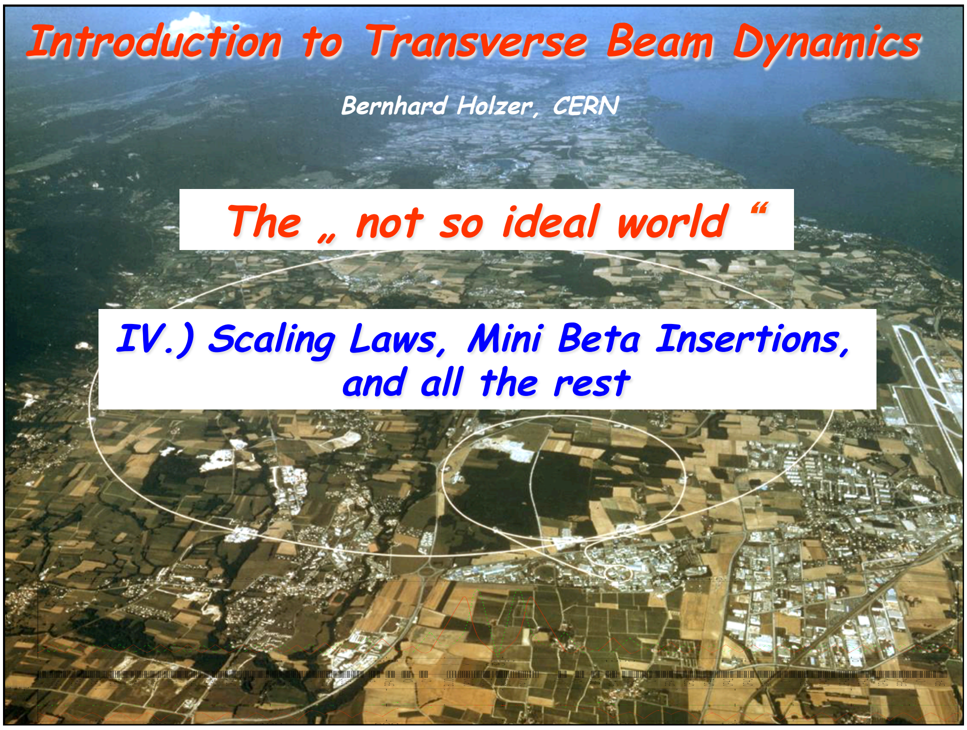


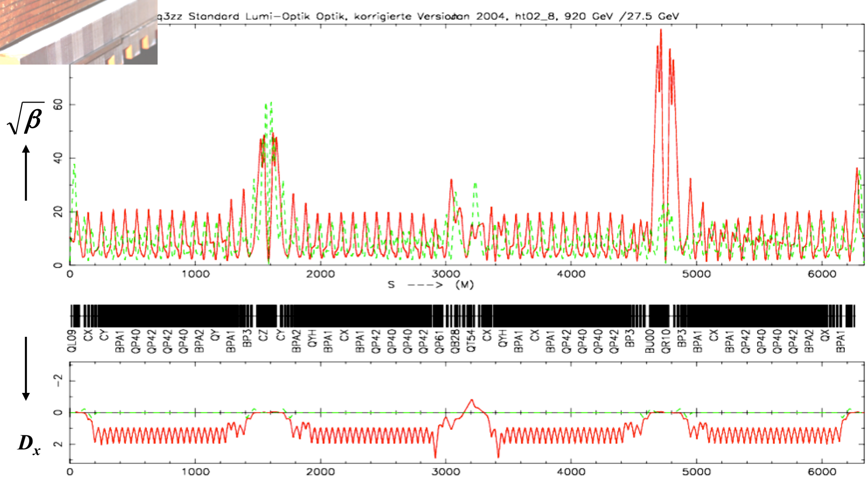
Introduction to Transverse Beam Dynamics

Bernhard Holzer, CERN

The „ not so ideal world “

IV.) Scaling Laws, Mini Beta Insertions, and all the rest

17.) Quadrupole Errors



Quadrupole Errors

go back to Lecture I, page 1
single particle trajectory

$$\begin{pmatrix} x \\ x' \end{pmatrix}_2 = M_{QF} * \begin{pmatrix} x \\ x' \end{pmatrix}_1$$

Solution of equation of motion

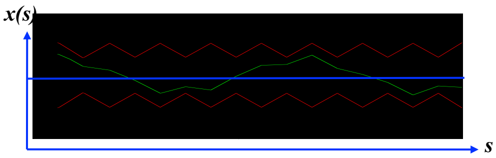
$$x = x_0 \cos(\sqrt{k}\, l_q) + x_0' \frac{1}{\sqrt{k}} \sin(\sqrt{k}\, l_q)$$

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{k}\, l_q) & \frac{1}{\sqrt{k}} \sin(\sqrt{k}\, l_q) \\ -\sqrt{k} \sin(\sqrt{k}\, l_q) & \cos(\sqrt{k}\, l_q) \end{pmatrix} \quad , \quad M_{thinlens} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

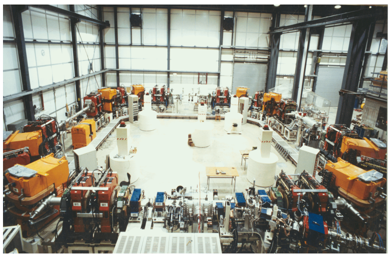
$$M_{turn} = M_{QF} * M_{D1} * M_{QD} * M_{D2} * M_{QF} \ \ldots$$

