Electron dynamics with Synchrotron Radiation

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Radiation is emitted into a narrow cone

\[ \theta = \frac{1}{\gamma} \cdot \theta_c \]

\[ v \sim c \]
Synchrotron radiation power

Power emitted is proportional to:

\[ P \propto E^2 B^2 \]

\[ P = \frac{c C_\gamma E^4}{2\pi \rho^2} \]

\[ C_\gamma = \frac{4\pi}{3} \frac{r_e}{(m_e c^2)^3} = 8.858 \cdot 10^{-5} \frac{m}{(\text{GeV})^3} \]

Energy loss per turn:

\[ U_0 = C_\gamma \frac{E^4}{\rho} \]

\[ \alpha = \frac{1}{137} \]

\[ \hbar c = 197 \text{ Mev} \cdot \text{fm} \]

\[ P_{\text{tot}} = \frac{2}{3} \hbar c^2 \alpha \frac{\gamma^4}{\rho^2} \]

\[ \omega_c = \frac{3 c \gamma^3}{2 \rho} \]

\[ S(x) = \frac{9\sqrt{3}}{8\pi} x \int_x^\infty K_{\gamma^3}(x')dx' \]

\[ \int_0^\infty S(x')dx' = 1 \]

\[ G_1(x) = x \int_x^\infty K_{\gamma^3}(x')dx' \]

\[ \varepsilon_{[\text{eV}]} = 665 E^2 (\text{GeV}) B(T) \]

\[ x = \omega/\omega_c \]

\[ \sim 2.1 x^{2/3} \]

\[ \sim 1.3 \sqrt{x} e^{-x} \]
Radiation effects in electron storage rings

Average radiated power restored by RF

- Electron loses energy each turn to synchrotron radiation
- RF cavities accelerate electrons back to the nominal energy

Radiation damping

- Average rate of energy loss produces **DAMPING** of electron oscillations in all three degrees of freedom (if properly arranged!)

Quantum fluctuations

- Statistical fluctuations in energy loss (from quantized emission of radiation) produce **RANDOM EXCITATION** of these oscillations

Equilibrium distributions

- The balance between the damping and the excitation of the electron oscillations determines the equilibrium distribution of particles in the beam

\[ U_0 \approx 10^{-3} \text{ of } E_0 \]

\[ V_{RF} > U_0 \]


Radiation damping

Transverse oscillations
Average energy loss and gain per turn

- Every turn electron radiates small amount of energy
  \[ E_1 = E_0 - U_0 = E_0 \left(1 - \frac{U_0}{E_0}\right) \]
- Only the amplitude of the momentum changes
  \[ P_1 = P_0 - \frac{U_0}{c} = P_0 \left(1 - \frac{U_0}{E_0}\right) \]
- Only the longitudinal component of the momentum is increased in the RF cavity
- Energy of betatron oscillation
  \[ E_0 \propto A^2 \]
  \[ A_1^2 = A_0^2 \left(1 - \frac{U_0}{E_0}\right) \quad \text{or} \quad A_1 \approx A_0 \left(1 - \frac{U_0}{2E_0}\right) \]

Damping of vertical oscillations

- But this is just the exponential decay law!
  \[ \Delta A = \frac{U_0}{2E} \]
  \[ A = A_0 e^{-t/\tau} \]

The oscillations are exponentially damped with the damping time (milliseconds!),

\[ \tau = \frac{2E}{T_0} \]

the time it would take particle to ‘lose all of its energy’

- In terms of radiation power
  \[ \tau \propto \frac{1}{E^3} \]
  \[ P_\gamma \propto E^4 \]
Adiabatic damping in linear accelerators

In a linear accelerator:

\[ x' = \frac{p_1}{p} \text{ decreases } \propto \frac{1}{E} \]

In a storage ring beam passes many times through same RF cavity

- Clean loss of energy every turn (no change in \( x' \))
- Every turn is re-accelerated by RF (\( x' \) is reduced)
- Particle energy on average remains constant

