Collective Effects I

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CERN Accelerator – School

Budapest, 2-14 / 10 / 2016

Disclaimer: not all in this handouts will be presented

Type of fields

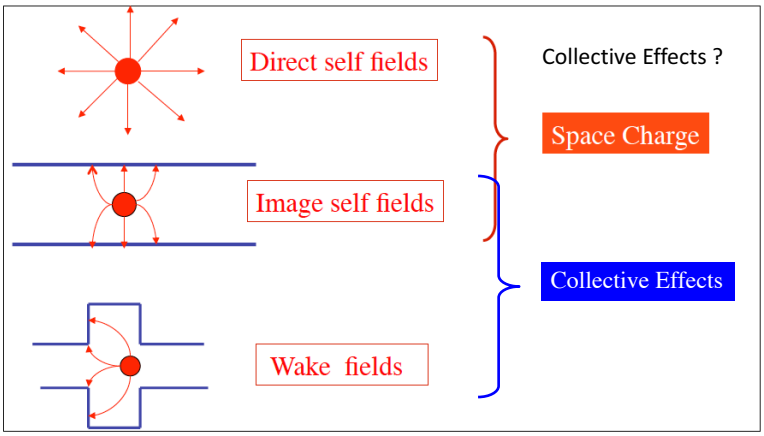
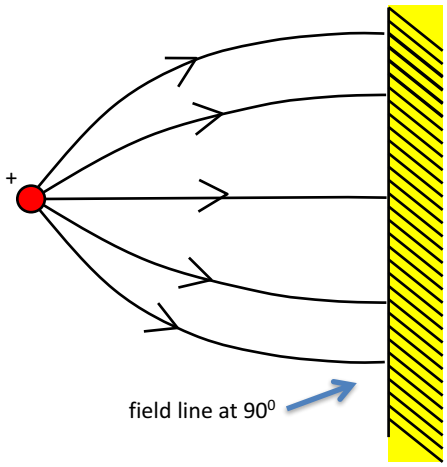


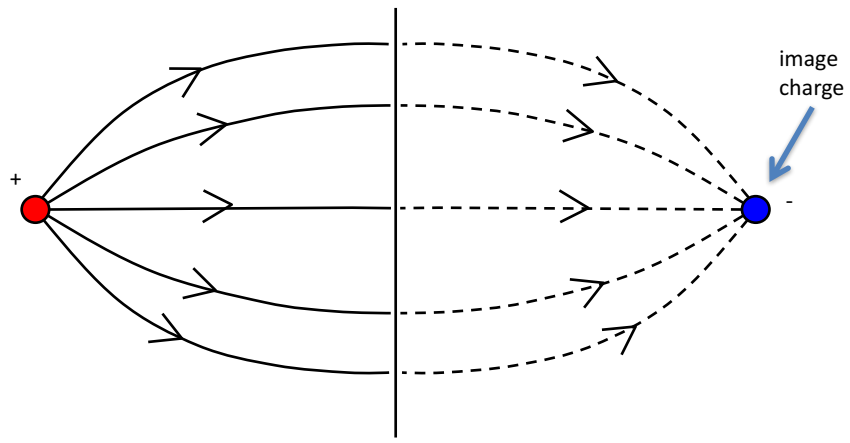
Image charges

Influence of the chamber wall



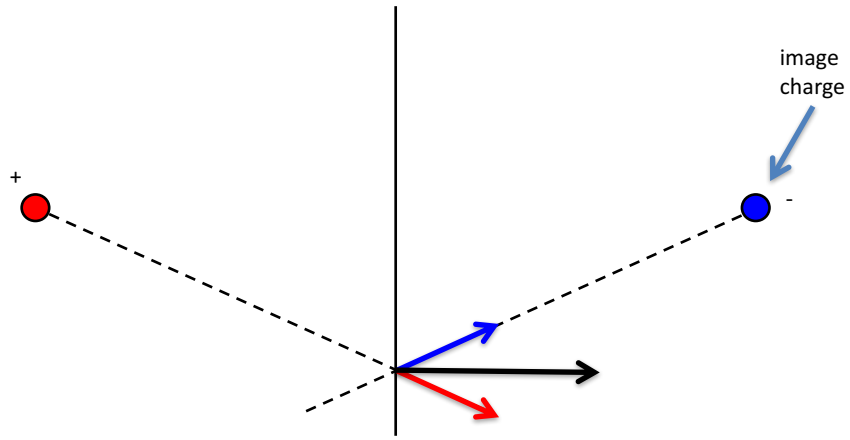
the electron in the metal quickly travel on the surface of the metal until the electric field parallel to the surface is zero

Image charge



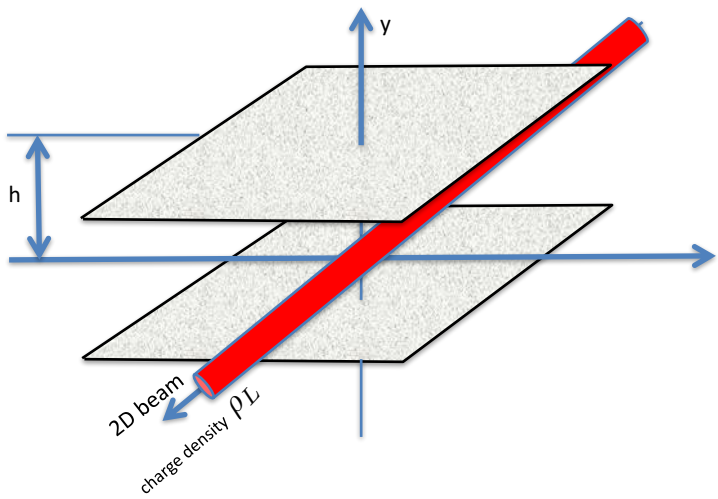
the image charge is a reflection of the particle with exchanged sign

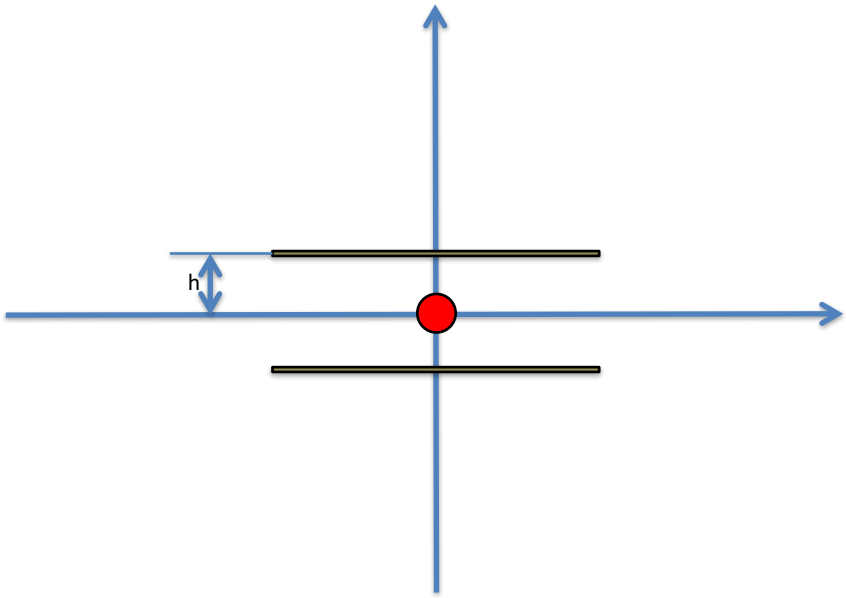
Image charge

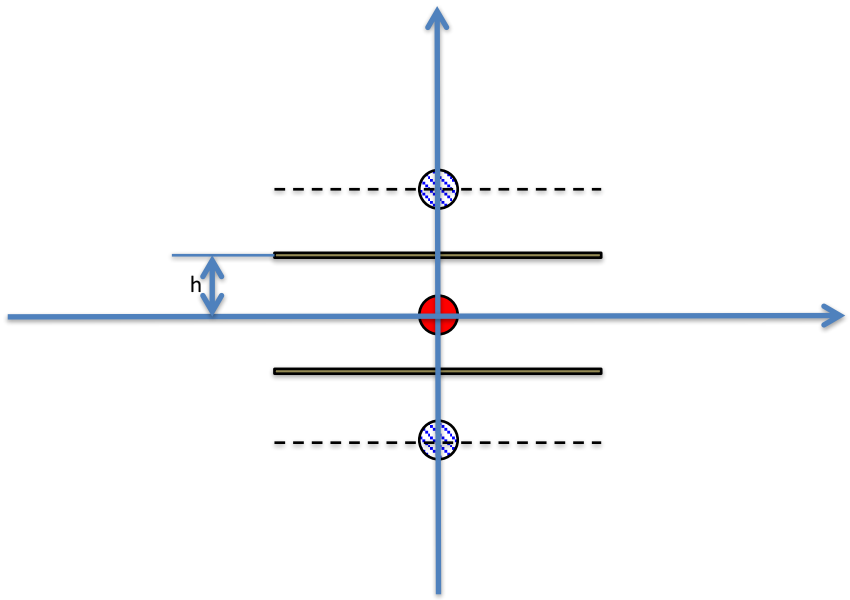


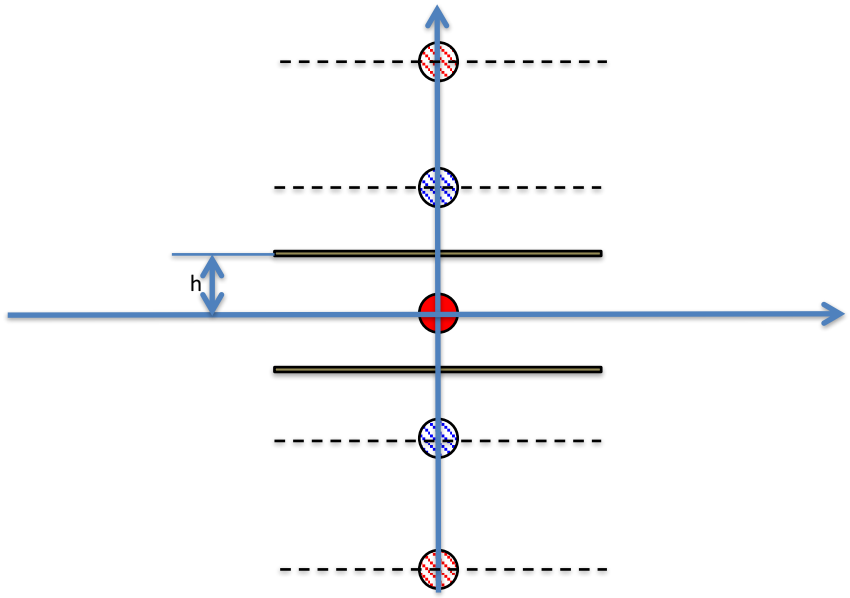
the image charge is a reflection of the particle with exchanged sign