Definition: *phase advance of the particle oscillation per revolution in units of 2π is called tune*

$$Q = \frac{\psi_{turn}}{2\pi}$$



Matrix in Twiss Form



Transfer Matrix from point „0" in the lattice to point „s":

$$M(s) = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}} (\cos\psi_s + \alpha_0 \sin\psi_s) & \sqrt{\beta_s \beta_0} \sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos(\psi_s - (1 + \alpha_0 \alpha_s)\sin\psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}} (\cos(\psi_s - \alpha_0 \sin\psi_s) \end{pmatrix}$$

For one complete turn the Twiss parameters have to obey periodic bundary conditions:

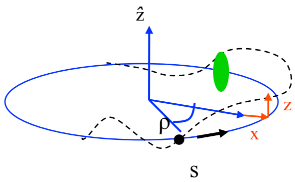
$$\beta(s+L) = \beta(s)$$
$$\alpha(s+L) = \alpha(s)$$
$$\gamma(s+L) = \gamma(s)$$

$$M(s) = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_s & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

Quadrupole Error in the Lattice

optic perturbation described by thin lens quadrupole

$$M_{dist} = M_{\Delta k} \cdot M_0 = \underbrace{\begin{pmatrix} 1 & 0 \\ \Delta k ds & 1 \end{pmatrix}}_{\text{quad error}} \cdot \underbrace{\begin{pmatrix} \cos\psi_{turn} + \alpha \sin\psi_{turn} & \beta \sin\psi_{turn} \\ -\gamma \sin\psi_{turn} & \cos\psi_{turn} - \alpha \sin\psi_{turn} \end{pmatrix}}_{\text{ideal storage ring}}$$



$$M_{dist} = \begin{pmatrix} \cos\psi_0 + \alpha \sin\psi_0 & \beta \sin\psi_0 \\ \Delta k ds (\cos\psi_0 + \alpha \sin\psi_0) - \gamma \sin\psi_0 & \Delta k ds\, \beta \sin\psi_0 + \cos\psi_0 - \alpha \sin\psi_0 \end{pmatrix}$$

rule for getting the tune

$$Trace(M) = 2\cos\psi = 2\cos\psi_0 + \Delta k ds \beta \sin\psi_0$$

Quadrupole error → Tune Shift

$$\psi = \psi_0 + \Delta\psi \quad \longrightarrow \quad \cos(\psi_0 + \Delta\psi) = \cos\psi_0 + \frac{\Delta k ds\, \beta \sin\psi_0}{2}$$

remember the old fashioned trigonometric stuff and assume that the error is small !!!

$$\cos\psi_0 \underbrace{\cos\Delta\psi}_{\approx 1} - \sin\psi_0 \underbrace{\sin\Delta\psi}_{\approx \Delta\psi} = \cos\psi_0 + \frac{k ds\, \beta \sin\psi_0}{2}$$

$$\Delta\psi = \frac{k ds\, \beta}{2}$$

and referring to Q instead of ψ:

$$\psi = 2\pi Q$$

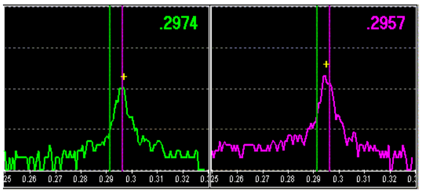
$$\Delta Q = \int_{s0}^{s0+l} \frac{\Delta k(s) \beta(s) ds}{4\pi}$$

! the tune shift is proportional to the β-function at the quadrupole

!! field quality, power supply tolerances etc are much tighter at places where β is large

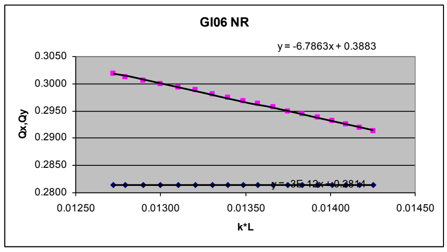
!!! mini beta quads: β ≈ 1900 m
arc quads: β ≈ 80 m

!!!! β is a measure for the sensitivity of the beam *a quadrupol error leads to a shift of the tune:*



$$\Delta Q = \int_{s0}^{s0+l} \frac{\Delta k \beta(s)}{4\pi} ds \approx \frac{\Delta k l_{quad} \bar{\beta}}{4\pi}$$