Emittance damping in linacs:

\[ \varepsilon \quad \frac{\varepsilon}{2} \quad \frac{\varepsilon}{4} \quad \frac{\varepsilon}{4} \]

\[ \Omega \quad \frac{\Omega}{2} \quad \frac{\Omega}{4} \]

\[ \gamma \quad 2\gamma \quad 4\gamma \]

\[ \varepsilon \propto \frac{1}{\gamma} \]

or

\[ \gamma \varepsilon = \text{const.} \]
Radiation damping

Longitudinal oscillations

Longitudinal motion: compensating radiation loss $U_0$

- RF cavity provides accelerating field with frequency
  - $h$ - harmonic number

- The energy gain:
  $$U_{RF} = eV_{RF}(\tau)$$

- Synchronous particle:
  - has design energy
  - gains from the RF on the average as it loses per turn $U_0$
**Longitudinal motion: phase stability**

- **Particle ahead of synchronous one**
  - gets too much energy from the RF
  - goes on a longer orbit (not enough B)
    - >> takes longer to go around
  - comes back to the RF cavity closer to synchronous part.

- **Particle behind the synchronous one**
  - gets too little energy from the RF
  - goes on a shorter orbit (too much B)
  - catches-up with the synchronous particle

**Longitudinal motion: energy-time oscillations**

energy deviation from the design energy, or the energy of the synchronous particle

longitudinal coordinate measured from the position of the synchronous electron
**Orbit Length**

Length element depends on x

\[ dl = (1 + \frac{x}{\rho}) ds \]

Horizontal displacement has two parts:

- To first order \( x_\beta \) does not change \( L \)
- \( x_\varepsilon \) - has the same sign around the ring

Length of the off–energy orbit

\[ L_\varepsilon = \int dl = \int (1 + \frac{x_\varepsilon}{\rho}) ds = L_0 + \Delta L \]

\[ \Delta L = \delta \int \frac{D(s)}{p(s)} ds \]

where \( \delta = \frac{\Delta p}{p} = \frac{\Delta E}{E} \)

\[ \frac{\Delta L}{L} = \alpha \cdot \delta \]

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**Something funny happens on the way around the ring...**

Revolution time changes with energy

\[ T_0 = \frac{L_0}{c\beta} \]

- Particle goes faster (not much!)
- while the orbit length increases (more!)

\[ \frac{d\beta}{\beta} = \frac{1}{\gamma^2} \cdot \frac{dp}{p} \] (relativity)

\[ \frac{\Delta T}{T} = \frac{\Delta L}{L} - \frac{\Delta \beta}{\beta} \]

- The “slip factor” \( \eta \cong \alpha \) since \( \alpha \gg \frac{1}{\gamma^2} \)

\[ \frac{\Delta T}{T} = (\alpha - \frac{1}{\gamma^2}) \cdot \frac{dp}{p} = \eta \cdot \frac{dp}{p} \]

- Ring is above “transition energy”

isochronous ring: \( \eta = 0 \) or \( \gamma = \gamma_n \)

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Not only accelerators work above transition

Dante Aligieri
Divine Comedy

**RF Voltage**

\[ V(\tau) = \hat{V} \sin(h \omega_0 \tau + \psi_s) \]

*here the synchronous phase*

\[ \psi_s = \arcsin \left( \frac{U_0}{e \hat{V}} \right) \]
Momentum compaction factor

$$\alpha = \frac{1}{L} \int D(s) \rho(s) ds$$

Like the tunes $Q_x, Q_y$ - $\alpha$ depends on the whole optics

- A quick estimate for separated function guide field:

$$\alpha = \frac{1}{L_0 \rho_0} \int_{mag} D(s) ds = \frac{1}{L_0 \rho_0} \langle D \rangle \cdot L_{mag}$$

- But $L_{mag} = 2 \pi \rho_0$

- Since dispersion is approximately

$$D \approx \frac{R}{Q^2} \Rightarrow \alpha \approx \frac{1}{Q^2}$$

and the orbit change for $\sim 1\%$ energy deviation

$$\frac{\Delta L}{L} = \frac{1}{Q^2} \cdot \delta \approx 10^{-4}$$

Energy balance

Energy gain from the RF system: $U_{RF} = eV_{RF} \tau = U_0 + eV_{RF} \cdot \tau$

