Collective Effects I

G. Franchetti, GSI
CERN Accelerator – School
Budapest, 2-14 / 10 / 2016

Disclaimer: not all in this handouts will be presented

Type of fields

- Direct self fields
- Image self fields
- Wake fields
- Space Charge
- Collective Effects

Collective Effects ?
Image charges

Influence of the chamber wall

the electron in the metal quickly travel on the surface of the metal until the electric field parallel to the surface is zero

field line at $90^\circ$
The image charge is a reflection of the particle with exchanged sign.
Conducting plates

\[ \text{h} \]

2D beam
charge density \( \rho_L \)

\[ \text{h} \]
Consider a particle of the beam

Incoherent motion

Summing image charge contribution in pairs

\[ E_{y,n} = \frac{\rho L}{2\pi \epsilon_0} (-1)^n \left( \frac{1}{2nh + y} - \frac{1}{2nh - y} \right) \]

\[ E_{y,n} = -\frac{\rho L y}{4\pi \epsilon_0 h^2} (-1)^n \frac{n}{n^2} \]

Total electric field

\[ E_y = \sum_{n=1}^{\infty} E_{y,n} = \frac{\rho L y}{\pi \epsilon_0} \frac{\pi^2}{48h^2} \]
Equation of motion

In the equation of motion

\[ \frac{d^2 y}{ds^2} + k_y y = \frac{e}{m\gamma^3 v_0^2} E_{b,y} + \frac{e}{m\gamma v_0^2} E_{i,y} \]

\[ \frac{d^2 y}{ds^2} + k_y y = \frac{2K}{Y(X + Y)} y + K\gamma^2 \frac{\pi^2}{24h^2} y \]

as \ \nabla \cdot \vec{E} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial x} = -\frac{\partial E_y}{\partial y} \\

\[ \frac{d^2 y}{ds^2} + k_y y - \frac{2K}{Y(X + Y)} \left[ 1 + \gamma^2 \frac{\pi^2}{48} \frac{Y(X + Y)}{h^2} \right] y = 0 \]

\[ \frac{d^2 x}{ds^2} + k_x x - \frac{2K}{X(X + Y)} \left[ 1 - \gamma^2 \frac{\pi^2}{48} \frac{X(X + Y)}{h^2} \right] x = 0 \]

Laslett Tuneshift

\[ \Delta Q_y \simeq -\frac{R_m^2}{Q_{y0}} \frac{K}{Y(X + Y)} \left[ 1 + \gamma^2 \frac{\pi^2}{48} \frac{Y(X + Y)}{h^2} \right] \]

\[ \Delta Q_x \simeq -\frac{R_m^2}{Q_{x0}} \frac{K}{X(X + Y)} \left[ 1 - \gamma^2 \frac{\pi^2}{48} \frac{X(X + Y)}{h^2} \right] \]
Image currents

\[ B_{1n} = B_{2n}, \quad \mu_1 \ll \mu_2 \]

\[ \frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2} \quad B_{1,t} \simeq 0 \]
Ferromagnetic Boundaries

\[ B_x = \frac{\mu_0 I}{2\pi} \frac{1}{2g - y} \]
\[ B_x = \mu_0 I \sum_{n=1}^{\infty} \left( \frac{1}{2ng - y} - \frac{1}{2ng + y} \right) \]

for \( g \gg y \)

\[ B_x = \frac{\mu_0 I y \pi^2}{4\pi g^2} \frac{1}{6} \]

In the equation of motion

\[ \frac{d^2y}{ds^2} + k_y y = \frac{2K}{Y(X + Y)} y - \frac{1}{m'v_0^2} v_x B_x \]

therefore

\[ \frac{d^2y}{ds^2} + k_y y = \frac{2K}{Y(X + Y)} y + K \frac{2\gamma^2 \beta^2 \pi^2}{24g^2} y \]

incoherent SC

ferromagnetic induced image current (coherent force)

Tune-shift!
Coherent Motion

Coherent motion
Coherent motion

all image charges moves

Force on the beam
NO FORCE CREATED ON THE BEAM

\[ E_{y,n} = -\frac{\rho L}{2\pi \epsilon_0} \left[ \frac{1}{2nh + 2y} - \frac{1}{2nh - 2y} \right] \]

\[ n = 1, 3, 5, 7, \ldots \]

\[ E_{y,n} = \frac{\rho L}{2\pi \epsilon_0 h^2} \frac{4y}{(2nh)^2 - (2y)^2} \]

\[ n = 1, 3, 5, 7, \ldots \]
therefore

\[ E_{y,n} = \frac{\rho_L}{4\pi\epsilon_0 h^2 y} \left[ \frac{1}{n^2} - \frac{(-1)^n}{n^2} \right] \]  

