LONGITUDINAL beam DYNAMICS in circular accelerators

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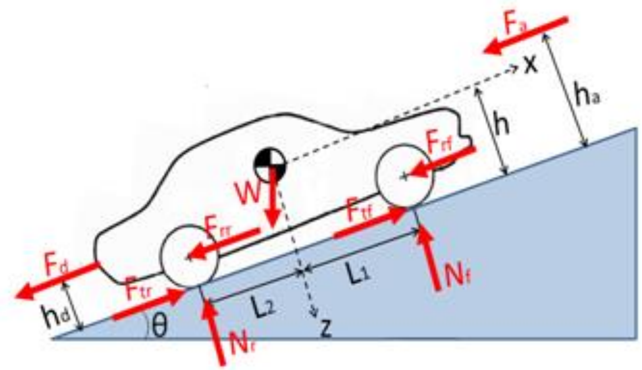


Introduction to Accelerator Physics
Budapest, 2-14/10/2016

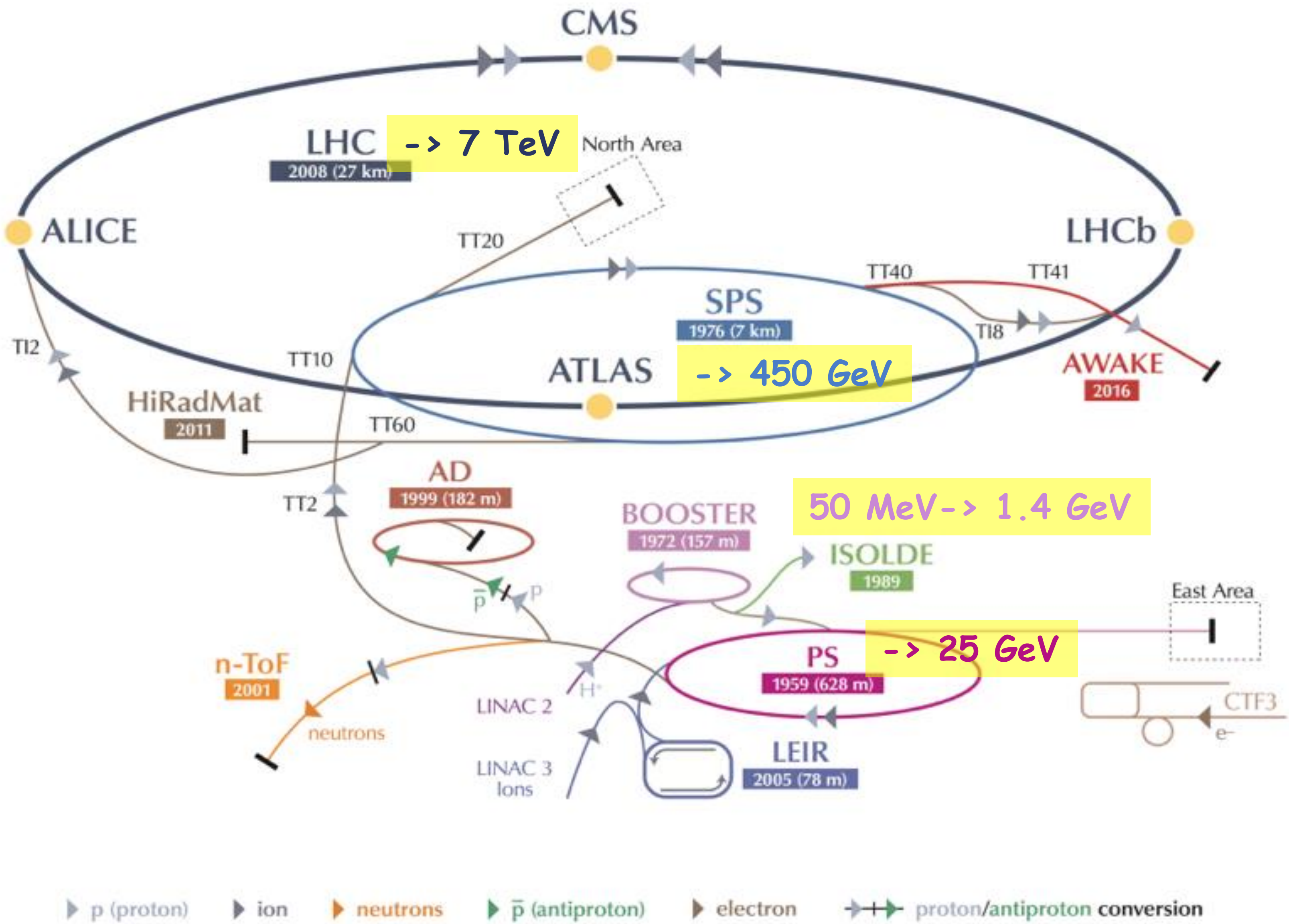
Summary of the 2 lectures:

- Introduction
- Circular accelerators: Cyclotron / Synchrotron
- Dispersion Effects in Synchrotron
- Stability and Longitudinal Phase Space Motion
- Stationary Bucket
- Injection Matching
- RF manipulations in the PS

More related lectures:

- Linacs – David Alesini
- Cyclotrons - Mike Seidel
- RF Systems - myself
- Electron Beam Dynamics - Lenny Rivkin

The CERN Accelerator Complex



Particle types and acceleration

The accelerating system will depend upon the evolution of the particle velocity along the system

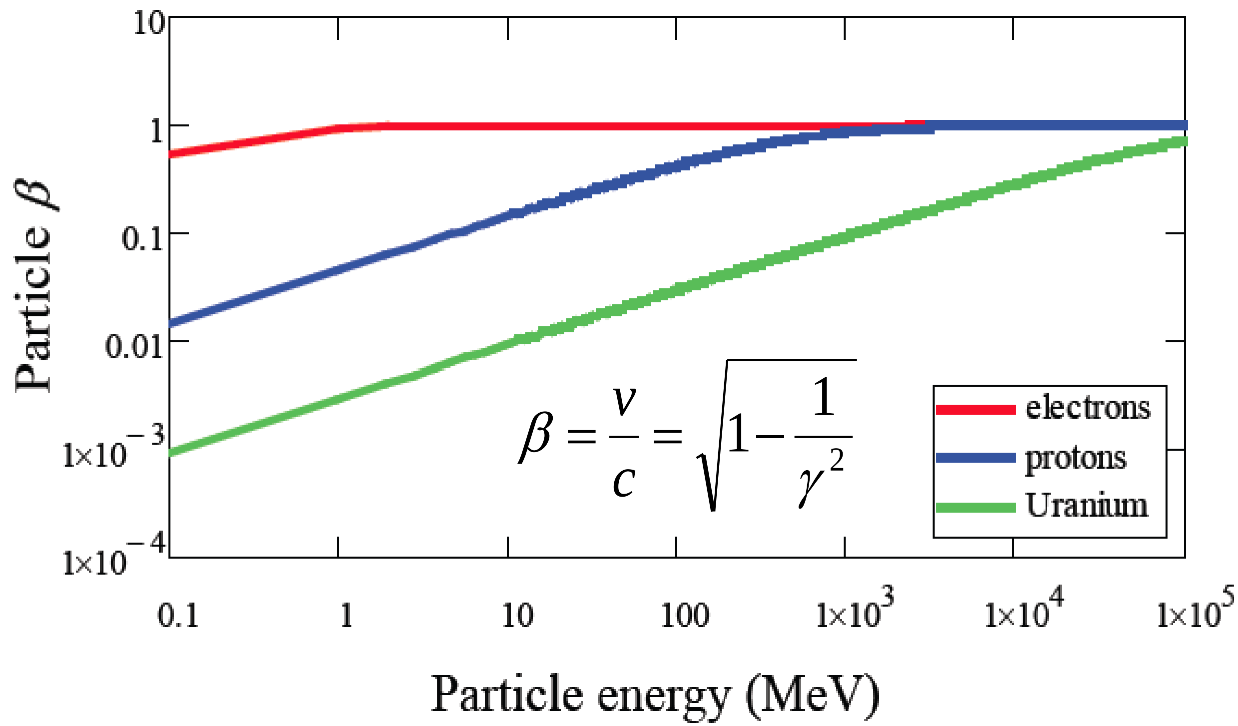
- electrons reach a constant velocity at relatively low energy
- heavy particles reach a constant velocity only at very high energy
  - -> we need different types of resonators, optimized for different velocities
  - -> the **revolution frequency will vary**, so the RF frequency will be changing

Particle rest mass:

electron	0.511	MeV
proton	938	MeV
^{239}U	~220000	MeV

Relativistic gamma factor:

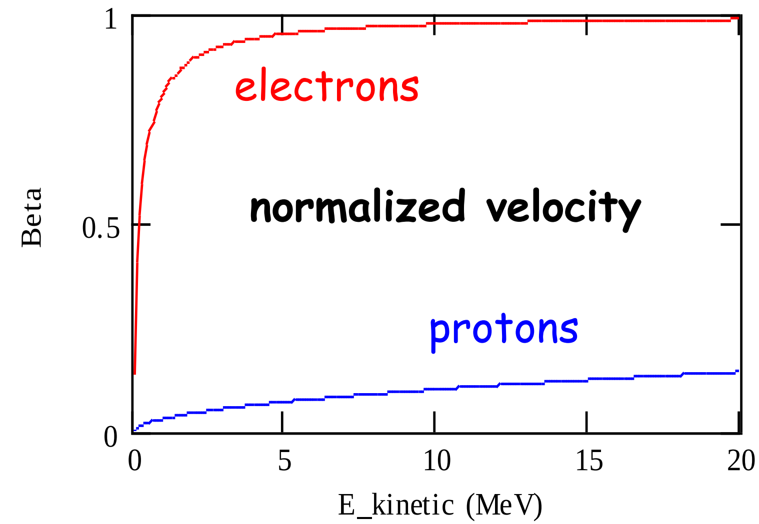
$$g = \frac{E}{E_0} = \frac{m}{m_0}$$



Velocity, Energy and Momentum

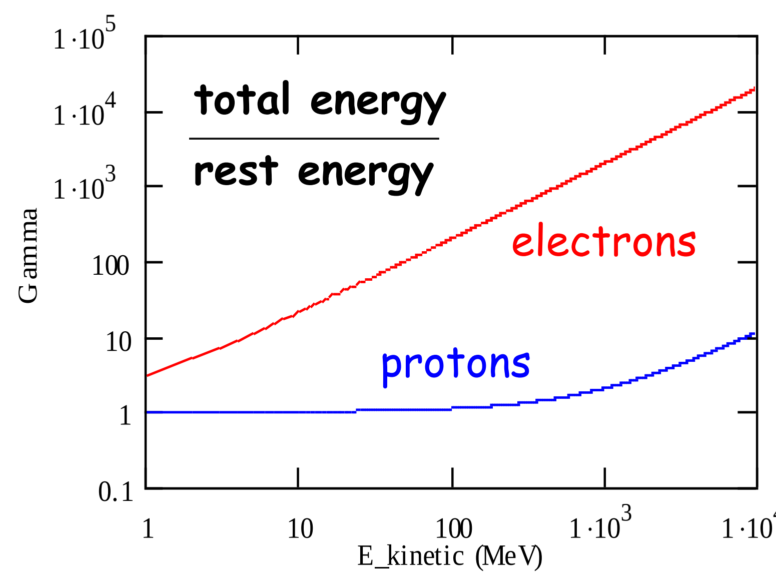
normalized velocity $\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}}$

=> electrons almost reach the speed of light very quickly (few MeV range)



$\frac{\text{total energy}}{\text{rest energy}}$ $\quad E = g m_0 c^2$

$$\gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1-\beta^2}}$$



Momentum $\quad p = mv = \frac{E}{c^2} bc = b\frac{E}{c} = bg m_0 c$

=> Magnetic field needs to follow the momentum increase

Acceleration: May the force be with you

To accelerate, we need a **force in** the **direction of motion**!

Newton-Lorentz Force on a charged particle:

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{dt}} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

2nd term always perpendicular to motion => no acceleration

Hence, it is necessary to have an electric field E (preferably) along the direction of the initial momentum (z), which changes the momentum of the particle.

$$\boxed{\frac{dp}{dt} = eE_z}$$

The 2nd term - larger at high velocities – is used for:

- BENDING: generated by a magnetic field perpendicular to the plane of the particle trajectory. The bending radius ρ obeys to the relation :

$$\boxed{\frac{p}{e} = B\rho}$$

in practical units: $$B\rho\,[\mathrm{Tm}] \approx \frac{p\,[\mathrm{GeV/c}]}{0.3}$$

- FOCUSING: the bending effect is used to bring the particles trajectory closer to the axis, hence to increase the beam density.

Energy Gain

The acceleration increases the **momentum**, providing **kinetic energy** to the charged particles.

In relativistic dynamics, total energy E and momentum p are linked by

$$E^2 = E_0^2 + p^2c^2 \qquad (E = E_0 + W) \qquad W \text{ kinetic energy}$$

Hence: $\quad dE = v\,dp \qquad \left(2E\,dE = 2c^2 p\,dp \Leftrightarrow dE = c^2 mv / E\; dp = v\,dp\right)$

The rate of energy gain per unit length of acceleration (along z) is then:

$$\frac{dE}{dz} = v\frac{dp}{dz} = \frac{dp}{dt} = eE_z$$

and the kinetic energy gained from the field along the z path is:

$$dW = dE = eE_z\,dz \quad \rightarrow \quad W = e\int E_z\,dz = eV$$

where V is just a potential.

Unit of Energy

Today's accelerators and future projects work/aim at the TeV energy range.

LHC: 7 TeV -> 14 TeV
CLIC: 380 GeV -> 3 TeV
HE-LHC/FCC: 33/100 TeV

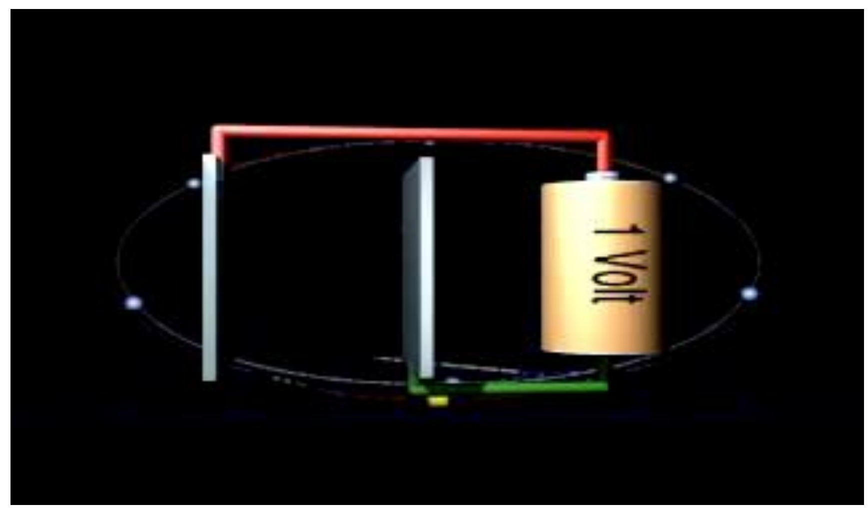
In fact, this energy unit comes from acceleration:

1 eV (electron Volt) is the energy that 1 elementary charge e (like one electron or proton) gains when it is accelerated in a potential (voltage) difference of 1 Volt.

Basic Unit: **eV (electron Volt)**

keV = 1000 eV = 10^3 eV
MeV = 10^6 eV
GeV = 10^9 eV
TeV = 10^{12} eV

LHC = ~450 Million km of batteries!!!
3x distance Earth-Sun



Methods of Acceleration: Time varying fields

Electrostatic field is limited by insulation problems,
the magnetic field does not accelerate at all.

Circular machine: DC acceleration impossible since $\oint \vec{E} \cdot \mathrm{d}\vec{s} = 0$

From Maxwell's Equations:

$$\vec{E} = -\vec{\nabla} f - \frac{\partial \vec{A}}{\partial t} \qquad \text{or} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{B} = m\vec{H} = \vec{\nabla} \times \vec{A}$$

The electric field is derived from a scalar potential φ and a vector potential A
The time variation of the magnetic field H generates an electric field E

The solution: => time varying electric fields
- Induction
- RF frequency fields

$$\oint \vec{E} \cdot \mathrm{d}\vec{s} = -\iint \frac{\partial \vec{B}}{\partial t} \cdot \mathrm{d}\vec{A}$$

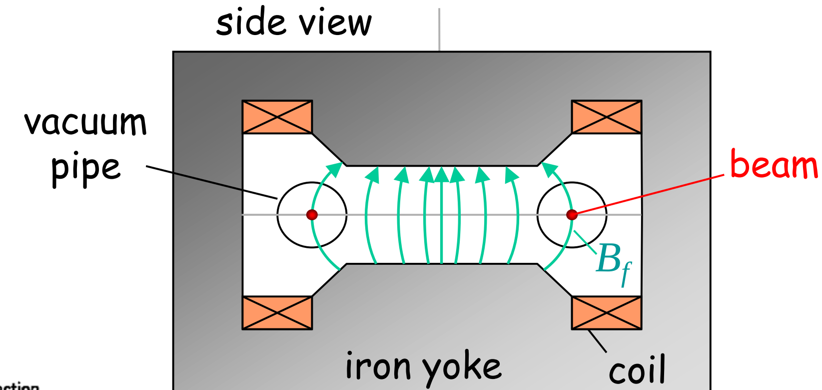
Acceleration by Induction: The Betatron

It is based on the principle of a transformer:
- primary side: large electromagnet   - secondary side: electron beam.

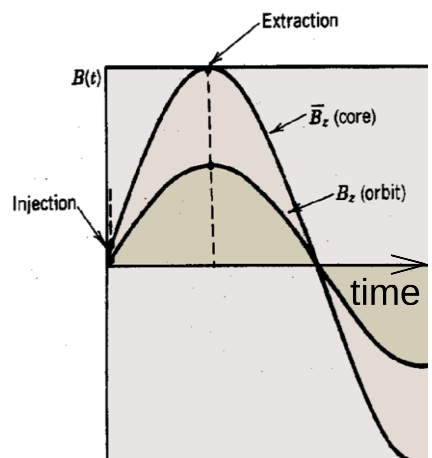
The ramping magnetic field is used to guide particles on a circular trajectory as well as for acceleration.