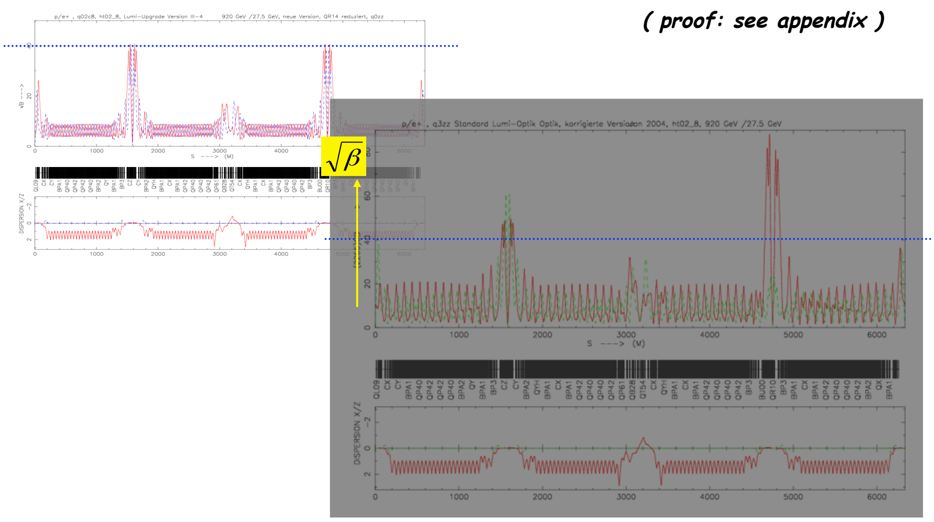
Example: measurement of β in a storage ring:
tune spectrum



Quadrupole error: Beta Beat

$$\Delta\beta(s_0) = \frac{\beta_0}{2\sin 2\pi Q} \int_{s1}^{s1+l} \beta(s_1)\Delta K \cos\left(2|\psi_{s1} - \psi_{s0}| - 2\pi Q\right) ds$$

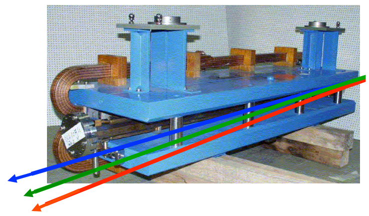
(proof: see appendix)



18.) Chromaticity:
A Quadrupole Error for $\Delta p/p \neq 0$

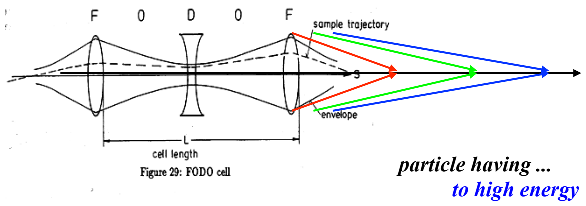
Influence of external fields on the beam: *prop. to magn. field & prop. zu 1/p*

dipole magnet $\alpha = \dfrac{\int B\,dl}{p/e}$



$x_D(s) = D(s)\dfrac{\Delta p}{p}$

focusing lens $k = \dfrac{g}{p/e}$



particle having ...
to high energy
to low energy
ideal energy

Chromaticity: Q'

$$k = \frac{g}{p/e} \qquad\qquad p = p_0 + \Delta p$$

in case of a momentum spread:

$$k = \frac{eg}{p_0 + \Delta p} \approx \frac{e}{p_0}\left(1 - \frac{\Delta p}{p_0}\right) g = k_0 + \Delta k$$

$$\Delta k = -\frac{\Delta p}{p_0} k_0$$

... which acts like a quadrupole error in the machine and leads to a tune spread:

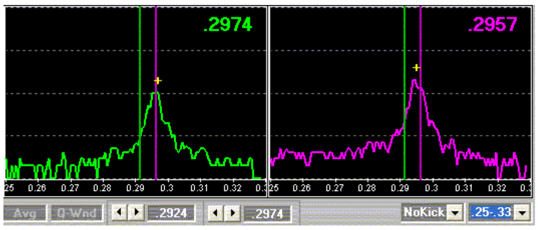
$$\Delta Q = -\frac{1}{4\pi}\frac{\Delta p}{p_0} k_0 \beta(s) ds$$

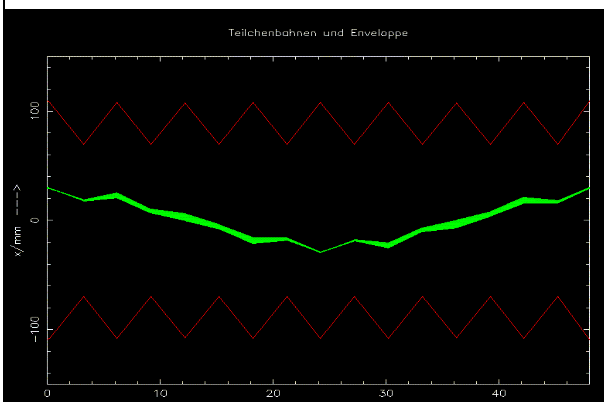
definition of chromaticity:

$$\Delta Q = Q' \frac{\Delta p}{p} \quad ; \qquad Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)ds$$

Where is the Problem ?

Tunes and Resonances





avoid resonance conditions:

$$m\, Q_x + n\, Q_y + l\, Q_s = integer$$

… for example: 1 $Q_x = 1$

… and now again about Chromaticity:

Problem: chromaticity is generated by the lattice itself !!