(trick!)

with \( n = 1, 2, 3, 4, 5, 6, \ldots \)

\[ E_y = \sum_{n=1}^{\infty} E_{y,n} = \frac{\rho_L}{4\pi\epsilon_0 h^2 y} \left[ \frac{\pi^2}{6} + \frac{\pi^2}{12} \right] = \frac{\rho_L}{16\pi\epsilon_0 h^2 \pi^2 y} \]

The electric field \( E_y \) due to coherent shift is zero on the center of mass 😊

---

equation of motion

\[ \frac{d^2 y_c}{ds^2} + k_y y_c = \frac{e}{m\gamma v_0^2} \frac{\rho_L}{16\pi\epsilon_0 h^2 \pi^2 y_c} \]

but \( I = v_z \rho_L \approx v_0 \rho_L \)

\[ \frac{d^2 y_c}{ds^2} + k_y y_c = K \frac{\gamma^2 \pi^2}{8h^2} y_c \]

\[ \frac{d^2 y_c}{ds^2} + \left( k_y - 2K \frac{\gamma^2 \pi^2}{16h^2} \right) y_c = 0 \]
Coherent detuning

\[ \Delta Q_{y,c} \simeq -\frac{R_m^2}{Q_y} K \gamma^2 \pi^2 \frac{1}{16h^2} \]

The Collective Effects

Thanks to Oliver Boine-Frankenheim, I. Hofmann, U. Niedermayer, D. Brandt
Interaction of the beam with the environment

Effect on the dynamics
Resistive wall effect: Finite conductivity

Narrow-band resonators: Cavity-like objects

Broad-band resonators: Tapers, other non-resonant structures

Wake Field
Cavities

MODEL
MODEL

Resistance

I

All together

L

C

R

beam

beam
## RLC Features

\[ \omega_r = \frac{1}{\sqrt{LC}} \]

Quality factor
\[ Q = R \sqrt{\frac{C}{L}} \]

Damping rate
\[ \alpha = \frac{\omega_r}{2Q} \]

### Meaning

\[ V(t) = e^{-\alpha t} \left[ A \cos \left( \omega_r \sqrt{1 - \frac{1}{4Q^2}} t \right) + B \sin \left( \omega_r \sqrt{1 - \frac{1}{4Q^2}} t \right) \right] \]
Response to one particle

What happens when one particle goes through the cavity?

Before

\[ I(t) = q\delta(t) \]

\[ V(0^-) = 0 \]

\[ \dot{V}(0^-) = 0 \]

After

\[ V(0^+) = \frac{q}{C} \]

\[ \dot{V}(0^+) = -\frac{2\omega_r k_{pm}}{Q} q \]
Pulse Response

\[ V(t) = 2qk_{pm}e^{-\alpha t} \left[ \cos \left( \omega_r \sqrt{1 - \frac{1}{4Q^2}} t \right) - \frac{\sin \left( \omega_r \sqrt{1 - \frac{1}{4Q^2}} t \right)}{2Q \sqrt{1 - \frac{1}{4Q^2}}} \right] \]

This is the potential in the cavity.

Green or wake function

\[ G(t) = \frac{V(t)}{q} \]

\[ G(t)q = -\int_{z_1}^{z_2} E_z(z, t)dz \]
The wake function tells us what is the longitudinal field experienced by another particle passing through the cavity later.
Summary

The wake function tells us what is the longitudinal field experienced by another particle passing through the cavity later.
Summary

The wake function tells us what is the longitudinal field experienced by another particle passing through the cavity later.

