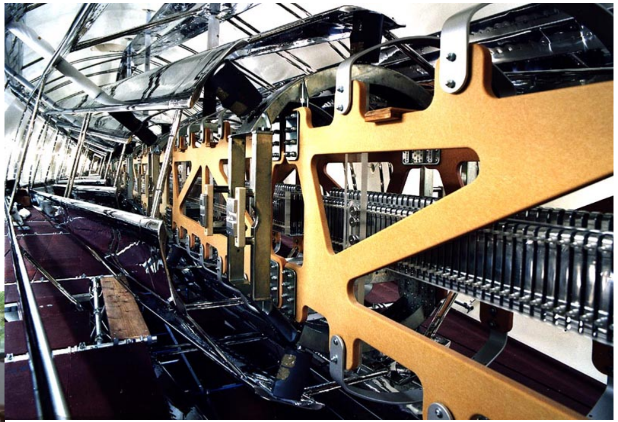
14.) The „ $\Delta p / p \neq 0$" Problem

ideal accelerator: *all particles will see the same accelerating voltage.*

$\rightarrow \Delta p / p = 0$

„nearly ideal" accelerator: *Cockroft Walton or van de Graaf*

$\Delta p / p \approx 10^{-5}$

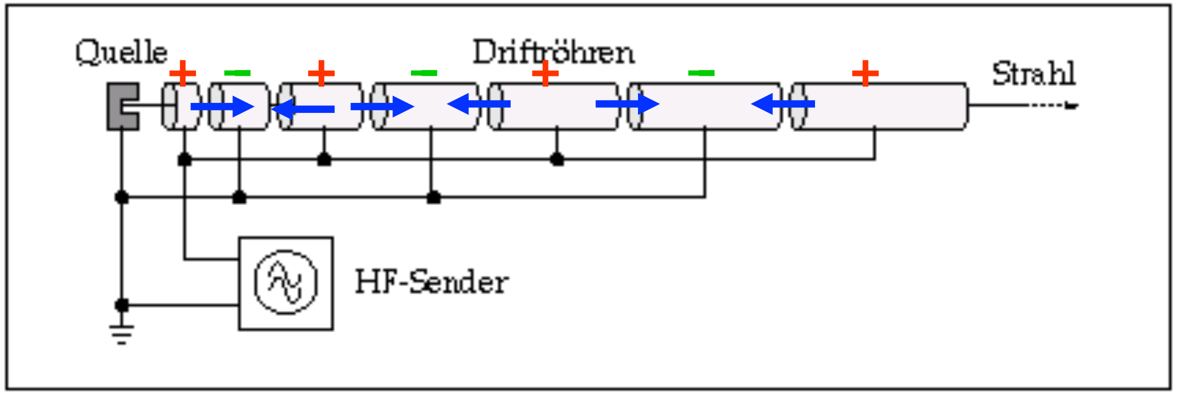


Vivitron, Straßbourg, inner structure of the acc. section

MP Tandem van de Graaf Accelerator at MPI for Nucl. Phys. Heidelberg

RF Acceleration

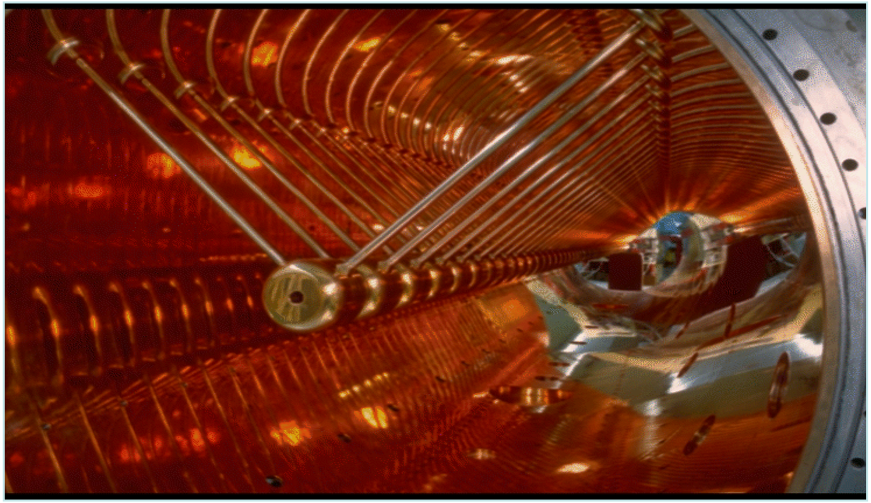
1928, Wideroe



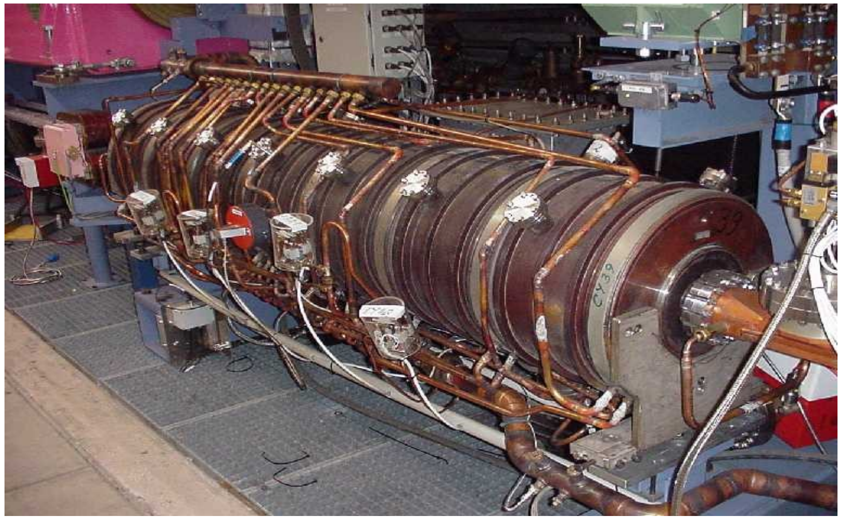
Energy Gain per „Gap":

$$\boxed{W = q\,U_0 \sin \omega_{RF} t}$$

drift tube structure at a proton linac (GSI Unilac)