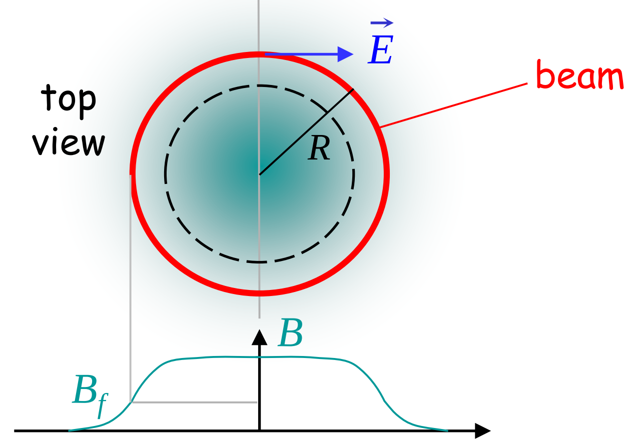
Limited by saturation in iron (~300 MeV e-)

Used in industry and medicine, as they are compact accelerators for electrons



Donald Kerst with the first betatron, invented at the University of Illinois in 1940





Summary: Relativity + Energy Gain

Newton-Lorentz Force $\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{dt}} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$

2[nd] term always perpendicular to motion => no acceleration

Relativistics Dynamics

$$\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \qquad \gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1-\beta^2}}$$

$$p = mv = \frac{E}{c^2}\beta c = \beta\frac{E}{c} = \beta\gamma m_0 c$$

$$E^2 = E_0^2 + p^2c^2 \longrightarrow dE = v\,dp$$

$$\frac{dE}{dz} = v\frac{dp}{dz} = \frac{dp}{dt} = eE_z$$

$$dE = dW = eE_z\,dz \rightarrow W = e\int E_z\,dz$$

RF Acceleration

$$E_z = \hat{E}_z \sin\omega_{RF}t = \hat{E}_z \sin\phi(t)$$

$$\int \hat{E}_z\,dz = \hat{V}$$

$$W = e\hat{V}\sin\phi$$

(neglecting transit time factor)

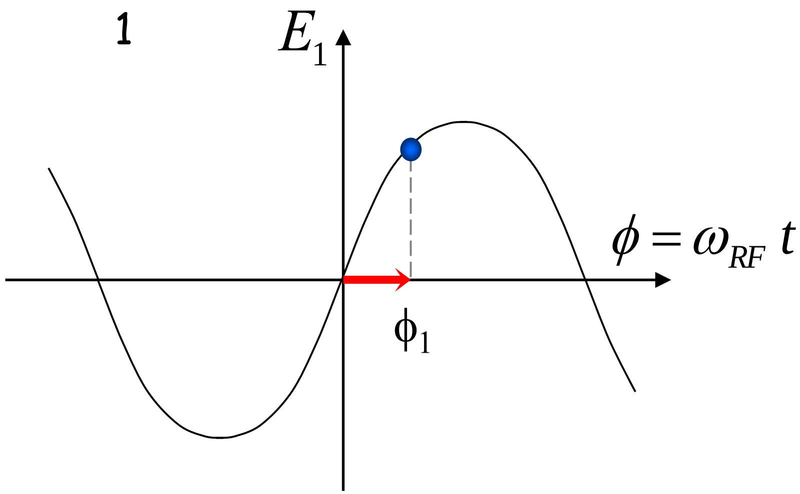
The field will change during the passage of the particle through the cavity
=> effective energy gain is lower

Common Phase Conventions

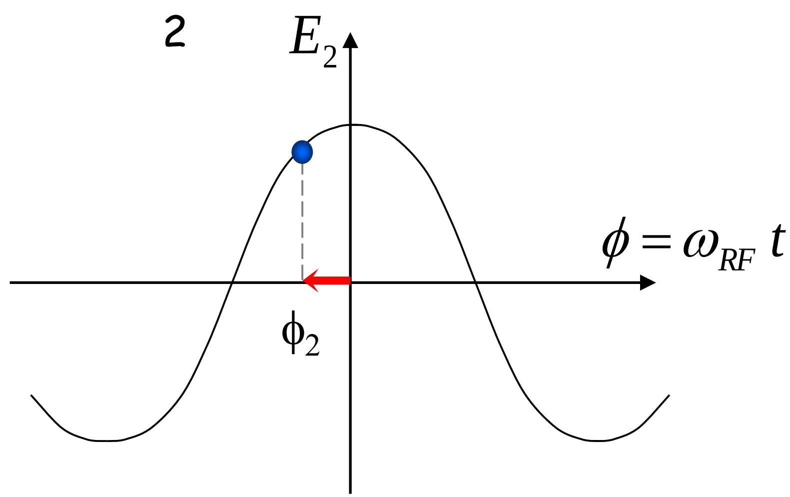
1. For circular accelerators, the origin of time is taken at the zero crossing of the RF voltage with positive slope

2. For linear accelerators, the origin of time is taken at the positive crest of the RF voltage

Time t= 0 chosen such that:



$$E_1(t) = E_0 \sin\left(W_{RF}\, t\right)$$

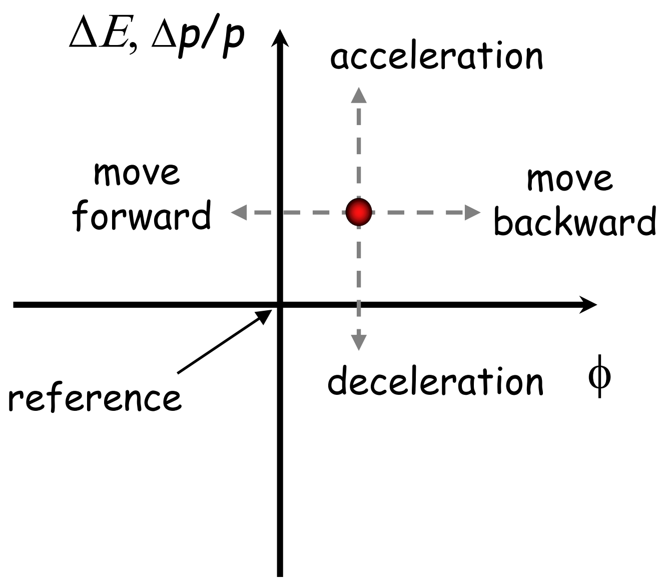


$$E_2(t) = E_0 \cos\left(W_{RF}\, t\right)$$

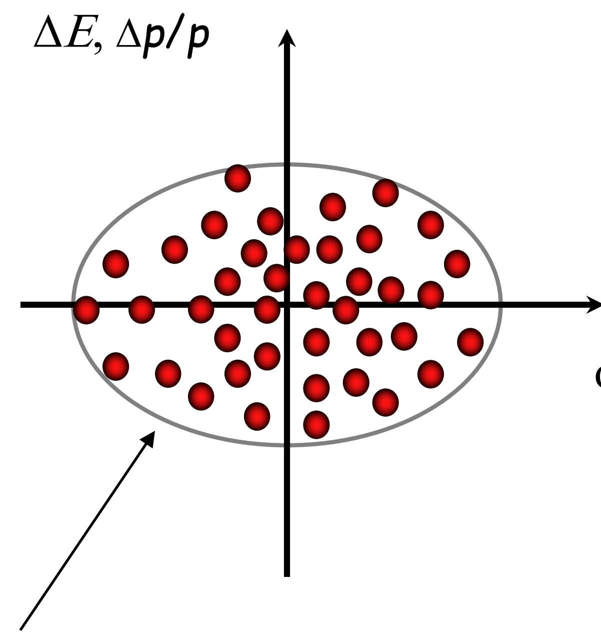
3. I will stick to **convention 1** in the following to avoid confusion

Longitudinal phase space

The energy – phase oscillations can be drawn in phase space:



The particle trajectory in the phase space ($\Delta p/p$, ϕ) describes its longitudinal motion.



Emittance: phase space area including all the particles

NB: if the emittance contour correspond to a possible orbit in phase space, its shape does not change with time (matched beam)

Circular accelerators

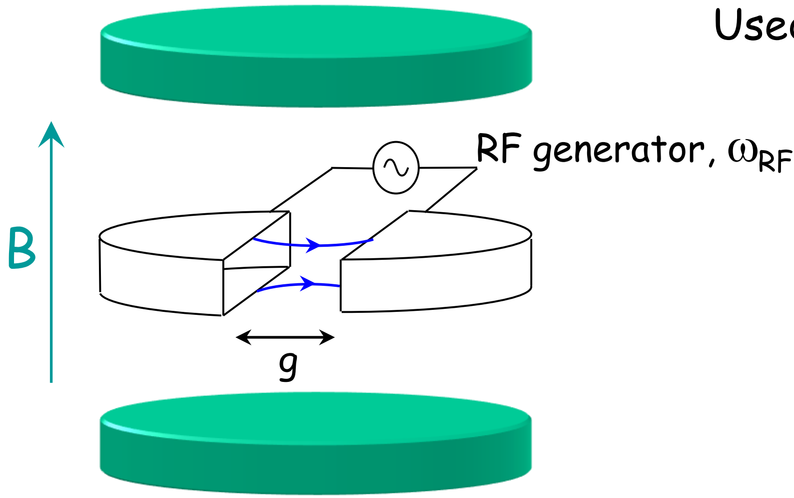
Cyclotron

Synchrotron

Circular accelerators: Cyclotron

Courtesy: EdukiteLearning, https://youtu.be/cNnNM2ZqIsc

Circular accelerators: Cyclotron



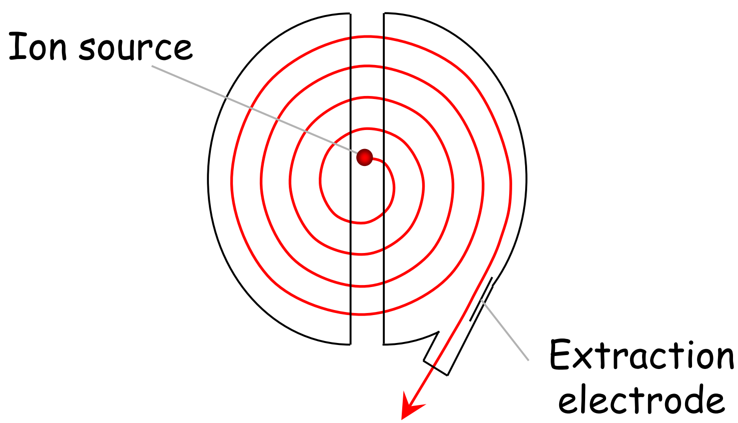
Used for protons, ions

B = constant

ω_{RF} = constant

Synchronism condition

$$\boxed{\begin{aligned} \omega_s &= \omega_{RF} \\ 2\pi\ \rho &= v_s\ T_{RF} \end{aligned}}$$



Cyclotron frequency $\omega = \dfrac{q\,B}{m_0\,\gamma}$

1. γ increases with the energy
   $\Rightarrow$ no exact synchronism
2. if $v \ll c \Rightarrow \gamma \cong 1$

Cyclotron Animation

Animation: http://www.sciences.univ-nantes.fr/sites/genevieve_tulloue/Meca/Charges/cyclotron.html

Circular accelerators: Cyclotron

Courtesy Berkeley Lab,
https://www.youtube.com/watch?v=cutKuFxeXmQ

Cyclotron / Synchrocyclotron

TRIUMF 520 MeV cyclotron Vancouver - Canada

CERN 600 MeV synchrocyclotron

Synchrocyclotron: Same as cyclotron, except a modulation of ω_{RF}

B = constant

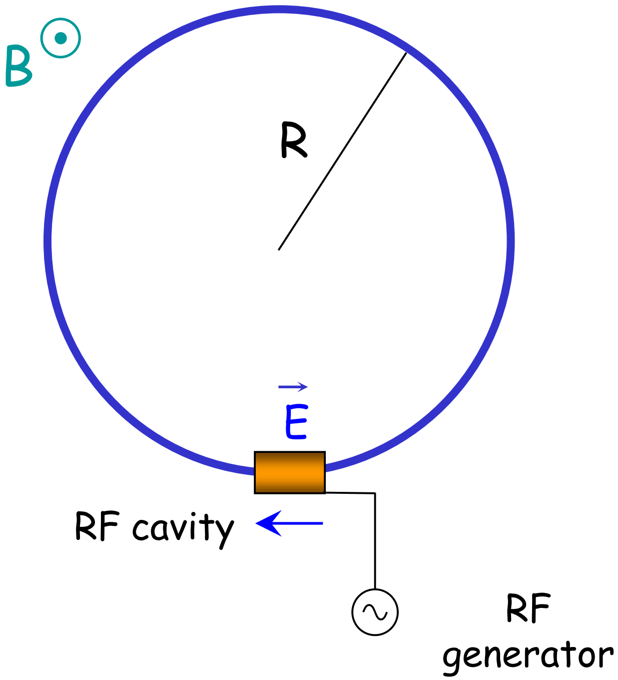
$\gamma\ \omega_{RF}$ = constant ω_{RF} decreases with time

The condition:

$$\omega_s(t) = \omega_{RF}(t) = \frac{q\,B}{m_0\,\gamma(t)}$$

Allows to go beyond the non-relativistic energies

Circular accelerators: The Synchrotron



1. Constant orbit during acceleration
2. To keep particles on the closed orbit, B should increase with time
3. ω and ω_{RF} increase with energy

RF frequency can be multiple of revolution frequency

$$\omega_{RF} = h\omega$$

Synchronism condition

$$T_s = h\,T_{RF}$$

$$\frac{2\pi R}{v_s} = h\,T_{RF}$$

h integer,
harmonic number:
number of RF cycles per revolution

Circular accelerators: The Synchrotron

LEAR (CERN)
Low Energy Antiproton Ring

EPA (CERN)
Electron Positron Accumulator

PS (CERN)
Proton Synchrotron

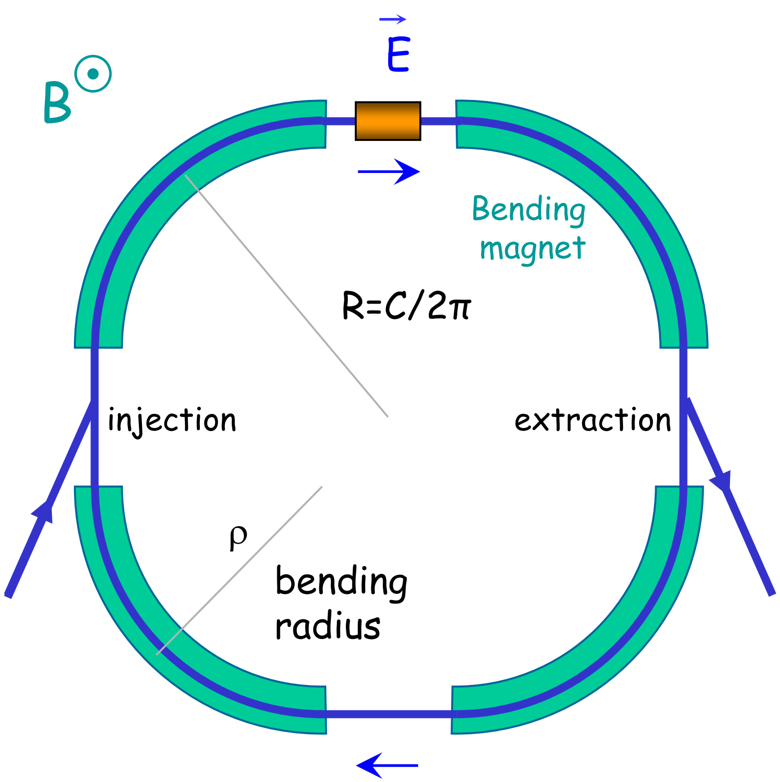
© CERN Geneva

Examples of different proton and electron synchrotrons at CERN

+ LHC (of course!)

The Synchrotron

The synchrotron is a synchronous accelerator since there is a synchronous RF phase for which the energy gain fits the increase of the magnetic field at each turn. That implies the following operating conditions:



$e\hat{V}\sin f \longrightarrow$ **Energy gain per turn**

$f = f_s = cte \longrightarrow$ **Synchronous particle**

$\omega_{RF} = h\omega \longrightarrow$ **RF synchronism (h - harmonic number)**

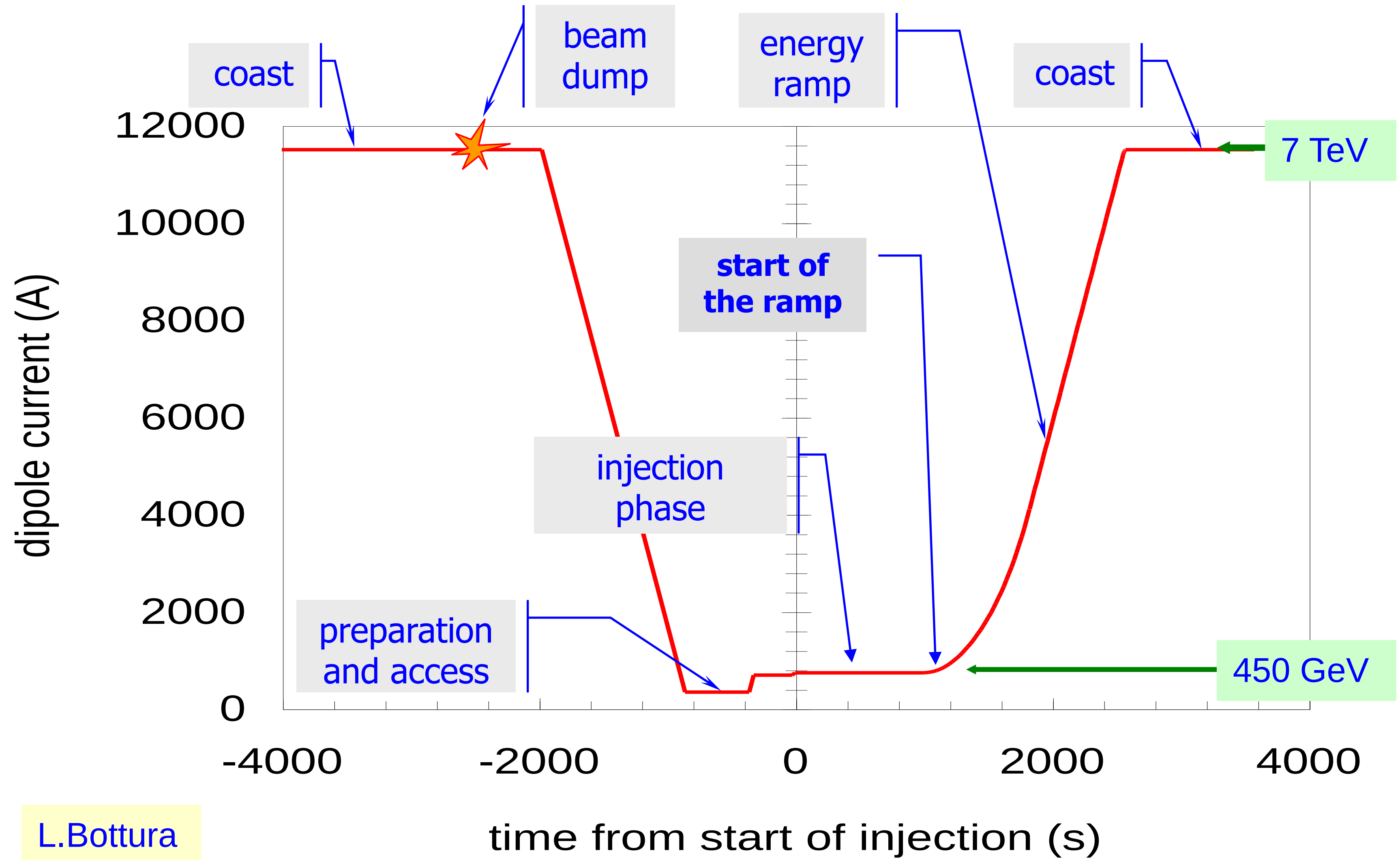
$r = cte \quad R = cte \rightarrow$ **Constant orbit**

$Br = {}^{P}\!/_{e}\ \text{Þ}\ B \longrightarrow$ **Variable magnetic field**

If v≈c, ω hence ω_{RF} remain constant (ultra-relativistic e^-)

The Synchrotron – LHC Operation Cycle

The magnetic field (dipole current) is increased during the acceleration.



L.Bottura

The Synchrotron – Energy ramping

Energy ramping by increasing the B field (frequency has to follow v):

$$p = eB\rho \quad \Rightarrow \quad \frac{dp}{dt} = e\rho\,\dot{B} \quad \Rightarrow (\Delta p)_{turn} = e\rho\,\dot{B}T_r = \frac{2\pi e\rho R\dot{B}}{v}$$

Since: $$E^2 = E_0^2 + p^2c^2 \quad \Rightarrow \quad \Delta E = v\Delta p$$

$$\boxed{\left(\Delta E\right)_{turn} = \left(\Delta W\right)_s = 2\pi e\rho R\dot{B} = e\hat{V}\sin\phi_s}$$

Stable phase φ_s changes during energy ramping

$$\sin\phi_s = 2\pi\rho R\,\frac{\dot{B}}{\hat{V}_{RF}} \quad \Rightarrow \quad \boxed{\phi_s = \arcsin\left(2\pi\rho R\,\frac{\dot{B}}{\hat{V}_{RF}}\right)}$$

- The number of stable synchronous particles is equal to the harmonic number h. They are equally spaced along the circumference.
- Each synchronous particle satisfies the relation $p=eB\rho$. They have the nominal energy and follow the nominal trajectory.

The Synchrotron – Frequency change

During the energy ramping, the RF frequency increases to follow the increase of the revolution frequency :

$$\omega = \frac{\omega_{RF}}{h} = \omega(B, R_s)$$

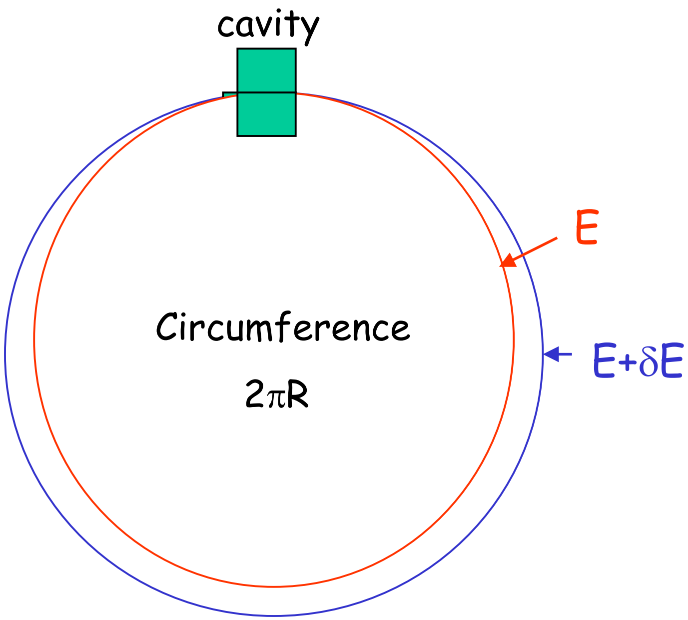
Hence: $\dfrac{f_{RF}(t)}{h} = \dfrac{v(t)}{2\pi R_s} = \dfrac{1}{2\pi}\dfrac{ec^2}{E_s(t)}\dfrac{\rho}{R_s}B(t)$ (using $p(t) = eB(t)\rho$, $E = mc^2$)

Since $E^2 = (m_0c^2)^2 + p^2c^2$ the RF frequency must follow the variation of the B field with the law

$$\boxed{\frac{f_{RF}(t)}{h} = \frac{c}{2\pi R_s}\left\{\frac{B(t)^2}{(m_0c^2/ec\rho)^2 + B(t)^2}\right\}^{1/2}}$$

This asymptotically tends towards $f_r \rightarrow \dfrac{c}{2\pi R_s}$ when B becomes large compared to $m_0c^2/(ec\rho)$ which corresponds to $v \rightarrow c$

Dispersion Effects in a Synchrotron



If a particle is slightly shifted in momentum it will have a different orbit and the orbit length is different.