Q' is a number indicating the size of the tune spot in the working diagram,
Q' is always created if the beam is focussed
→ *it is determined by the focusing strength k of all quadrupoles*

$$Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)ds$$

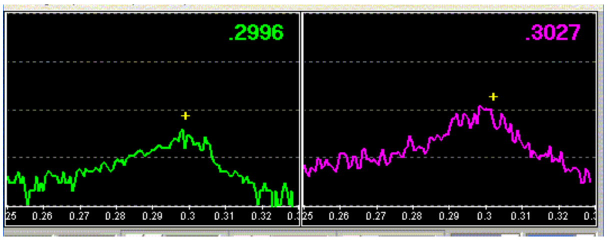
k = quadrupole strength
β = betafunction indicates the beam size … and even more the sensitivity of the beam to external fields

Example: LHC

Q' = 250
*Δ p/p = +/- 0.2 *10⁻³*
Δ Q = 0.256 … 0.36

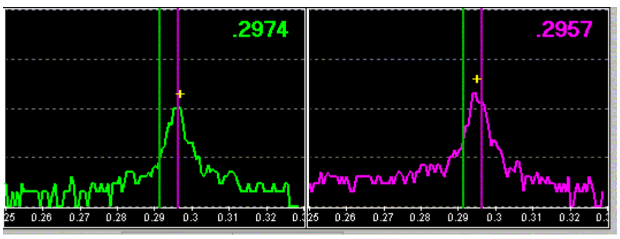
→ *Some particles get very close to resonances and are lost*

in other words: the tune is not a point it is a pancake



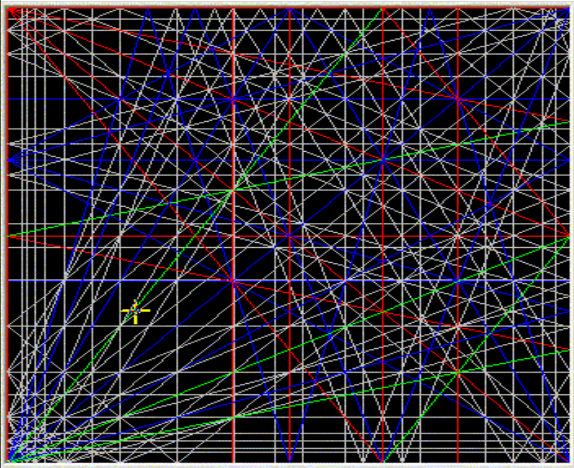
Tune signal for a nearly uncompensated cromaticity (Q' ≈ 20)

Ideal situation: cromaticity well corrected, (Q' ≈ 1)



Tune and Resonances

$$m^*Q_x + n^*Q_y + l^*Q_s = \text{integer}$$



RA e Tune diagram up to 3rd order

… and up to 7th order

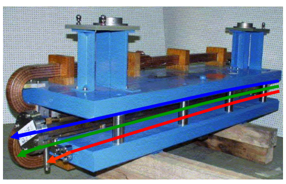
Homework for the operateurs: find a nice place for the tune where against all probability the beam will survive

Correction of Q':

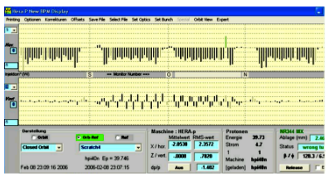
Need: additional quadrupole strength for each momentum deviation Δp/p

1.) sort the particles acording to their momentum

$$x_D(s) = D(s)\frac{\Delta p}{p}$$



… using the dispersion function



2.) apply a magnetic field that rises quadratically with x (sextupole field)

$$B_x = \tilde{g}xy$$

$$B_y = \frac{1}{2}\tilde{g}(x^2 - y^2)$$

$$\frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x} = \tilde{g}x$$

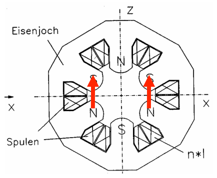
linear amplitude dependent „gradient":

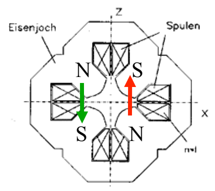
Correction of Q':

k_1 normalised quadrupole strength

k_2 normalised sextupole strength

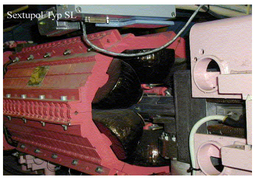
Sextupole Magnets:





$$k_1(sext) = \frac{\tilde{g}x}{p/e} = k_2 * x$$

$$= k_2 * D\frac{\Delta p}{p}$$



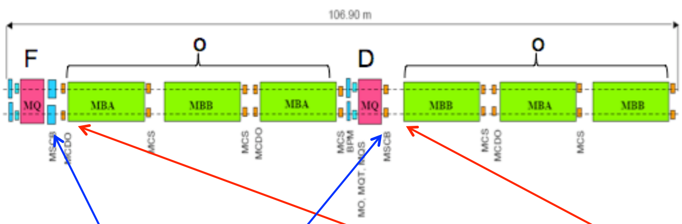
Combined effect of „natural chromaticity" and Sextupole Magnets:

$$Q' = -\frac{1}{4\pi}\left\{\int k_1(s)\beta(s)ds + \int k_2 * D(s)\beta(s)ds\right\}$$

You only should not forget to correct Q ' in both planes ...
and take into account the contribution from quadrupoles of both polarities.