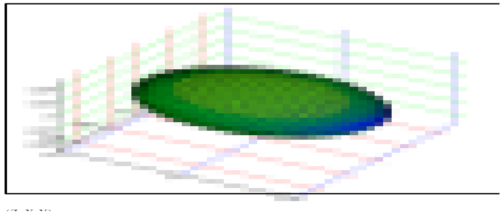
500 MHz cavities in an electron storage ring



* *RF Acceleration:* ***multiple application of the same acceleration voltage; brillant idea to gain higher energies***

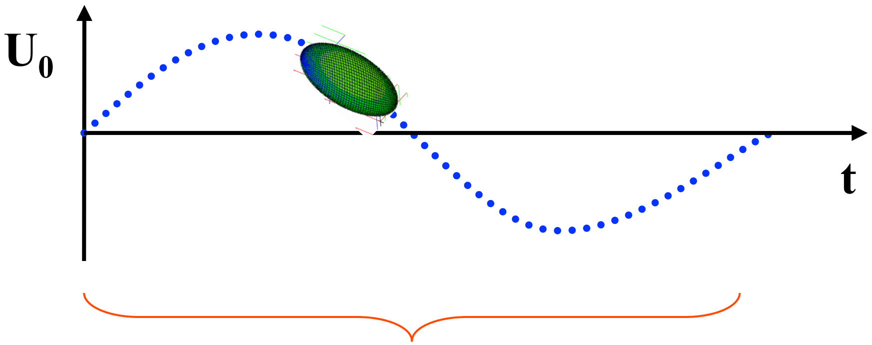
Problem: panta rhei !!!

(Heraklit: 540-480 v. Chr.)



Bunch length of Electrons ≈ 1cm

Example: HERA RF:



$\lambda = 60\ cm$

$$\left.\begin{array}{l} \nu = 500 MHz \\ c = \lambda \nu \end{array}\right\} \quad \lambda = 60\ cm$$

$\sin(90^o) = 1$

$\sin(84^o) = 0.994$

$$\frac{\Delta U}{U} = 6.0\ \ 10^{-3}$$

typical momentum spread of an electron bunch:

$$\frac{\Delta p}{p} \approx 1.0\ \ 10^{-3}$$

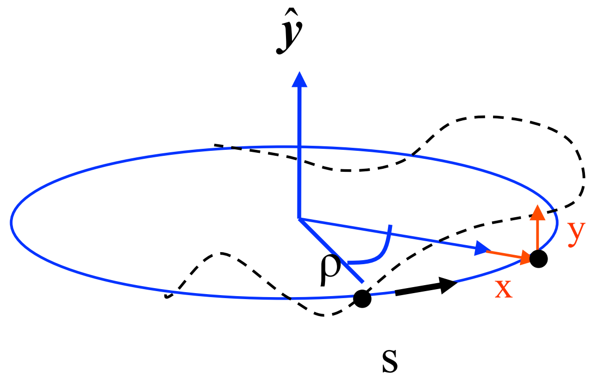
Dispersive and Chromatic Effects: $\Delta p/p \neq 0$

Are there any Problems ???
Sure there are !!!

font colors due to pedagogical reasons

15.) Dispersion: trajectories for $\Delta p / p \neq 0$

Question: ***do you remember last session, page 12 ? … sure you do***



Force ***acting on the particle***

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v$$

remember: ***$x \approx mm$, $\rho \approx m$ …*** → ***develop for small x***

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}\left(1-\frac{x}{\rho}\right) = eB_y v$$

consider only linear fields, and change independent variable: $t \rightarrow s$

$$B_y = B_0 + x\frac{\partial B_y}{\partial x}$$

$$x'' - \frac{1}{\rho}\left(1-\frac{x}{\rho}\right) = \frac{e\,B_0}{mv} + \frac{e\,x\,g}{mv}$$

$p = p_0 + \Delta p$

… but now take a small momentum error into account !!!

Dispersion:

develop for small momentum error $\Delta p << p_0 \Rightarrow \frac{1}{p_0 + \Delta p} \approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2}$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} \approx \underbrace{\frac{e\ B_0}{p_0}}_{-\frac{1}{\rho}} - \frac{\Delta p}{p_0^2} eB_0 + \underbrace{\frac{xeg}{p_0}}_{k*x} - \underbrace{xeg\frac{\Delta p}{p_0^2}}_{\approx 0}$$

$$x'' + \frac{x}{\rho^2} \approx \frac{\Delta p}{p_0} * \underbrace{\frac{(-eB_0)}{p_0}}_{\frac{1}{\rho}} + k * x = \frac{\Delta p}{p_0} * \frac{1}{\rho} + k * x$$

$$x'' + \frac{x}{\rho^2} - kx = \frac{\Delta p}{p_0}\frac{1}{\rho} \longrightarrow \boxed{x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p_0}\frac{1}{\rho}}$$

Momentum spread of the beam adds a term on the r.h.s. of the equation of motion.
→ *inhomogeneous differential equation.*

Dispersion:

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \dfrac{1}{\rho} \cdot \dfrac{\Delta p}{p} \end{cases}$$

Normalise with respect to Δp/p:

$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Dispersion function D(s)