Conducting plates

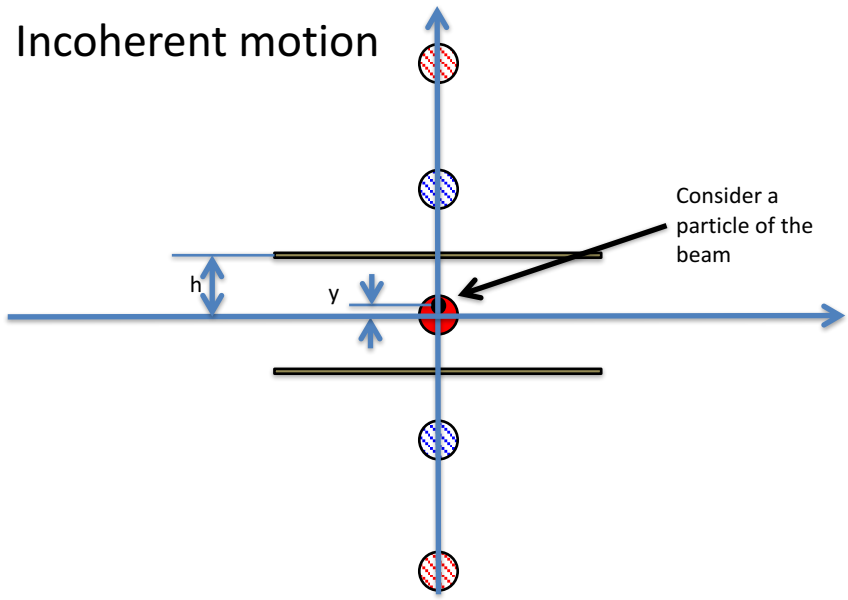




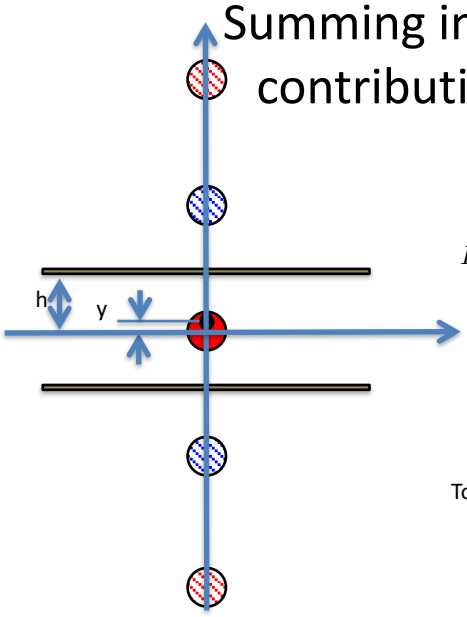

h


h

Incoherent motion



Summing image charge contribution in pairs



$$E_{y,n} = \frac{\rho_L}{2\pi\epsilon_0}(-1)^n \left(\frac{1}{2nh+y} - \frac{1}{2nh-y} \right)$$

$$h \gg y$$

$$E_{y,n} = -\frac{\rho_L y}{4\pi\epsilon_0 h^2} \frac{(-1)^n}{n^2}$$

Total electric field

$$E_y = \sum_{n=1}^{\infty} E_{y,n} = \frac{\rho_L y}{\pi\epsilon_0} \frac{\pi^2}{48h^2}$$

Equation of motion

In the equation of motion

$$\frac{d^2y}{ds^2} + k_y y = \frac{e}{m\gamma^3 v_0^2} E_{b,y} + \frac{e}{m\gamma v_0^2} E_{i,y}$$

$$\frac{d^2y}{ds^2} + k_y y = \frac{2K}{Y(X+Y)} y + K\gamma^2 \frac{\pi^2}{24h^2} y$$

as $\nabla \cdot \vec{E} = 0$ ⇒ $\frac{\partial E_x}{\partial x} = -\frac{\partial E_y}{\partial y}$

$$\frac{d^2y}{ds^2} + k_y y - \frac{2K}{Y(X+Y)} \left[1 + \gamma^2 \frac{\pi^2}{48} \frac{Y(X+Y)}{h^2}\right] y = 0$$

$$\frac{d^2x}{ds^2} + k_x x - \frac{2K}{X(X+Y)} \left[1 - \gamma^2 \frac{\pi^2}{48} \frac{X(X+Y)}{h^2}\right] x = 0$$

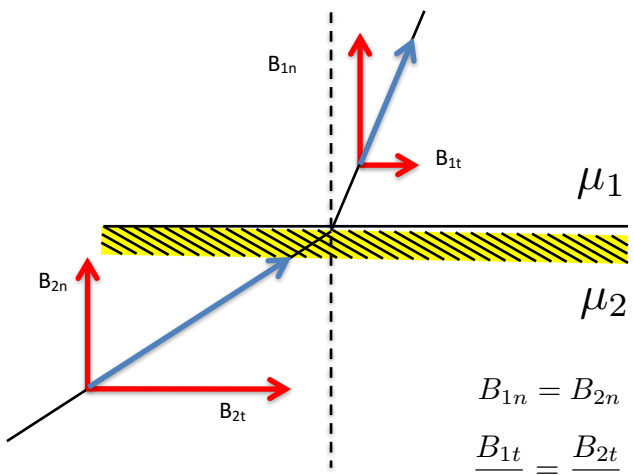
Laslett Tuneshift

$$\Delta Q_y \simeq -\frac{R_m^2}{Q_{y0}} \frac{K}{Y(X+Y)} \left[1 + \gamma^2 \frac{\pi^2}{48} \frac{Y(X+Y)}{h^2}\right]$$

$$\Delta Q_x \simeq -\frac{R_m^2}{Q_{x0}} \frac{K}{X(X+Y)} \left[1 - \gamma^2 \frac{\pi^2}{48} \frac{X(X+Y)}{h^2}\right]$$

Image currents

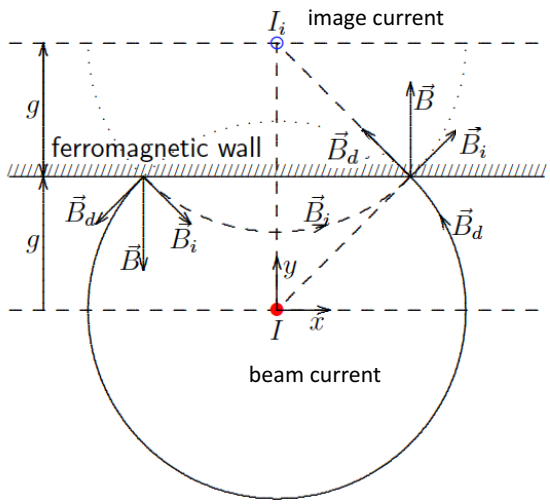
Ferromagnetic Boundaries

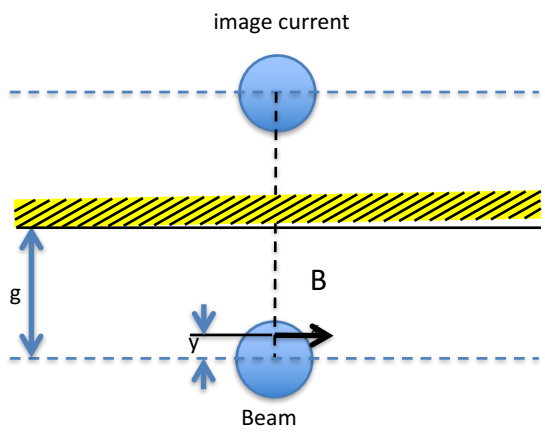


$$B_{1n} = B_{2n} \qquad \mu_1 \ll \mu_2$$

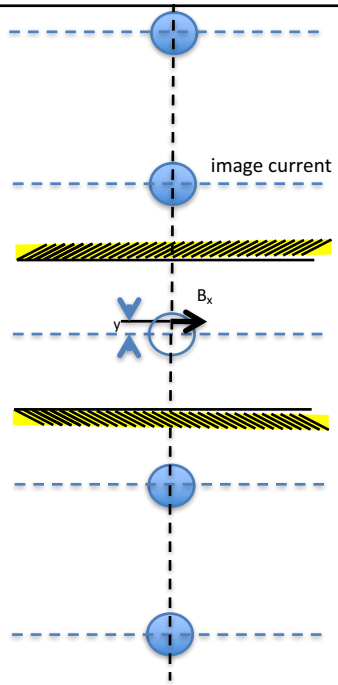
$$\frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2} \qquad B_{1,t} \simeq 0$$

Ferromagnetic Boundaries





$$B_x = \frac{\mu_0 I}{2\pi} \frac{1}{2g - y}$$



$$B_x = \frac{\mu_0 I}{2\pi} \sum_{n=1}^{\infty} \left(\frac{1}{2ng - y} - \frac{1}{2ng + y} \right)$$

for $g \gg y$

$$B_x = \frac{\mu_0 I y}{4\pi g^2} \frac{\pi^2}{6}$$

In the equation of motion

$$\frac{d^2 y}{ds^2} + k_y y = \frac{2K}{Y(X+Y)} y - \frac{1}{m\gamma v_0^2} v_z B_x$$

therefore

$$\frac{d^2 y}{ds^2} + k_y y = \frac{2K}{Y(X+Y)} y + K \frac{2\gamma^2 \beta^2 \pi^2}{24 g^2} y$$