corrected chromaticity

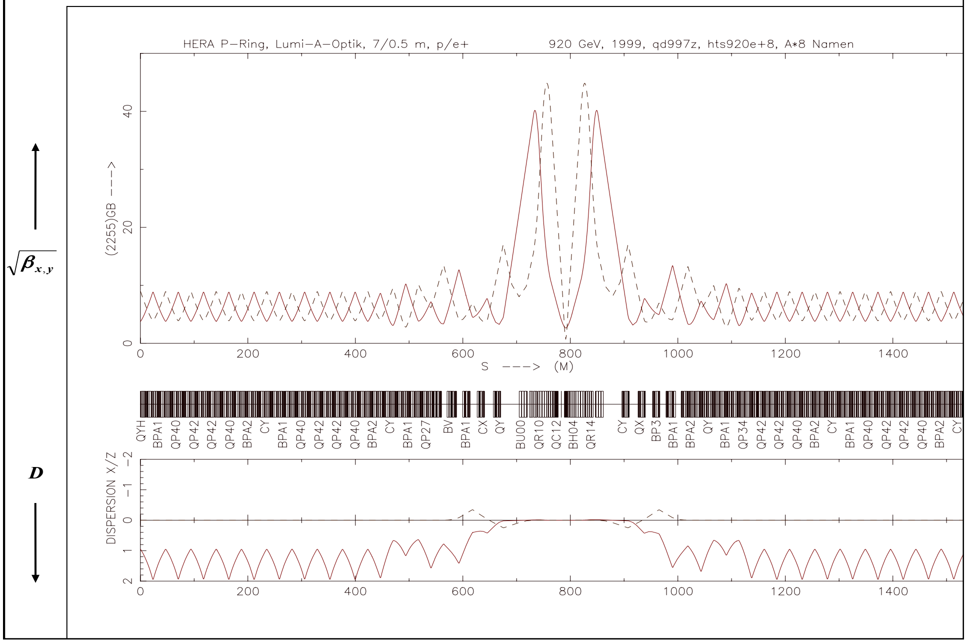
considering an arc built out of single cells:



$$Q'_x = -\frac{1}{4\pi}\left\{\sum_{F\,quad} k_{qf}\hat{\beta}_x l_{qf} - \sum_{D\,quad} k_{qd}\breve{\beta}_x l_{qd}\right\} + \frac{1}{4\pi}\sum_{F\,sext} k_2^F l_{sext} D_x^F \beta_x^F - \frac{1}{4\pi}\sum_{D\,sext} k_2^D l_{sext} D_x^D \beta_x^D$$

$$Q'_y = -\frac{1}{4\pi}\left\{-\sum_{F\,quad} k_{qf}\breve{\beta}_y l_{qf} + \sum_{D\,quad} k_{qd}\hat{\beta}_y l_{qd}\right\} - \frac{1}{4\pi}\sum_{F\,sext} k_2^F l_{sext} D_x^F \beta_x^F + \frac{1}{4\pi}\sum_{D\,sext} k_2^D l_{sext} D_x^D \beta_x^D$$

20.) Insertions



Insertions

... the most complicated one: ***the drift space***

Question to the audience: ***what will happen to the beam parameters α, β, γ if we stop focusing for a while …?***

$$\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_S = \begin{pmatrix}C^2 & -2SC & S^2\\ -CC' & SC'+S'C & -SS'\\ C'^2 & -2S'C' & S'^2\end{pmatrix} * \begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_0$$

transfer matrix for a drift:

$$M = \begin{pmatrix}C & S\\ C' & S'\end{pmatrix} = \begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix} \longrightarrow$$

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$
$$\alpha(s) = \alpha_0 - \gamma_0 s$$
$$\gamma(s) = \gamma_0$$

β-Function in a Drift:

let 's assume we are at a symmetry point in the center of a drift.

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

as $\alpha_0 = 0, \quad \rightarrow \quad \gamma_0 = \frac{1+\alpha_0^2}{\beta_0} = \frac{1}{\beta_0}$

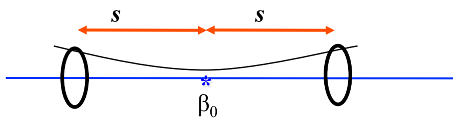
and we get for the β function in the neighborhood of the symmetry point

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

! ! !

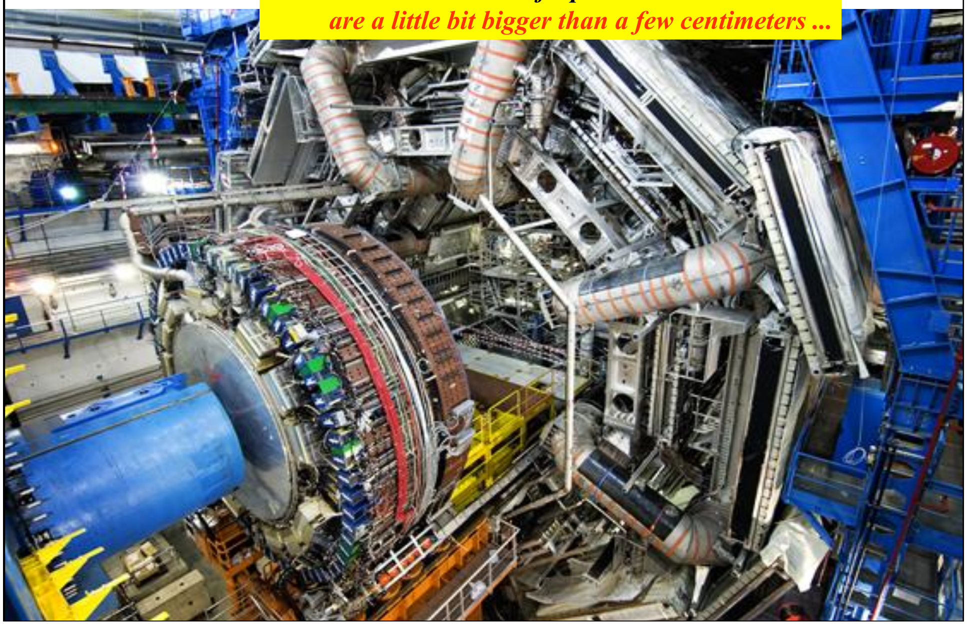
At the end of a long symmetric drift space the beta function reaches its maximum value in the complete lattice. -> here we get the largest beam dimension.