The "momentum compaction factor" is defined as:

$$\alpha_c = \frac{dL/L}{dp/p} \qquad \boxed{\alpha_c = \frac{p}{L}\frac{dL}{dp}}$$

If the particle is shifted in momentum it will have also a different velocity.
As a result of both effects the revolution frequency changes with a "slip factor":

$$\eta = \frac{\mathrm{d}f_r/f_r}{\mathrm{d}p/p} \Rightarrow \qquad \boxed{\eta = \frac{p}{f_r}\frac{df_r}{dp}}$$

p=particle momentum

R=synchrotron physical radius

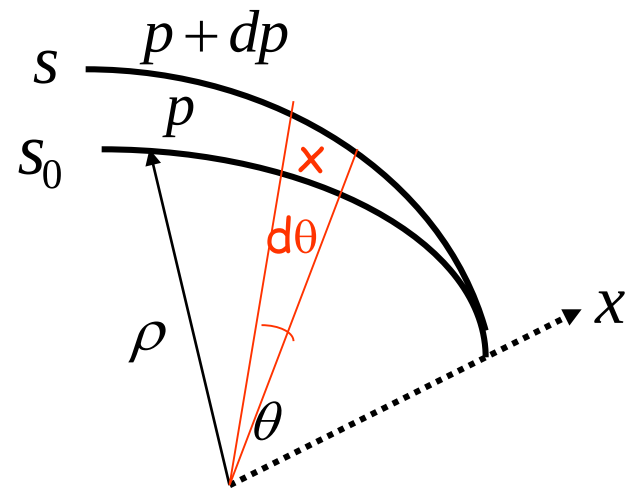
f_r=revolution frequency

Momentum Compaction Factor

$$\alpha_c = \frac{p}{L}\frac{dL}{dp}$$

$$ds_0 = \rho\, d\theta$$

$$ds = (\rho + x)\, d\theta$$



The elementary path difference
from the two orbits is:

definition of dispersion D_x

$$\frac{dl}{ds_0} = \frac{ds - ds_0}{ds_0} = \frac{x}{\rho} = \frac{D_x}{\rho}\frac{dp}{p}$$

leading to the total change in the circumference:

$$dL = \oint_C dl = \oint \frac{x}{\rho}\, ds_0 = \oint \frac{D_x}{\rho}\frac{dp}{p}\, ds_0$$

$$\alpha_c = \frac{1}{L}\int_C \frac{D_x(s)}{\rho(s)}\, ds_0$$

With ρ=∞ in straight sections we get:

$$\alpha_c = \frac{\langle D_x \rangle_m}{R}$$

$\langle\ \rangle_m$ means that the average is considered over the bending magnet only

Dispersion Effects – Revolution Frequency

There are two effects changing the revolution frequency:
the orbit length and the velocity of the particle

$$f_r = \frac{bc}{2pR} \quad Þ \quad \frac{df_r}{f_r} = \frac{db}{b} - \frac{dR}{R} \underset{\uparrow}{=} \frac{db}{b} - a_c\frac{dp}{p}$$

definition of momentum
compaction factor

$$p = mv = bg\frac{E_0}{c} \quad Þ \quad \frac{dp}{p} = \frac{db}{b} + \frac{d\left(1 - b^2\right)^{-1/2}}{\left(1 - b^2\right)^{-1/2}} = \underbrace{\left(1 - b^2\right)^{-1}}_{g^2}\frac{db}{b}$$

$$\frac{df_r}{f_r} = \left(\frac{1}{\gamma^2} - \alpha_c\right)\frac{dp}{p} \quad \xrightarrow{\frac{df_r}{f_r} = h\frac{dp}{p}} \quad \eta = \frac{1}{\gamma^2} - \alpha_c$$

η=0 at the transition energy

$$\gamma_t = \frac{1}{\sqrt{\alpha_c}}$$

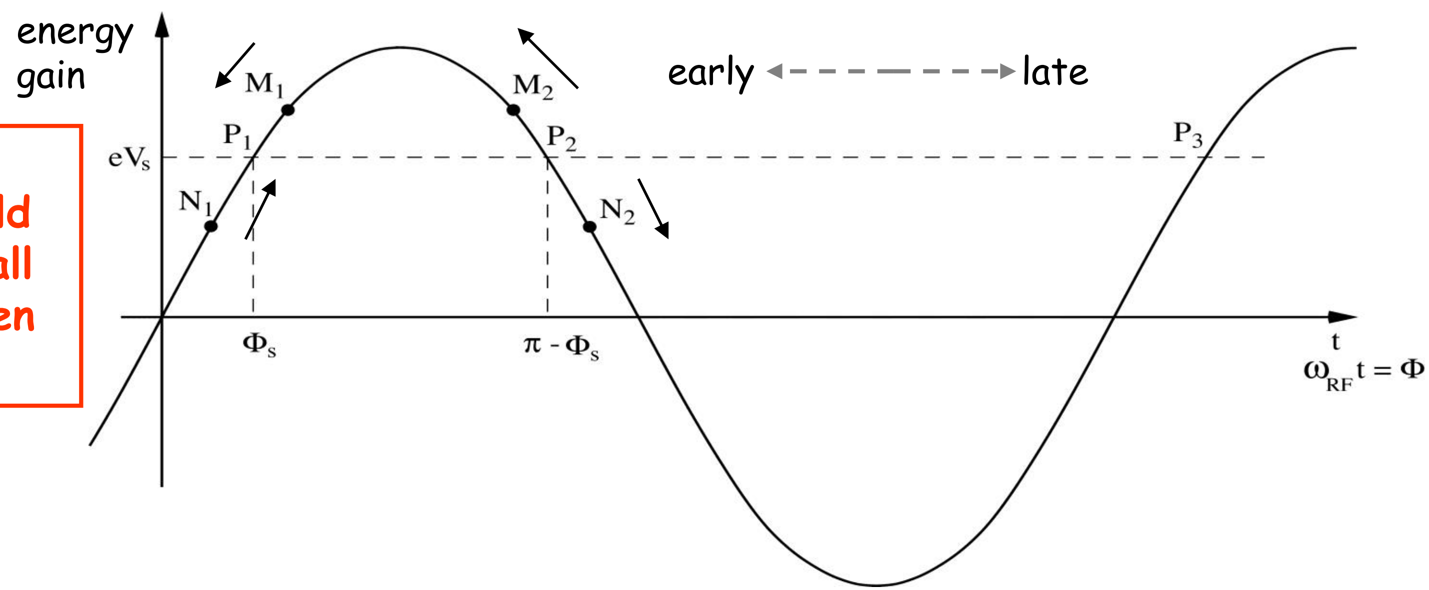
RECAP: Principle of Phase Stability (Linac)

Let's consider a succession of accelerating gaps, operating in the 2π mode, for which the synchronism condition is fulfilled for a phase Φ_s .

$$eV_s = e\hat{V}\sin\Phi_s$$

is the energy gain in one gap for the particle to reach the next gap with the same RF phase: P_1 ,P_2, …… are fixed points.

For a 2π mode, the electric field is the same in all gaps at any given time.



If an energy increase is transferred into a velocity increase =>

M_1 & N_1 will move towards P_1 => stable

M_2 & N_2 will go away from P_2 => unstable

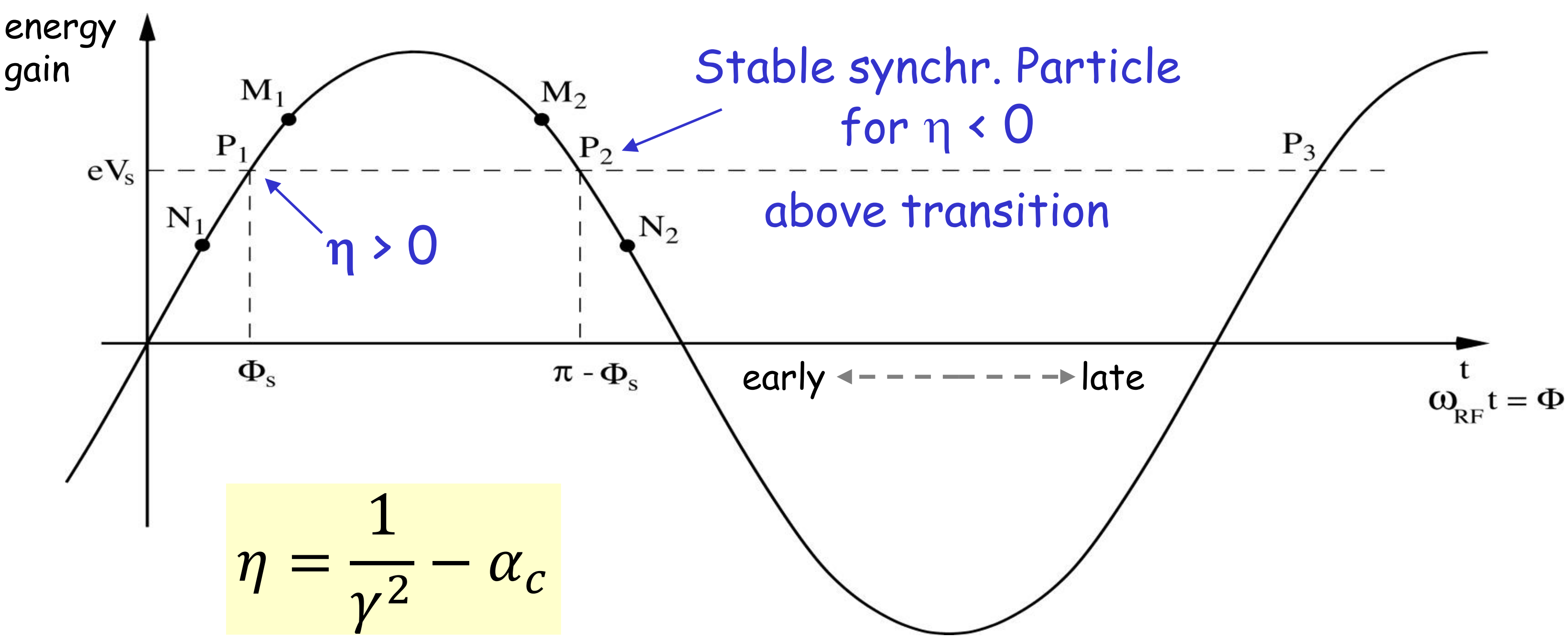
(Highly relativistic particles have no significant velocity change)

Phase Stability in a Synchrotron

From the definition of η it is clear that an increase in momentum gives

- below transition ($\eta > 0$) a higher revolution frequency (increase in velocity dominates) while

- above transition ($\eta < 0$) a lower revolution frequency ($v \approx c$ and longer path) where the momentum compaction (generally > 0) dominates.



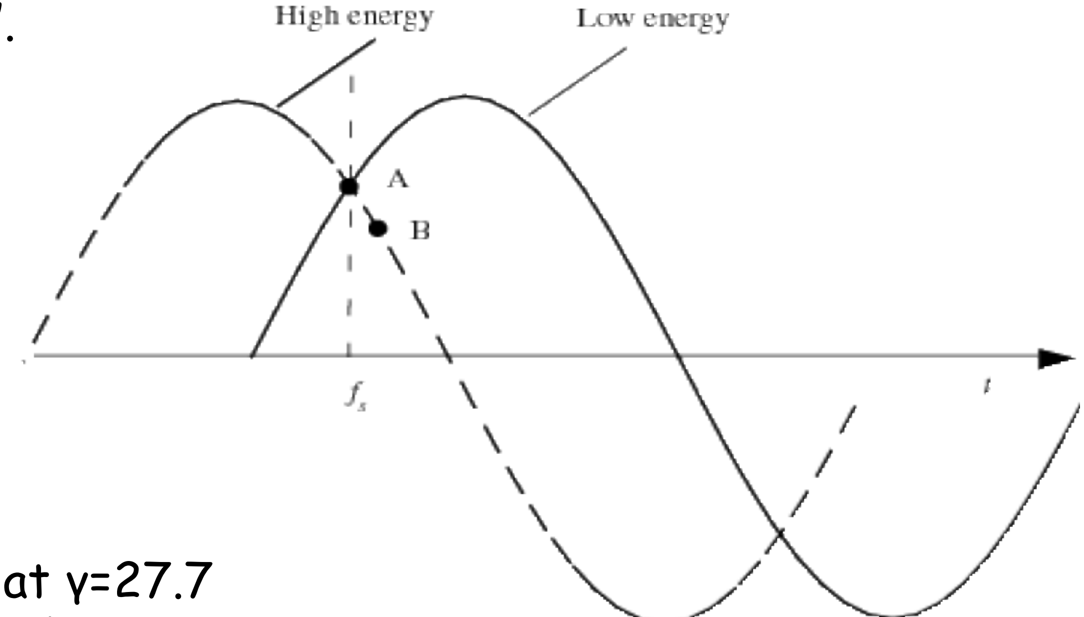
$$\eta = \frac{1}{\gamma^2} - \alpha_c$$

Crossing Transition

At transition, the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.

Crossing transition during acceleration makes the previous stable synchronous phase unstable. The RF system needs to make a rapid change of the RF phase, a 'phase jump'.

$$\alpha_c \sim \frac{1}{Q_x^2} \qquad \gamma_t = \frac{1}{\sqrt{\alpha_c}} \sim Q_x$$



In the PS: γ_t is at ~6 GeV
In the SPS: γ_t= 22.8, injection at γ=27.7
=> no transition crossing!
In the LHC: γ_t is at ~55 GeV, also far below injection energy

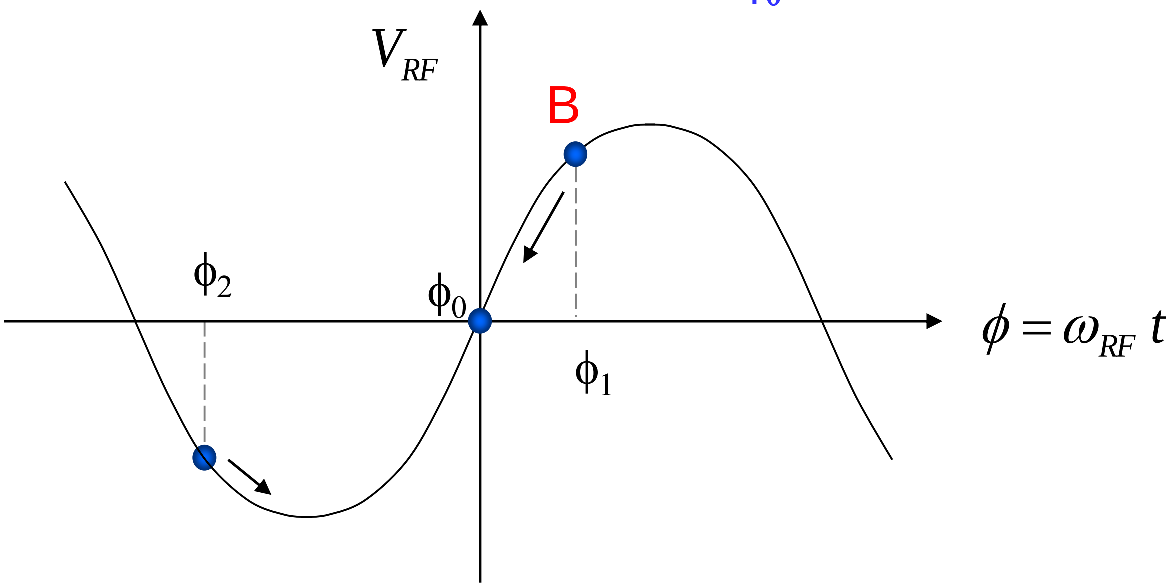
Transition crossing not needed in leptons machines, why?

Dynamics: Synchrotron oscillations

Simple case (no accel.): B = const., below transition $\gamma < \gamma_t$

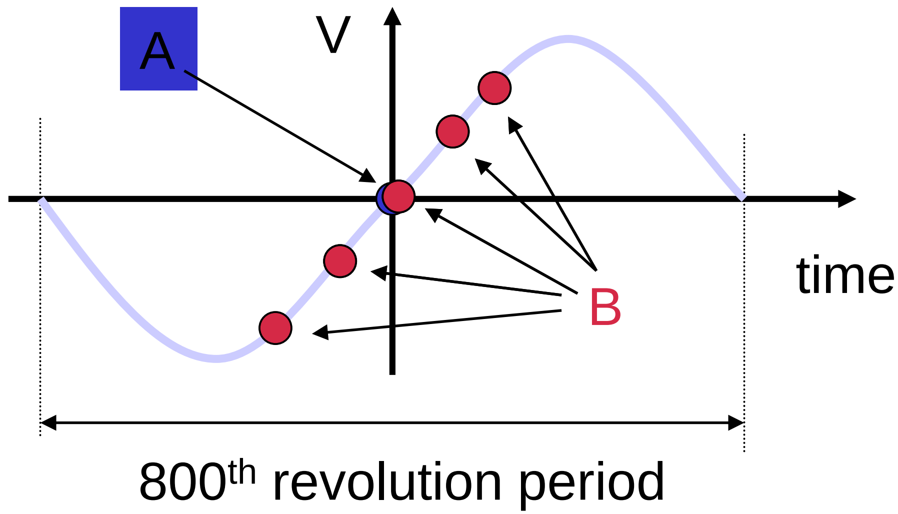
The phase of the synchronous particle must therefore be $\phi_0 = 0$.

Φ_1 - The particle B is accelerated
- Below transition, an energy increase means an increase in revolution frequency
- The particle arrives earlier – tends toward ϕ_0

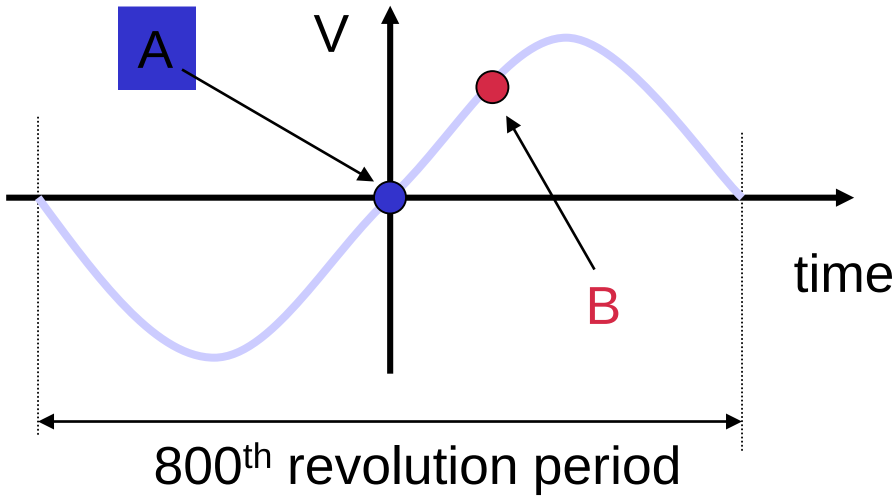


ϕ_2 - The particle is decelerated
- decrease in energy - decrease in revolution frequency
- The particle arrives later – tends toward ϕ_0

Synchrotron oscillations



Synchrotron oscillations



Particle B has made one full oscillation around particle A.