Impedance
Impedance

It is a quantity that relates $V$ and $I$

$$
\begin{align*}
\omega &= 0 & V &= RI \\
\omega &> 0 & V(t) &= \hat{I}R \cos(\omega t) + Q \frac{\omega^2 - \omega_r^2}{\omega_r \omega} \sin(\omega t) \\
& & & \quad 1 + Q^2 \left( \frac{\omega^2 - \omega_r^2}{\omega_r \omega} \right)^2
\end{align*}
$$

Impedance

$$
V(t) = Z_r(\omega)\hat{I}\cos(\omega t) - Z_i(\omega)\hat{I}\sin(\omega t)
$$

$$
\begin{align*}
Z_r(\omega) &= R \frac{1}{1 + Q^2 \left( \frac{\omega^2 - \omega_r^2}{\omega_r \omega} \right)^2} \\
Z_i(\omega) &= -R \frac{Q \omega^2 - \omega_r^2}{\omega_r \omega} \left( \frac{\omega^2 - \omega_r^2}{\omega_r \omega} \right)^2
\end{align*}
$$
Properties

At \( \omega = \omega_r \), \( \begin{cases} Z_i(\omega_r) & \text{is zero} \\ Z_r(\omega_r) & \text{is maximum} \end{cases} \)

- For \( 0 < \omega < \omega_r \), \( Z_i(\omega) > 0 \), inductive
- For \( \omega > \omega_r \), \( Z_i(\omega) < 0 \), capacitive

\[ Z_r(\omega) = Z_r(-\omega) \quad Z_i(\omega) = -Z_i(-\omega) \]
Power dissipated

\[ V(t)I(t) = \bar{I}^2 R \frac{\cos^2(\omega t) + Q \frac{\omega^2 - \omega_r^2}{\omega_r \omega} \sin(\omega t) \cos(\omega t)}{1 + Q^2 \left( \frac{\omega^2 - \omega_r^2}{\omega_r \omega} \right)^2} \]

\[ V(t)I(t) = \bar{I}^2 Z_r(\omega) \cos^2(\omega t) + \bar{I}^2 Z_i(\omega) \sin(\omega t) \cos(\omega t) \]

The power dissipated depends on the resistive impedance

\[ < V(t)I(t) >_{cycle} = \frac{1}{2} \bar{I}^2 Z_r(\omega) \]

Complex notation

Complex notation \[ Z(\omega) = Z_r(\omega) + iZ_i(\omega) \]

If \( Q \) is very large only for \( \omega \) close to \( \omega_r \).

\[ \frac{\omega^2 - \omega_r^2}{\omega \omega_r} = \frac{(\omega - \omega_r)(\omega + \omega_r)}{\omega \omega_r} \approx 2 \Delta \omega \]

\[ Z(\omega) = R \frac{1 - iQ \frac{\omega^2 - \omega_r^2}{\omega \omega_r}}{1 + Q^2 \left( \frac{\omega^2 - \omega_r^2}{\omega \omega_r} \right)^2} = R \frac{1 - i2Q \frac{\Delta \omega}{\omega_r}}{1 + 4Q^2 \left( \frac{\Delta \omega}{\omega_r} \right)^2} \]
Wake potential $\leftrightarrow$ Impedance

Charge through the cavity at $t'$
\[ dq(t') = I(t')dt' \]

Consider now the wake at $t > t' > 0$

The wake of that charge at time $t$ is $G(t - t')$

The potential in the cavity at time $t$ due to the charge passing at $t'$ is
\[ dq(t')G(t - t') \]

The total potential due to all charges passing through the cavity in $t > t' > 0$ is
\[ V(t) = \int_{0}^{t} dq(t')G(t - t') \]
If now the current $I$ is

$$I(t') = \hat{I} e^{i\omega t'}$$

then

$$V(t) = \int_{0}^{t} \hat{I} e^{i\omega t'} G(t - t') dt'$$

with some change of variable

$$V(t) = I(t) \int_{0}^{t} e^{-i\omega \tau} G(\tau) d\tau$$

We wait long enough that transient effect disappears, hence

$$Z(\omega) = \frac{V(t)}{I(t)} = \int_{0}^{\infty} e^{-i\omega \tau} G(\tau) d\tau$$

Complicated geometries of the vacuum chamber give an effect on the beam which is described by the impedance $Z(\omega)$
Consequences of impedances

Energy loss on pipes → heating (important if you have a superconducting machine!)

Consequences of impedances

Feed-back to the beam as a hole: collective effects

We have seen the longitudinal impedance in a cavity

More types of impedances ...