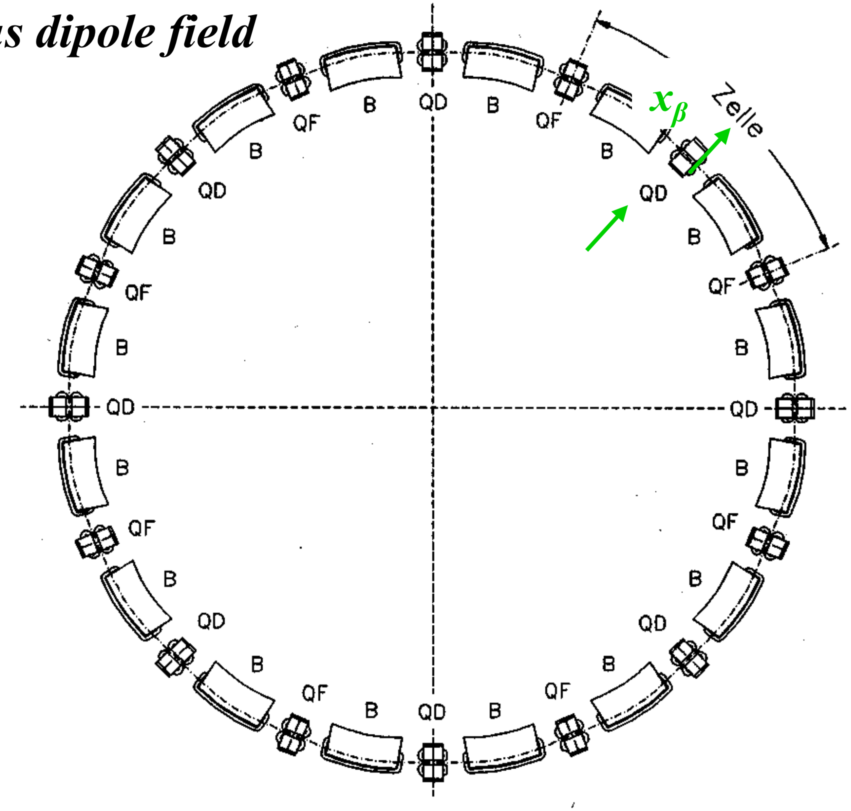
*** is that special orbit, an ideal particle would have for Δp/p = 1**

*** the orbit of any particle is the sum of the well known x_β and the dispersion**

*** as D(s) is just another orbit it will be subject to the focusing properties of the lattice**

Dispersion

Example: homogeneous dipole field



it for Δp/p > 0

$$: D(s)\cdot\frac{\Delta p}{p}$$

Matrix formalism:

$$x(s) = x_\beta(s) + D(s)\cdot\frac{\Delta p}{p}$$

$$x(s) = C(s)\cdot x_0 + S(s)\cdot x_0' + D(s)\cdot\frac{\Delta p}{p}$$

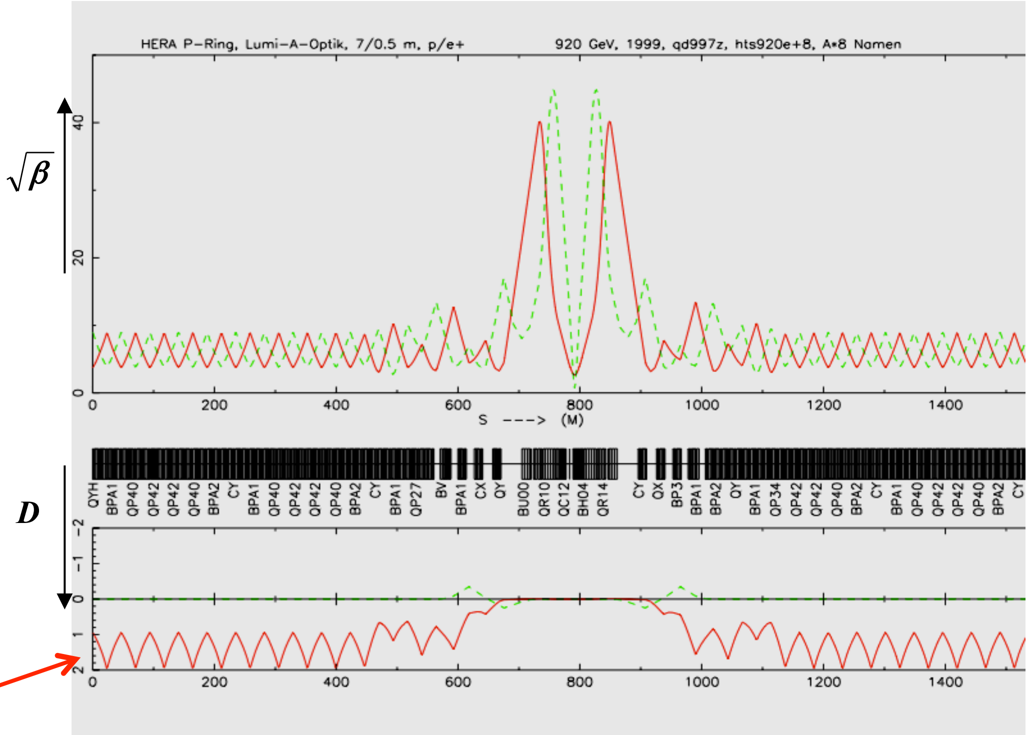
$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p}\begin{pmatrix} D \\ D' \end{pmatrix}$$

$$C = \cos(\sqrt{|k|}s) \qquad S = \frac{1}{\sqrt{|k|}}\sin(\sqrt{|k|}s)$$

$$C' = \frac{dC}{ds} \qquad S' = \frac{dS}{ds}$$

or expressed as 3x3 matrix

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$



Example

$$x_\beta = 1 \ldots 2\, mm$$

$$D(s) \approx 1 \ldots 2\, m$$

$$\Delta p/p \approx 1 \cdot 10^{-3}$$

Amplitude of Orbit oscillation contribution due to Dispersion ≈ beam size

→ Dispersion must vanish at the collision point

!