- synchronous particle ($\tau = 0$) will get exactly the energy loss per turn
- we consider only linear oscillations

$$V_{RF} = \left. \frac{dV_{RF}}{d\tau} \right|_{\tau = 0}$$

- Each turn electron gets energy from RF and loses energy to radiation within one revolution time $T_0$

$$\Delta \varepsilon = (U_0 + eV_{RF} \cdot \tau) - (U_0 + U' \cdot \varepsilon)$$

$$d\varepsilon \over dt = {1 \over T_0} (eV_{RF} \cdot \tau - U' \cdot \varepsilon)$$

- An electron with an energy deviation will arrive after one turn at a different time with respect to the synchronous particle

$$d\tau \over dt = -\alpha \frac{\varepsilon}{E_0}$$
Synchrotron oscillations: damped harmonic oscillator

Combining the two equations

\[
\frac{d^2 \varepsilon}{dt^2} + 2\alpha_\varepsilon \frac{d\varepsilon}{dt} + \Omega^2 \varepsilon = 0
\]

where the oscillation frequency

\[
\Omega^2 \equiv \frac{\alpha e V_{RF}}{T_0 E_0}
\]

the damping is slow:

\[
\alpha_\varepsilon \equiv \frac{U'}{2T_0}
\]

typically \( \alpha_\varepsilon \ll \Omega \)

the solution is then:

\[
\varepsilon(t) = \varepsilon_0 e^{-\alpha_\varepsilon t} \cos (\Omega t + \theta_\varepsilon)
\]

similarly, we can get for the time delay:

\[
\tau(t) = \tau_0 e^{-\alpha_\varepsilon t} \cos (\Omega t + \theta_\tau)
\]

Synchrotron (time - energy) oscillations

The ratio of amplitudes at any instant

\[
\hat{\varepsilon} = \frac{\varepsilon_0}{\Omega E_0}
\]

Oscillations are 90 degrees out of phase

\[
\theta_\varepsilon = \theta_\tau + \frac{\pi}{2}
\]

The motion can be viewed in the phase space of conjugate variables
During one period of synchrotron oscillation:

- When the particle is in the upper half-plane, it loses more energy per turn, its energy gradually reduces.
- When the particle is in the lower half-plane, it loses less energy per turn, but receives $U_0$ on the average, so its energy deviation gradually reduces.

The synchrotron motion is damped:

- The phase space trajectory is spiraling towards the origin.
Robinson theorem: Damping partition numbers

- Transverse betatron oscillations are damped with
  \[ \tau_x = \tau_z = \frac{2ET_0}{U_0} \]
- Synchrotron oscillations are damped twice as fast
  \[ \tau_e = \frac{ET_0}{U_0} \]
- The total amount of damping (Robinson theorem) depends only on energy and loss per turn
  \[ \frac{1}{\tau_x} + \frac{1}{\tau_y} + \frac{1}{\tau_e} = \frac{2U_0}{ET_0} = \frac{U_0}{2ET_0}(J_x + J_y + J_e) \]
  the sum of the partition numbers \[ J_x + J_z + J_e = 4 \]

Radiation loss

Displaced off the design orbit particle sees fields that are different from design values

- energy deviation \( \varepsilon \)
  - different energy: \( P_\gamma \propto E^2 \)
  - different magnetic field \( B \)
    particle moves on a different orbit, defined by the off-energy or dispersion function \( D_x \)
    both contribute to linear term in \( P_\gamma(\varepsilon) \)

- betatron oscillations: zero on average
Radiation loss

To first order in $\varepsilon$

$$U_{\text{rad}} = U_0 + U' \cdot \varepsilon$$

electron energy changes slowly, at any instant it is moving on an orbit defined by $D_x$

after some algebra one can write

$$U' = \frac{U_0}{E_0} (2 + D)$$

$D \neq 0$ only when $\frac{k}{\rho} \neq 0$

Damping partition numbers

$$J_x + J_z + J_\varepsilon = 4$$

- Typically we build rings with no vertical dispersion

  $$J_z = 1$$

  $$J_x + J_\varepsilon = 3$$

- Horizontal and energy partition numbers can be modified via $D$:

  $$J_x = 1 - D$$

  $$J_\varepsilon = 2 + D$$

- Use of combined function magnets

- Shift the equilibrium orbit in quads with RF frequency
Equilibrium beam sizes

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Quantum fluctuations
- Statistical fluctuations in energy loss (from quantized emission of radiation) produce RANDOM EXCITATION of these oscillations

