Collective Effects II

G. Franchetti, GSI

CERN Accelerator – School

Budapest, 2-14 / 10 / 2016

Type of fields

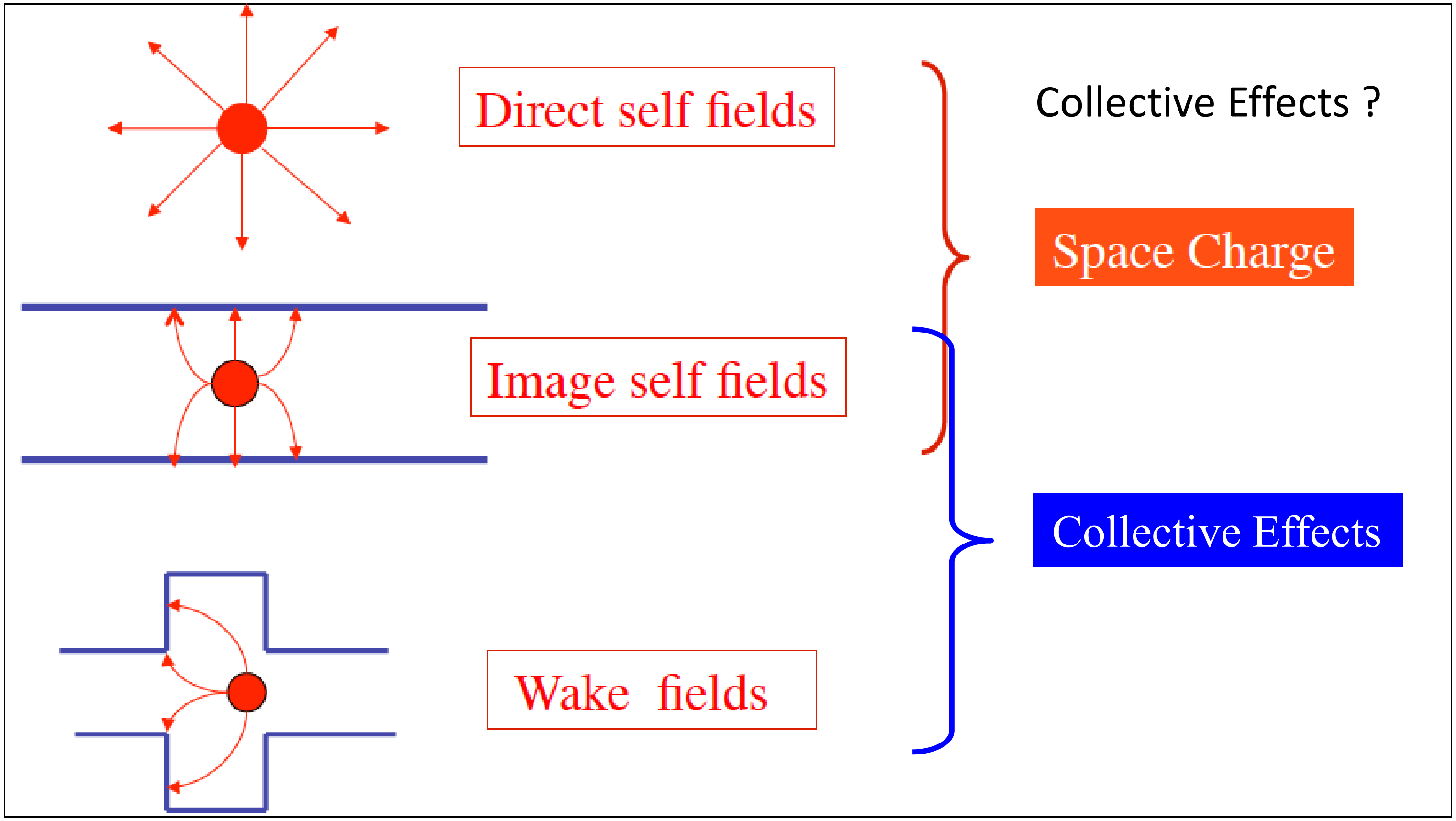
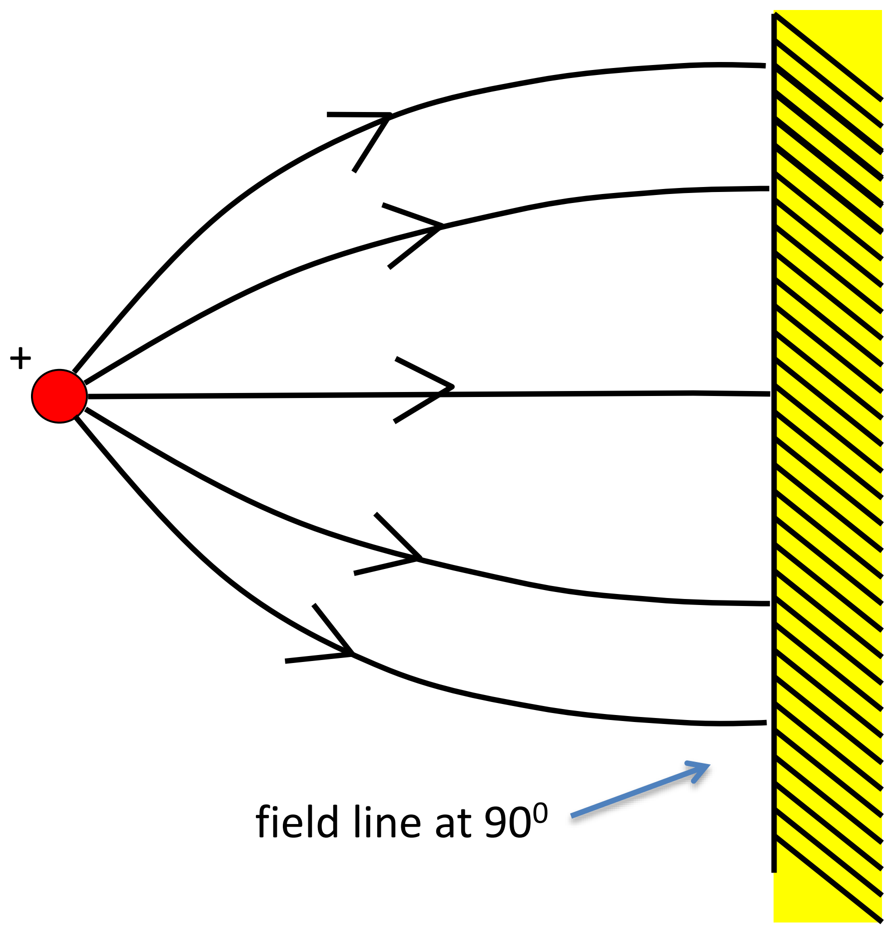