Calculate D, D´: ... takes a couple of sunny Sunday evenings !

$$D(s) = S(s) \int_{s_0}^{s_1} \frac{1}{\rho} C(\tilde{s})\, d\tilde{s} - C(s) \int_{s_0}^{s_1} \frac{1}{\rho} S(\tilde{s})\, d\tilde{s}$$

(proof see appendix)

Example: Drift

$$M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

$$D(s) = S(s) \underbrace{\int_{s_0}^{s_1} \frac{1}{\rho} C(\tilde{s}) d\tilde{s}}_{=0} - C(s) \underbrace{\int_{s_0}^{s_1} \frac{1}{\rho} S(\tilde{s}) d\tilde{s}}_{=0}$$

Example: Dipole

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s \\ -\sqrt{|K|} \sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

$$\boldsymbol{K} = \frac{1}{\rho^2} \cancel{-k}$$

$$\boldsymbol{s} = \boldsymbol{l_B}$$

$$M_{Dipole} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho \sin\frac{l}{\rho} \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} \end{pmatrix} \quad \rightarrow$$

$$D(s) = \rho \cdot (1 - \cos\frac{l}{\rho})$$

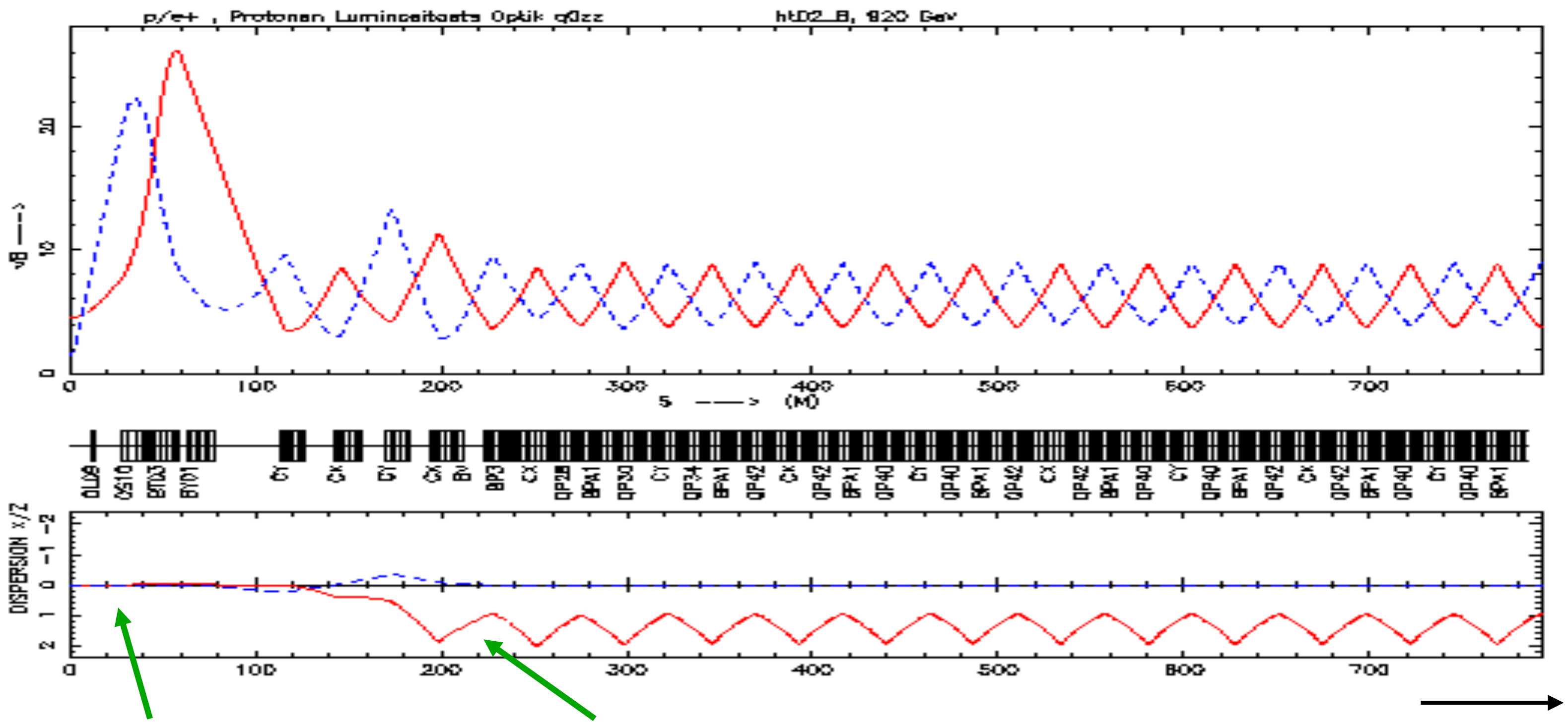
$$D'(s) = \sin\frac{l}{\rho}$$

Example: Dispersion, calculated by an optics code for a real machine

$$x_D = D(s)\,\frac{\Delta p}{p}$$

** D(s) is created by the dipole magnets*

… and afterwards focused by the quadrupole fields

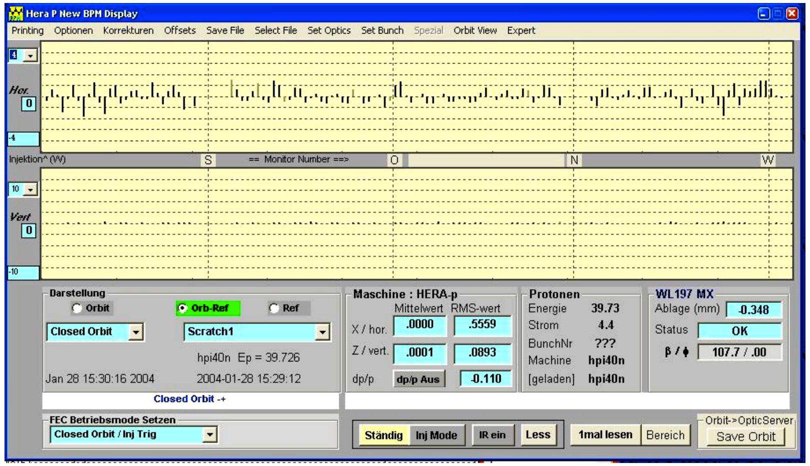


$\longrightarrow s$

Mini Beta Section,
→ no dipoles !!!