Equilibrium distributions
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\[ U_0 \approx 10^{-3} \text{ of } E_0 \]
Quantum nature of synchrotron radiation

Damping only

- If damping was the whole story, the beam emittance (size) would shrink to microscopic dimensions!*
- Lots of problems! (e.g. coherent radiation)

- How small? On the order of electron wavelength

\[ E = \gamma mc^2 = h\nu = \frac{hc}{\lambda_e} \Rightarrow \lambda_e = \frac{1}{\gamma} \frac{h}{mc} = \frac{\lambda_C}{\gamma} \]

\[ \lambda_C = 2.4 \cdot 10^{-12} m \quad \text{– Compton wavelength} \]

Diffraction limited electron emittance

\[ \epsilon \geq \frac{\lambda_C}{4\pi\gamma} (\times N^{1/3} \quad \text{fermions}) \]

Quantum nature of synchrotron radiation

Quantum fluctuations

- Because the radiation is emitted in quanta, radiation itself takes care of the problem!

- It is sufficient to use quasi-classical picture:
  - Emission time is very short
  - Emission times are statistically independent
    (each emission - only a small change in electron energy)

Purely stochastic (Poisson) process
Visible quantum effects

I have always been somewhat amazed that a purely quantum effect can have gross macroscopic effects in large machines; and, even more, that Planck’s constant has just the right magnitude needed to make practical the construction of large electron storage rings. A significantly larger or smaller value of \( \hbar \) would have posed serious -- perhaps insurmountable -- problems for the realization of large rings.

Mathew Sands

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Quantum excitation of energy oscillations

Photons are emitted with typical energy \( u_{ph} \approx \hbar \omega_{typ} = \hbar c \gamma^3 \) at the rate (photons/second) \( \mathcal{N} \approx \frac{P}{u_{ph}} \)

**Fluctuations in this rate excite oscillations**

During a small interval \( \Delta t \) electron emits photons

\[ N = \mathcal{N} \cdot \Delta t \]

losing energy of

\[ N \cdot u_{ph} \]

Actually, because of fluctuations, the number is

\[ N \pm \sqrt{N} \]

resulting in **spread in energy loss**

\[ \pm \sqrt{N} \cdot u_{ph} \]

For large time intervals RF compensates the energy loss, providing damping towards the design energy \( E_0 \)

**Steady state:** typical deviations from \( E_0 \)

\[ \approx \text{typical fluctuations in energy during a damping time} \ t_\varepsilon \]
Equilibrium energy spread: rough estimate

We then expect the rms energy spread to be
\[ \sigma_e \approx \sqrt{N \cdot \tau_e \cdot u_{ph}} \]
and since
\[ \tau_e \approx \frac{E_0}{P_{\gamma}} \]
and
\[ P_{\gamma} = N \cdot u_{ph} \]

Relative energy spread can be written then as:
\[ \frac{\sigma_e}{E_0} \approx \gamma \sqrt{\frac{\lambda_e}{\rho}} \]

\[ \lambda_e = \frac{h}{m_e c} \cdot 4 \cdot 10^{-13} m \]

it is roughly constant for all rings

- typically \( \rho \propto E^2 \)

Equilibrium energy spread

More detailed calculations give

- for the case of an ‘isomagnetic’ lattice
  \[ \rho(s) = \rho_0 \quad \text{in dipoles} \]
  \[ \rho(s) = \infty \quad \text{elsewhere} \]

\[ \langle \sigma_e \rangle^2 = \frac{C_q E^2}{J_e P_0} \]

with
\[ C_q = \frac{55}{32 \sqrt{3}} \left( \frac{h}{m_e c^2} \right)^3 = 1.468 \cdot 10^{-6} \left( \frac{m}{[\text{GeV}]} \right) \]