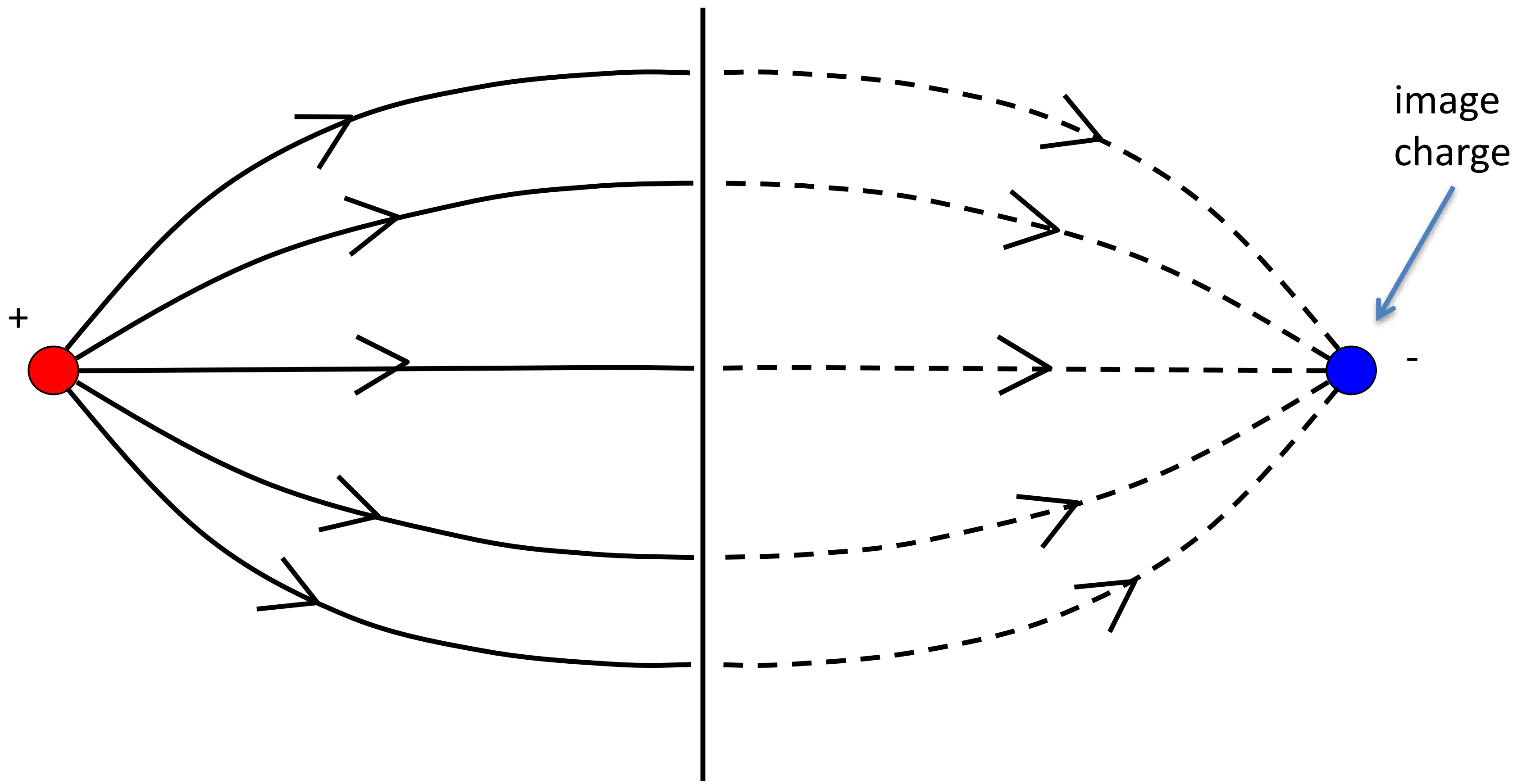
Image charges

Influence of the chamber wall



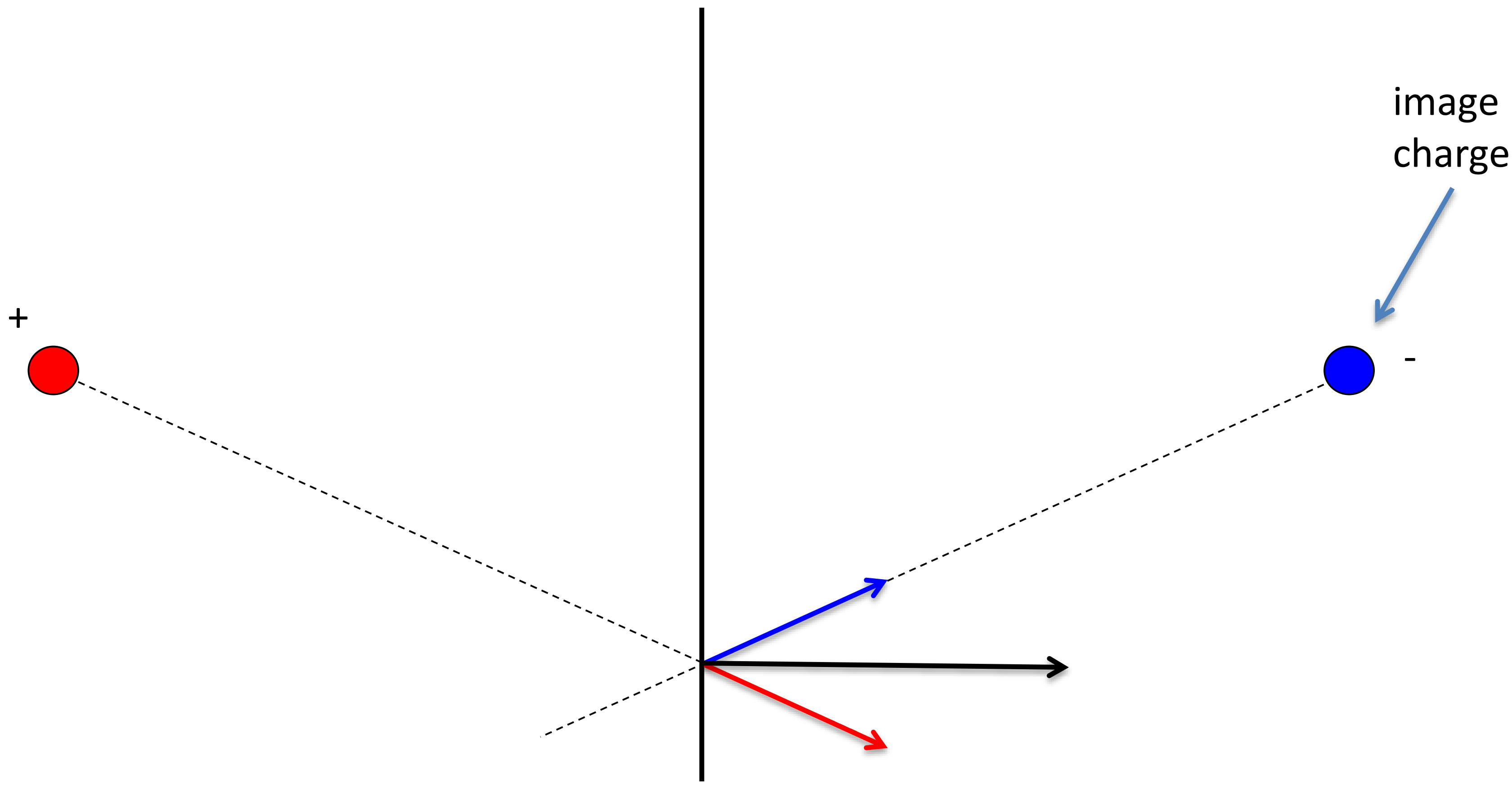
the electron in the metal quickly travel on the surface of the