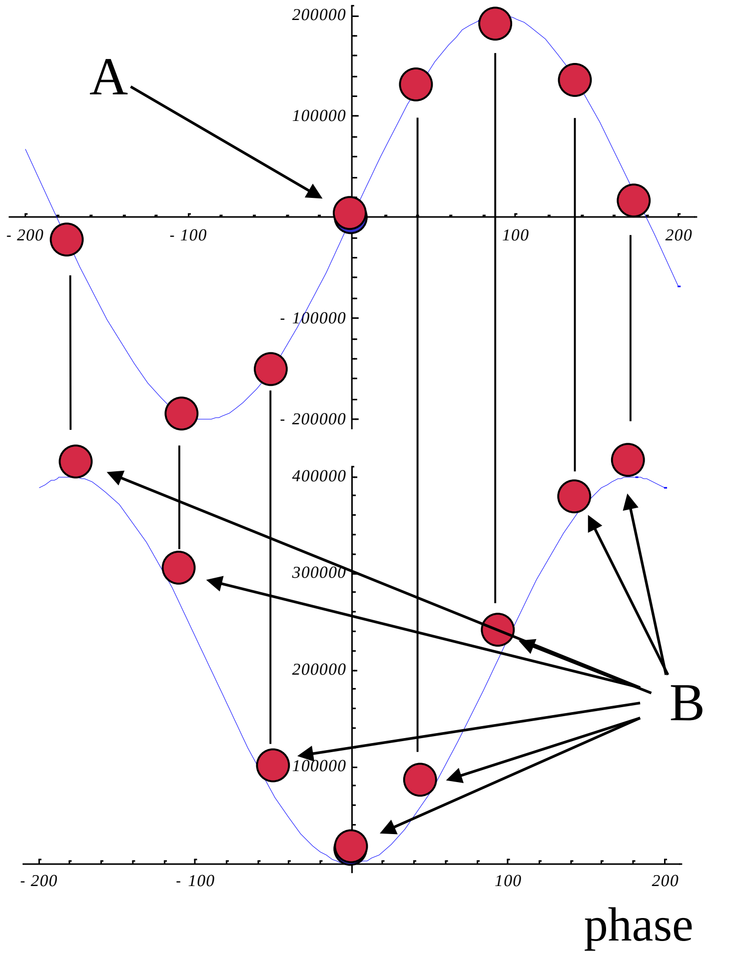
The amplitude depends on the initial phase and energy.

Exactly like the pendulum

This oscillation is called:

Synchrotron Oscillation

The Potential Well



Cavity voltage

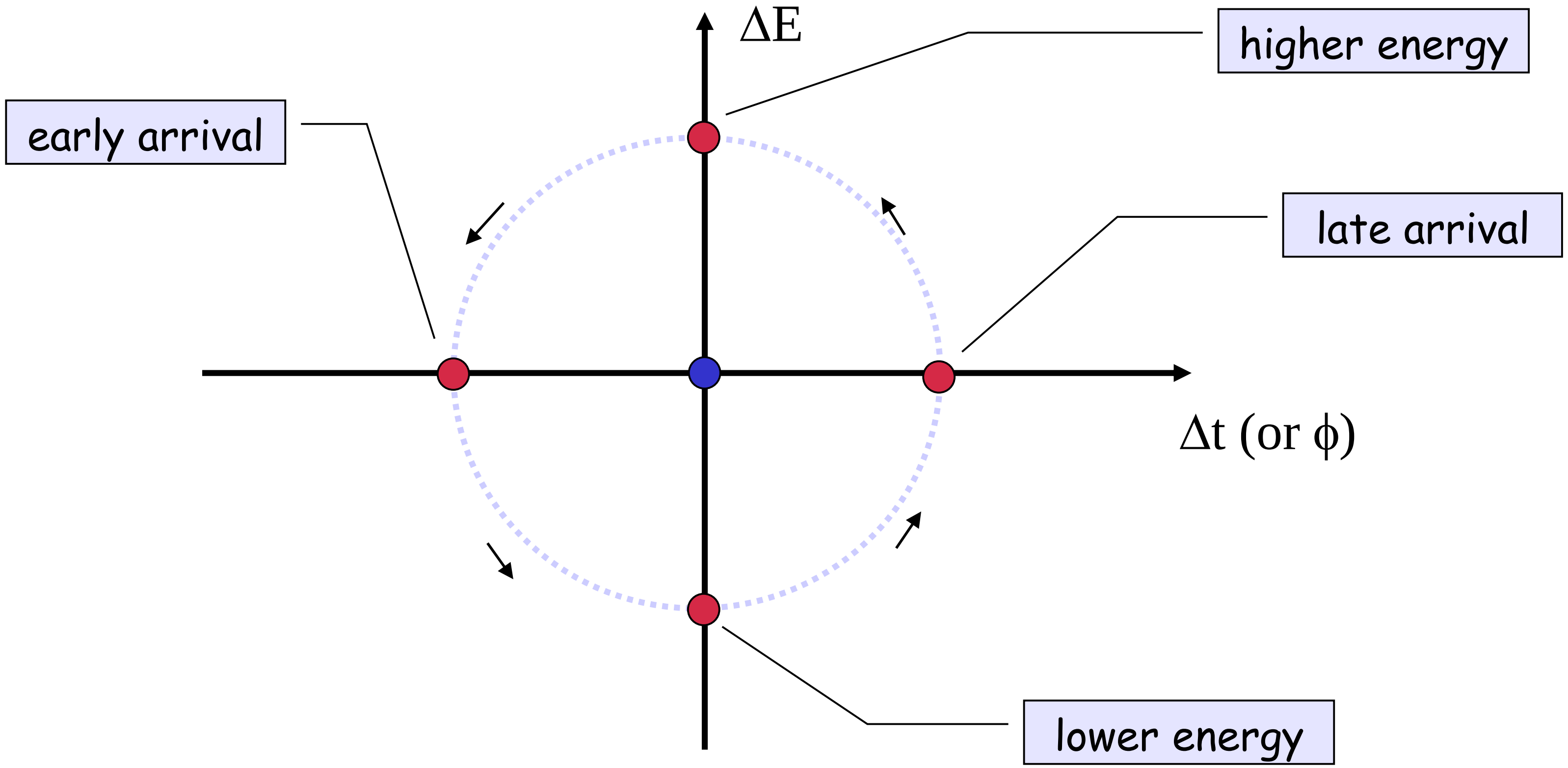
Potential well

Longitudinal Phase Space Motion

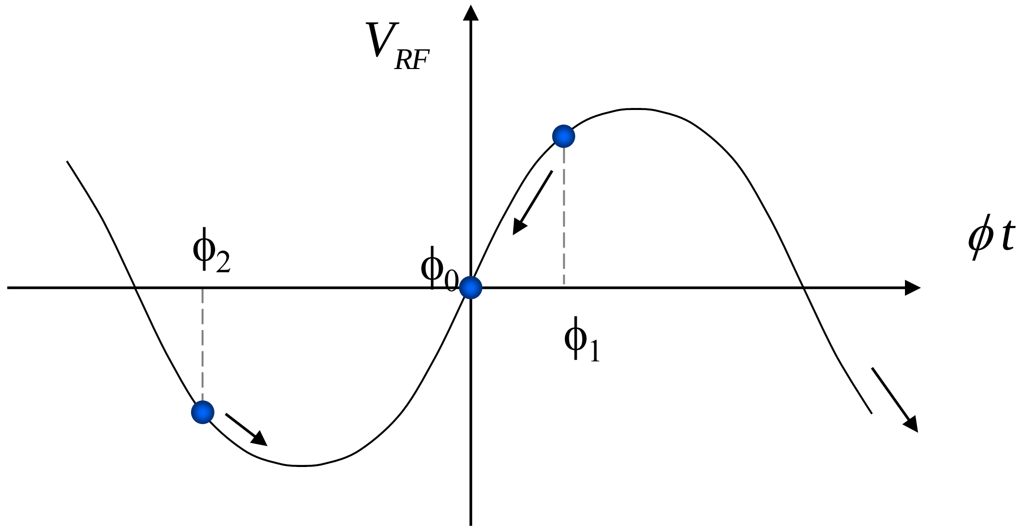
Particle B oscillates around particle A

This is a synchrotron oscillation

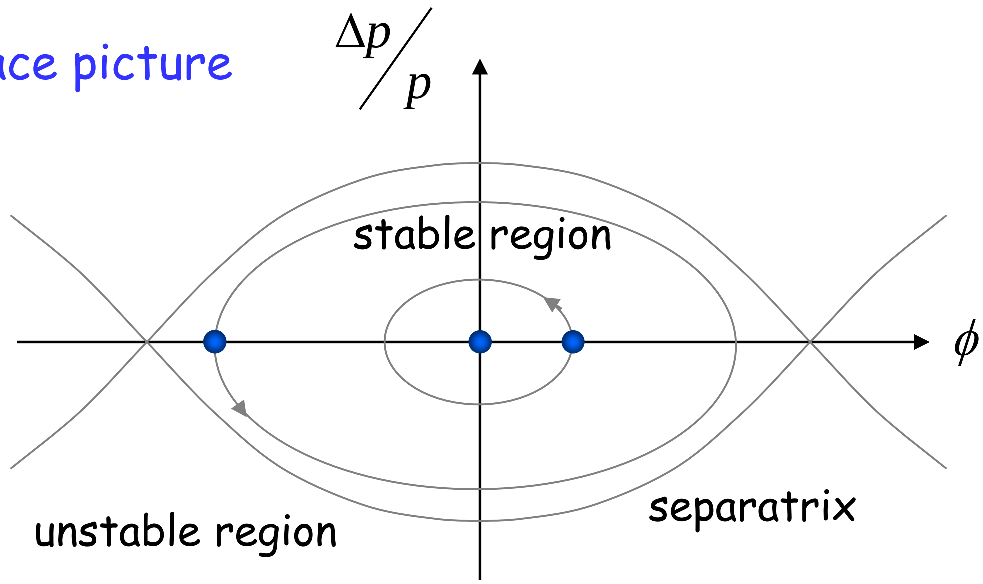
Plotting this motion in longitudinal phase space gives:



Synchrotron oscillations – No acceleration



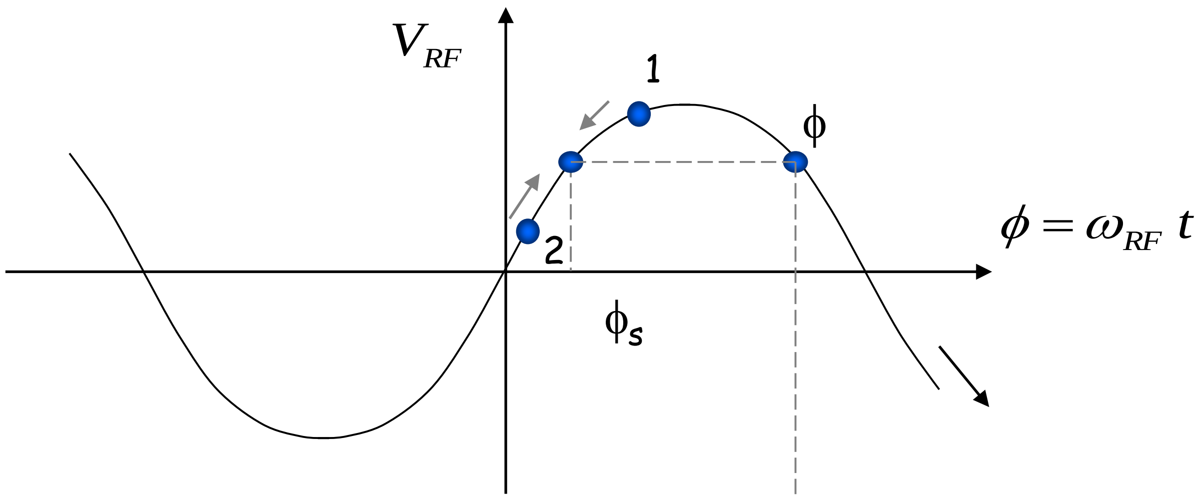
Phase space picture



Synchrotron oscillations (with acceleration)

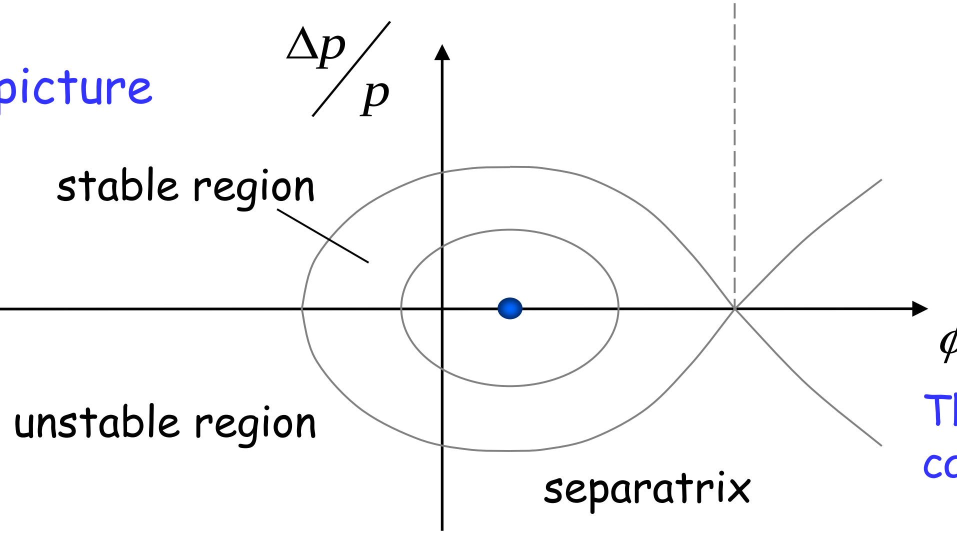
Case with acceleration *B increasing*

$\gamma < \gamma_t$



$$\phi_s < \phi < \pi - \phi_s$$

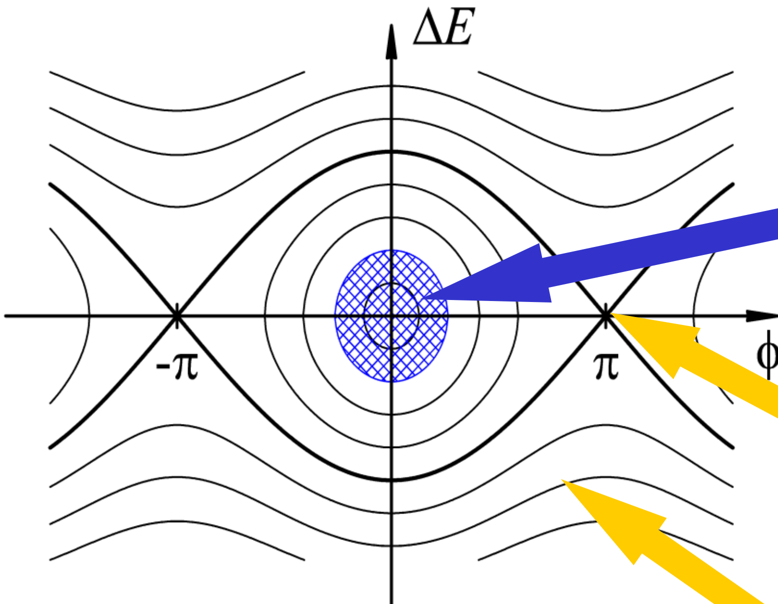
Phase space picture



The symmetry of the case B = const. is lost

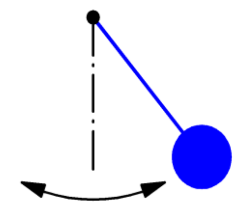
Synchrotron motion in phase space

ΔE-ϕ **phase space** of a stationary bucket
(when there is no acceleration)



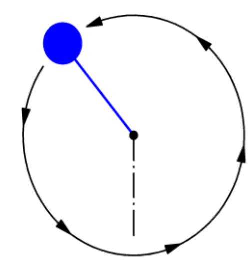
Dynamics of a particle
Non-linear, conservative oscillator → e.g. pendulum

Particle **inside the separatrix:**



Particle at the **unstable fix-point**

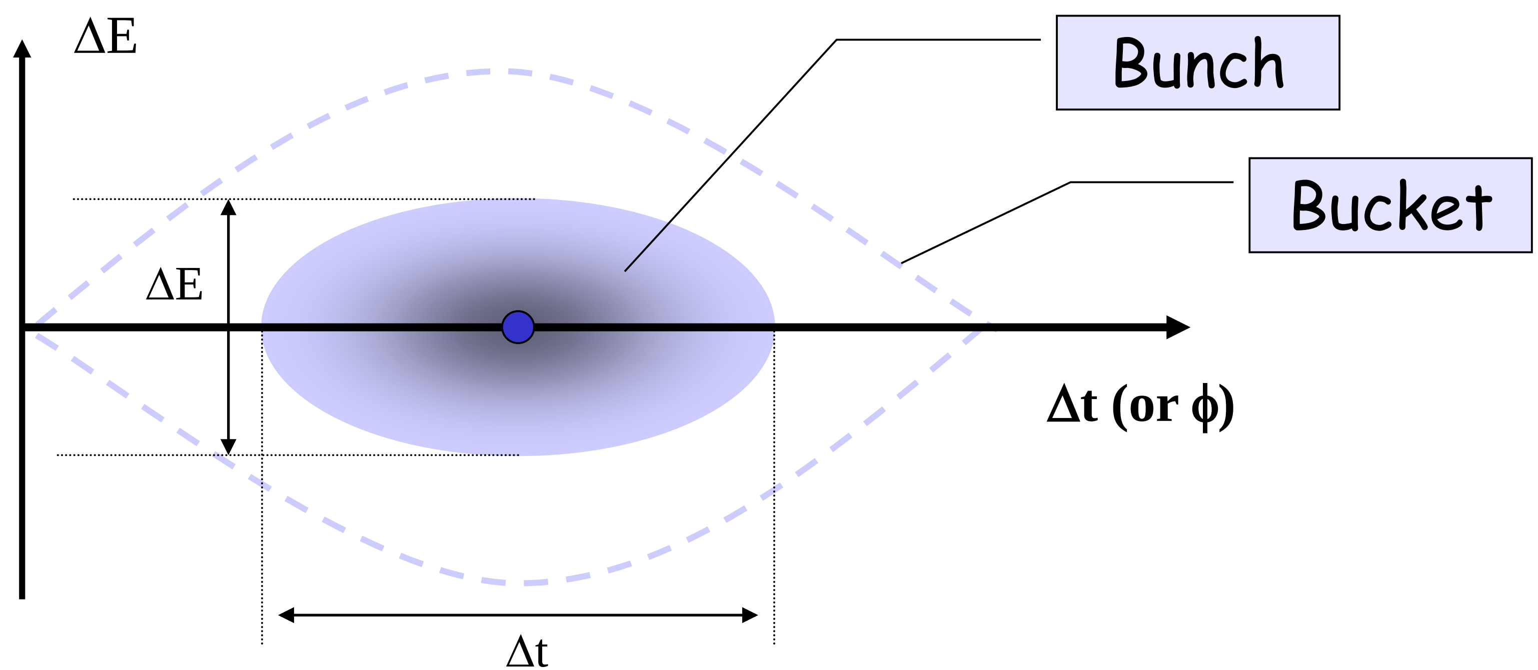
Particle **outside the separatrix:**



Bucket area: area enclosed by the separatrix
The area covered by particles is the longitudinal emittance

(Stationary) Bunch & Bucket

The bunches of the beam fill usually a part of the bucket area.