Dynamics of the all beam is affected
Longitudinal dynamics

synchronous orbit

\[ T_0 \quad \omega_0 \quad p_0 \quad E_0 \]
Longitudinal dynamics

\[ \frac{\delta C}{C} = \alpha_c \frac{\delta p}{p} \]

This property comes from the magnets

\[
\begin{align*}
C + \delta C \\
p + \delta p
\end{align*}
\]

revolution time

\[
\begin{align*}
C + \delta C \\
p + \delta p \\
v + dv
\end{align*}
\]
Nobody can go faster than light

\[ p + \delta p \quad \text{If this is large} \]

\[ v + dv \quad \text{this velocity will always be less than “c”} \]

Therefore at a certain point the circumference will growth, but the particle speed remains “c”

It takes longer to make one turn!

\[
\frac{\delta T}{T_0} = \frac{1}{T_0} \delta \left( \frac{L}{v} \right) = \left( \alpha_c - \frac{1}{\gamma^2} \right) \frac{\delta p}{p} = \eta \frac{\delta p}{p}
\]

If \( \alpha_c = \frac{1}{\gamma^2} \) we are at the transition energy \( E_T \)

If \( E < E_T \) increasing energy \( \Rightarrow \) revolution time shorter

If \( E > E_T \) increasing energy \( \Rightarrow \) revolution time longer!!
The synchronous particle has energy $E$ and goes through the cavity at time $t_s$.

Voltage in the cavity

$$V = \hat{V} \sin(h\omega_0 t_s)$$

$\phi_s = h\omega_0 t_s$ this is the phase of the synchronous particle.

This is a phase we know each time the particle goes through the cavity.
Non synchronous particle

Voltage on the particle

\[ V = \hat{V} \sin(\phi_s + h\omega_0\tau) \]

Gain of energy

\[ \delta E = e\hat{V} \sin(\phi_s + h\omega_0\tau) \]

Now we include an energy loss per turn an per particle \(U\)

\[ \delta E = e\hat{V} \sin(\phi_s + h\omega_0\tau) - U \]

Define relative energy

\[ \epsilon = \Delta E / E_0 \]

\[ \delta \epsilon = \frac{e\hat{V}}{E_0} \sin(\phi_s + h\omega_0\tau) - \frac{U}{E_0} \]
\[
\frac{\delta \epsilon}{T_0} = \frac{e \hat{V}}{T_0 E_0} \sin(\phi_s + h \omega_0 \tau) - \frac{U}{T_0 E_0}
\]

If \( \frac{\delta \epsilon}{T_0} \) is small, than this term is equal to the time derivative of \( \epsilon \).

\[
\dot{\epsilon} = \frac{e \hat{V} \omega_0}{2\pi E_0} \sin(\phi_s + h \omega_0 \tau) - \frac{\omega_0 U}{2\pi E_0}
\]

but \( U \), depends on \( \epsilon \), and \( \tau \) \to U(\epsilon, \tau)

\[\text{These two terms are equal for the synchronous particle}\]

We remain with the equation

\[
\dot{\epsilon} = \frac{e \hat{V} \omega_0^2}{2\pi E_0} \cos(\phi_s) \tau - \frac{\omega_0 U}{2\pi E_0} \frac{\partial U}{\partial E} \epsilon - \frac{\omega_0}{2\pi E_0} \frac{\partial U}{\partial t} \tau
\]

In addition at high energy

\[
\frac{\delta T}{T} \approx \eta \frac{\delta E}{E} \quad \Rightarrow \quad \dot{\tau} = \eta \epsilon
\]
\[ \ddot{\tau} = \eta \frac{e \dot{V} h \omega_0^2}{2\pi E_0} \cos(\psi) \tau - \eta \frac{\omega_0}{2\pi E_0} \frac{\partial U}{\partial t} - \eta \frac{\omega_0}{2\pi E_0} \frac{\partial U}{\partial t} \tau \]

\[ \omega_{s0}^2 = -\eta \frac{e \dot{V} h \omega_0^2}{2\pi E_0} \cos(\psi) \quad \alpha_s = \frac{\omega_0}{2\pi E_0} \frac{\partial U}{\partial \psi} \]

Final equation of motion (in tau)

\[ \ddot{\tau} + 2\alpha_s \dot{\tau} + \left[ \omega_{s0}^2 + \eta \frac{\omega_0}{2\pi E_0} \frac{\partial U}{\partial t} \right] \tau = 0 \]

\[ \tau \propto e^{\lambda t} \rightarrow \lambda^2 + 2\alpha_s \lambda + \left[ \omega_{s0}^2 + \eta \frac{\omega_0}{2\pi E_0} \frac{\partial U}{\partial t} \right] = 0 \]