metal until the electric field parallel to the surface is zero

Image charge



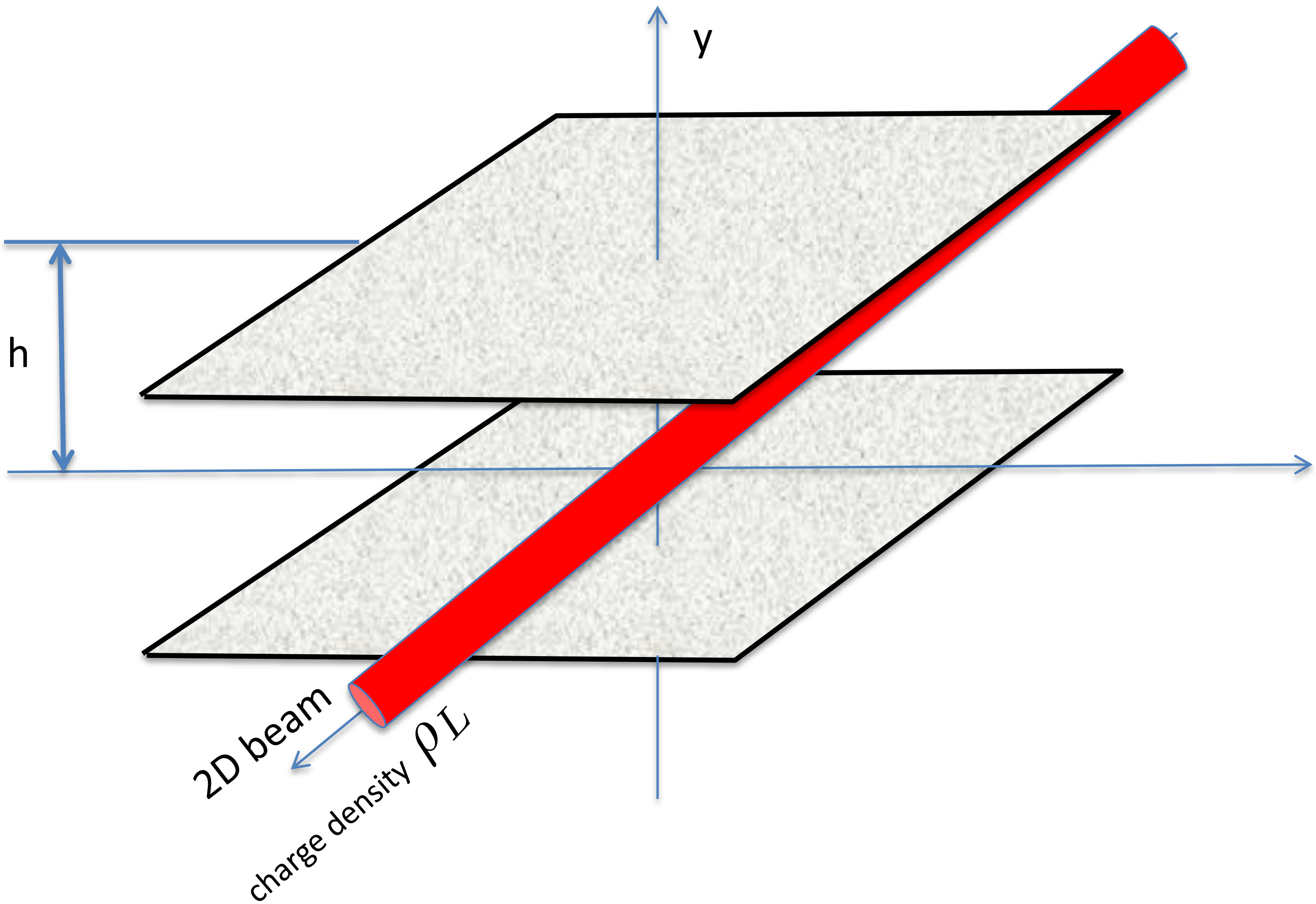
the image charge is a reflection of the particle with exchanged sign