It is difficult to obtain energy spread < 0.1%

- limit on undulator brightness!
Equilibrium bunch length

Bunch length is related to the energy spread
- Energy deviation and time of arrival (or position along the bunch) are **conjugate variables** (synchrotron oscillations)
- recall that

\[ \sigma_\tau = \frac{\alpha}{\Omega_s \sqrt{E}} \]

Two ways to obtain short bunches:
- RF voltage (power!)
  \[ \sigma_\tau \propto \sqrt{V_{RF}} \]
- Momentum compaction factor in the limit of \( \alpha = 0 \)

**Isochronous ring**: particle position along the bunch is frozen

\[ \sigma_\tau \propto \alpha \]

Excitation of betatron oscillations

\[ x = x_\beta + x_e \]

\[ x_e = D \cdot \frac{E}{E} \]

\[ \Delta x = \Delta x_\beta + \Delta x_e = 0 \]

\[ \Delta x_\beta = -D \cdot \frac{e}{E} \quad \text{Courant Snyder invariant} \]

\[ \Delta x_\beta' = -D' \cdot \frac{e'}{E} \]

\[ \Delta e = \gamma \Delta x_\beta^2 + 2\alpha \Delta x_\beta \Delta x_\beta' + \beta \Delta x_\beta'^2 = \left[\gamma D^2 + 2\alpha DD' + \beta D'^2\right] \left(\frac{e'}{E}\right)^2 \]
Excitation of betatron oscillations

Electron emitting a photon
• at a place with non-zero dispersion
• starts a betatron oscillation around a new reference orbit

\[ x_\beta \approx D \cdot \frac{\varepsilon_y}{E} \]

Horizontal oscillations: equilibrium

Emission of photons is a random process
- Again we have random walk, now in \( x \). How far particle will wander away is limited by the radiation damping
- The balance is achieved on the time scale of the damping time \( \tau_x = 2 \tau_e \)

\[ \sigma_{x\beta} \approx \sqrt{\mathcal{N} \cdot \tau_x} \cdot D \cdot \frac{\varepsilon_y}{E} = \sqrt{2} \cdot D \cdot \frac{\sigma_e}{E} \]

- Typical horizontal beam size \( \sim 1 \text{ mm} \)

Quantum effect visible to the naked eye!

- Vertical size - determined by coupling
**Beam emittance**

**Betatron oscillations**
- Particles in the beam execute betatron oscillations with different amplitudes.

**Transverse beam distribution**
- Gaussian (electrons)
- “Typical” particle: $1 - \sigma$ ellipse
  (in a place where $\alpha = \beta' = 0$)

**Emittance**
$$\text{Emittance} \equiv \frac{\sigma_x^2}{\beta}$$

**Units of $\varepsilon$** \(m \cdot \text{rad}\)

$$\varepsilon = \sigma_x \cdot \sigma_{x'}$$

$$\beta = \frac{\sigma_x}{\sigma_{x'}}$$

**Equilibrium horizontal emittance**

**Detailed calculations for isomagnetic lattice**

$$\varepsilon_{x0} \equiv \frac{\sigma_{x\beta}^2}{\beta} = \frac{C_q E^2}{J_x} \cdot \frac{\langle \mathcal{H} \rangle_{\text{mag}}}{\rho}$$

where

$$\mathcal{H} = \gamma D^2 + 2\alpha D D' + \beta D'^2$$
$$= \frac{1}{\beta} \left[ D^2 + (\beta D' + \alpha D)^2 \right]$$

and $\langle \mathcal{H} \rangle_{\text{mag}}$ is average value in the bending magnets.
2-D Gaussian distribution

Electron rings emittance definition

- 1 - $\sigma$ ellipse

$$n(x)dx = \frac{1}{\sqrt{2\pi}\sigma}e^{-x^2/2\sigma^2}dx$$

Area = $\pi \varepsilon_x$

- Probability to be inside 1-$\sigma$ ellipse

$$P_1 = 1 - e^{-1/2} = 0.39$$

- Probability to be inside n-$\sigma$ ellipse

$$P_n = 1 - e^{-n^2/2}$$

FODO cell lattice
**FODO lattice emittance**

\[ H \sim \frac{D^2}{\beta} \sim \frac{R}{Q^3} \]

\[ \epsilon_{x0} \approx \frac{C E^2}{J_x} \cdot \frac{R}{\beta} \cdot \frac{1}{Q^3} \]

\[ \epsilon \propto \frac{E^2}{J_x} \theta^3 F_{\text{FODO}(\mu)} \]