Solving for lambda:

\[ \lambda = -\alpha_s \pm \sqrt{\alpha_s^2 - (\omega_{s0}^2 + ...)} \]

that is \( \lambda = -\alpha_s \pm i\omega_1 \) with

\[ \omega_{s1}^2 = \omega_{s0}^2 + \eta \frac{\omega_0}{2\pi E_0} \frac{\partial U}{\partial t} - \alpha_s^2 \]

\[ \tau = \hat{\tau} e^{-\alpha_s t} \cos(\omega_{s1} t) \rightarrow \text{if } \alpha_s > 0 \text{ Solution stable} \]
**Interpretation**

For **No Energy Loss**:
- \( E < E_T \)
- \( \epsilon \)
- \( T \)

For **With Energy Loss**:
- \( E < E_T \)
- \( \partial U / \partial E > 0 \)
- \( \epsilon < 0 \)
- \( \partial U / \partial E < 0 \)
Bunch Lengthening

\[ V = -L \frac{dI_b}{dt} \]

L is the integrated inductance
\[ \rho \propto Q \left( 1 - \frac{z^2}{z_0^2} \right) \]

Parabolic bunch

\[ I = \frac{3\pi I_0}{2\omega \hat{\tau}} \left( 1 - \frac{\tau^2}{\hat{\tau}^2} \right) \]

Parabolic bunch
Voltage induced

\[ V = -L \frac{dI}{dz} \]

If we compare with RF

\[ V = -L \frac{dI}{dz} \]

This creates a “longitudinal detuning”
By using a bunch with the same longitudinal emittance a reduction of longitudinal focusing strength produces a bunch lengthening.

The bunch becomes matched with the effective voltage slope.

Effective voltage

\[ V = \hat{V} \sin(\phi_s + h\omega_0 \tau) + \frac{3\pi I_0 L}{\omega_0 \tau^3} \tau \]

Linearizing in tau

\[ V = \hat{V} \sin(\phi_s) + \hat{V} \cos(\phi_s) h\omega_0 \tau + \frac{3\pi I_0 L}{\omega_0 \tau^3} \tau \]

induced voltage

focusing from RF
defocusing from impedance
\[ \dot{\varepsilon} = \frac{e \hat{V} \omega_0}{2\pi E_0} \cos(\phi_s) h \omega_0 \tau + e \frac{\omega_0}{2\pi E_0} \frac{3\pi I_0 L}{\omega_0 \hat{\tau}^3} \tau \]

But \( \dot{\tau} = \eta \varepsilon \) therefore

\[ \ddot{\tau} = \frac{\eta e \hat{V} \omega_0}{2\pi E_0} \cos(\phi_s) h \omega_0 \tau + e \frac{\eta \omega_0}{2\pi E_0} \frac{3\pi I_0 L}{\omega_0 \hat{\tau}^3} \tau \]

\[ \frac{\left| Z \right|}{n} \bigg|_0 = L \omega_0 \]

Therefore

\[ \ddot{\tau} = \frac{\eta e \hat{V} \omega_0^2}{2\pi E_0} \cos(\phi_s) \left[ 1 + \frac{1}{V \cos(\phi_s)} \frac{3\pi I_0}{h \omega_0^3 \hat{\tau}^3} \left| Z \right| \bigg|_0 \right] \tau \]

\[ \omega_{s0}^2 = -\frac{\eta e \hat{V} \omega_0^2}{2\pi E_0} \cos(\phi_s) \]

is the longitudinal strength in absence of impedance.
\[ \omega_s^2 = \omega_{s0}^2 \left[ 1 + \frac{1}{V \cos(\phi_s)} \frac{3\pi I_0}{h\omega_0^3 \tau^3} \right] \]

Therefore the relative change in omega is

\[ \frac{\Delta \omega_s}{\omega_{s0}} = \frac{1}{2} \frac{1}{V \cos(\phi_s)} \frac{3\pi I_0}{h\omega_0^3 \tau^3} \frac{Z}{n} \left| \right|_0 \]

For protons \( \hat{\tau} \hat{\epsilon} = \text{constant} \)

\[ \frac{\Delta \hat{\tau}}{\tau} \approx -\frac{\Delta \omega_s}{2\omega_s} \]

**Observation**

The effect of the impedance is local, hence the voltage induced by impedance do not effect the center of mass (like for the space charge)
Summary

1) Wall charges creates detuning $\rightarrow$ incoherent tunes
2) Ferromagnetic material creates image currents:
   Coherent motion $\rightarrow$ coherent tunes
3) Concept of Wake field
4) Impedance of a cavity, Wake $\leftrightarrow$ impedance
5) Energy loss
6) Longitudinal dynamics, effect of energy loss
7) Bunch lengthening