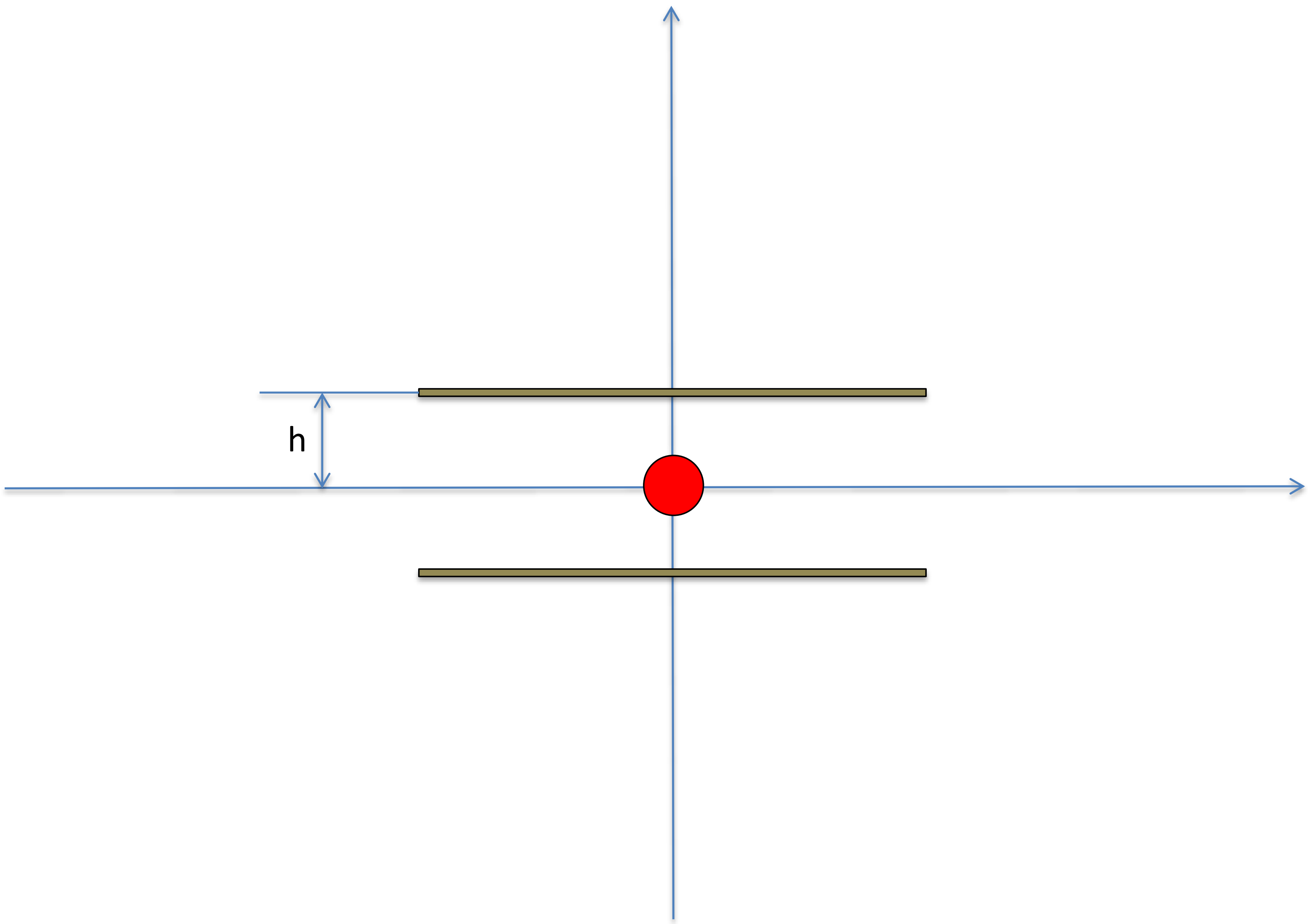
Image charge

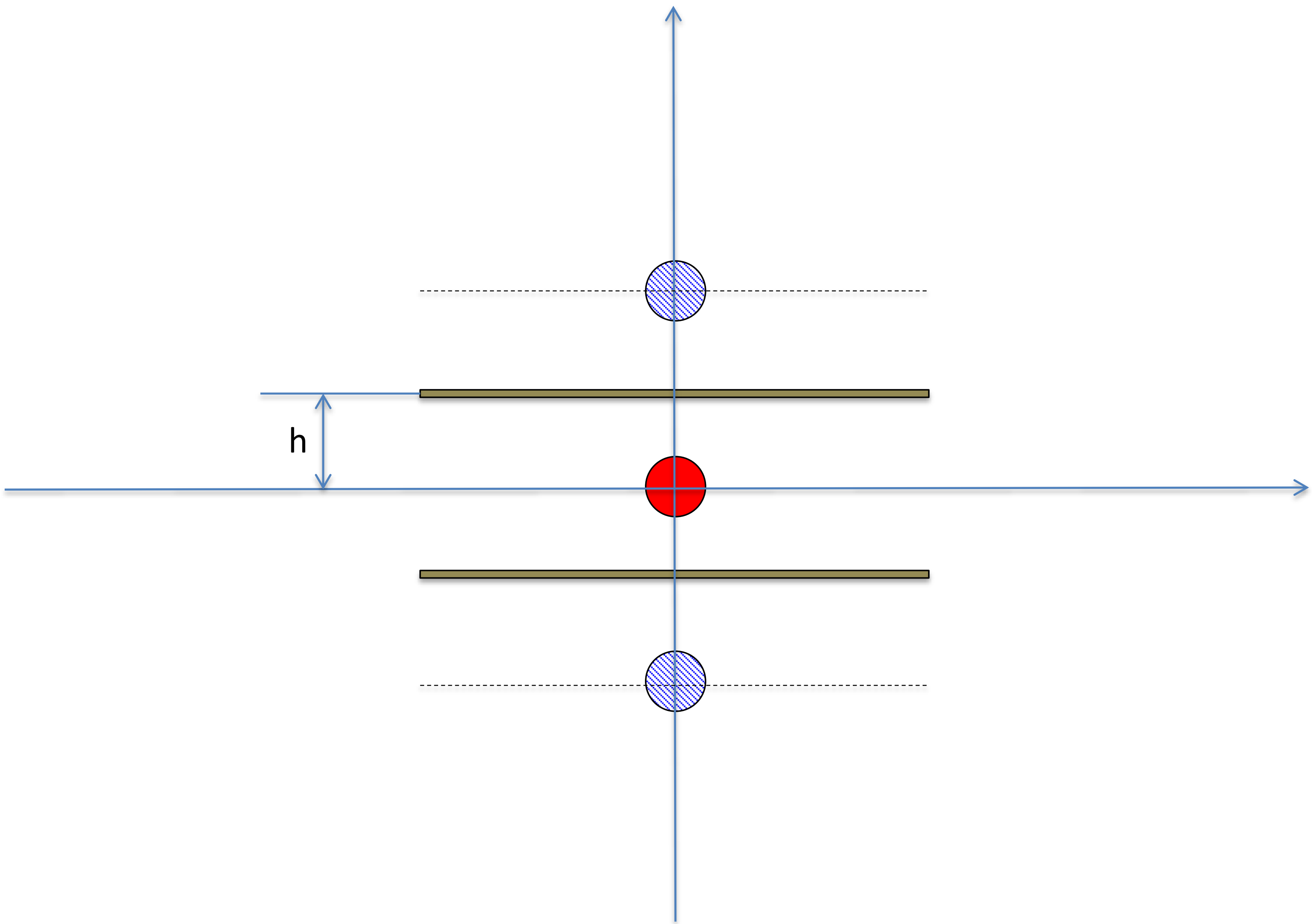