Bucket area = <u>longitudinal Acceptance</u> [eVs]

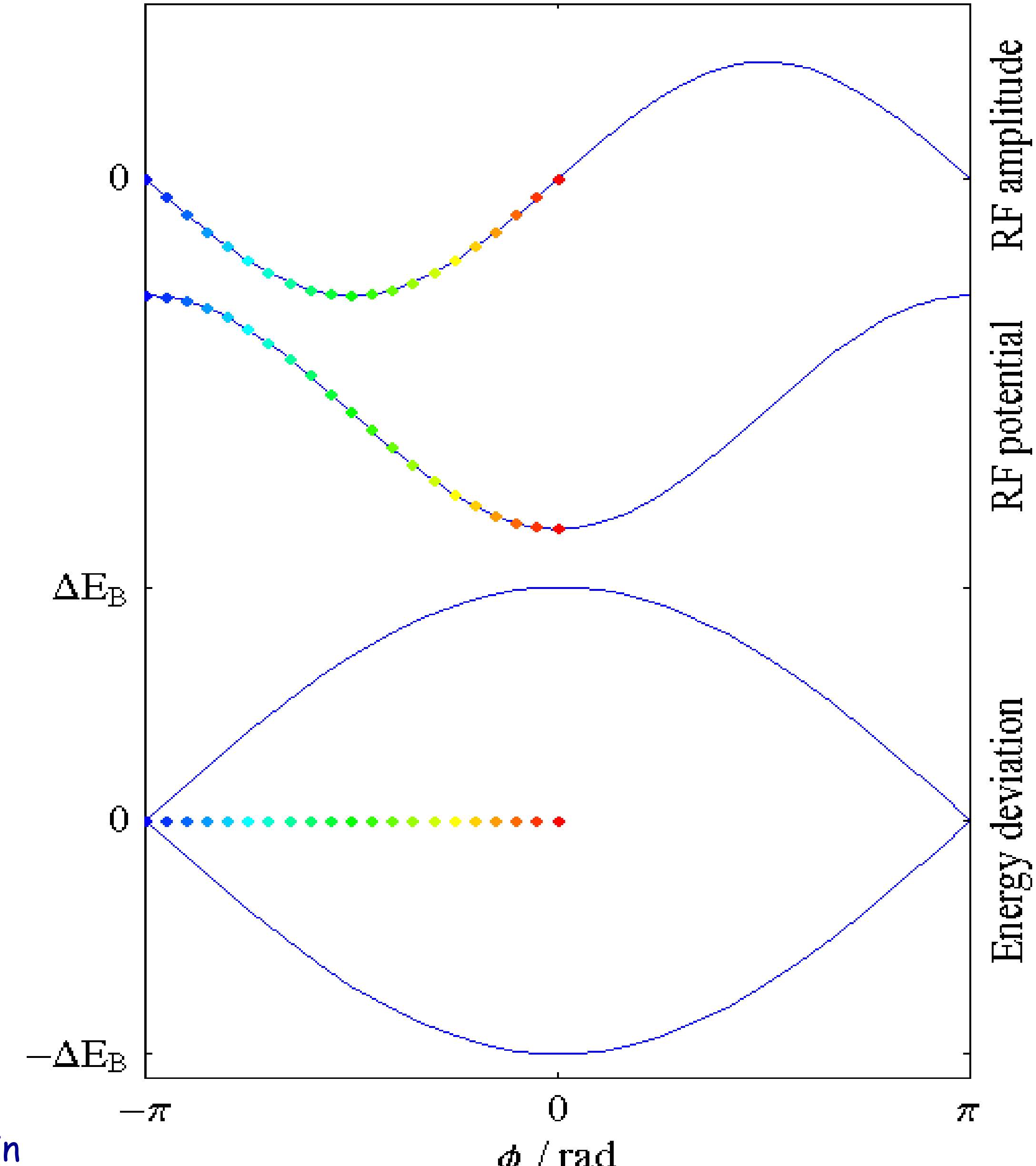
Bunch area = <u>longitudinal beam emittance</u> = $4\pi\, \sigma_E\, \sigma_t$ **[eVs]**

Attention: Different definitions are used!

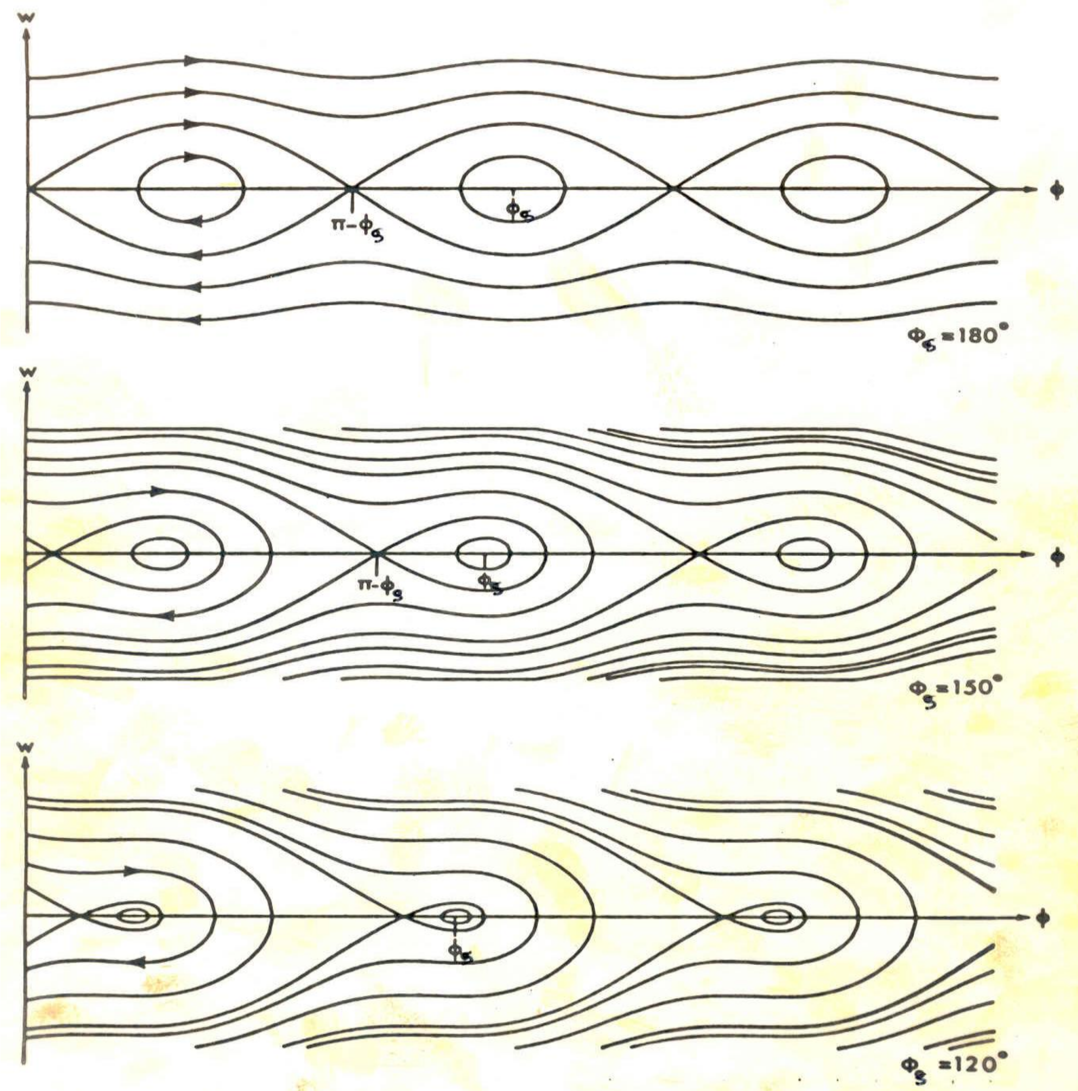
Synchrotron motion in phase space

The restoring force is non-linear.
⇒ speed of motion depends on position in phase-space

(here shown for a stationary bucket)



RF Acceptance versus Synchronous Phase



The areas of stable motion (closed trajectories) are called "BUCKET".

As the synchronous phase gets closer to 90° the buckets gets smaller.

The number of circulating buckets is equal to "h".

The phase extension of the bucket is maximum for ϕ_s =180° (or 0°) which correspond to no acceleration . The RF acceptance increases with the RF voltage.

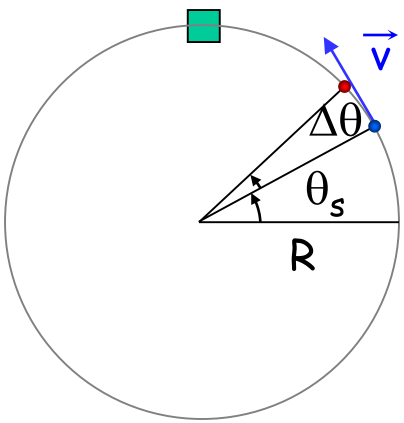
Longitudinal Dynamics in Synchrotrons

It is also often called "synchrotron motion".

The RF acceleration process clearly emphasizes two coupled variables, the energy gained by the particle and the RF phase experienced by the same particle. Since there is a well defined synchronous particle which has always the same phase ϕ_s, and the nominal energy E_s, it is sufficient to follow other particles with respect to that particle.
So let's introduce the following reduced variables:

revolution frequency	:	$\Delta f_r = f_r - f_{rs}$
particle RF phase	:	$\Delta\phi = \phi - \phi_s$
particle momentum	:	$\Delta p = p - p_s$
particle energy	:	$\Delta E = E - E_s$
azimuth angle	:	$\Delta\theta = \theta - \theta_s$

First Energy-Phase Equation



$$f_{RF} = h f_r \quad \Rightarrow \quad \Delta f = -h\,\Delta\theta \quad \textit{with} \quad \theta = \int \omega\, dt$$

particle ahead arrives earlier
=> smaller RF phase

For a given particle with respect to the reference one:

$$\Delta\omega = \frac{d}{dt}(\Delta\theta) = -\frac{1}{h}\frac{d}{dt}(\Delta\phi) = -\frac{1}{h}\frac{d\phi}{dt}$$

Since: $\eta = \dfrac{p_s}{\omega_{rs}}\left(\dfrac{d\omega}{dp}\right)_s$ and

$$E^2 = E_0^2 + p^2c^2$$

$$\Delta E = v_s \Delta p = \omega_{rs} R_s \Delta p$$

one gets:

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}}\frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}}\dot{\phi}$$

Second Energy-Phase Equation

The rate of energy gained by a particle is: $$\frac{dE}{dt} = e\hat{V}\sin\phi\,\frac{\omega_r}{2\pi}$$

The rate of relative energy gain with respect to the reference particle is then:

$$2\pi\Delta\left(\frac{\dot{E}}{\omega_r}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

Expanding the left-hand side to first order:

$$\Delta\left(\dot{E}T_r\right) \cong \dot{E}\,\Delta T_r + T_{rs}\,\Delta\dot{E} = \Delta E\,\dot{T}_r + T_{rs}\,\Delta\dot{E} = \frac{d}{dt}\left(T_{rs}\,\Delta E\right)$$

leads to the second energy-phase equation:

$$2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}\left(\sin\phi - \sin\phi_s\right)$$

Equations of Longitudinal Motion

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}} \frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}} \dot{\phi}$$

$$2\pi \frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

deriving and combining

$$\frac{d}{dt}\left[\frac{R_s p_s}{h\eta\omega_{rs}} \frac{d\phi}{dt}\right] + \frac{e\hat{V}}{2\pi}(\sin\phi - \sin\phi_s) = 0$$

This second order equation is non linear. Moreover the parameters within the bracket are in general slowly varying with time.

We will study some cases in the following…

Small Amplitude Oscillations

Let's assume constant parameters R_s, p_s, ω_s and η:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

with

$$\Omega_s^2 = \frac{h\eta\omega_{rs} e\hat{V}\cos\phi_s}{2\pi R_s p_s}$$

Consider now small phase deviations from the reference particle:

$$\sin\phi - \sin\phi_s = \sin\left(\phi_s + \Delta\phi\right) - \sin\phi_s \cong \cos\phi_s \Delta\phi$$

(for small $\Delta\phi$)

and the corresponding linearized motion reduces to a harmonic oscillation:

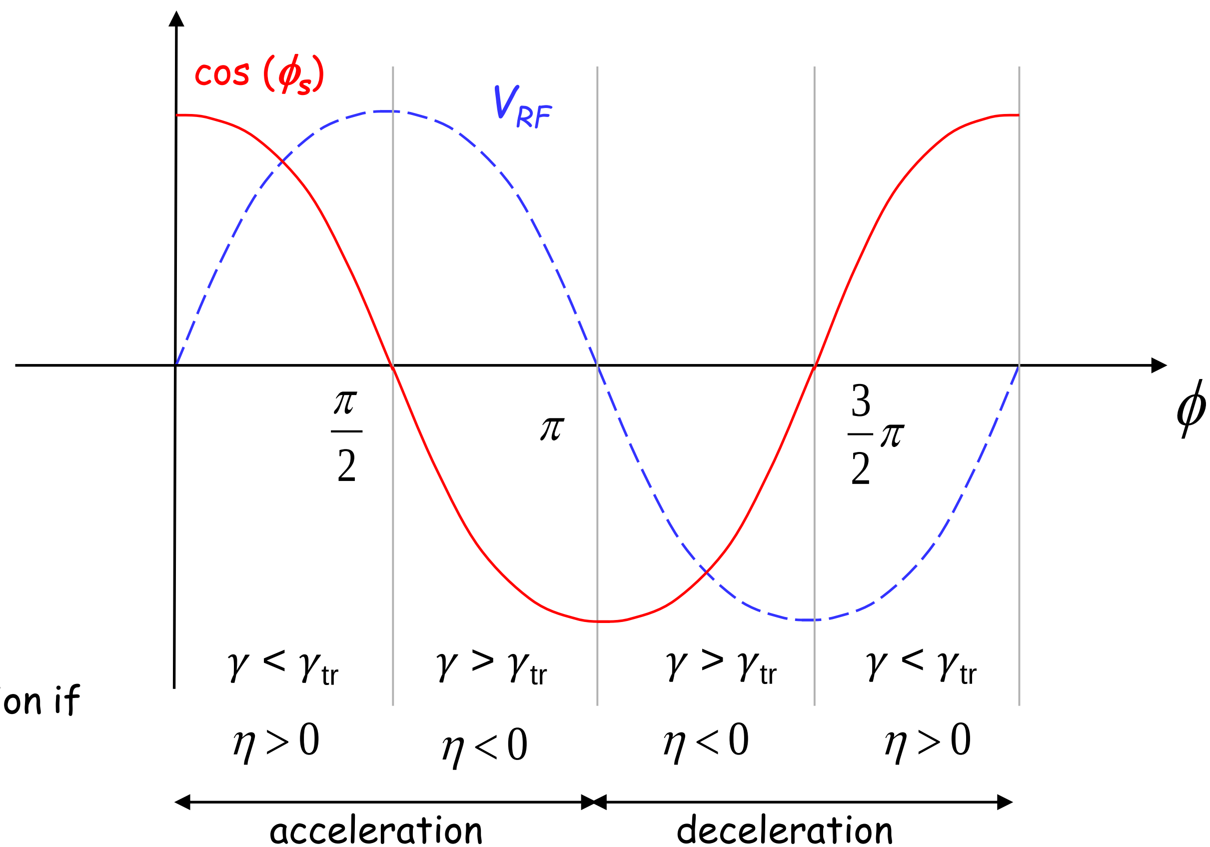
$$\ddot{f} + W_s^2 Df = 0$$

where Ω_s is the synchrotron angular frequency

Stability condition for ϕ_s

Stability is obtained when Ω_s is real and so Ω_s^2 positive:

$$\boxed{\Omega_s^2 = \frac{e\hat{V}_{RF}\, h\, \eta\, \omega_s}{2\pi R_s p_s}\cos\phi_s} \quad \Rightarrow \quad \Omega_s^2 > 0 \quad \Leftrightarrow \quad \boxed{\eta\cos\phi_s > 0}$$



Stable in the region if

Large Amplitude Oscillations

For larger phase (or energy) deviations from the reference the second order differential equation is non-linear:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

(Ω_s as previously defined)

Multiplying by $\dot{\phi}$ and integrating gives an invariant of the motion:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I$$

which for small amplitudes reduces to:

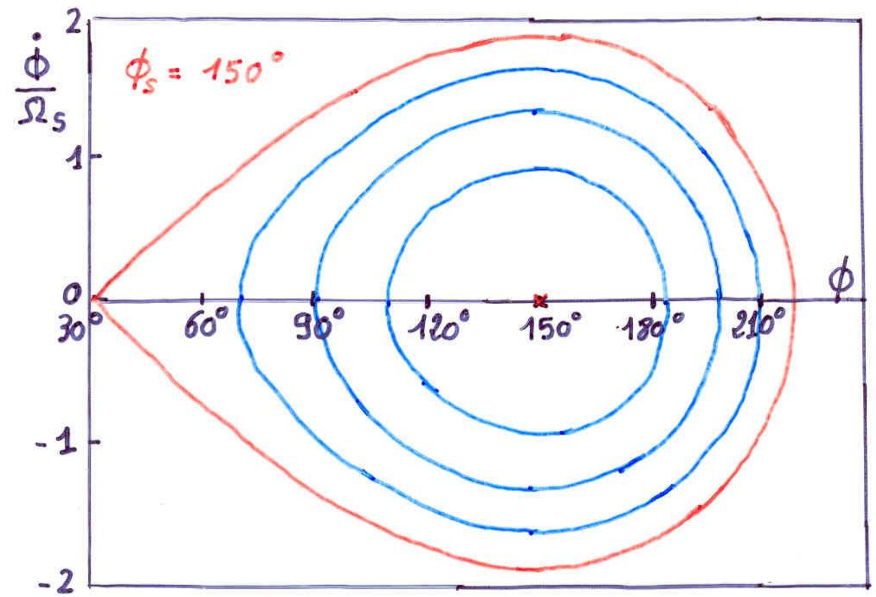
$$\frac{\dot{\phi}^2}{2} + \Omega_s^2\frac{(\Delta\phi)^2}{2} = I'$$

(the variable is $\Delta\phi$, and ϕ_s is constant)

Similar equations exist for the second variable : $\Delta E \propto d\phi/dt$

Large Amplitude Oscillations (2)

When ϕ reaches π-ϕ_s the force goes to zero and beyond it becomes non restoring.
Hence π-ϕ_s is an extreme amplitude for a stable motion which in the phase space($\frac{\dot{\phi}}{\Omega_s}, \Delta\phi$) is shown as closed trajectories.



Equation of the separatrix:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = -\frac{\Omega_s^2}{\cos\phi_s}\left(\cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s\right)$$

Second value ϕ_m where the separatrix crosses the horizontal axis:

$$\cos\phi_m + \phi_m\sin\phi_s = \cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s$$

Energy Acceptance

From the equation of motion it is seen that $\dot{\phi}$ reaches an extreme when $\ddot{\phi} = 0$, hence corresponding to $\phi = \phi_s$.

Introducing this value into the equation of the separatrix gives:

$$\dot{\phi}^2_{\max} = 2\Omega_s^2 \left\{ 2 + \left(2\phi_s - \pi \right) \tan\phi_s \right\}$$

That translates into an acceptance in energy:

$$\left(\frac{\Delta E}{E_s} \right)_{\max} = \mp \beta \sqrt{ -\frac{e\hat{V}}{\pi h \eta E_s} G\left(\phi_s\right) }$$

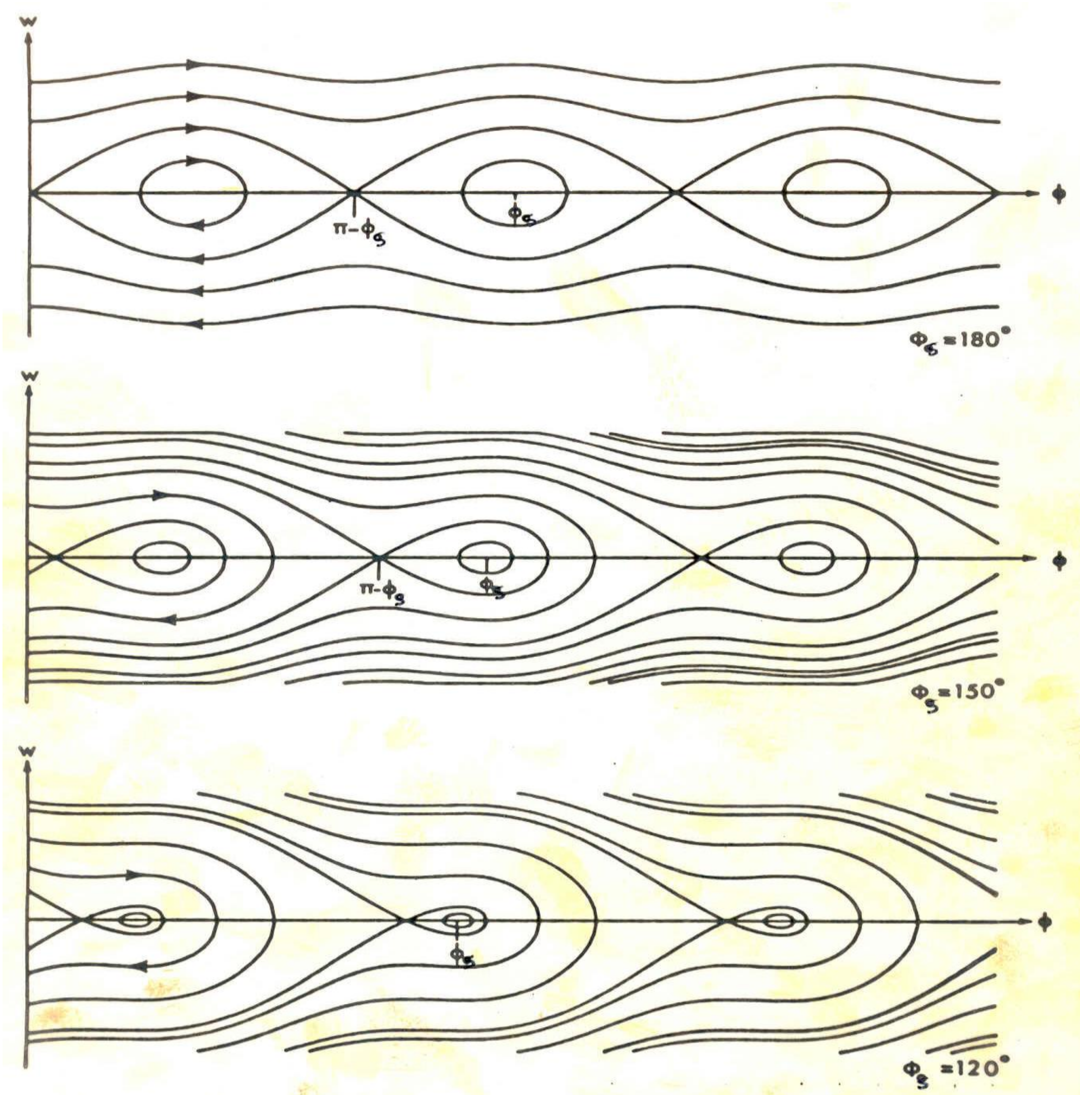
$$G\left(\phi_s\right) = \left[2\cos\phi_s + \left(2\phi_s - \pi \right) \sin\phi_s \right]$$

This "RF acceptance" depends strongly on ϕ_s and plays an important role for the capture at injection, and the stored beam lifetime.