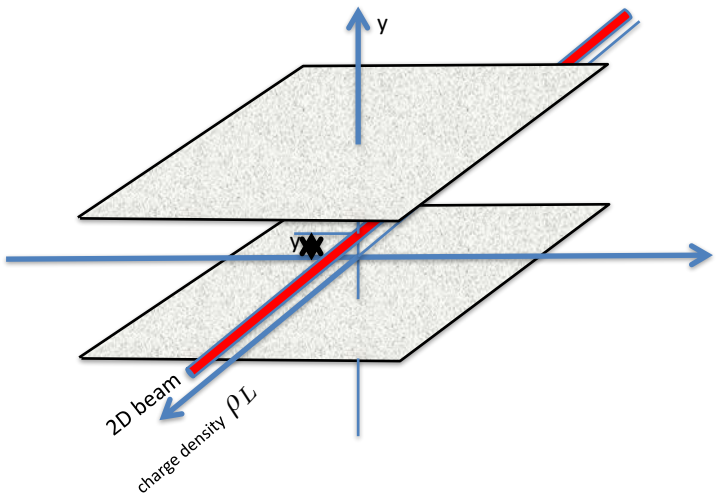
incoherent SC

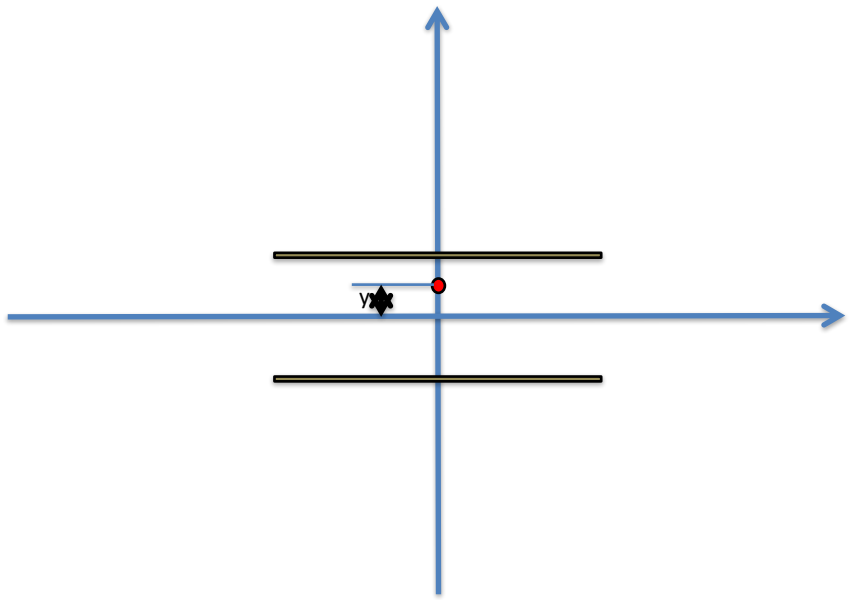
ferromagnetic induced image current (coherent force)

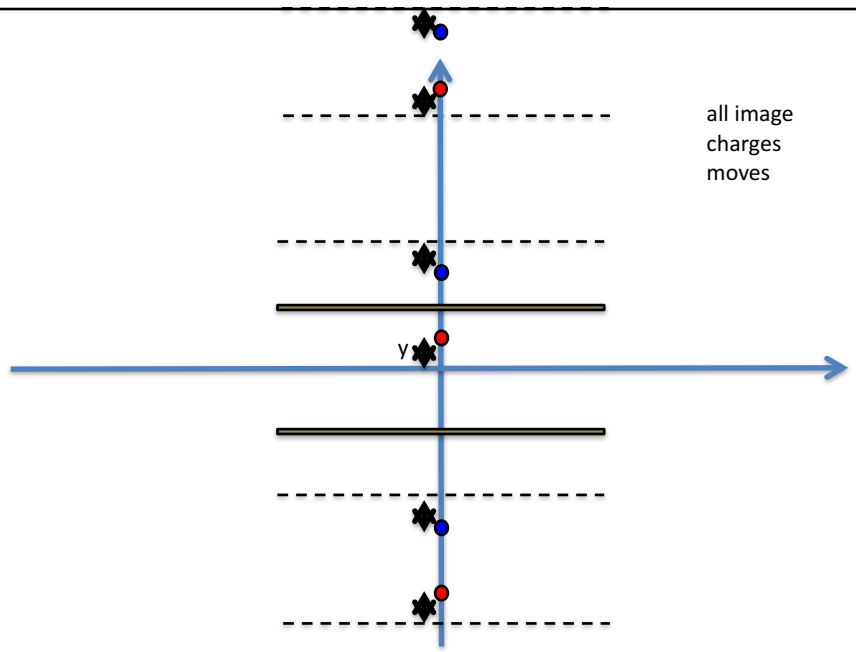
Tune-shift !

Coherent Motion

Coherent motion

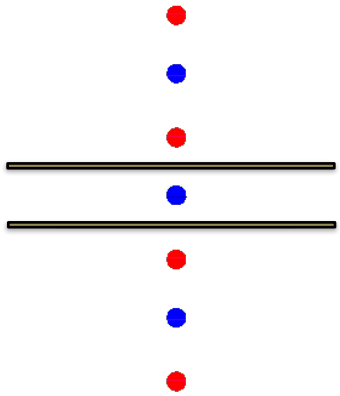


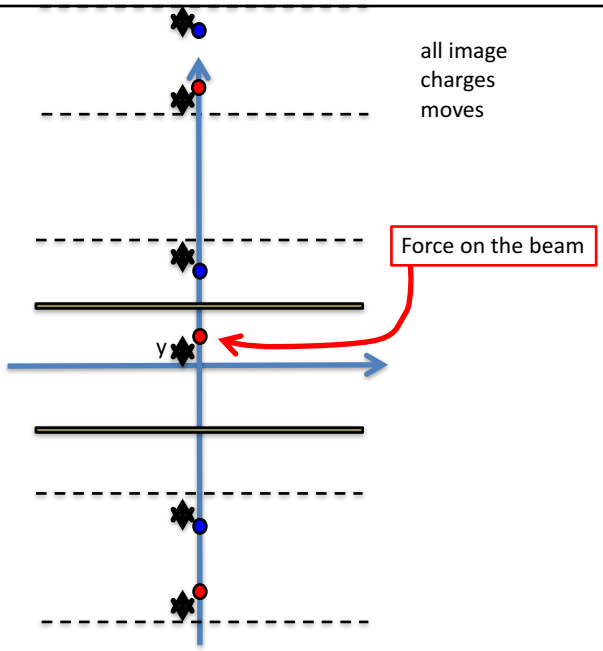


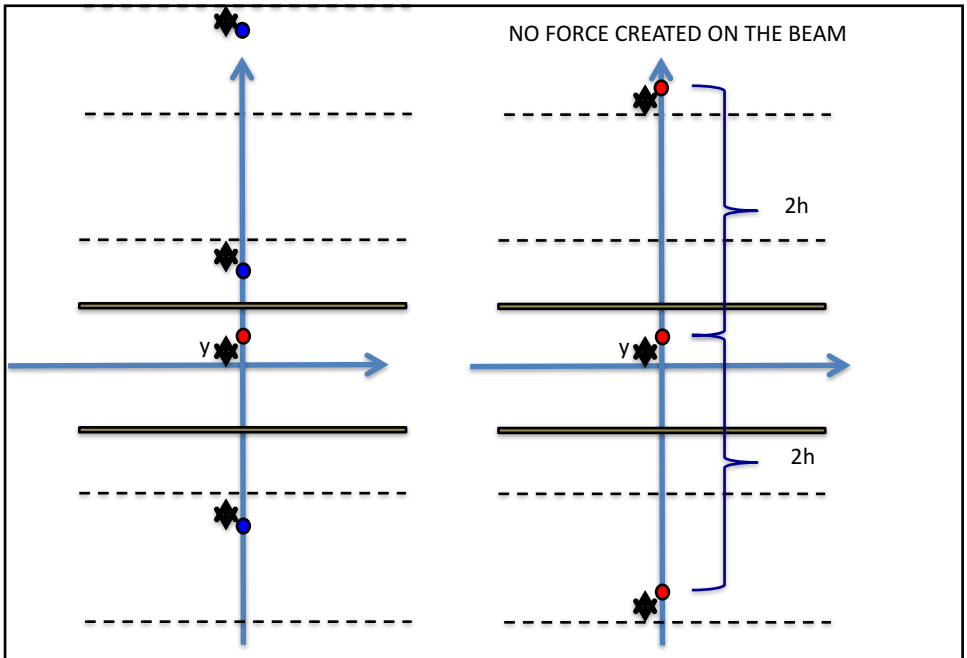


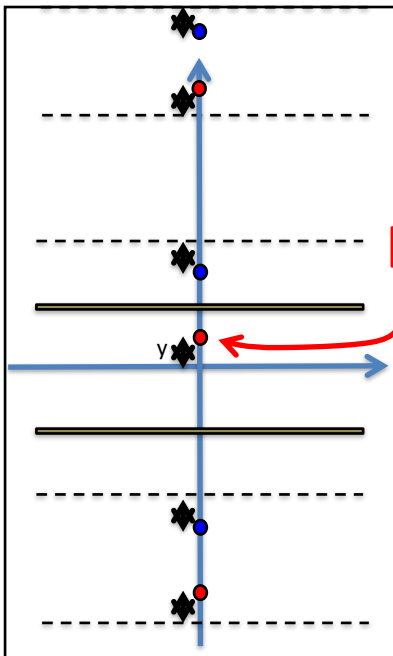
all image charges moves

Coherent motion









all image charges moves

Force on the beam

$$E_{y,n} = -\frac{\rho_L}{2\pi\epsilon_0}\left[\frac{1}{2nh+2y} - \frac{1}{2nh-2y}\right]$$

n = 1, 3, 5, 7, ...

$$E_{y,n} = \frac{\rho_L}{2\pi\epsilon_0}\frac{4y}{(2nh)^2-(2y)^2}$$

$$E_{y,n} = \frac{\rho_L}{2\pi\epsilon_0 h^2} y \frac{1}{n^2}$$

n = 1, 3, 5, 7, ...

therefore

$$E_{y,n} = \frac{\rho_L}{4\pi\epsilon_0 h^2} y \left[\frac{1}{n^2} - \frac{(-1)^n}{n^2} \right]$$ (trick!)

with n = 1, 2, 3, 4, 5, 6, …

$$E_y = \sum_{n=1}^{\infty} E_{y,n} = \frac{\rho_L}{4\pi\epsilon_0 h^2} y \left[\frac{\pi^2}{6} + \frac{\pi^2}{12} \right] = \frac{\rho_L}{16\pi\epsilon_0 h^2} \pi^2 y$$