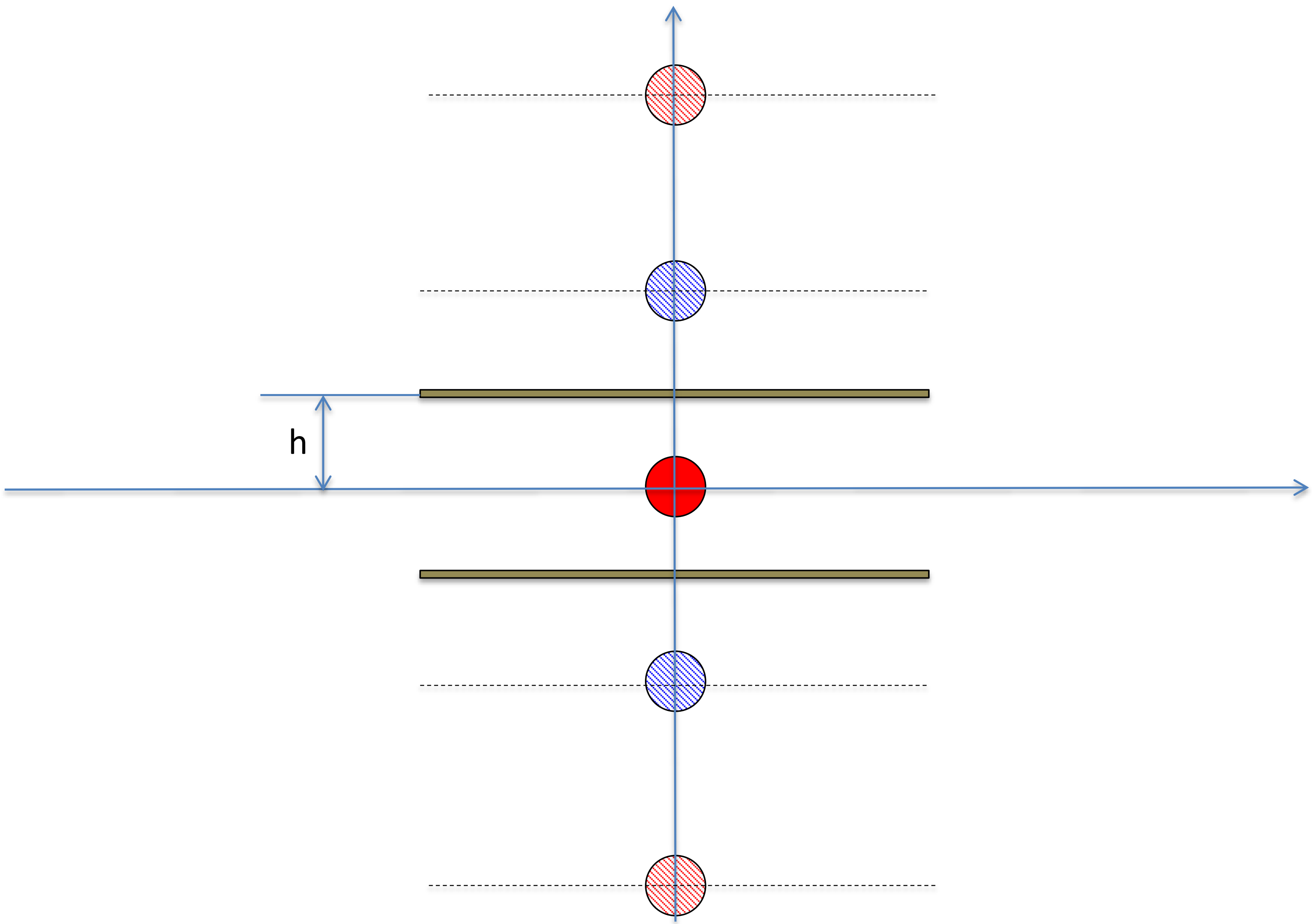
the image charge is a reflection of the particle with exchanged sign