-> keep l as small as possible

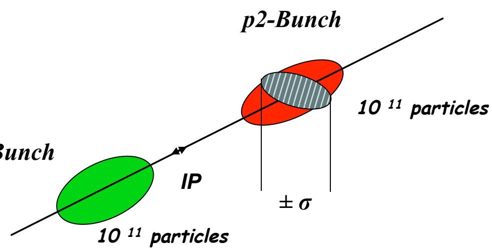


... clearly there is an

But: ... unfortunately ... in general high energy detectors that are installed in that drift spaces are a little bit bigger than a few centimeters ...



21.) Luminosity



Example: Luminosity run at LHC

$\beta_{x,y} = 0.55\ m$ $\quad f_0 = 11.245\ kHz$

$\varepsilon_{x,y} = 5*10^{-10}\ rad\ m$ $\quad n_b = 2808$

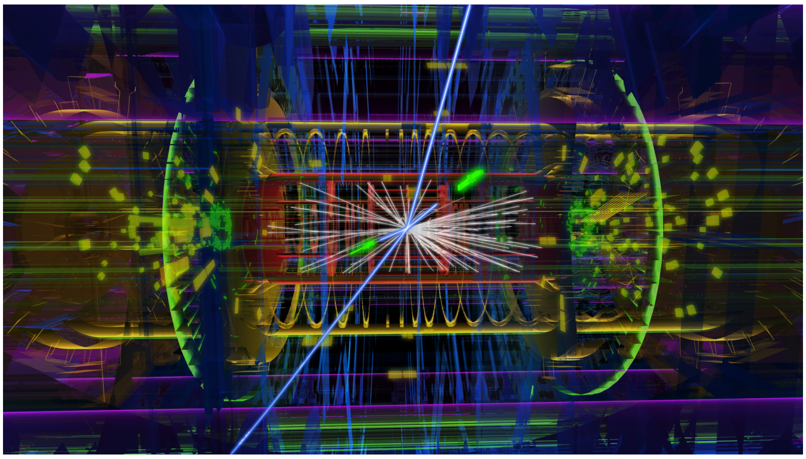
$\sigma_{x,y} = 17\ \mu m$

$$L = \frac{1}{4\pi e^2 f_0 n_b} * \frac{I_{p1} I_{p2}}{\sigma_x \sigma_y}$$

$I_p = 584\ mA$

$L = 1.0*10^{34}\ {}^{1}\!/_{cm^2 s}$

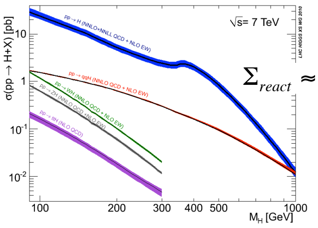
High Light of the HEP-Year 2012 / 13 naturally the HIGGS



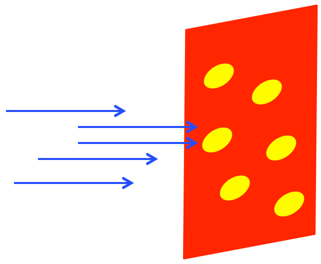
ATLAS event display: Higgs => two electrons & two muons

Problem: Our particles are VERY small !!

Overall cross section of the Higgs:



$\Sigma_{react} \approx 1\,pb$

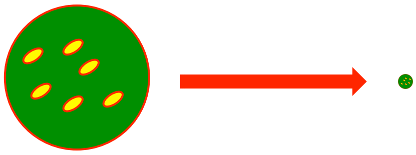


$1b \; = 10^{-24} cm^2$

$1pb = \; 10^{-12} * 10^{-24} cm^2 = 1/mio * 1/mio * 1/mio * 1/mio * 1/10000\, mm^2$

The particles are "very small"

The only chance we have:
compress the transverse beam size … at the IP



LHC typical:

σ = 0.1 mm → 16 μm

Mini-β Insertions: some guide lines

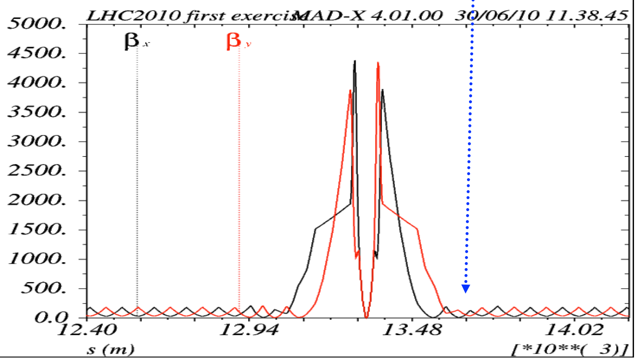
* calculate the periodic solution in the arc
* introduce the drift space needed for the insertion device (detector ...)
* put a quadrupole doublet (triplet ?) as close as possible
* introduce additional quadrupole lenses to match the beam parameters to the values at the beginning of the arc structure

parameters to be optimised & matched to the periodic solution:

$\alpha_x, \beta_x \qquad D_x, D_x'$

$\alpha_y, \beta_y \qquad Q_x, Q_y$

8 individually powered quad magnets are needed to match the insertion (... at least)



Mini-β Insertions: Betafunctions

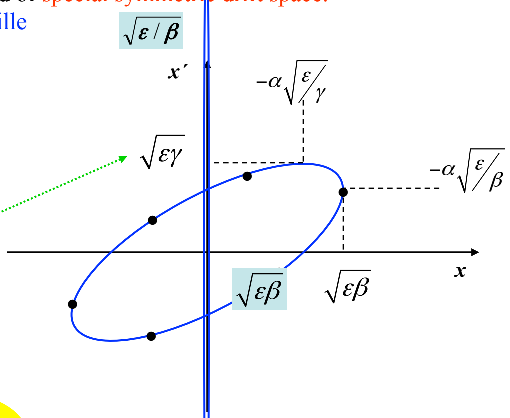
A mini-β insertion is always a kind of special symmetric drift space.

→ greetings from Liouville

$$\alpha^* = 0$$

$$\gamma^* = \frac{1+\alpha^2}{\beta} = \frac{1}{\beta^*}$$

$$\sigma'^* = \sqrt{\frac{\varepsilon}{\beta^*}}$$



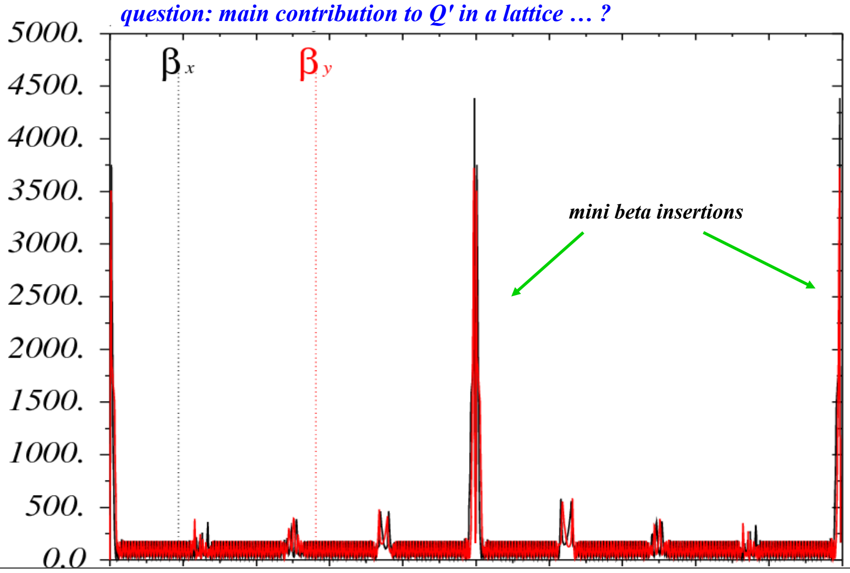
$$\beta^* = \frac{\sigma^*}{\sigma'^*}$$

at a symmetry point β is just the ratio of beam dimension and beam divergence.

… and now back to the Chromaticity

$$Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)ds$$

question: main contribution to Q' in a lattice … ?



Resume´:

quadrupole error: *tune shift*

$$\Delta Q \approx \int_{s0}^{s0+l} \frac{\Delta k(s)\,\beta(s)}{4\pi}\,ds \approx \frac{\Delta k(s)\ l_{quad}\ \bar{\beta}}{4\pi}$$

beta beat

$$\Delta\beta(s_0) = \frac{\beta_0}{2\sin 2\pi Q} \int_{s1}^{s1+l} \beta(s_1)\Delta k \cos\left(2(\psi_{s1} - \psi_{s0}) - 2\pi Q\right) ds$$

chromaticity

$$\Delta Q = Q' \frac{\Delta p}{p}$$

$$Q' = -\frac{1}{4\pi} \oint k(s)\beta(s)\,ds$$

momentum compaction

$$\frac{\delta l_\varepsilon}{L} = \alpha_p \frac{\Delta p}{p}$$

$$\alpha_p \approx \frac{2\pi}{L} \langle D \rangle \approx \frac{\langle D \rangle}{R}$$

beta function in a symmateric drift

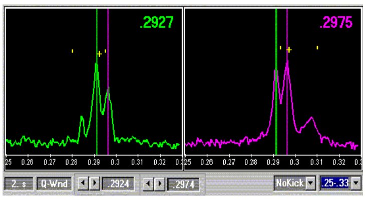
$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

Appendix:

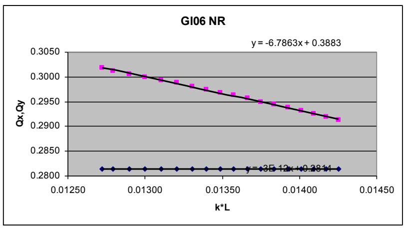
Quadrupole Error and Beta Function

a change of quadrupole strength in a synchrotron leads to tune sift:

$$\Delta Q \approx \int_{s0}^{s0+l} \frac{\Delta k(s)\,\beta(s)}{4\pi}\,ds \approx \frac{\Delta k(s) * l_{quad} * \bar{\beta}}{4\pi}$$



tune spectrum …



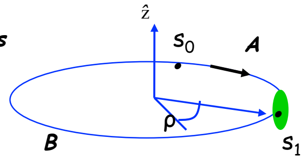
tune shift as a function of a gradient change

But we should expect an error in the β-function as well …
… shouldn´t we ???