D(s) ≈ 1 … 2 m

Dispersion is visible



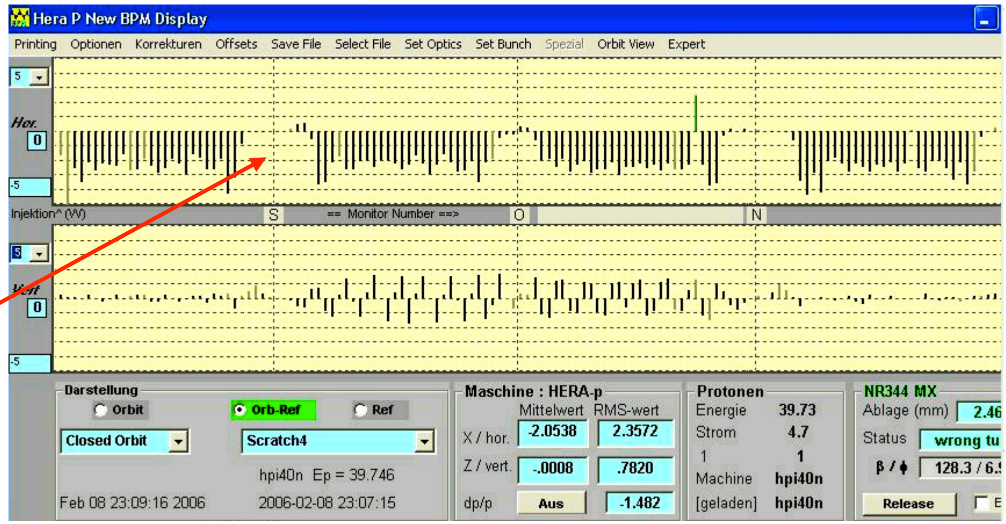
HERA Standard Orbit

HERA Dispersion Orbit

dedicated energy change of the stored beam

→ closed orbit is moved to a dispersions trajectory

$$x_D = D(s) * \frac{\Delta p}{p}$$



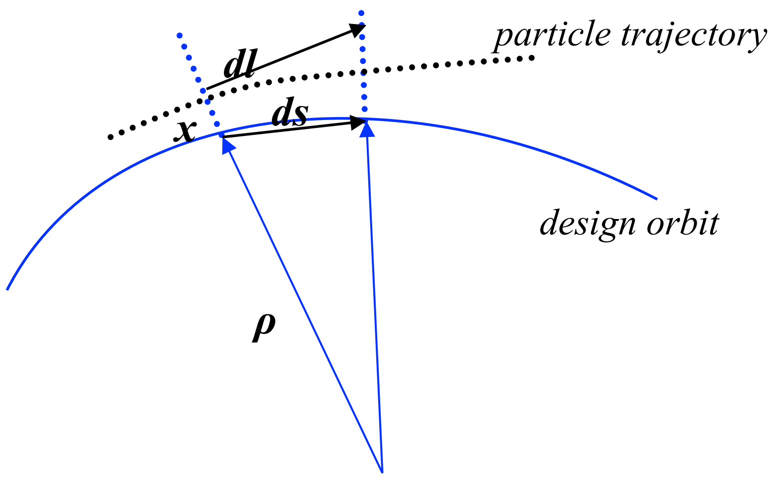
Attention: at the Interaction Points we require D=D′= 0

16.) Momentum Compaction Factor: α_p

particle with a displacement x to the design orbit
→ path length dl ...

$$\frac{dl}{ds} = \frac{\rho + x}{\rho}$$

$$\rightarrow \; dl = \left(1 + \frac{x}{\rho(s)}\right) ds$$



circumference of an off-energy closed orbit

$$l_{\Delta E} = \oint dl = \oint \left(1 + \frac{x_{\Delta E}}{\rho(s)}\right) ds$$

remember:

$$x_{\Delta E}(s) = D(s)\frac{\Delta p}{p}$$

$$\delta l_{\Delta E} = \frac{\Delta p}{p} \oint \left(\frac{D(s)}{\rho(s)}\right) ds$$

**** The lengthening of the orbit for off-momentum particles is given by the dispersion function and the bending radius.***

Definition:

$$\frac{\delta l_\varepsilon}{L} = \alpha_p \frac{\Delta p}{p}$$

$$\boxed{\rightarrow \alpha_p = \frac{1}{L}\oint \left(\frac{D(s)}{\rho(s)}\right) ds}$$

For first estimates assume:

$$\frac{1}{\rho} = const.$$

$$\int_{dipoles} D(s)\, ds \approx l_{\Sigma(dipoles)} \cdot \langle D \rangle_{dipole}$$

$$\alpha_p = \frac{1}{L}\, l_{\Sigma(dipoles)} \cdot \langle D \rangle \frac{1}{\rho} = \frac{1}{L}\, 2\pi\rho \cdot \langle D \rangle \frac{1}{\rho} \quad \rightarrow \quad \boxed{\alpha_p \approx \frac{2\pi}{L} \langle D \rangle \approx \frac{\langle D \rangle}{R}}$$

Assume: $v \approx c$

$$\boxed{\rightarrow \quad \frac{\delta T}{T} = \frac{\delta l_\varepsilon}{L} = \alpha_p \frac{\Delta p}{p}}$$

α_p combines via the dispersion function the momentum spread with the longitudinal motion of the particle.

Resume´:

transfer matrix in Twiss form

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$

... and for the periodic case

$$\boldsymbol{M}(s) = \begin{pmatrix} \cos\boldsymbol{\psi}_{turn} + \boldsymbol{\alpha}_s \sin\boldsymbol{\psi}_{turn} & \boldsymbol{\beta}_s \sin\boldsymbol{\psi}_{turn} \\ -\boldsymbol{\gamma}_s \sin\boldsymbol{\psi}_{turn} & \cos\boldsymbol{\psi}_{turn} - \boldsymbol{\alpha}_s \sin\boldsymbol{\psi}_{turn} \end{pmatrix}$$

beam emittance during acceleration

$$\varepsilon \propto \frac{1}{\beta\gamma}$$

dispersion

$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Appendix:

Dispersion: Solution of the inhomogenious equation of motion

Ansatz:

$$D(s) = S(s)\int_{s_0}^{s_1}\frac{1}{\rho}C(\tilde{s})d\tilde{s} - C(s)\int_{s_0}^{s_1}\frac{1}{\rho}S(\tilde{s})d\tilde{s}$$

$$D'(s) = S'\int_{s_0}^{s_1}\frac{1}{\rho}C\,d\tilde{s} + \cancel{S\frac{1}{\rho}C} - C'\int_{s_0}^{s_1}\frac{1}{\rho}S\,d\,\tilde{s} - \cancel{S\frac{1}{\rho}C}$$

$$D'(s) = S'\int\frac{C}{\rho}d\tilde{s} - C'\int\frac{S}{\rho}d\,\tilde{s}$$

$$D''(s) = S''*\int\frac{C}{\rho}d\tilde{s} + S'\frac{C}{\rho} - C''*\int\frac{S}{\rho}d\tilde{s} - C'\frac{S}{\rho}$$

$$= S''*\int\frac{C}{\rho}d\tilde{s} - C''*\int\frac{S}{\rho}d\tilde{s} + \frac{1}{\rho}\underbrace{\left(CS' - S\,C'\right)}_{=\,det\,M\,=\,1}$$

---

remember: for Cs) and S(s) to be independent solutions the Wronski determinant has to meet the condition

$$W = \begin{vmatrix} C & S \\ C' & S' \end{vmatrix} \neq 0$$

and as it is independent of the variable „s"

$$\frac{dW}{ds} = \frac{d}{ds}(CS' - SC') = CS'' - SC'' = -K(CS - SC) = 0$$

we get for the initital conditions that we had chosen ...

$$\left.\begin{matrix} C_0 = 1, & C_0' = 0 \\ S_0 = 0, & S_0' = 1 \end{matrix}\right\} \quad W = \begin{vmatrix} C & S \\ C' & S' \end{vmatrix} = 1$$

$$D'' = S'' * \int \frac{C}{\rho}\, d\tilde{s} - C'' * \int \frac{S}{\rho}\, d\tilde{s} + \frac{1}{\rho}$$

remember: S & C are solutions of the homog. equation of motion:

$$S'' + K * S = 0$$

$$C'' + K * C = 0$$

$$D'' = -K * S * \int \frac{C}{\rho}\, d\tilde{s} + K * C * \int \frac{S}{\rho}\, d\tilde{s} + \frac{1}{\rho}$$

$$D'' = -K * \underbrace{\left\{ S \int \frac{C}{\rho}\, d\tilde{s} + C \int \frac{S}{\rho}\, d\tilde{s} \right\}}_{=D(s)} + \frac{1}{\rho}$$

$$D'' = -K * D + \frac{1}{\rho}$$

... or

$$D'' + K * D = \frac{1}{\rho}$$

qed

1.) Dipole Errors / Quadrupole Misalignment

The Design Orbit is defined by the strength and arrangement of the dipoles.
Under the influence of dipole imperfections and quadrupole misalignments we obtain a "Closed Orbit" which is hopefully still closed and not too far away from the design.

Dipole field error:

$$\theta = \frac{dl}{\rho} = \frac{\int B\, dl}{B\rho}$$

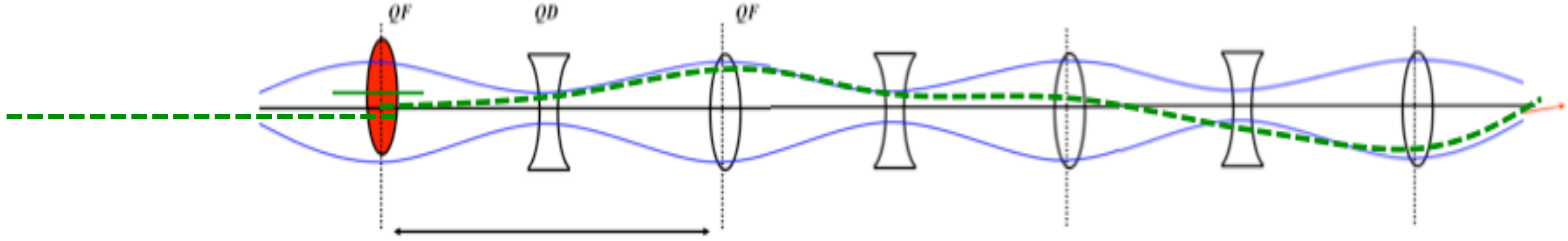
Quadrupole offset:

$$g = \frac{dB}{dx} \rightarrow \quad \Delta x \cdot g \quad = \Delta x \frac{dB}{dx} = \Delta B$$

misaligned quadrupoles (or orbit offsets in quadrupoles) create dipole effects that lead to a distorted "closed orbit"

normalised to p/e:

$$\Delta x \cdot k = \Delta x \cdot \frac{g}{B\rho} = \frac{1}{\rho} \qquad \begin{pmatrix} x \\ x' \end{pmatrix}_i = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ x' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{l}{\rho} \end{pmatrix}$$



In a Linac – starting with a perfect orbit – the misaligned quadrupole creates an oscillation that is transformed from now on downstream via

$$\begin{pmatrix} x \\ x' \end{pmatrix}_f = M \begin{pmatrix} x \\ x' \end{pmatrix}_i$$

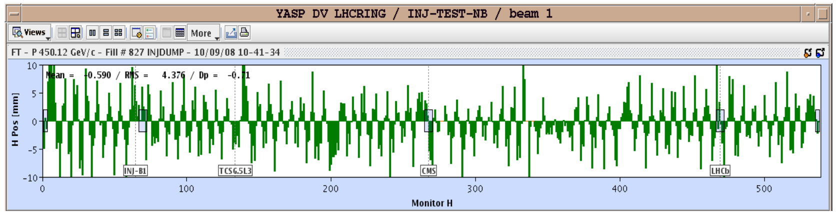
... and in a circular machine ??

we have to obey the periodicity condition.
The orbit is closed !! ... even under the influence of a orbit kick.