Conducting plates

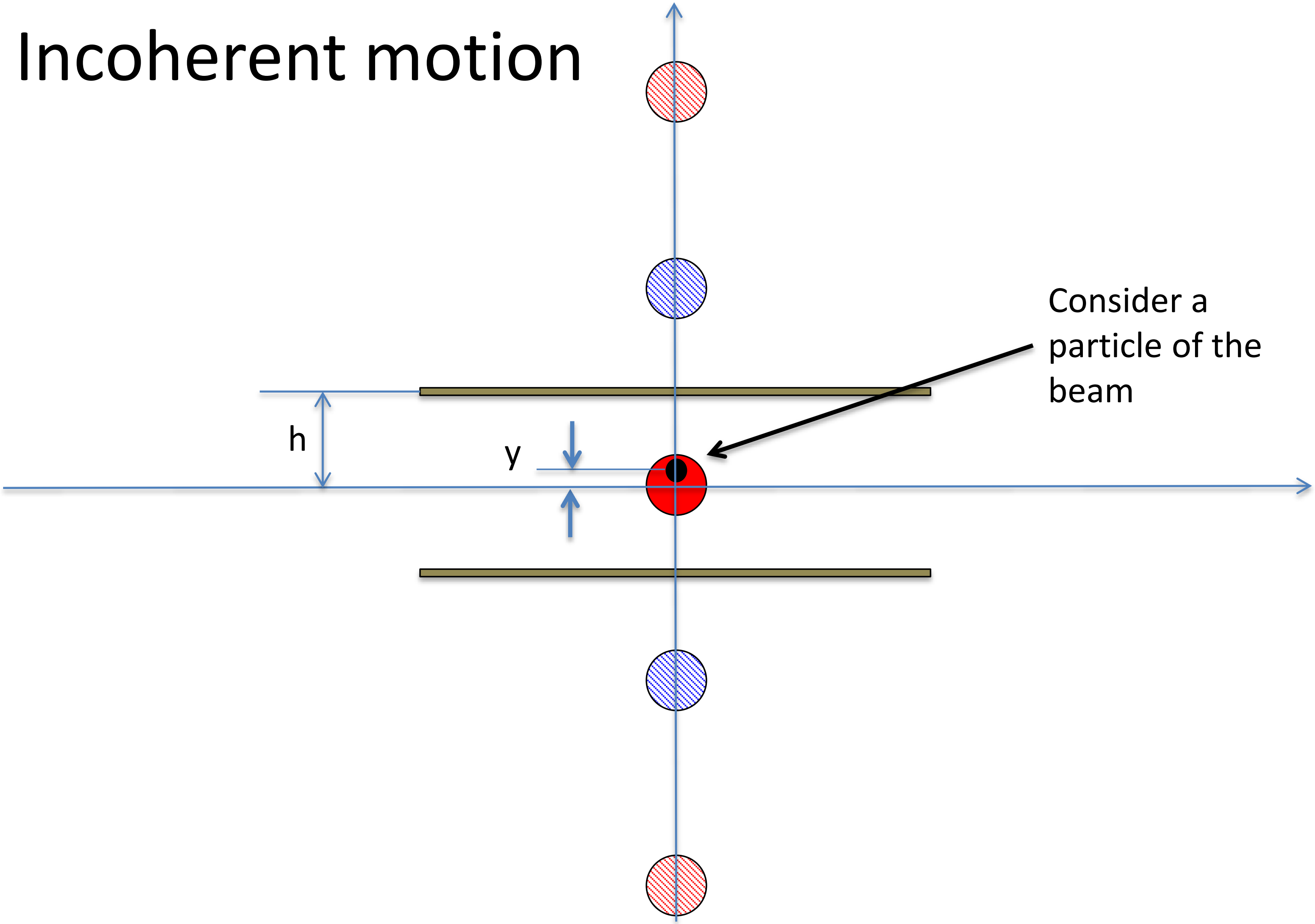



h

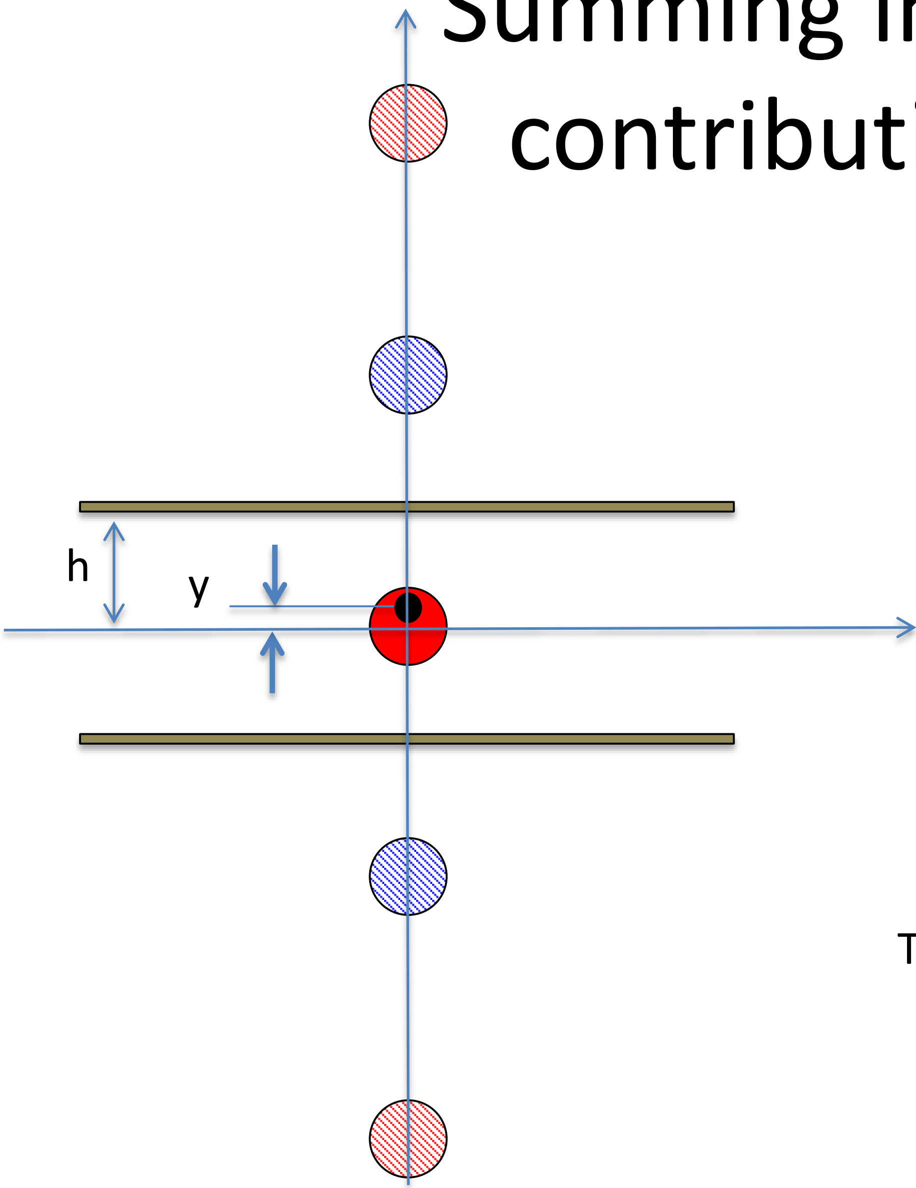

h


h

Incoherent motion



Summing image charge contribution in pairs



$$E_{y,n} = \frac{\rho_L}{2\pi\epsilon_0}(-1)^n \left(\frac{1}{2nh+y} - \frac{1}{2nh-y} \right)$$

$$h \gg y$$

$$E_{y,n} = -\frac{\rho_L y}{4\pi\epsilon_0 h^2}\frac{(-1)^n}{n^2}$$

Total electric field

$$E_y = \sum_{n=1}^{\infty} E_{y,n} = \frac{\rho_L y}{\pi\epsilon_0}\frac{\pi^2}{48h^2}$$

Equation of motion

In the equation of motion

$$\frac{d^2y}{ds^2} + k_y y = \frac{e}{m\gamma^3 v_0^2} E_{b,y} + \frac{e}{m\gamma v_0^2} E_{i,y}$$

$$\frac{d^2y}{ds^2} + k_y y = \frac{2K}{Y(X+Y)} y + K\gamma^2 \frac{\pi^2}{24h^2} y$$