The electric field E_x due to coherent shift is zero on the center of mass ☺

equation of motion

$$\frac{d^2 y_c}{ds^2} + k_y y_c = \frac{e}{m\gamma v_0^2} \frac{\rho_L}{16\pi\epsilon_0 h^2} \pi^2 y_c$$

but $I = v_z \rho_L \simeq v_0 \rho_L$

$$\frac{d^2 y_c}{ds^2} + k_y y_c = K \frac{\gamma^2 \pi^2}{8h^2} y_c$$

$$\frac{d^2 y_c}{ds^2} + \left(k_y - 2K \frac{\gamma^2 \pi^2}{16h^2} \right) y_c = 0$$

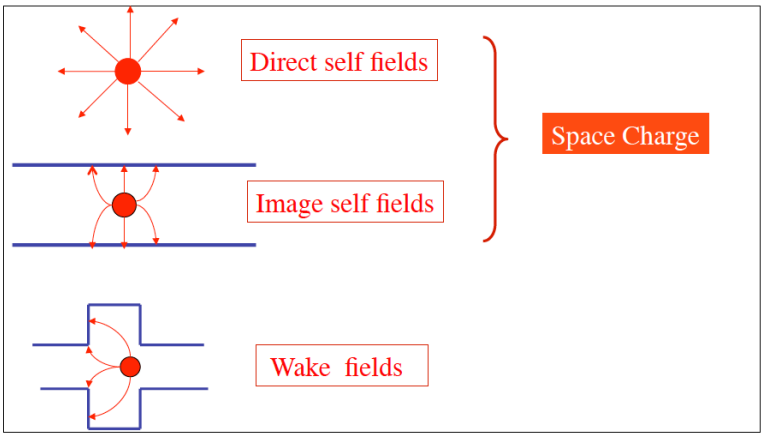
Coherent detuning

$$\Delta Q_{y,c} \simeq -\frac{R_m^2}{Q_y} K \frac{\gamma^2 \pi^2}{16 h^2}$$

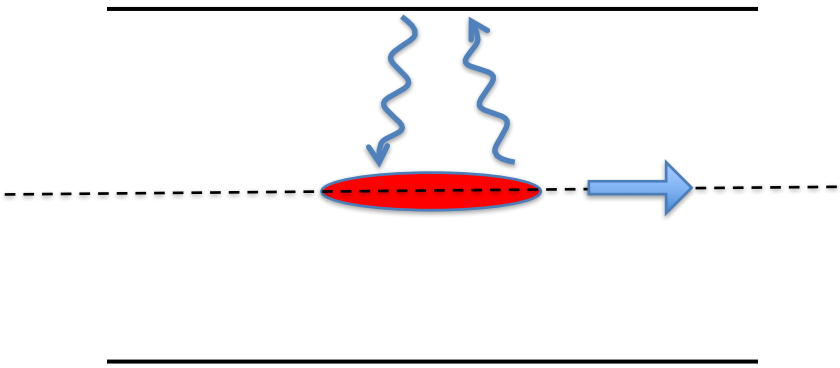
The Collective Effects

Thanks to Oliver Boine-Frankenheim, I. Hofmann, U. Niedermayer, D. Brandt

Interaction of the beam with the environment

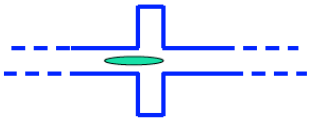


Effect on the dynamics



Resistive wall effect:
Finite conductivity

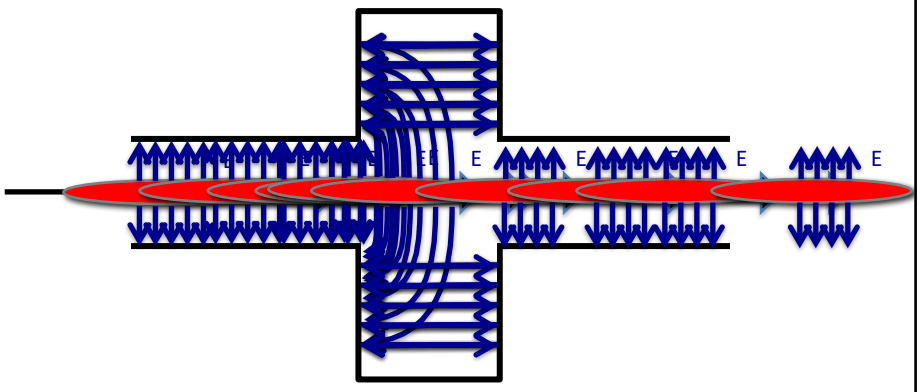
Narrow-band resonators:
Cavity-like objects



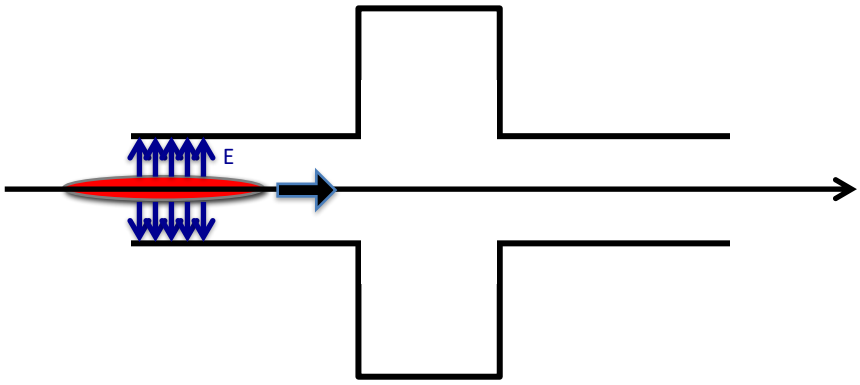
Broad-band resonators:
Tapers, other non-resonant structures

Wake Field

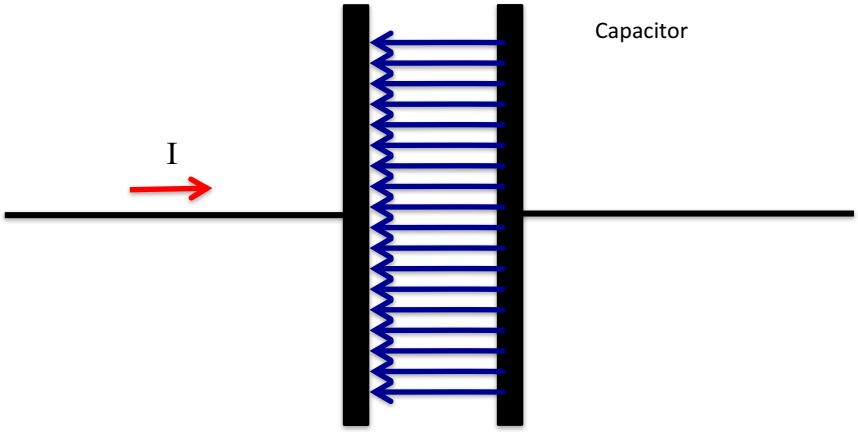
Cavities



MODEL



MODEL



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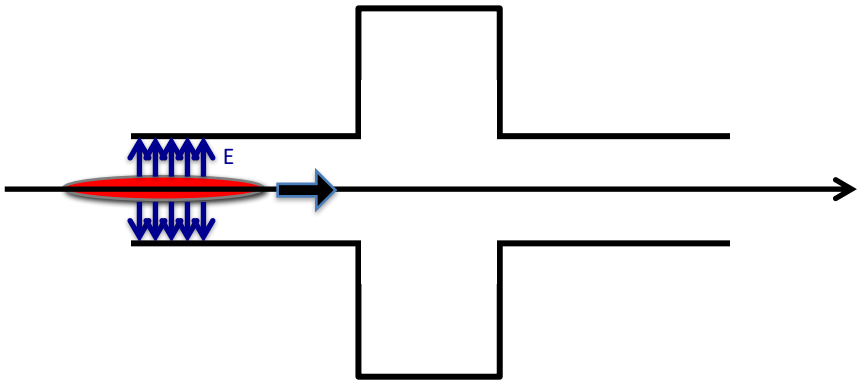

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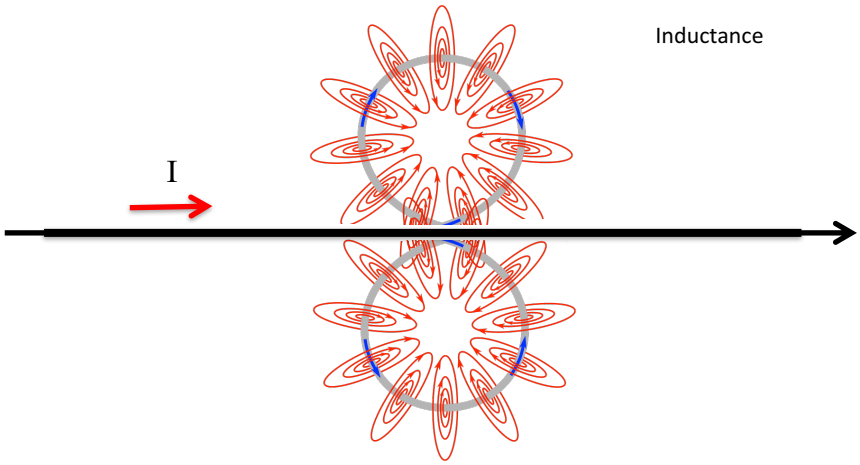