It's largest for ϕ_s=0 and ϕ_s=π (no acceleration, depending on η).

Need a higher RF voltage for higher acceptance.

RF Acceptance versus Synchronous Phase



The areas of stable motion (closed trajectories) are called "BUCKET".

As the synchronous phase gets closer to 90° the buckets gets smaller.

The number of circulating buckets is equal to "h".

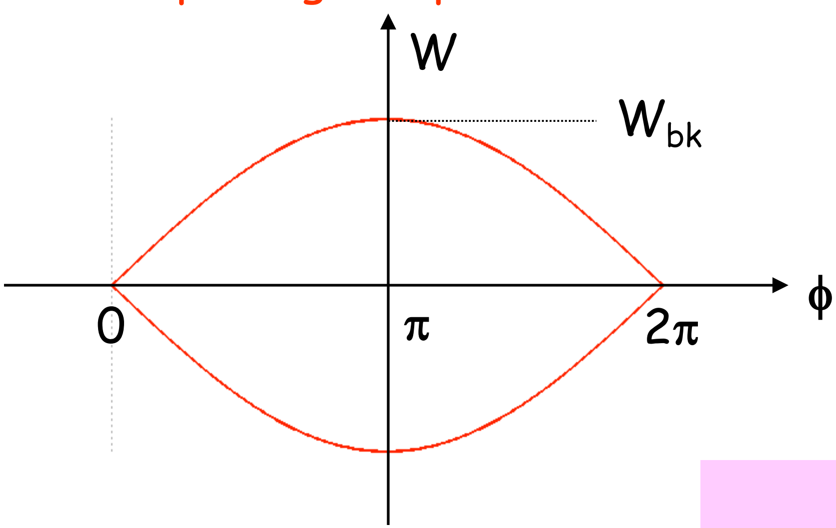
The phase extension of the bucket is maximum for ϕ_s =180° (or 0°) which correspond to no acceleration . The RF acceptance increases with the RF voltage.

Stationnary Bucket - Separatrix

This is the case $\sin\phi_s=0$ (no acceleration) which means $\phi_s=0$ or π . The equation of the separatrix for $\phi_s=\pi$ (above transition) becomes:

$$\frac{\dot{\phi}^2}{2}+\Omega_s^2\cos\phi=\Omega_s^2 \qquad\qquad \frac{\dot{\phi}^2}{2}=2\Omega_s^2\sin^2\frac{\phi}{2}$$

Replacing the phase derivative by the (canonical) variable W:



$$W=\frac{\Delta E}{\omega_{rf}}=-\frac{p_s R_s}{h\eta\omega_{rf}}\dot{\phi}$$

and introducing the expression for Ω_s leads to the following equation for the separatrix:

$$W=\pm\frac{C}{\pi hc}\sqrt{\frac{-e\hat{V}E_s}{2\pi h\eta}}\sin\frac{\phi}{2}=\pm W_{bk}\sin\frac{\phi}{2}$$

with $C=2\pi R_s$

Stationnary Bucket (2)

Setting ϕ=π in the previous equation gives the height of the bucket:

$$W_{bk} = \frac{C}{phc}\sqrt{\frac{-e\hat{V}E_s}{2phh}}$$

This results in the maximum energy acceptance:

$$DE_{\max} = w_{rf}W_{bk} = b_s\sqrt{2\frac{-e\hat{V}_{RF}E_s}{phh}}$$

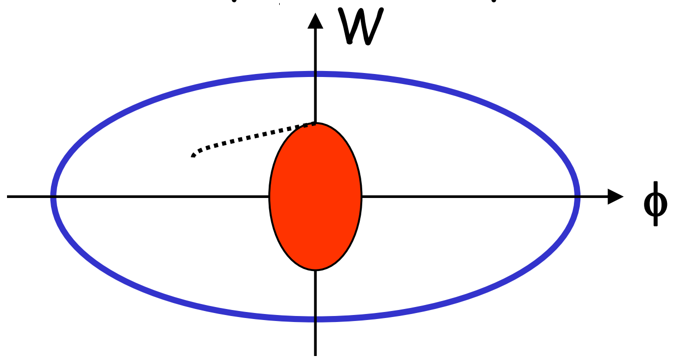
The area of the bucket is: $A_{bk} = 2\int_0^{2\pi} W\,d\phi$

Since: $\int_0^{2\pi}\sin\frac{\phi}{2}\,d\phi = 4$

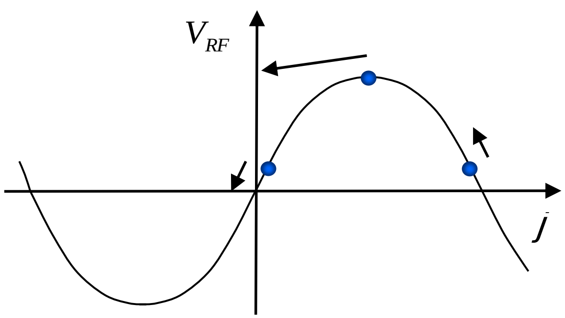
one gets: $A_{bk} = 8W_{bk} = 8\frac{C}{phc}\sqrt{\frac{-e\hat{V}E_s}{2phh}}$ $\longrightarrow$ $W_{bk} = \frac{A_{bk}}{8}$

Effect of a Mismatch

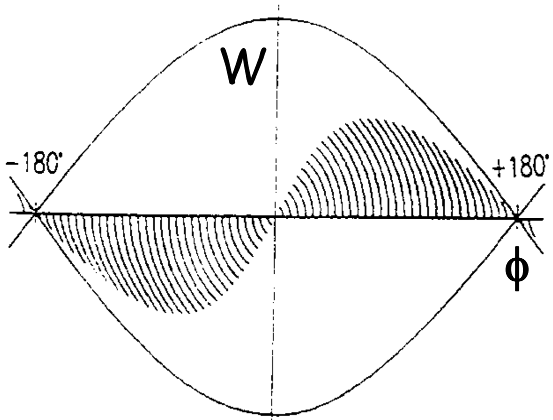
Injected bunch: short length and large energy spread
after 1/4 synchrotron period: longer bunch with a smaller energy spread.



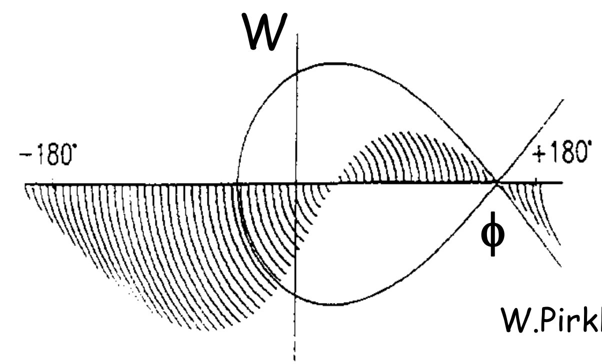
For larger amplitudes, the angular phase space motion is slower
(1/8 period shown below) => can lead to filamentation and emittance growth



restoring force is non-linear



stationary bucket

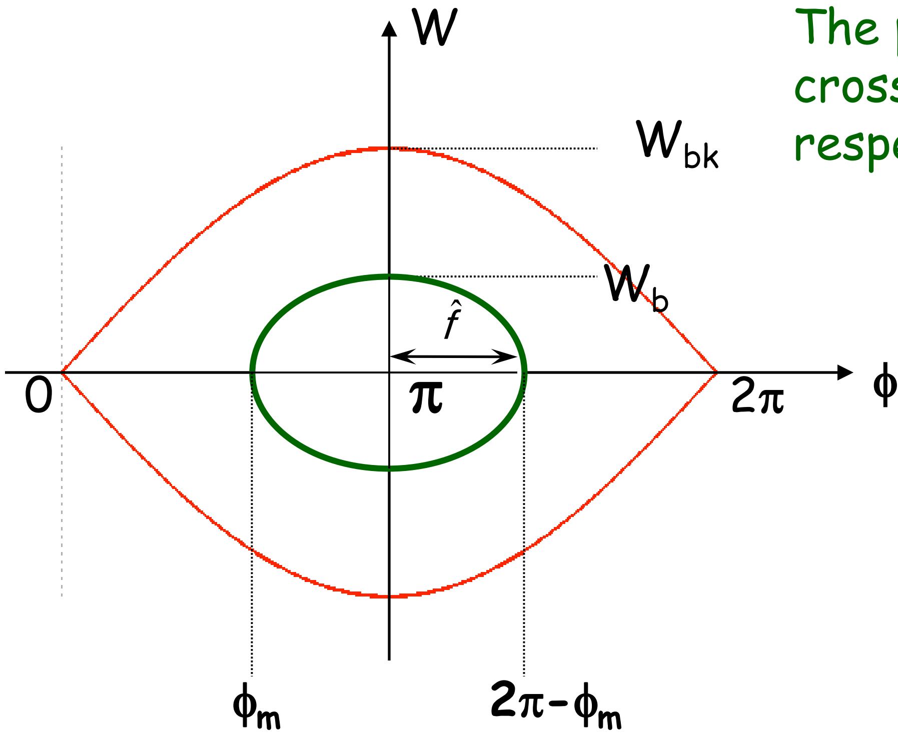


accelerating bucket

Bunch Matching into a Stationnary Bucket

A particle trajectory inside the separatrix is described by the equation:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I \quad \xrightarrow{\phi_s = \pi} \quad \boxed{\frac{\dot{\phi}^2}{2} + \Omega_s^2\cos\phi = I}$$



The points where the trajectory crosses the axis are symmetric with respect to $\phi_s = \pi$

$$\frac{\dot{\phi}^2}{2} + \Omega_s^2\cos\phi = \Omega_s^2\cos\phi_m$$

$$\dot{\phi} = \pm\Omega_s\sqrt{2\left(\cos\phi_m - \cos\phi\right)}$$

$$\boxed{W = \pm W_{bk}\sqrt{\cos^2\frac{j_m}{2} - \cos^2\frac{j}{2}}}$$

$$\cos(f) = 2\cos^2\frac{f}{2} - 1$$

Bunch Matching into a Stationnary Bucket (2)

Setting $\phi = \pi$ in the previous formula allows to calculate the bunch height:

$$W_b = W_{bk} \cos\frac{\phi_m}{2} = W_{bk} \sin\frac{\hat{\phi}}{2}$$

or:

$$W_b = \frac{A_{bk}}{8} \cos\frac{\phi_m}{2}$$

$$\longrightarrow \quad \left(\frac{\Delta E}{E_s}\right)_b = \left(\frac{\Delta E}{E_s}\right)_{RF} \cos\frac{\phi_m}{2} = \left(\frac{\Delta E}{E_s}\right)_{RF} \sin\frac{\hat{\phi}}{2}$$

This formula shows that for a given bunch energy spread the proper matching of a shorter bunch (ϕ_m close to π, $\hat{\phi}$ small) will require a bigger RF acceptance, hence a higher voltage

For small oscillation amplitudes the equation of the ellipse reduces to:

$$W = \frac{A_{bk}}{16}\sqrt{\hat{\phi}^2 - (\Delta\phi)^2} \quad \longrightarrow \quad \left(\frac{16\,W}{A_{bk}\hat{\phi}}\right)^2 + \left(\frac{\Delta\phi}{\hat{\phi}}\right)^2 = 1$$

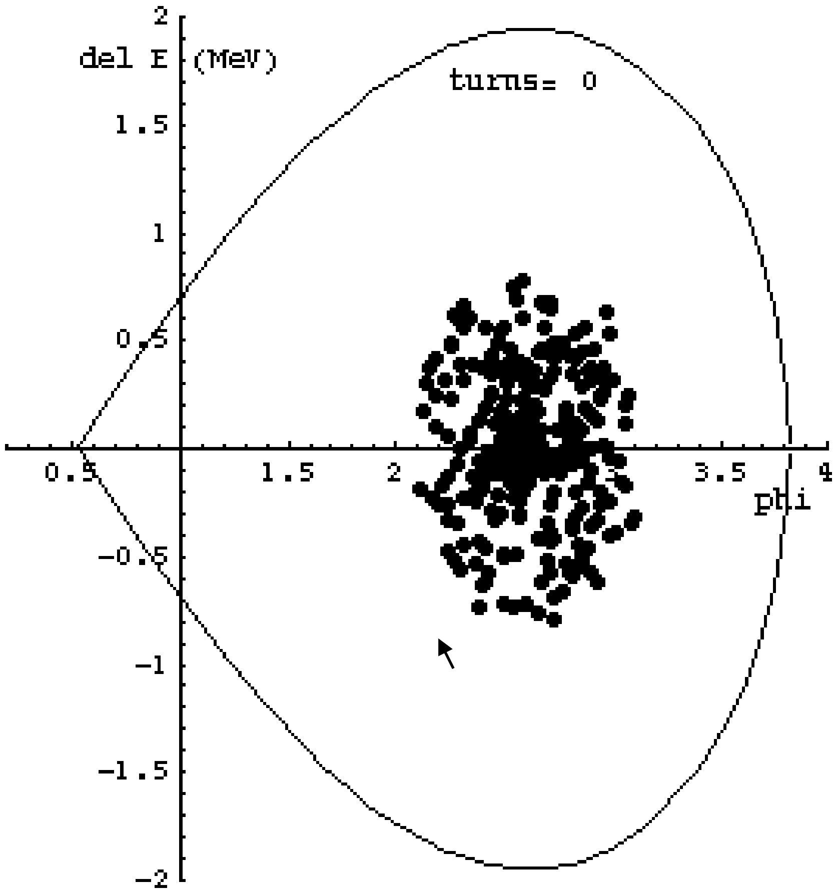
Ellipse area is called longitudinal emittance

$$A_b = \frac{\pi}{16} A_{bk} \hat{\phi}^2$$

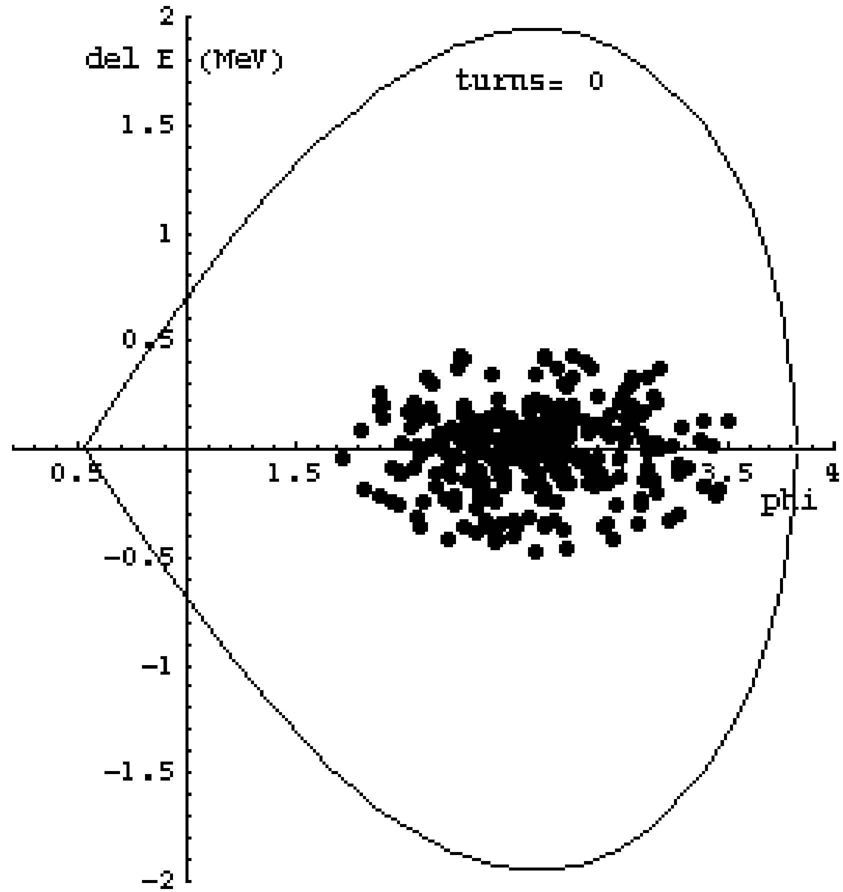
Effect of a Mismatch (2)

Evolution of an injected beam for the first 100 turns.

For a matched transfer, the emittance does not grow (left).



matched beam

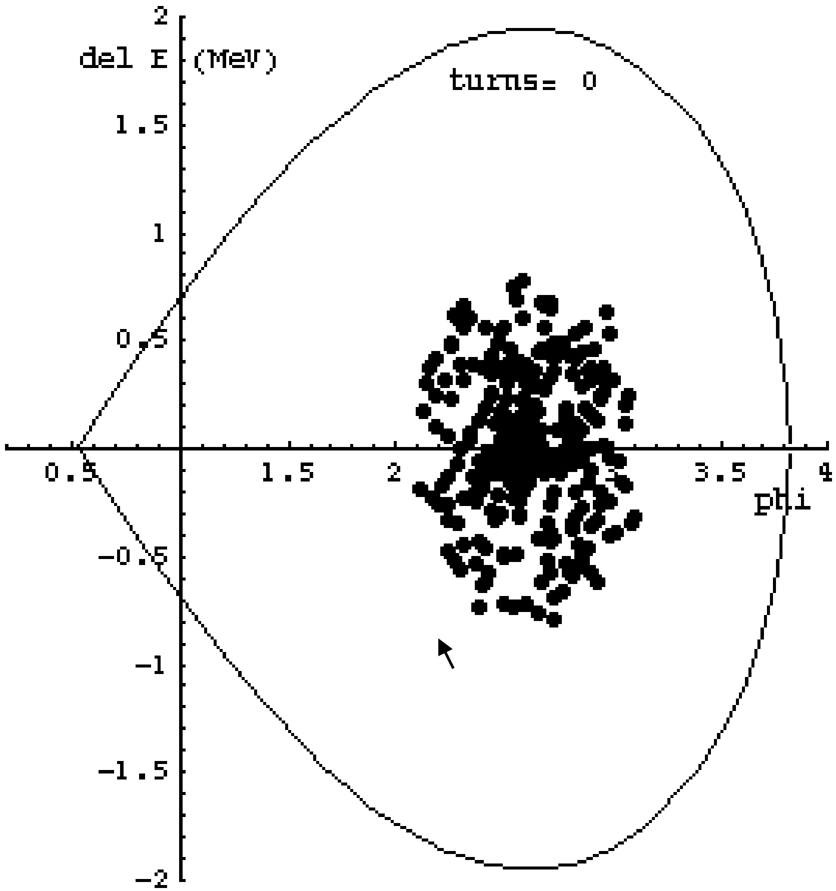


mismatched beam – bunch length

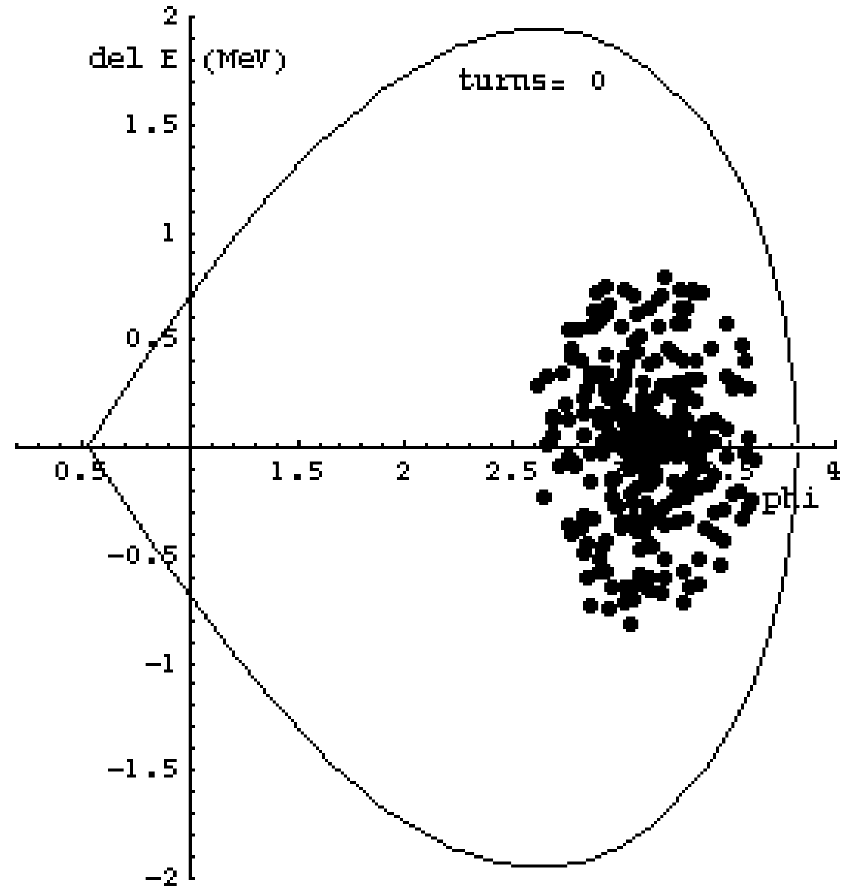
Effect of a Mismatch (3)

Evolution of an injected beam for the first 100 turns.