as $\nabla \cdot \vec{E} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial x} = -\frac{\partial E_y}{\partial y}$

$$\frac{d^2y}{ds^2} + k_y y - \frac{2K}{Y(X+Y)}\left[1 + \gamma^2 \frac{\pi^2}{48}\frac{Y(X+Y)}{h^2}\right] y = 0$$

$$\frac{d^2x}{ds^2} + k_x x - \frac{2K}{X(X+Y)}\left[1 - \gamma^2 \frac{\pi^2}{48}\frac{X(X+Y)}{h^2}\right] x = 0$$

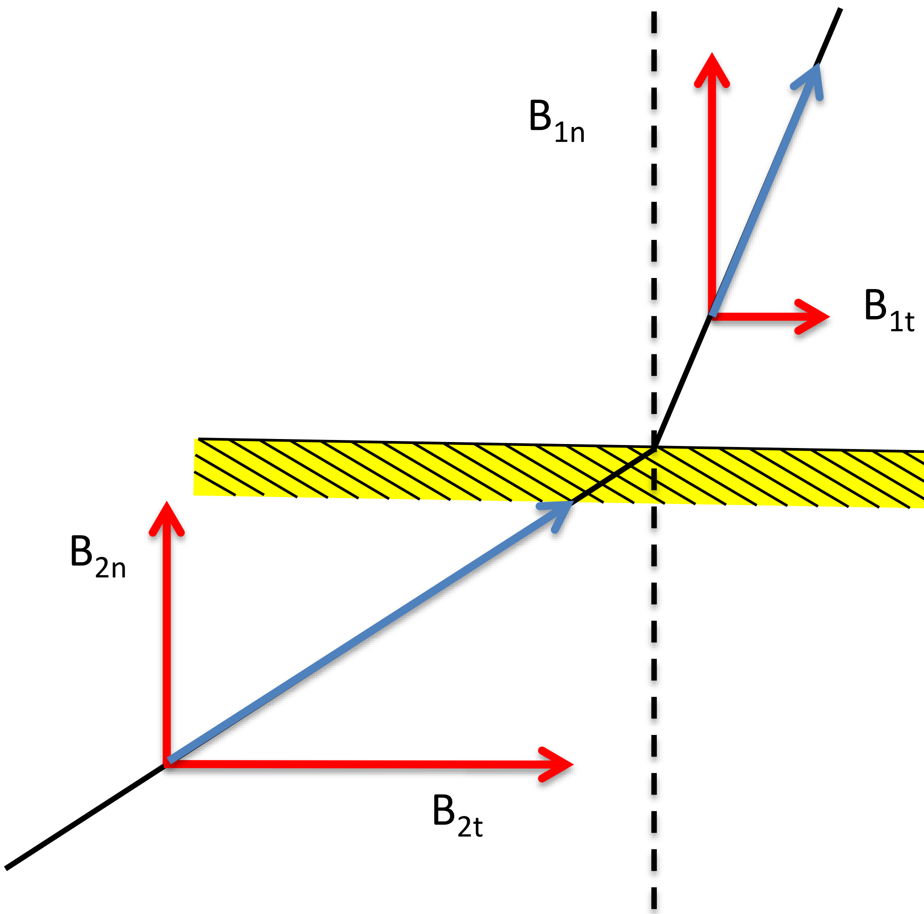
Laslett Tuneshift

$$\Delta Q_y \simeq -\frac{R_m^2}{Q_{y0}}\frac{K}{Y(X+Y)}\left[1 + \gamma^2 \frac{\pi^2}{48}\frac{Y(X+Y)}{h^2}\right]$$

$$\Delta Q_x \simeq -\frac{R_m^2}{Q_{x0}}\frac{K}{X(X+Y)}\left[1 - \gamma^2 \frac{\pi^2}{48}\frac{X(X+Y)}{h^2}\right]$$