Calculation of the new closed orbit:
the general orbit will always be a solution of Hill, so ...

$$x(s) = a \cdot \sqrt{\beta} \cos(\psi(s) + \varphi$$

We set at the location of the error s=0, Ψ(s)=0
and require as 1st boundary condition:
periodic amplitude



$$x(s+L) = x(s)$$

$$\cancel{a} \cdot \cancel{\sqrt{\beta(s+L)}} \cdot \cos(\psi(s) + 2\pi Q - \varphi) = \cancel{a} \cdot \cancel{\sqrt{\beta(s)}} \cdot \cos(\psi(s) - \varphi)$$

$$\cos(2\pi Q - \varphi) = \cos(-\varphi) = \cos(\varphi)$$

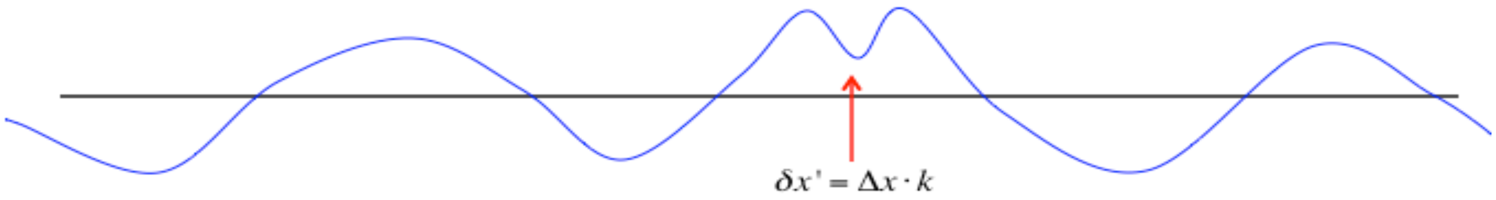
$$\boxed{\rightarrow \varphi = \pi Q}$$

$$\beta(s+L) = \beta(s)$$
$$\psi(s=0) = 0$$
$$\psi(s+L) = 2\pi Q$$

Misalignment error in a circular machine

2nd boundary condition: $x'(s+L) + \delta x' = x'(s)$

we have to close the orbit



$$x(s) = a \cdot \sqrt{\beta}\cos(\psi(s) - \varphi)$$

$$x'(s) = a \cdot \sqrt{\beta}\left(-\sin(\psi(s)-\varphi)\right)\psi' + \frac{\beta'(s)}{2\sqrt{\beta}} a \cdot \cos(\psi(s)-\varphi)$$

$$x'(s) = -a \cdot \frac{1}{\sqrt{\beta}}\left(\sin(\psi(s)-\varphi)\right) + \frac{\beta'(s)}{2\sqrt{\beta}} a \cdot \cos(\psi(s)-\varphi)$$

$$\psi(s) = \int \frac{1}{\beta(s)} ds$$

$$\psi'(s) = \frac{1}{\beta(s)}$$

boundary condition: $x'(s+L) + \delta x' = x'(s)$

$$-a \cdot \frac{1}{\sqrt{\beta(\tilde{s}+L)}}\left(\sin(2\pi Q - \varphi)\right) + \frac{\beta'(\tilde{s}+L)}{2\beta(\tilde{s}+L)}\sqrt{\beta(\tilde{s}+L)}\, a \cdot \cos(2\pi Q - \varphi) + \frac{\Delta\tilde{s}}{\rho} =$$

$$= -a \cdot \frac{1}{\sqrt{\beta(\tilde{s})}}\left(\sin(-\varphi)\right) + \frac{\beta'(\tilde{s})}{2\beta(\tilde{s})}\sqrt{\beta(\tilde{s})}\, a \cdot \cos(-\varphi)$$

Nota bene: $\tilde{s}$ refers to the location of the kick

Misalignment error in a circular machine

Now we use: β(s+L) = β(s), φ=πQ

$$\frac{-a}{\sqrt{\beta(\tilde{s})}}\left(\sin(\pi Q)+\frac{\beta'(\tilde{s})}{2\beta(\tilde{s})}\sqrt{\beta(\tilde{s})}\,a\cdot\cos(\pi Q)+\frac{\Delta\tilde{s}}{\rho}\right. = \frac{a}{\sqrt{\beta(\tilde{s})}}\left(\sin(\pi Q)+\frac{\beta'(\tilde{s})}{2\beta(\tilde{s})}\sqrt{\beta(\tilde{s})}\,a\cdot\cos(\pi Q)\right.$$

$$\Rightarrow 2\,a\cdot\frac{\sin(\pi Q)}{\sqrt{\beta(\tilde{s})}}=\frac{\Delta\tilde{s}}{\rho} \quad\Rightarrow$$

$$a=\frac{\Delta\tilde{s}}{\rho}\cdot\sqrt{\beta(\tilde{s})}\,\frac{1}{2\sin(\pi Q)}$$

! this is the amplitude of the orbit oscillation resulting from a single kick

inserting in the equation of motion

$$x(s)=a\cdot\sqrt{\beta}\cos(\psi(s)+\varphi$$

$$x(s)=\frac{\Delta\tilde{s}}{\rho}\cdot\frac{\sqrt{\beta(\tilde{s})}\sqrt{\beta(s)}\cos(\psi(s)-\varphi)}{2\sin(\pi Q)}$$

! the distorted orbit depends on the kick strength,
! the local β function
! the β function at the observation point

!!! there is a resoncance denominator
→ watch your tune !!!

Misalignment error in a circular machine

For completeness:

if we do not set $\psi(s=0)=0$ *we have to write a bit more but finally we get:*

$$x(s)=\frac{\sqrt{\beta(s)}}{2\sin(\pi Q)}*\oint\sqrt{\beta(\tilde{s})}\,\frac{1}{\rho(\tilde{s})}\cos(|\psi(\tilde{s})-\psi(s)|-\pi Q)\,d\tilde{s}$$