For a mismatched transfer, the emittance increases (right).



matched beam

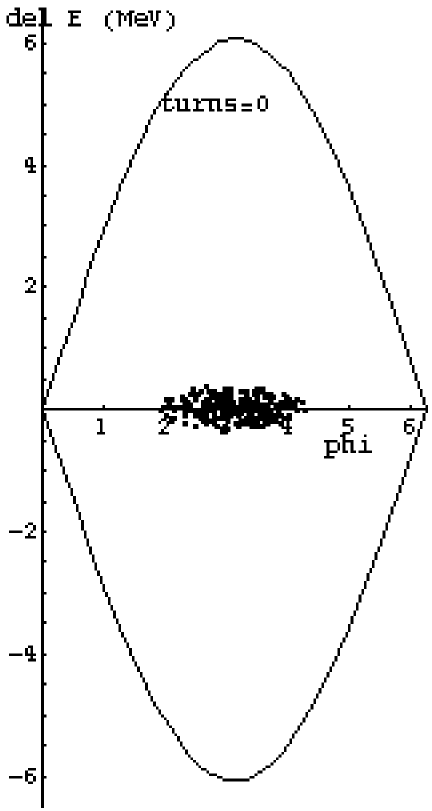


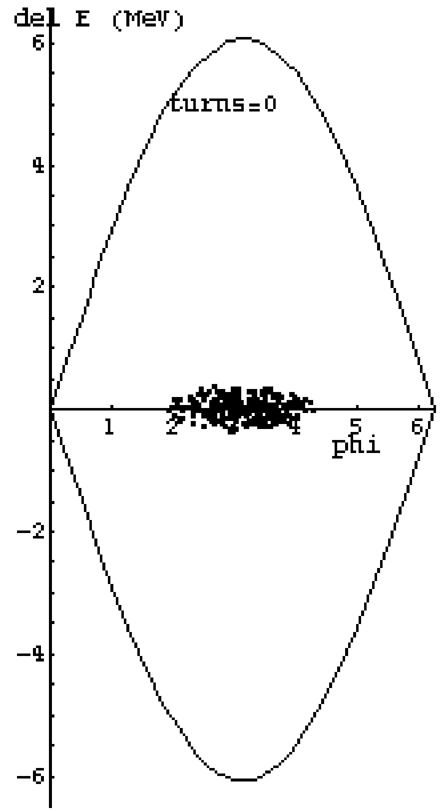
mismatched beam – phase error

Bunch Rotation

Phase space motion can be used to make short bunches.

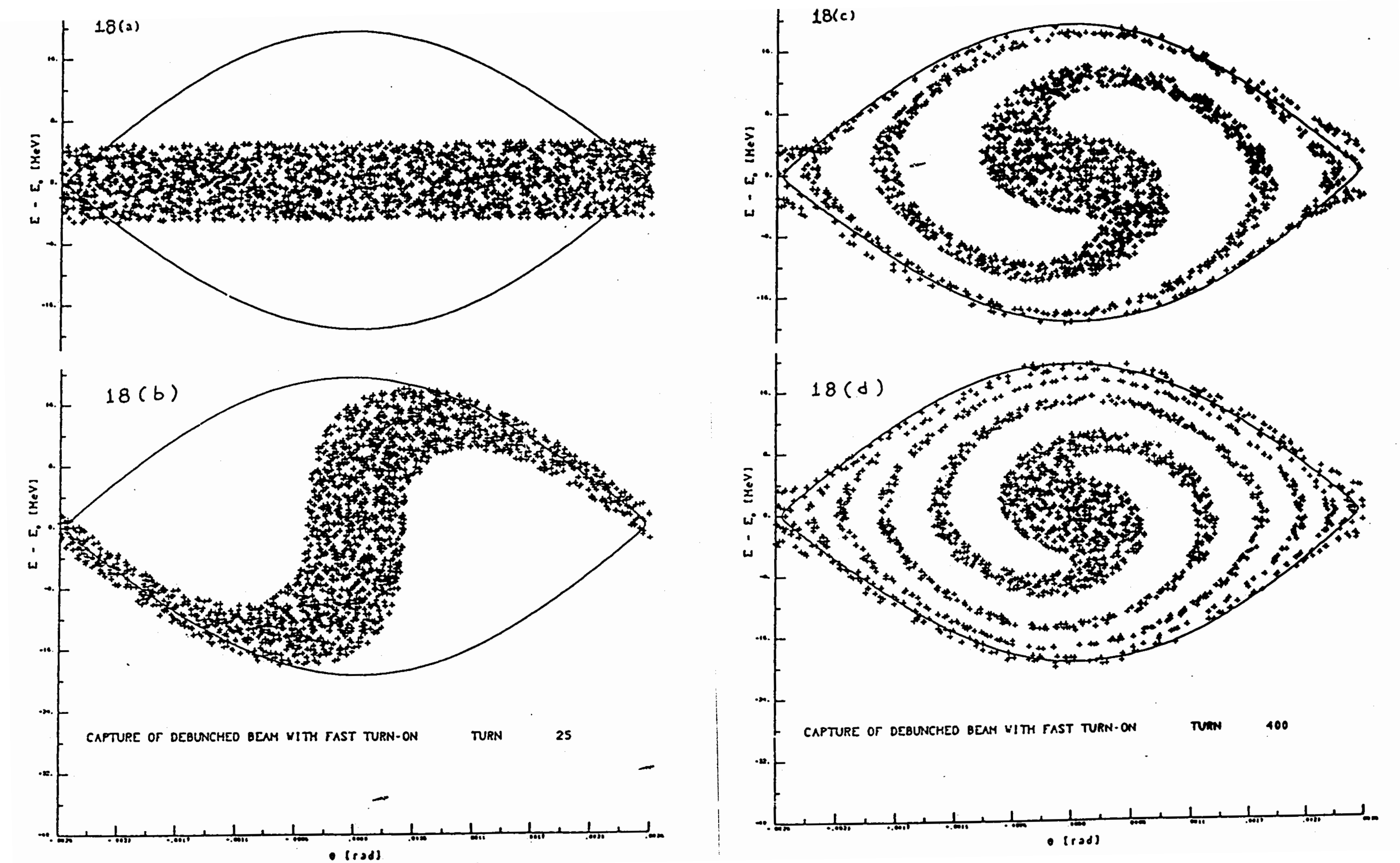
Start with a long bunch and extract or recapture when it's short.



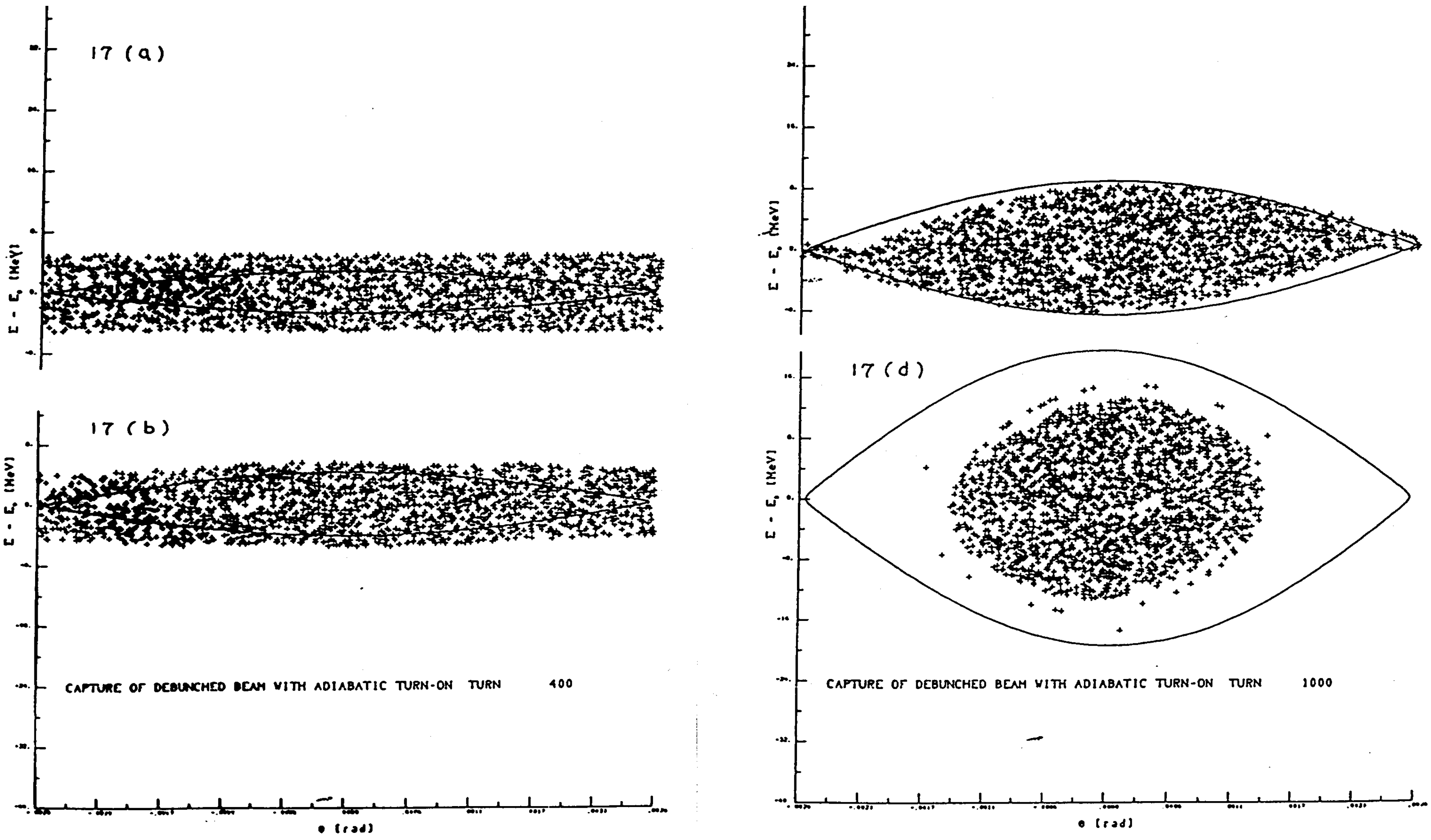


initial beam

Capture of a Debunched Beam with Fast Turn-On

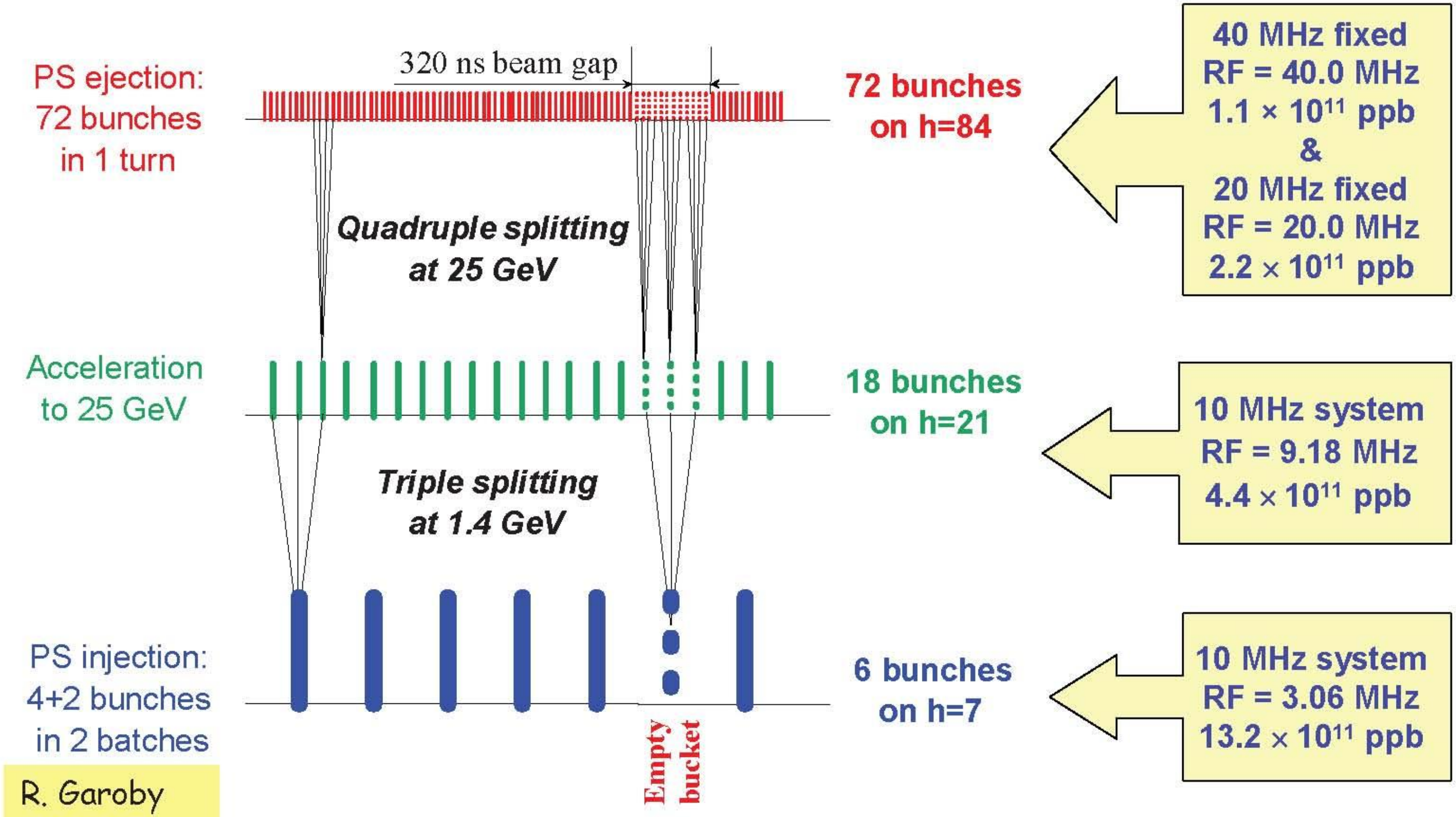


Capture of a Debunched Beam with Adiabatic Turn-On



Generating a 25ns LHC Bunch Train in the PS

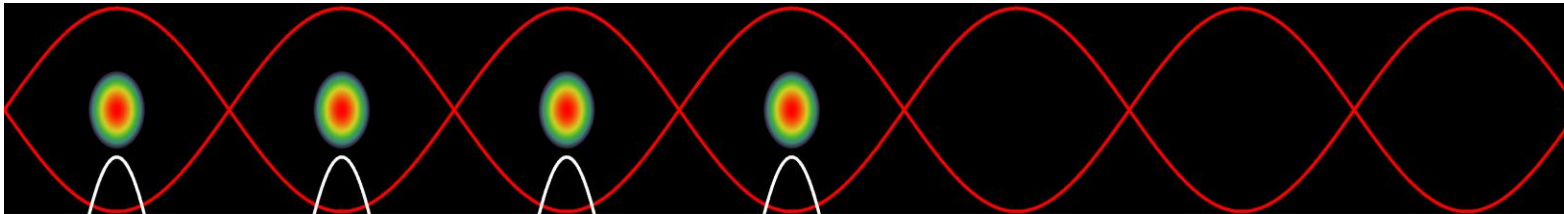
- **Longitudinal bunch splitting (basic principle)**
  - Reduce voltage on principal RF harmonic and simultaneously rise voltage on multiple harmonics (adiabatically with correct phase, etc.)