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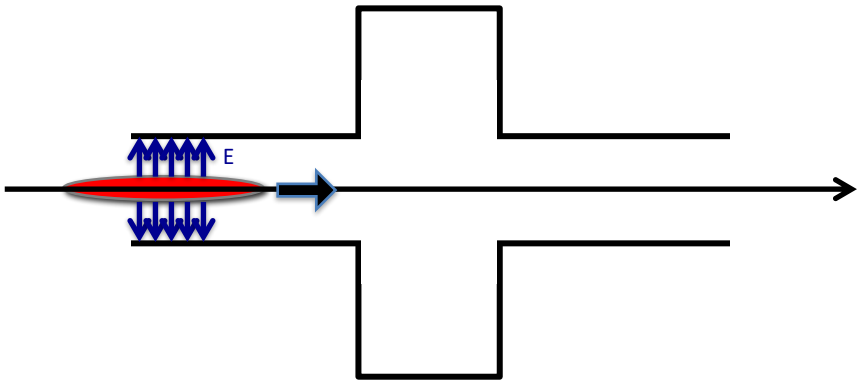
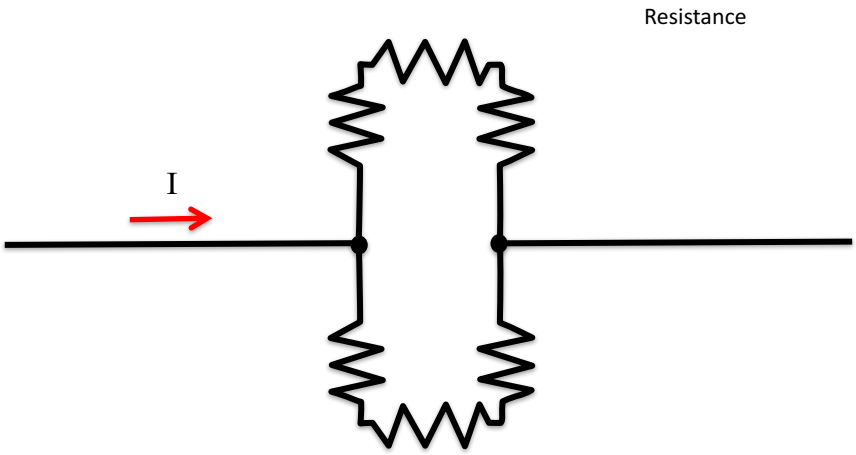
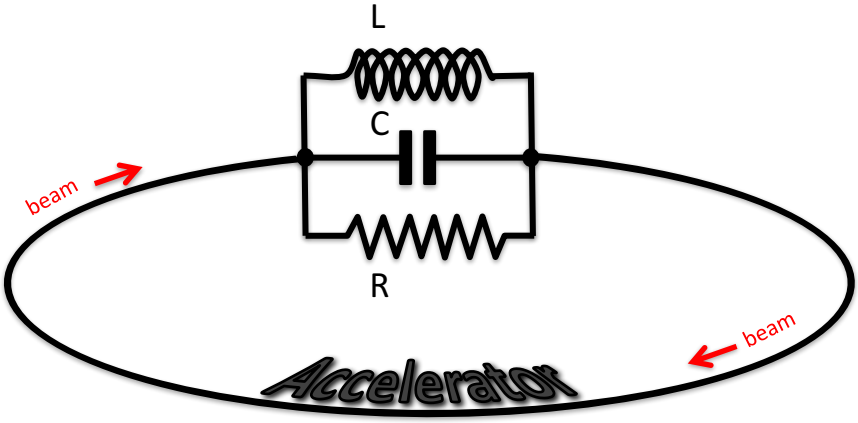


MODEL

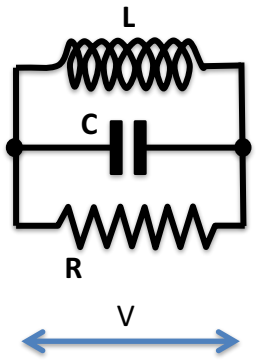


All together



RLC Features

Isolated RLC

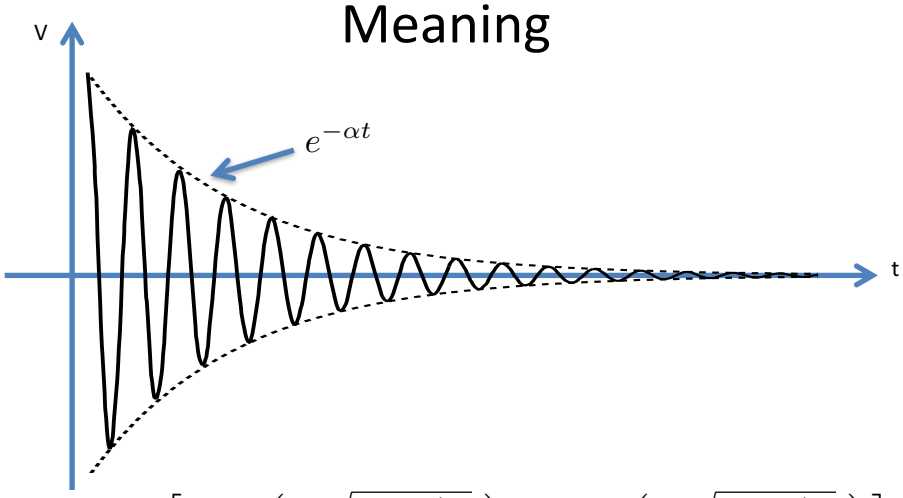


Resonance frequency $\omega_r = \frac{1}{\sqrt{LC}}$

Quality factor $Q = R\sqrt{\frac{C}{L}}$

Damping rate $\alpha = \frac{\omega_r}{2Q}$

Meaning



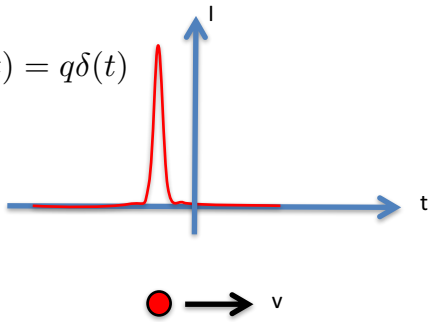
$$V(t) = e^{-\alpha t}\left[A\cos\left(\omega_r\sqrt{1-\frac{1}{4Q^2}}t\right) + B\sin\left(\omega_r\sqrt{1-\frac{1}{4Q^2}}t\right)\right]$$

Response to one particle

What happen when one particle goes through the cavity ?

Before

$$I(t) = q\delta(t)$$



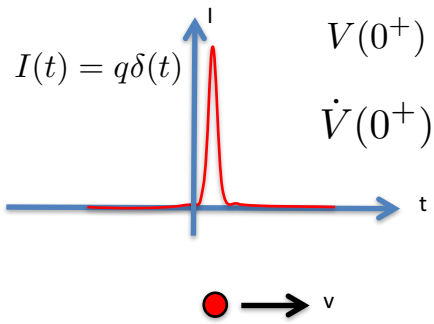
$$V(0^-) = 0$$

$$\dot{V}(0^-) = 0$$

Response to one particle

After

$$I(t) = q\delta(t)$$



$$V(0^+) = \frac{q}{C}$$

$$\dot{V}(0^+) = -\frac{2\omega_r k_{pm}}{Q} q$$

Pulse Response

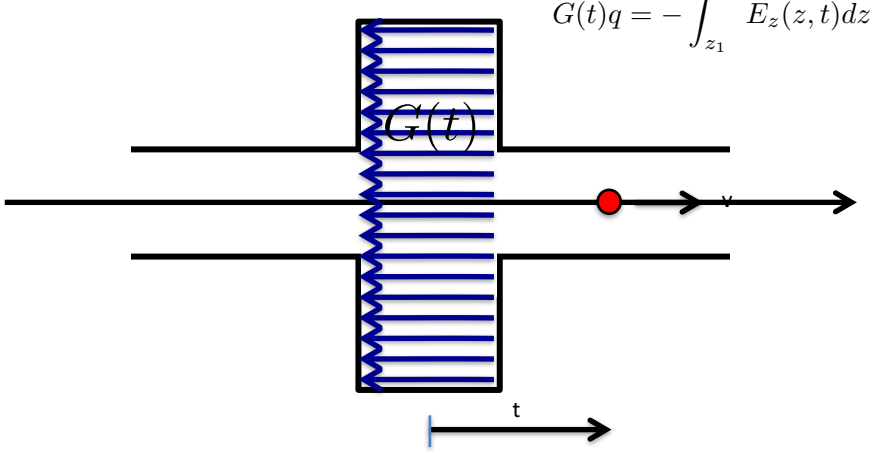
$$V(t) = 2qk_{pm}e^{-\alpha t}\left[\cos\left(\omega_r\sqrt{1-\frac{1}{4Q^2}}t\right) - \frac{\sin\left(\omega_r\sqrt{1-\frac{1}{4Q^2}}t\right)}{2Q\sqrt{1-\frac{1}{4Q^2}}}\right]$$

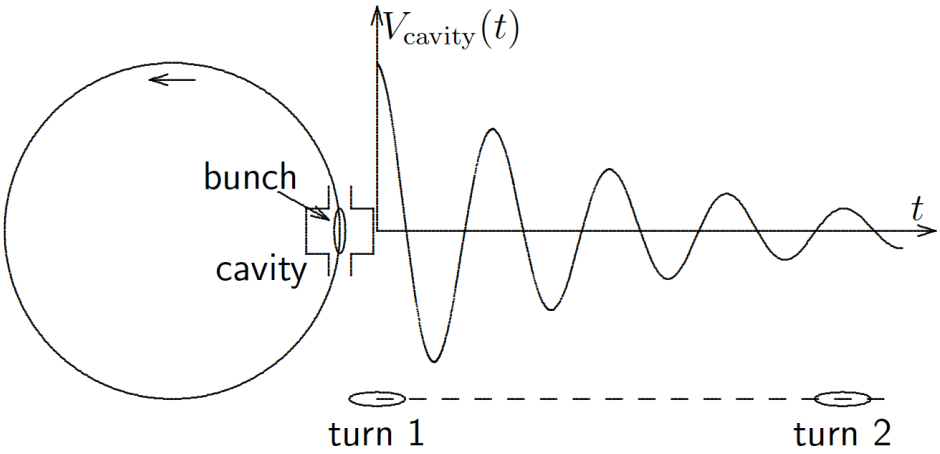
This is the potential in the cavity

Green or wake function

$$G(t) = \frac{V(t)}{q}$$

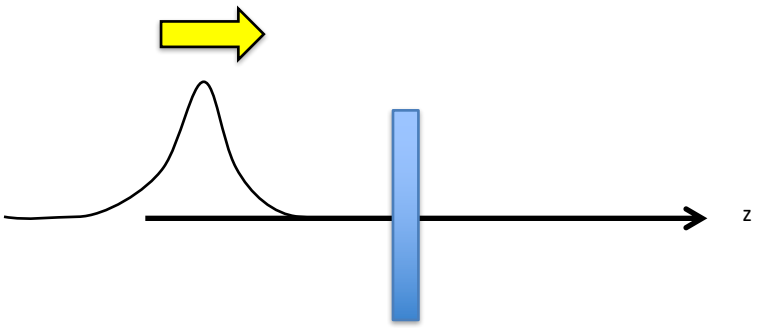
$$G(t)q = -\int_{z_1}^{z_2} E_z(z,t)dz$$





Summary

The wake function tells us what is the longitudinal field experienced by another particle passing through the cavity later