Quadrupole Errors and Beta Function

a quadrupole error will not only influence the oscillation frequency … „tune“
… but also the amplitude … „beta function“

split the ring into 2 parts, described by two matrices A and B



$$M_{turn} = B * A \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

matrix of a quad error between A and B

$$M_{dist} = \begin{pmatrix} m^*_{11} & m^*_{12} \\ m^*_{21} & m^*_{22} \end{pmatrix} = B \begin{pmatrix} 1 & 0 \\ -\Delta kds & 1 \end{pmatrix} A$$

$$M_{dist} = B \begin{pmatrix} a_{11} & a_{12} \\ -\Delta kdsa_{11} + a_{12} & -\Delta kdsa_{12} + a_{22} \end{pmatrix}$$

$$M_{dist} = \begin{pmatrix} \sim & b_{11}a_{12} + b_{12}(-\Delta kdsa_{12} + a_{22}) \\ \sim & \sim \end{pmatrix}$$

the beta function is usually obtained via the matrix element „m12“, which is in Twiss form for the undistorted case

$$m_{12} = \beta_0 \sin 2\pi Q$$

and including the error:

$$m^*_{12} = \underbrace{b_{11}a_{12} + b_{12}a_{22}}_{m_{12} = \beta_0 \sin 2\pi Q} - b_{12}a_{12}\Delta kds$$

$$(1) \quad m^*_{12} = \beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta kds$$

As M is still a matrix for one complete turn we still can express the element m_{12} in twiss form:*

$$(2) \quad m^*_{12} = (\beta_0 + d\beta) * \sin 2\pi(Q + dQ)$$

Equalising (1) and (2) and assuming a small error

$$\beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta kds = (\beta_0 + d\beta) * \sin 2\pi(Q + dQ)$$

$$\beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta kds = (\beta_0 + d\beta) * \sin 2\pi Q \underbrace{\cos 2\pi dQ}_{\approx 1} + \cos 2\pi Q \underbrace{\sin 2\pi dQ}_{\approx 2\pi dQ}$$

$$\beta_0 \sin 2\pi Q - a_{12} b_{12} \Delta k ds = \beta_0 \sin 2\pi Q + \beta_0 2\pi dQ \cos 2\pi Q + d\beta_0 \sin 2\pi Q + d\beta_0 2\pi dQ \cos 2\pi Q$$

ignoring second order terms

$$-a_{12} b_{12} \Delta k ds = \beta_0 2\pi dQ \cos 2\pi Q + d\beta_0 \sin 2\pi Q$$

remember: tune shift dQ due to quadrupole error: $dQ = \dfrac{\Delta k \beta_1 ds}{4\pi}$
(index „1" refers to location of the error)

$$-a_{12} b_{12} \Delta k ds = \frac{\beta_0 \Delta k \beta_1 ds}{2} \cos 2\pi Q + d\beta_0 \sin 2\pi Q$$

solve for dβ

$$d\beta_0 = \frac{-1}{2 \sin 2\pi Q} \{2 a_{12} b_{12} + \beta_0 \beta_1 \cos 2\pi Q\} \Delta k ds$$

express the matrix elements a_{12}, b_{12} in Twiss form

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(\cos\psi_s + \alpha_0 \sin\psi_s) & \sqrt{\beta_s \beta_0} \sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1 + \alpha_0 \alpha_s)\sin\psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}(\cos\psi_s - \alpha_s \sin\psi_s) \end{pmatrix}$$

$$d\beta_0 = \frac{-1}{2 \sin 2\pi Q} \{2 a_{12} b_{12} + \beta_0 \beta_1 \cos 2\pi Q\} \Delta k ds$$

$$a_{12} = \sqrt{\beta_0 \beta_1} \sin \Delta\psi_{0\to1}$$
$$b_{12} = \sqrt{\beta_1 \beta_0} \sin(2\pi Q - \Delta\psi_{0\to1})$$

$$d\beta_0 = \frac{-\beta_0 \beta_1}{2 \sin 2\pi Q} \{2 \sin \Delta\psi_{12} \sin(2\pi Q - \Delta\psi_{12}) + \cos 2\pi Q\} \Delta k ds$$

... after some TLC transformations ... $= \cos(2\Delta\psi_{01} - 2\pi Q)$

$$\Delta\beta(s_0) = \frac{-\beta_0}{2 \sin 2\pi Q} \int_{s1}^{s1+l} \beta(s_1) \Delta k \cos(2(\psi_{s1} - \psi_{s0}) - 2\pi Q) ds$$

Nota bene: *! the beta beat is proportional to the strength of the error Δk*

!! and to the β function at the place of the error ,

!!! and to the β function at the observation point, (... remember orbit distortion !!!)

!!!! there is a resonance denominator