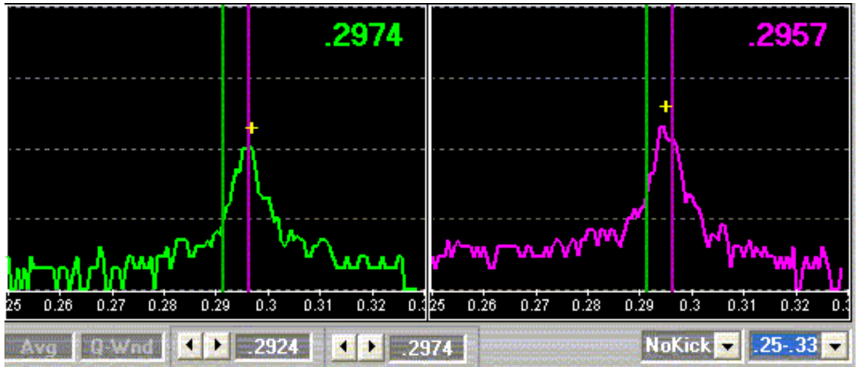
Reminder: LHC

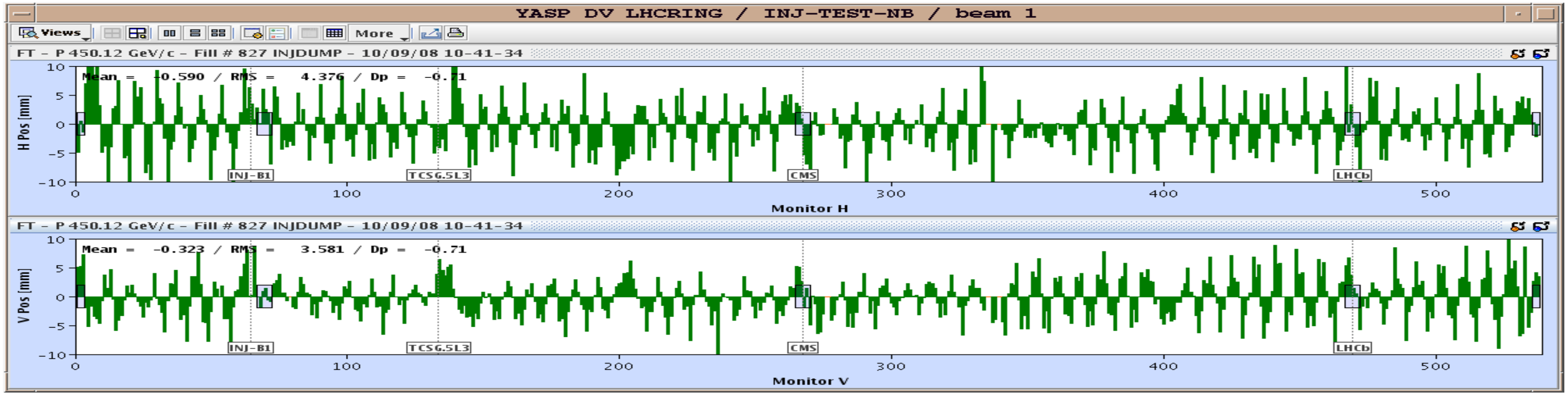
Tune: $Q_x = 64.31,\quad Q_y = 59.32$

Relevant for beam stability:

non integer part

avoid integer tunes

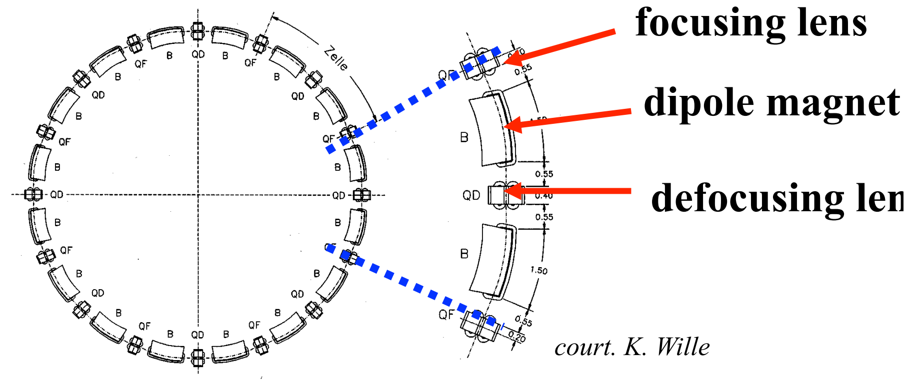




LHC First Turn Steering

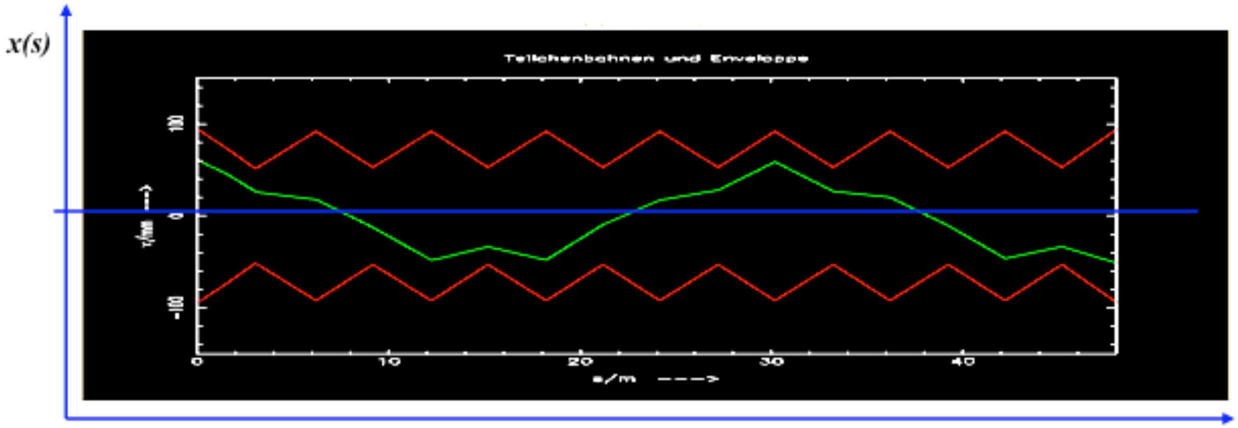
$$M_{total} = M_{QF} * M_{D} * M_{QD} * M_{Bend} * M_{D*.....}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s_2, s_1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$



court. K. Wille

in theory
nice harmonic oscillation



in reality:
effect of many localised orbit distortions

-> correct