Use double splitting at 25 GeV to generate 50ns bunch trains instead

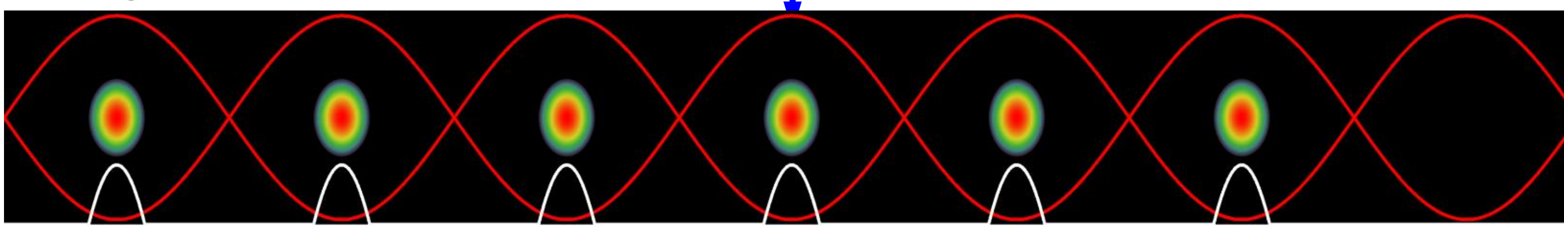
Production of the LHC 25 ns beam

1. Inject four bunches **~ 180 ns, 1.3 eVs**

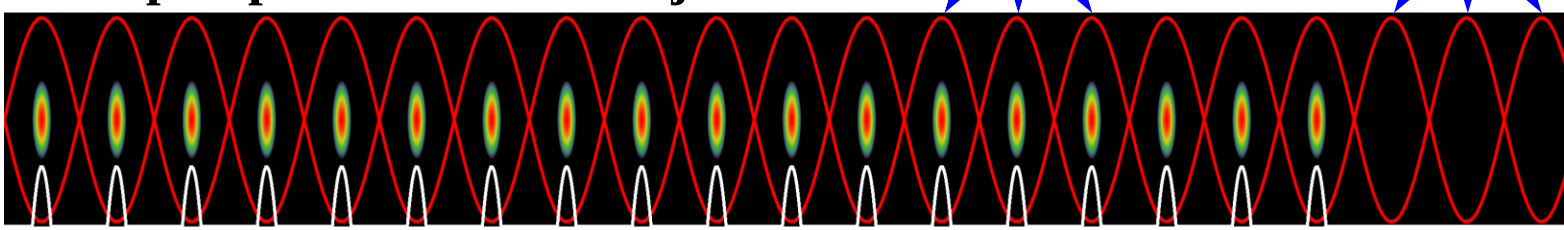


Wait 1.2 s for second injection

2. Inject two bunches



3. Triple split after second injection

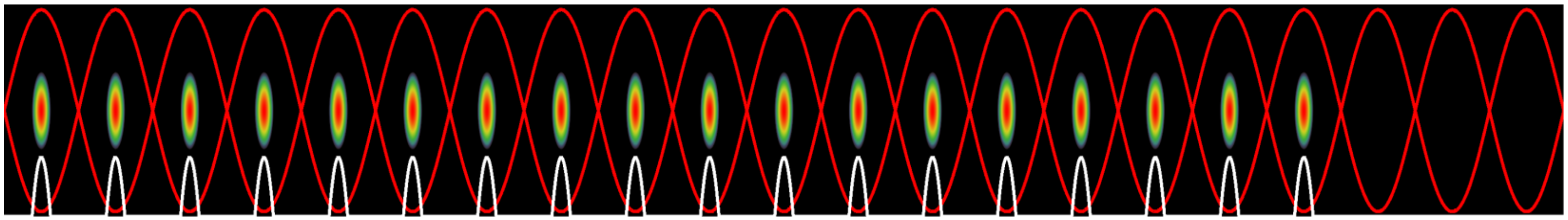


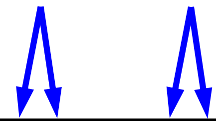
~ 0.7 eVs

4. Accelerate from 1.4 GeV (E_{kin}) to 26 GeV

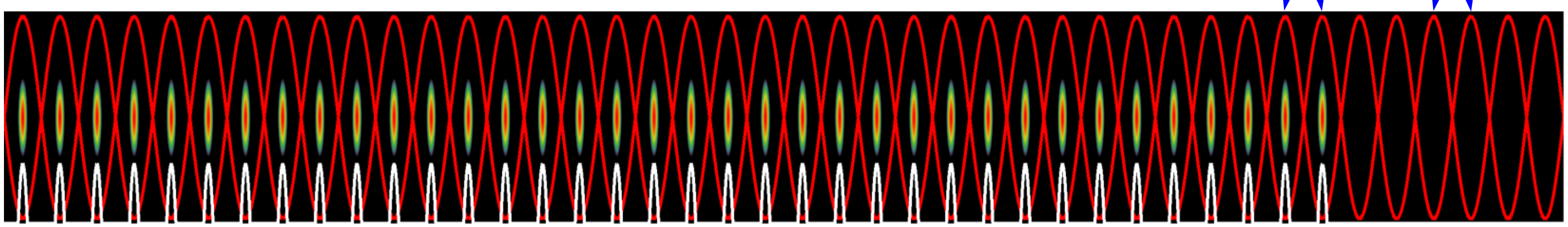
Production of the LHC 25 ns beam

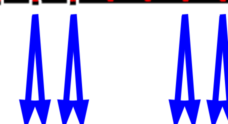
5. During acceleration: longitudinal emittance blow-up: 0.7 – 1.3 eVs



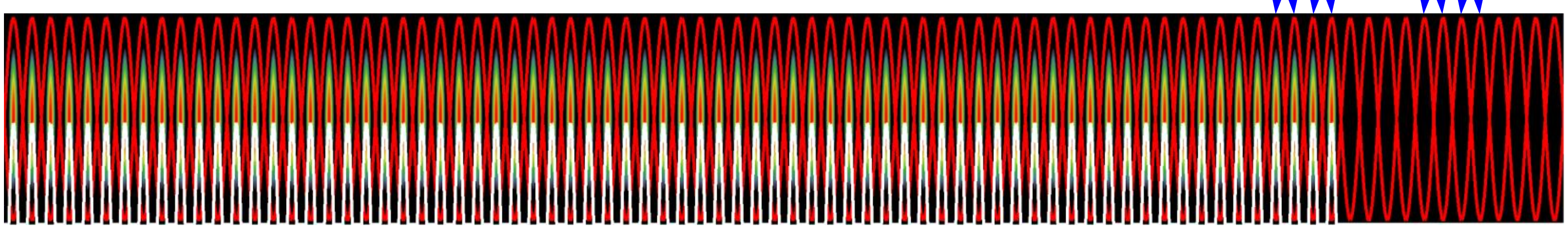


6. Double split ($h21 \rightarrow h42$)



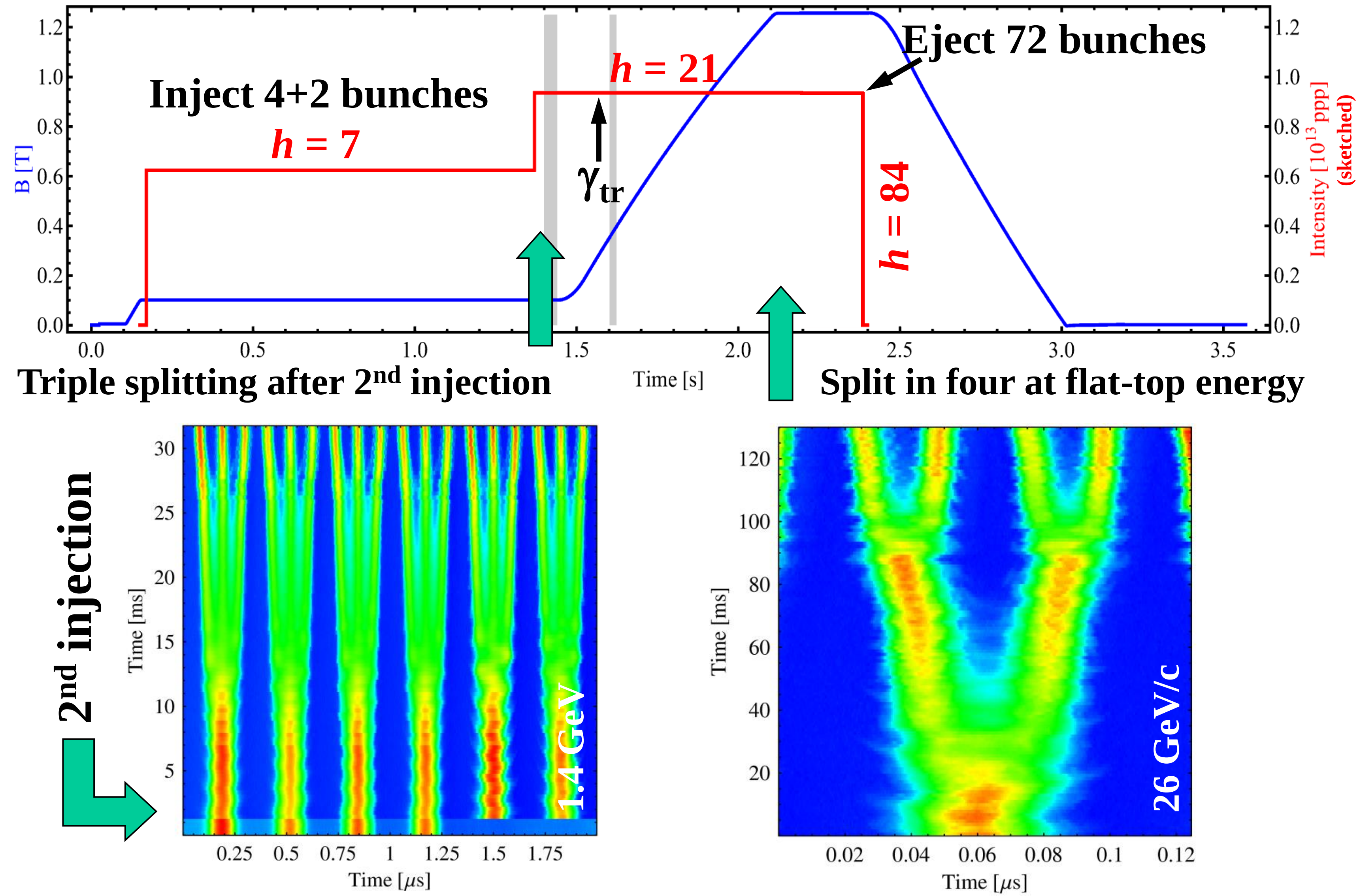


7. Double split ($h42 \rightarrow h84$) **~ 0.35 eVs, 4 ns**



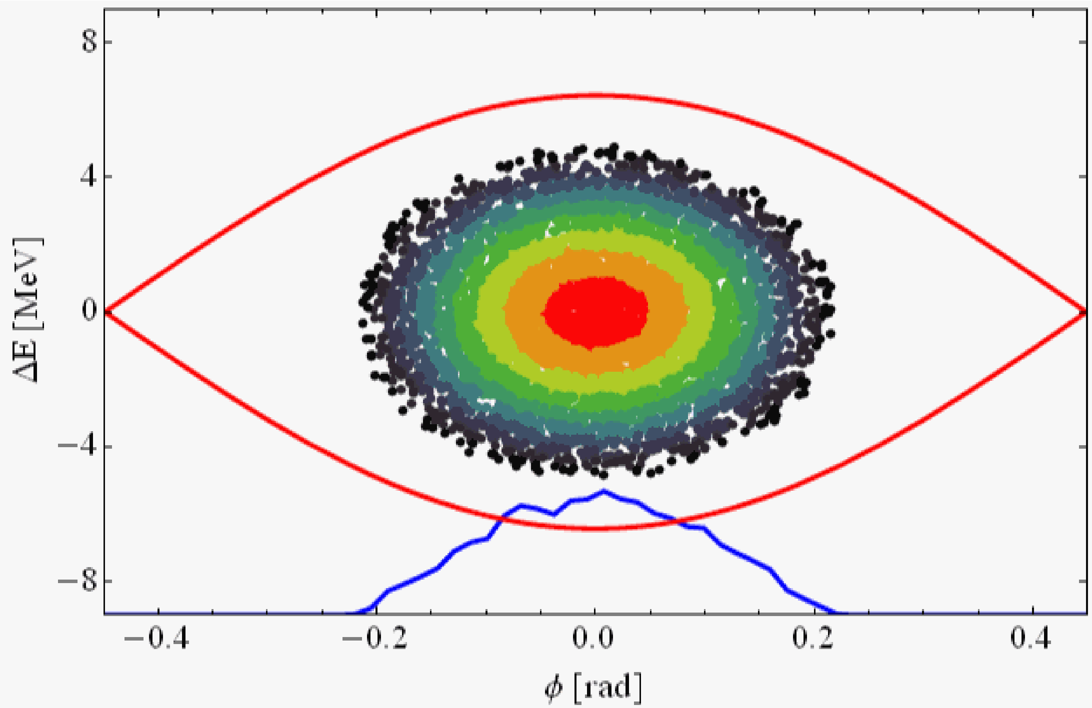
10. Fine synchronization, bunch rotation → Extraction!

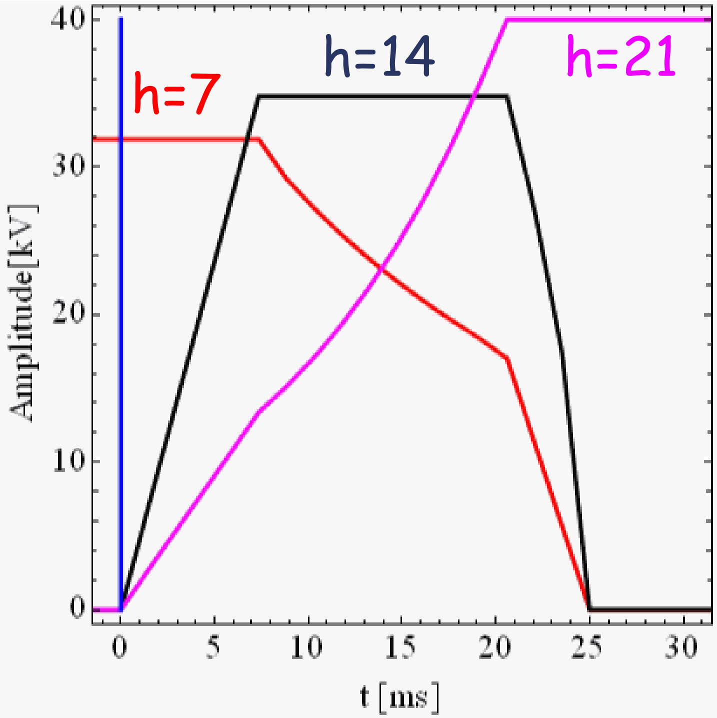
The LHC25 (ns) cycle in the PS



→ Each bunch from the Booster divided by 12 → 6 × 3 × 2 × 2 = 72

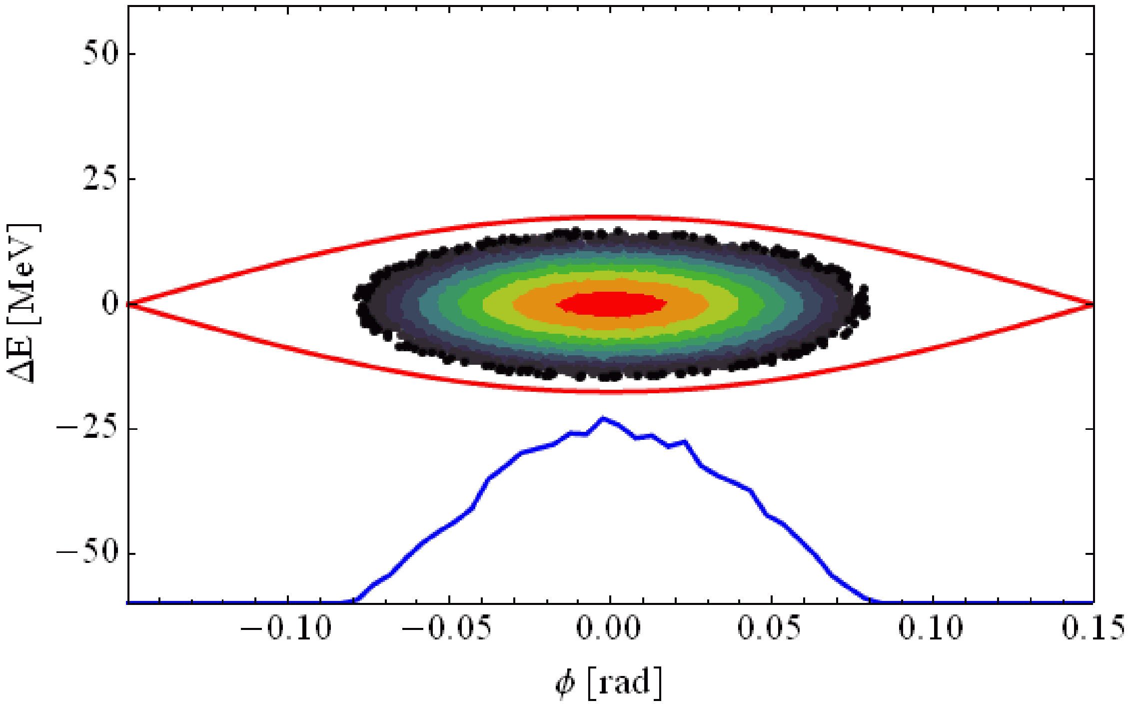
Triple splitting in the PS

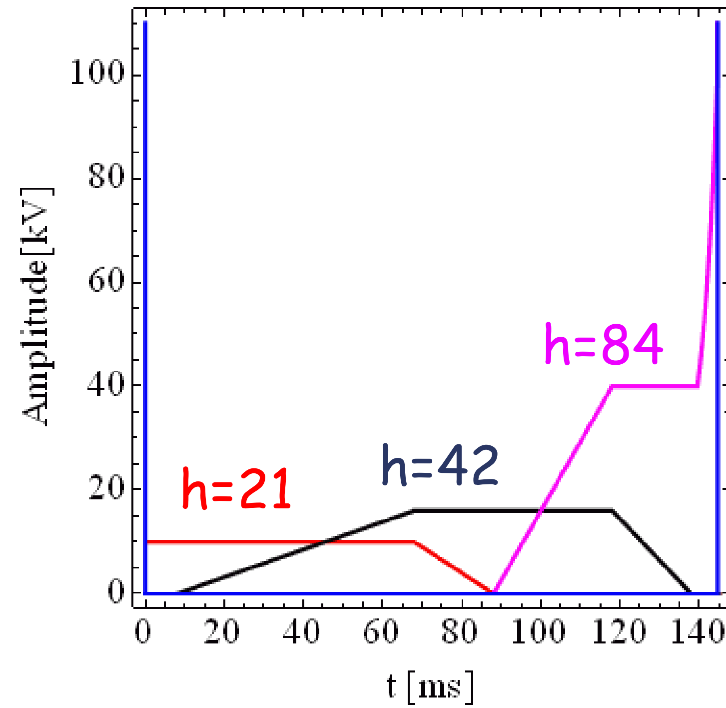




Two times double splitting in the PS

Two times double splitting and bunch rotation:





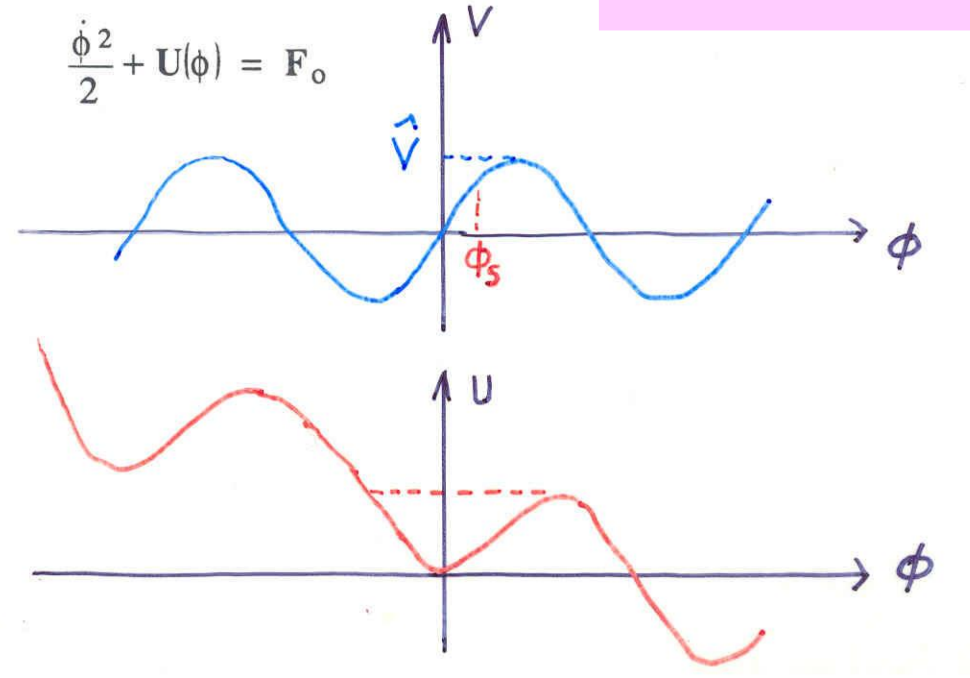
- Bunch is divided twice using RF systems at *h* = 21/42 (10/20 MHz) and *h* = 42/84 (20/40 MHz)
- Bunch rotation: first part *h*84 only + *h*168 (80 MHz) for final part

Potential Energy Function

The longitudinal motion is produced by a force that can be derived from a scalar potential:

$$\frac{d^2\phi}{dt^2} = F(\phi) \qquad F(\phi) = -\frac{\partial U}{\partial \phi}$$

$$U = -\int_0^\phi F(\phi)\,d\phi = -\frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) - F_0$$



The sum of the potential energy and kinetic energy is constant and by analogy represents the total energy of a non-dissipative system.

Hamiltonian of Longitudinal Motion

Introducing a new convenient variable, W, leads to the 1st order equations:

$$W = \frac{\Delta E}{\omega_{rf}} = 2\pi R_s \Delta p \longrightarrow \quad \frac{d\phi}{dt} = -\frac{h\eta\omega_{rf}}{p_s R_s} W \qquad \frac{dW}{dt} = \frac{1}{2\pi h} e\hat{V}\left(\sin\phi - \sin\phi_s\right)$$

The two variables ϕ,W are canonical since these equations of motion can be derived from a Hamiltonian H(ϕ,W,t):

$$\frac{d\phi}{dt} = \frac{\partial H}{\partial W} \qquad \frac{dW}{dt} = -\frac{\partial H}{\partial \phi}$$

$$H(\phi, W, t) = \frac{1}{2\pi h} e\hat{V}\left[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s\right] + \frac{1}{2}\frac{h\eta\omega_{rf}}{p_s R_s} W^2$$

Summary

- Cyclotrons/Synchrocylotrons for low energy
- Synchrotrons for high energies
  constant orbit, rising field and frequency
- Particles with higher energy have a longer orbit (normally) but a higher velocity
  - at low energies (below transition) velocity increase dominates
  - at high energies (above transition) velocity almost constant
- Particles perform oscillations around synchronous phase
  - synchronous phase depending on acceleration
  - below or above transition
- bucket is the region in phase space for stable oscillations
- matching the shape of the bunch to the bucket is important

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And CERN Accelerator Schools (CAS) Proceedings

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