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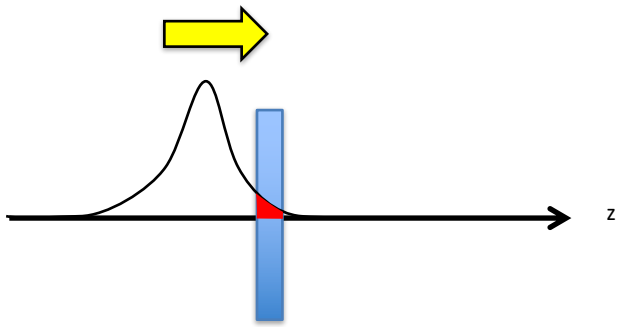

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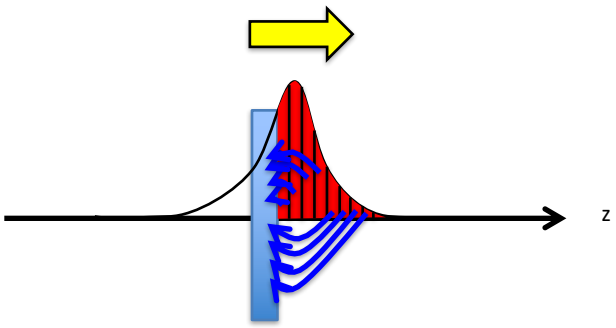

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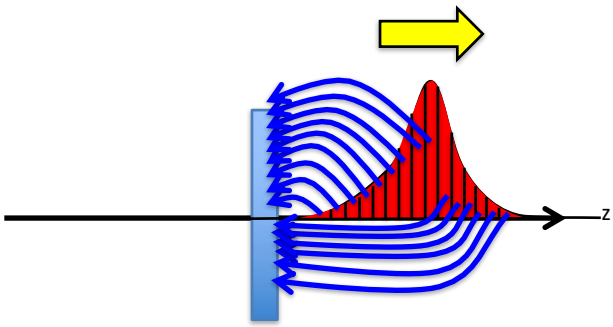
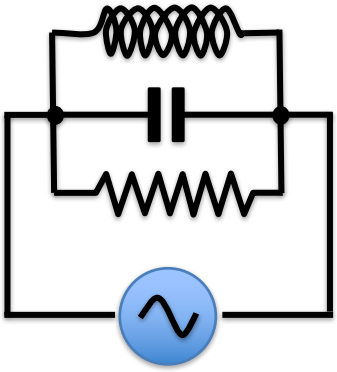


Impedance

Impedance



$I = \hat{I}\cos(\omega t)$

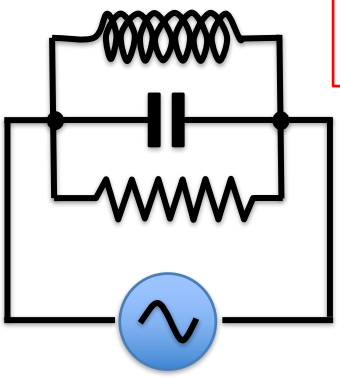
It is a quantity that relate V and I

$$\omega = 0 \Rightarrow V = RI$$

$$\omega > 0$$

$$V(t) = \hat{I}R\frac{\cos(\omega t) + Q\frac{\omega^2-\omega_r^2}{\omega_r\omega}\sin(\omega t)}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}$$

Impedance



$I = \hat{I}\cos(\omega t)$

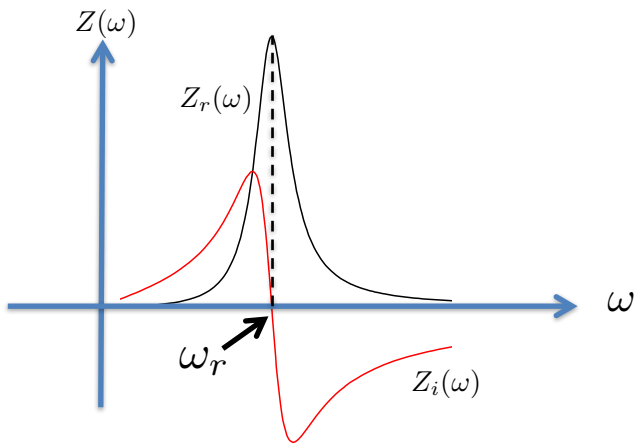
Impedance

$$V(t) = Z_r(\omega)\hat{I}\cos(\omega t) - Z_i(\omega)\hat{I}\sin(\omega t)$$

$$Z_r(\omega) = R\frac{1}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}$$

$$Z_i(\omega) = -R\frac{Q\frac{\omega^2-\omega_r^2}{\omega_r\omega}}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}$$

Properties



Properties

At $\omega = \omega_r$ $\begin{cases} Z_i(\omega_r) & \text{is zero} \\ Z_r(\omega_r) & \text{is maximum} \end{cases}$

$0 < \omega < \omega_r \Rightarrow Z_i(\omega) > 0$ inductive

$\omega > \omega_r \Rightarrow Z_i(\omega) < 0$ capacitive

$$Z_r(\omega) = Z_r(-\omega) \qquad Z_i(\omega) = -Z_i(-\omega)$$

Power dissipated

$$V(t)I(t) = \hat{I}^2 R \frac{\cos^2(\omega t) + Q\frac{\omega^2-\omega_r^2}{\omega_r\omega}\sin(\omega t)\cos(\omega t)}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}$$

$$V(t)I(t) = \hat{I}^2 Z_r(\omega)\cos^2(\omega t) + \hat{I}^2 Z_i(\omega)\sin(\omega t)\cos(\omega t)$$

The power dissipated depends on the resistive impedance

$$< V(t)I(t) >_{cycle} = \frac{1}{2}\hat{I}^2 Z_r(\omega)$$

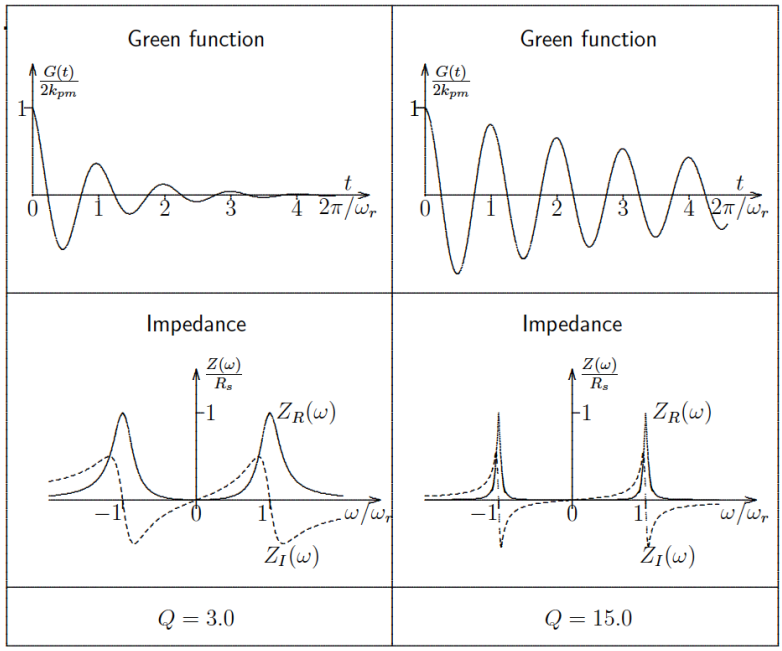
Complex notation

Complex notation $Z(\omega) = Z_r(\omega) + iZ_i(\omega)$

If Q is very large only for ω close to ω_r

$$\frac{\omega^2-\omega_r^2}{\omega_r\omega} = \frac{(\omega-\omega_r)(\omega+\omega_r)}{\omega_r\omega} \simeq \frac{2\Delta\omega}{\omega_r}$$

$$Z(\omega) = R\frac{1 - iQ\frac{\omega^2-\omega_r^2}{\omega_r\omega}}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2} = R\frac{1 - i2Q\frac{\Delta\omega}{\omega_r}}{1 + 4Q^2\left(\frac{\Delta\omega}{\omega_r}\right)^2}$$



Wake potential ←→ Impedance

Charge through the cavity at t' $\quad dq(t') = I(t')dt'$

Consider now the wake at $t > t' > 0$

The wake of that charge at time t is $G(t - t')$

The potential in the cavity at time *t* due to the charge passing at *t'* is

$$dq(t')G(t - t')$$

The total potential due to all charges passing through the cavity in $t > t' > 0$ is

$$V(t) = \int_0^t dq(t')G(t - t')$$

If now the current I is $I(t') = \hat{I} e^{i\omega t'}$

then $$V(t) = \int_0^t \hat{I} e^{i\omega t'} G(t - t') dt'$$

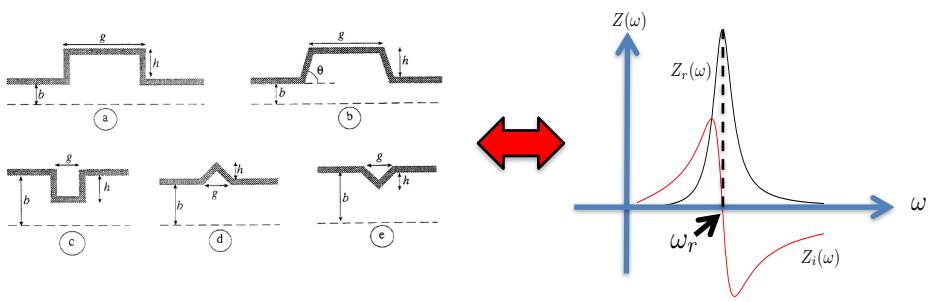
with some change of variable

$$V(t) = I(t) \int_0^t e^{-i\omega\tau} G(\tau) d\tau$$

We wait long enough that transient effect disappears, hence

$$Z(\omega) = \frac{V(t)}{I(t)} = \int_0^\infty e^{-i\omega\tau} G(\tau) d\tau$$

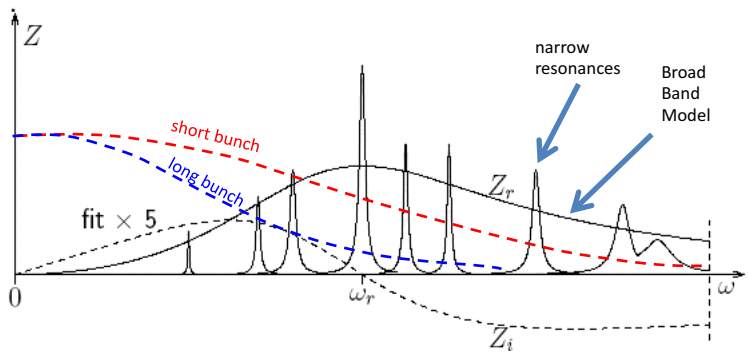
Complicated geometries of the vacuum chamber give an effect on the beam which is described by the impedance Z(ω)



$$Z(\omega) \Longleftrightarrow G(t)$$

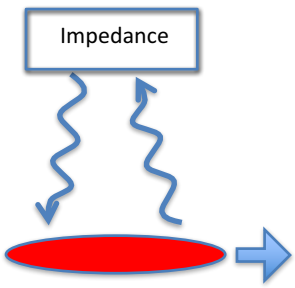
Consequences of impedances

Energy loss on pipes → heating (important if you have a superconducting machine!)