**Ionization cooling**

similar to radiation damping, but there is multiple scattering in the absorber that blows up the emittance

\[ \sigma' = \sqrt{\sigma_0'^2 + \sigma_{MS}'^2} \]

\[ \sigma_0' >> \sigma_{MS}' \] to minimize the blow up due to multiple scattering in the absorber we can focus the beam
Minimum emittance lattices

\[ \sigma_0 \to \sigma'_0 \]

\[ \mathcal{E}_{x0} = \frac{C_q E^2}{J_x} \cdot \theta^3 \cdot F_{\text{latt}} \]

\[ F_{\text{min}} = \frac{1}{12 \sqrt{15}} \]

Quantum limit on emittance

- Electron in a storage ring’s dipole fields is accelerated, interacts with vacuum fluctuations: «accelerated thermometers show increased temperature»
- Synchrotron radiation opening angle is \( \sim 1/ \gamma \) -> a lower limit on equilibrium vertical emittance
- Independent of energy

\[ \mathcal{E}_y = \frac{13}{55} C_q \int \beta_y(s) |G^3(s)| ds \int \frac{G^2(s) ds}{G(s)} = \text{curvature}, \; C_q = 0.384 \text{ pm} \]

- In case of SLS: 0.2 pm

Isomagnetic lattice

\[ \mathcal{E}_y = 0.09 \text{ pm} \cdot \frac{\langle \beta_y \rangle_{\text{Mag}}}{\rho} \]
Vertical emittance record

Beam size $3.6 \pm 0.6 \mu m$

Emittance $0.9 \pm 0.4 \text{ pm}$

SLS beam cross section compared to a human hair:

Summary of radiation integrals

Momentum compaction factor

$$\alpha = \frac{I_1}{2\pi R}$$

Energy loss per turn

$$U_0 = \frac{1}{2\pi} C_\gamma E^4 \cdot I_2$$

$$C_\gamma = \frac{4\pi}{3} \frac{r_e}{(m_c \gamma)^3} = 8.858 \cdot 10^{-5} \left[ \frac{m}{\text{GeV}^3} \right]$$
# Summary of radiation integrals (2)

- **Damping parameter**
  \[ D = \frac{I_4}{I_2} \]

- **Damping times, partition numbers**
  \[ J_e = 2 + D, \quad J_x = 1 - D, \quad J_y = 1 \]
  \[ \tau_i = \frac{\tau_0}{J_i}, \quad \tau_0 = \frac{2ET_0}{U_0} \]

- **Equilibrium energy spread**
  \[ \left( \frac{\sigma_e}{E} \right)^2 = C_q \frac{E^2}{J_x} \cdot \frac{I_1}{I_2} \]

- **Equilibrium emittance**
  \[ \epsilon_{x0} = \frac{\sigma_x}{\beta} = C_q \frac{E^2}{J_x} \cdot \frac{I_5}{I_2} \]

- **Equation**
  \[ C_q = \frac{55}{32 \sqrt{3}} \frac{\hbar c}{(m_e c^2)^3} = 1.468 \cdot 10^{-6} \left( \frac{m}{\text{GeV}^2} \right) \]
  \[ \mathcal{H} = \gamma D^2 + 2\alpha DD' + \beta D'^2 \]

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## Damping wigglers

Increase the radiation loss per turn \( U_0 \) with **WIGGLERS**

- **reduce damping time**
  \[ \tau = \frac{E}{P_\gamma + P_{\text{wig}}} \]

- **emittance control**

  - **wigglers at high dispersion:** blow-up emittance
    - e.g. storage ring colliders for high energy physics

  - **wigglers at zero dispersion:** decrease emittance
    - e.g. damping rings for linear colliders
    - e.g. synchrotron light sources (PETRAIII, 1 nm.rad)
END