Consequences of impedances

Feed-back to the beam as a hole: collective effects



Dynamics of the all beam is affected

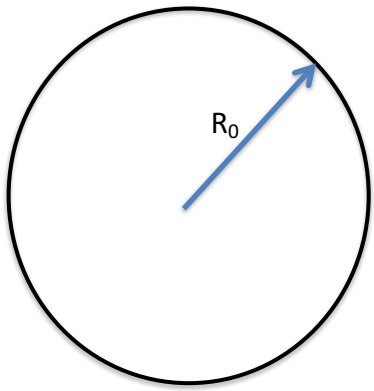
We have seen the longitudinal impedance in a cavity

More types of impedances …

Longitudinal dynamics

Longitudinal dynamics

synchronous orbit



T_0

ω_0

p_0

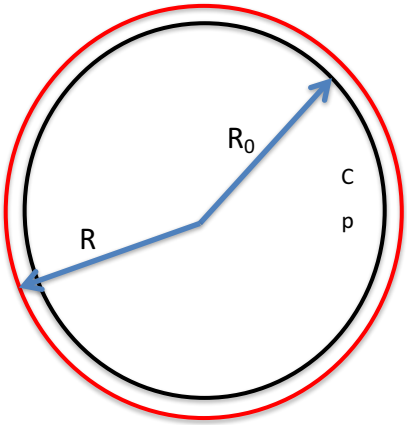
E_0

Longitudinal dynamics

synchronous orbit

C + δC

$p + \delta p$



$$\frac{\delta C}{C} = \alpha_c \frac{\delta p}{p}$$

This property comes from the magnets

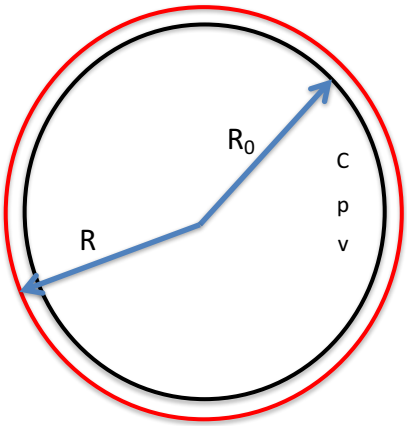
Longitudinal dynamics

revolution time

C + δC

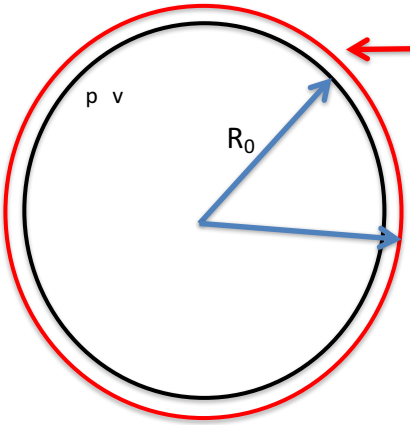
$p + \delta p$

v + dv



Nobody can go faster than light

revolution time



$p + \delta p$ If this is large

v + dv this velocity will always be less than "c"

Therefore at a certain point the circumference will growth, but the particle speed remains "c"

It takes longer to make one turn !

$$\frac{\delta T}{T_0} = \frac{1}{T_0}\delta\left(\frac{L}{v}\right) = \left(\alpha_c - \frac{1}{\gamma^2}\right)\frac{\delta p}{p} = \eta\frac{\delta p}{p}$$

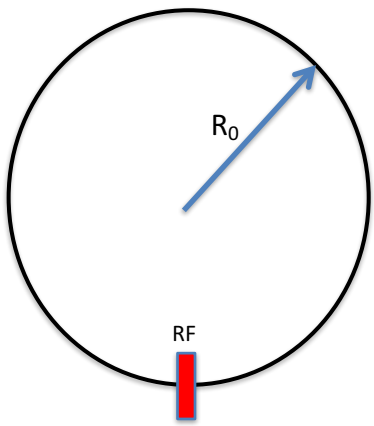
If $\alpha_c = \frac{1}{\gamma^2}$ we are at the transition energy E_T

If $E < E_T$ increasing energy → revolution time shorter

If $E > E_T$ increasing energy → revolution time longer !!

RF

synchronous orbit



T_0

ω_0

p_0

E_0

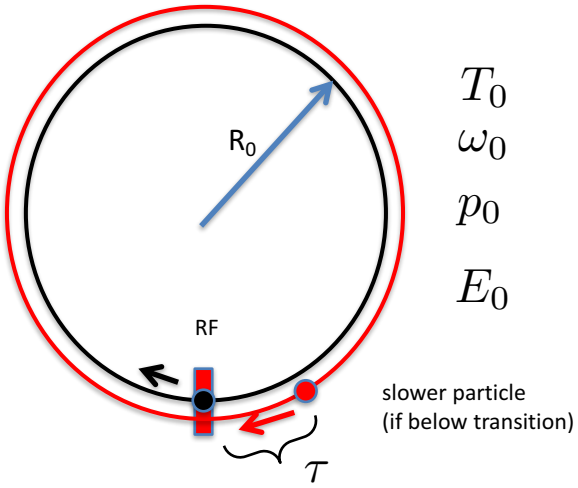
The synchronous particle has energy E and goes through the cavity at time t_s

Voltage in the cavity $V = \hat{V}\sin(h\omega_0 t_s)$

$\phi_s = h\omega_0 t_s$ this is the phase of the synchronous particle

This is a phase we know each time the particle goes through the cavity

Non synchronous particle



T_0
ω_0
p_0
E_0

Voltage on the particle

$$V = \hat{V}\sin(\phi_s + h\omega_0\tau)$$

Gain of energy $$\delta E = e\hat{V}\sin(\phi_s + h\omega_0\tau)$$

Now we include an energy loss per turn an per particle U

$$\delta E = e\hat{V}\sin(\phi_s + h\omega_0\tau) - U$$

Define relative energy $\epsilon = \Delta E/E_0$

$$\delta\epsilon = \frac{e\hat{V}}{E_0}\sin(\phi_s + h\omega_0\tau) - \frac{U}{E_0}$$

$$\frac{\delta\epsilon}{T_0} = \frac{e\hat{V}}{T_0 E_0}\sin(\phi_s + h\omega_0\tau) - \frac{U}{T_0 E_0}$$

If $\frac{\delta\epsilon}{T_0}$ is small, than this term is equal to the time derivative of ϵ

$$\dot{\epsilon} = \frac{e\hat{V}\omega_0}{2\pi E_0}\sin(\phi_s + h\omega_0\tau) - \frac{\omega_0 U}{2\pi E_0}$$

but U, depends on $\epsilon,\ \text{and}\ \tau$ ⇨ $U(\epsilon,\tau)$

If $\epsilon,\ \text{and}\ \tau$ are small we can expand

$$\dot{\epsilon} = \frac{e\hat{V}\omega_0}{2\pi E_0}\sin(\phi_s) + \frac{e\hat{V}\omega_0}{2\pi E_0}\cos(\phi_s)h\omega_0\tau - \frac{\omega_0 U_0}{2\pi E_0} - \frac{\omega_0}{2\pi E_0}\frac{\partial U}{\partial E}\epsilon - \frac{\omega_0}{2\pi E_0}\frac{\partial U}{\partial t}\tau$$

These two terms are equal for the synchronous particle

We remain with the equation

$$\dot{\epsilon} = \frac{e\hat{V}h\omega_0^2}{2\pi E_0}\cos(\phi_s)\tau - \frac{\omega_0}{2\pi E_0}\frac{\partial U}{\partial E}\epsilon - \frac{\omega_0}{2\pi E_0}\frac{\partial U}{\partial t}\tau$$

In addition at high energy $\frac{\delta T}{T} \simeq \eta\frac{\delta E}{E}$ ⇨ $\dot{\tau} = \eta\epsilon$

$$\ddot{\tau} = \eta \frac{e\hat{V}h\omega_0^2}{2\pi E_0}\cos(\phi_s)\tau - \eta\frac{\omega_0}{2\pi E_0}\frac{\partial U}{\partial E}\epsilon - \eta\frac{\omega_0}{2\pi E_0}\frac{\partial U}{\partial t}\tau$$

$$\omega_{s0}^2 = -\eta\frac{e\hat{V}h\omega_0^2}{2\pi E_0}\cos(\phi_s) \qquad \alpha_s = \frac{1}{2}\frac{\omega_0}{2\pi E}\frac{\partial U}{\partial E}$$

Final equation of motion (in tau)

$$\ddot{\tau} + 2\alpha_s\dot{\tau} + \left[\omega_{s0}^2 + \eta\frac{\omega_0}{2\pi E_0}\frac{\partial U}{\partial t}\right]\tau = 0$$

Solution

$\tau \propto e^{\lambda t}$ ⇒ $\lambda^2 + 2\alpha_s\lambda + \left[\omega_{s0}^2 + \eta\frac{\omega_0}{2\pi E_0}\frac{\partial U}{\partial t}\right] = 0$

Solving for lambda: $\lambda = -\alpha_s \pm \sqrt{\alpha_s^2 - (\omega_{s0}^2 + ...)}$

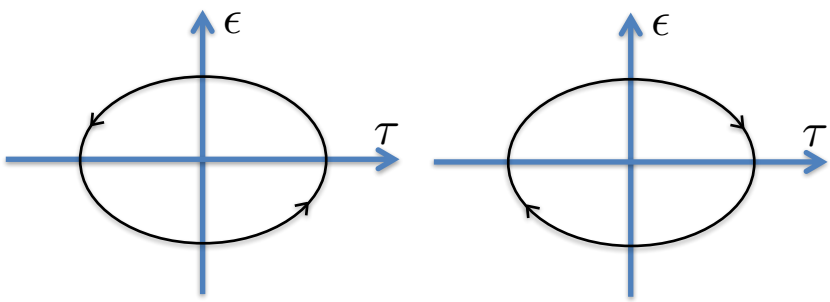
that is $\lambda = -\alpha_s \pm i\omega_{s1}$ with $\omega_{s1}^2 = \omega_{s0}^2 + \eta\frac{\omega_0}{2\pi E_0}\frac{\partial U}{\partial t} - \alpha_s^2$

$\boxed{\tau = \hat{\tau}e^{-\alpha_s t}\cos(\omega_{s1}t)}$ ⇒ if $\alpha_s > 0$ Solution stable

Interpretation

No Energy Loss

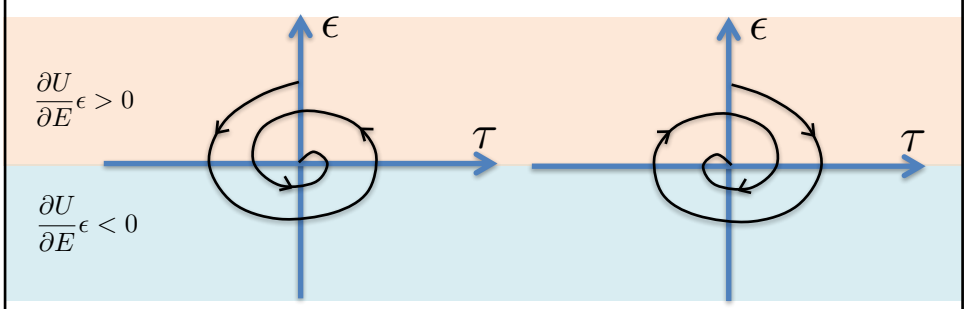
$E < E_T$ $\qquad$ $E > E_T$



Interpretation

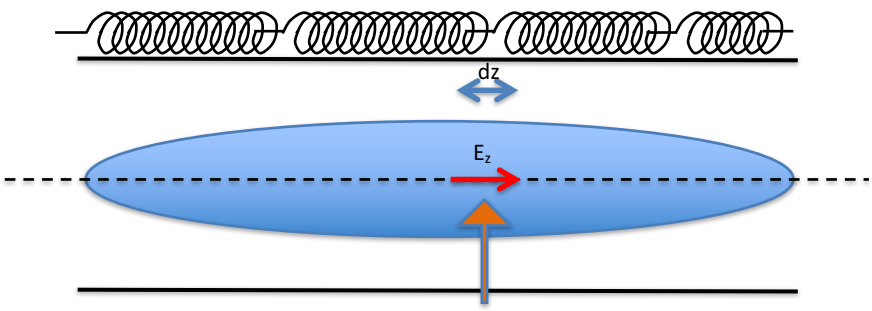
With Energy Loss

$E < E_T$ $\qquad$ $E > E_T$



Bunch Lengthening

Bunch lengthening



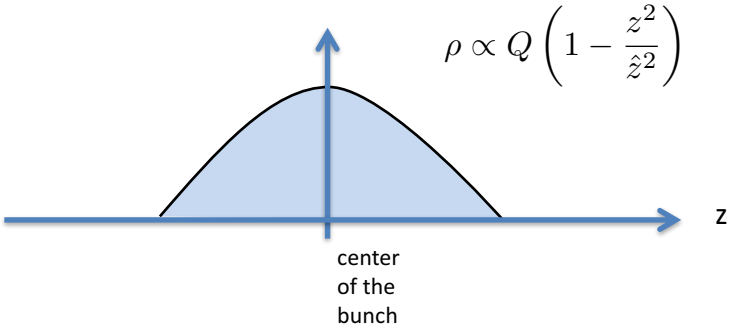
In one turn
change of energy
per charge

$$V = -L\frac{dI_b}{dt}$$

L is the integrated inductance

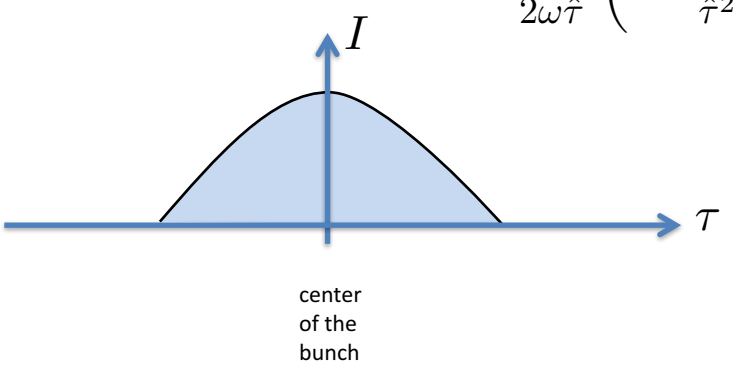
Parabolic bunch

$$\rho \propto Q\left(1-\frac{z^2}{\hat{z}^2}\right)$$

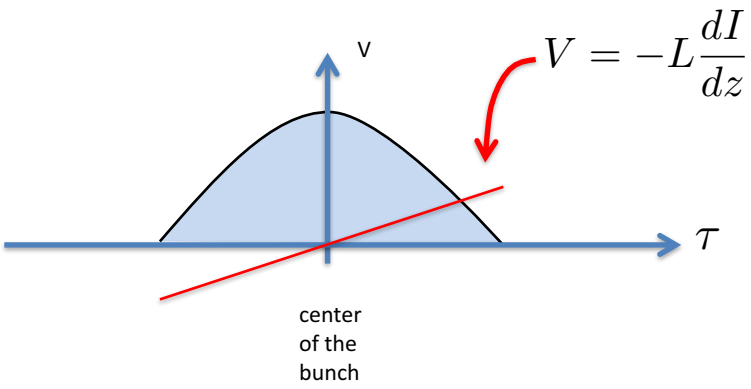


Parabolic bunch

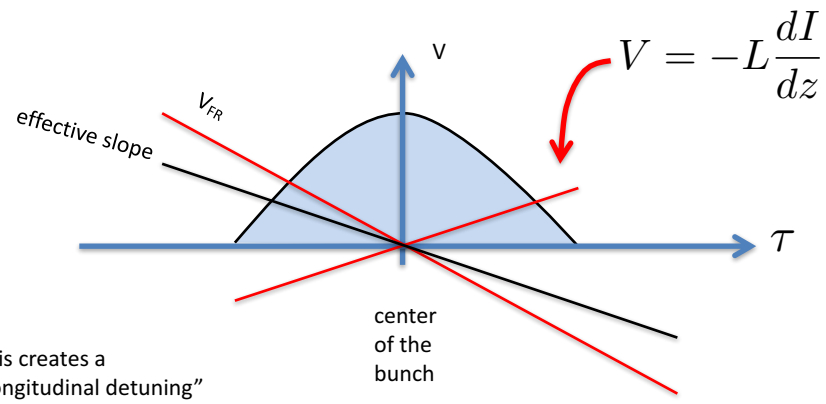
$$I = \frac{3\pi I_0}{2\omega\hat{\tau}}\left(1-\frac{\tau^2}{\hat{\tau}^2}\right)$$



Voltage induced



If we compare with RF



This creates a
"longitudinal detuning"

By using a bunch with the same longitudinal emittance a reduction of longitudinal focusing strength produces a bunch lengthening

The bunch becomes matched with the effective voltage slope

Effective voltage

$$V = \hat{V}\sin(\phi_s + h\omega_0\tau) + \frac{3\pi I_0 L}{\omega_0 \hat{\tau}^3}\tau$$

induced voltage

Linearizing in tau

$$V = \hat{V}\sin(\phi_s) + \boxed{\hat{V}\cos(\phi_s)h\omega_0\tau} + \boxed{\frac{3\pi I_0 L}{\omega_0 \hat{\tau}^3}\tau}$$

focusing from RF

defocusing from impedance

$$\dot{\epsilon} = \frac{e\hat{V}\omega_0}{2\pi E_0}\cos(\phi_s)h\omega_0\tau + e\frac{\omega_0}{2\pi E_0}\frac{3\pi I_0 L}{\omega_0\hat{\tau}^3}\tau$$

But $\dot{\tau} = \eta\epsilon$ therefore

$$\ddot{\tau} = \frac{\eta e\hat{V}\omega_0}{2\pi E_0}\cos(\phi_s)h\omega_0\tau + e\frac{\eta\omega_0}{2\pi E_0}\frac{3\pi I_0 L}{\omega_0\hat{\tau}}\tau$$

but $$\left|\frac{Z}{n}\right|_0 = L\omega_0$$

Therefore

$$\ddot{\tau} = \frac{\eta e h\hat{V}\omega_0^2}{2\pi E_0}\cos(\phi_s)\left[1 + \frac{1}{\hat{V}\cos(\phi_s)}\frac{3\pi I_0}{h\omega_0^3\hat{\tau}^3}\left|\frac{Z}{n}\right|_0\right]\tau$$

$$\omega_{s0}^2 = -\frac{\eta e h\hat{V}\omega_0^2}{2\pi E_0}\cos(\phi_s)$$

is the longitudinal strength in absence of impedance

$$\omega_s^2 = \omega_{s0}^2 \left[1 + \frac{1}{\hat{V}\cos(\phi_s)} \frac{3\pi I_0}{h\omega_0^3 \hat{\tau}^3} \left| \frac{Z}{n} \right|_0 \right]$$

Therefore the relative change in omega is

$$\frac{\Delta\omega_s}{\omega_{s0}} = \frac{1}{2} \frac{1}{\hat{V}\cos(\phi_s)} \frac{3\pi I_0}{h\omega_0^3 \hat{\tau}^3} \left| \frac{Z}{n} \right|_0$$

For protons $\hat{\tau}\hat{\epsilon} = \text{constant}$

$$\frac{\Delta\hat{\tau}}{\tau} \simeq -\frac{\Delta\omega_s}{2\omega_s}$$

Observation

The effect of the impedance is local, hence the voltage induced by impendence do not effect the center of mass (like for the space charge)

Summary

1) Wall charges creates detuning → incoherent tunes
2) Ferromagnetic material creates image currents:
   Coherent motion → coherent tunes
3) Concept of Wake field
4) Impedance of a cavity, Wake ←→ impedance
5) Energy loss
6) Longitudinal dynamics, effect of energy loss
7) Bunch lengthening