Image currents

Ferromagnetic Boundaries



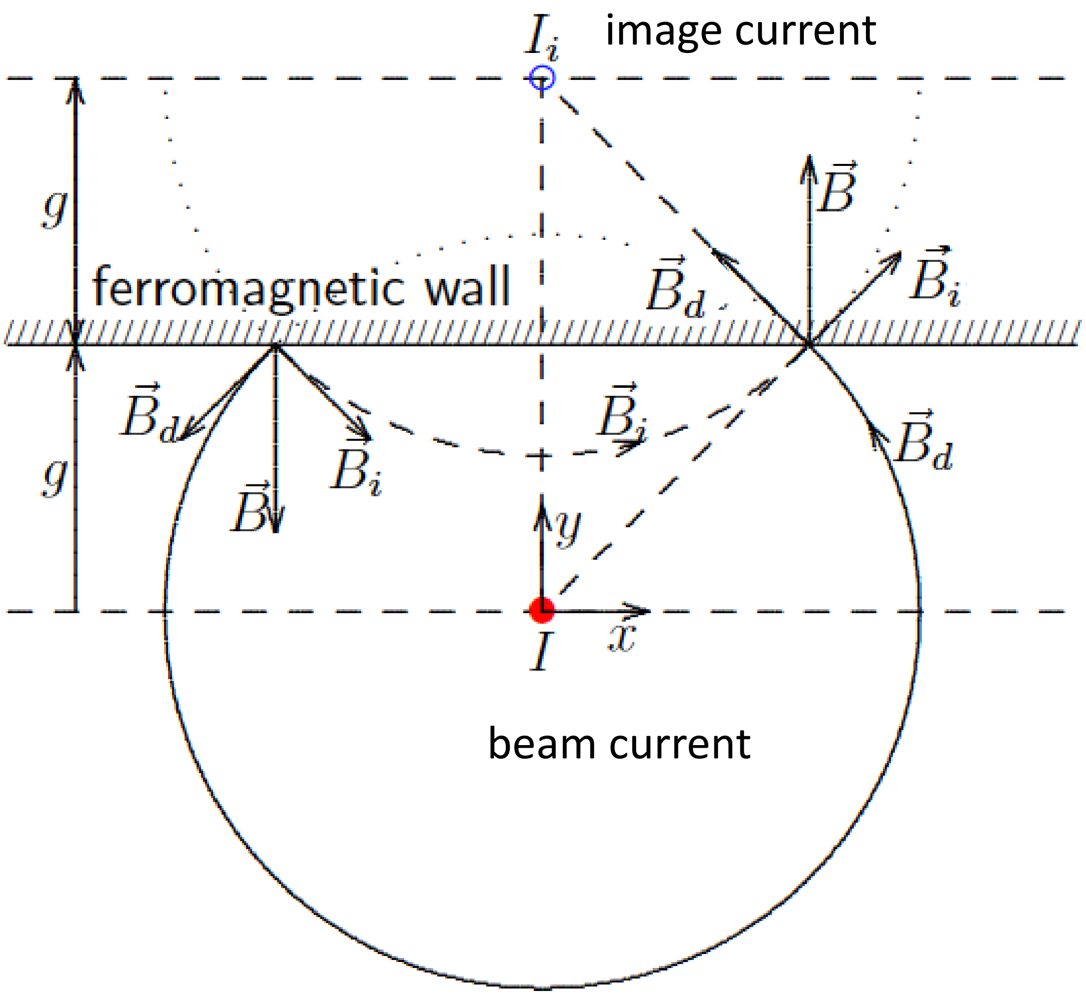
$$\mu_1$$

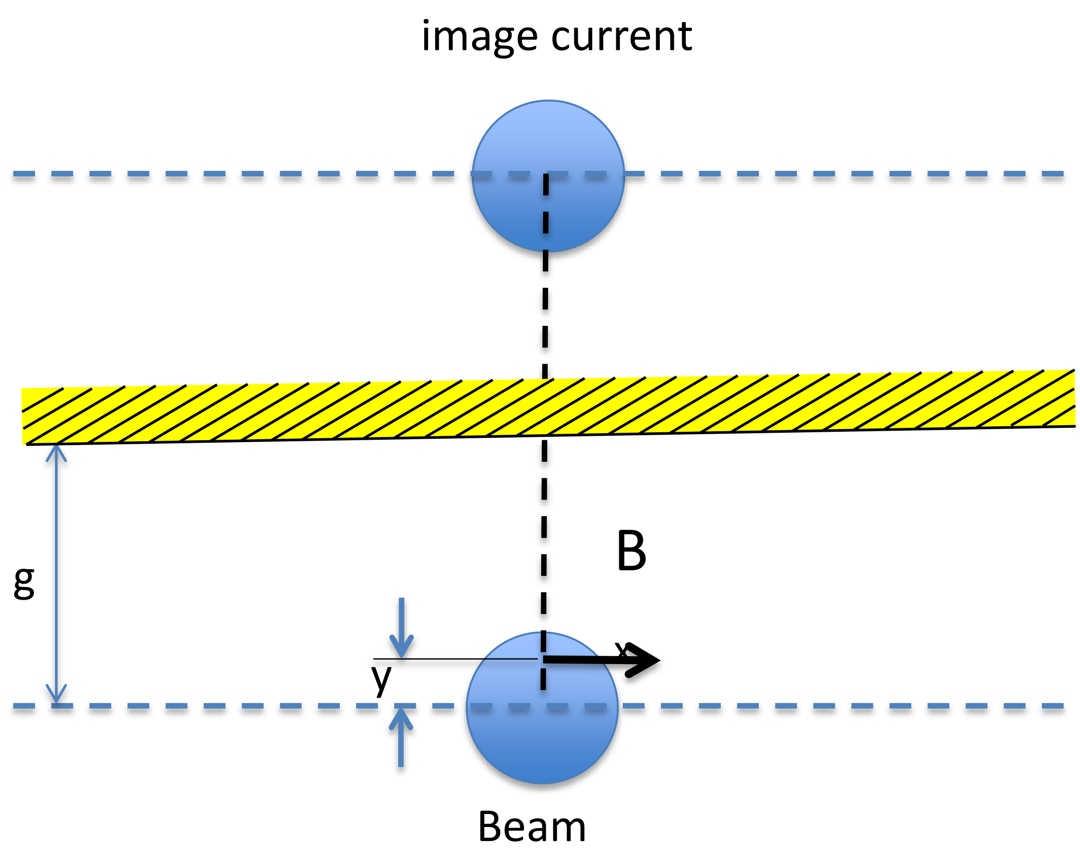
$$\mu_2$$

$$B_{1n} = B_{2n} \qquad \mu_1 \ll \mu_2$$

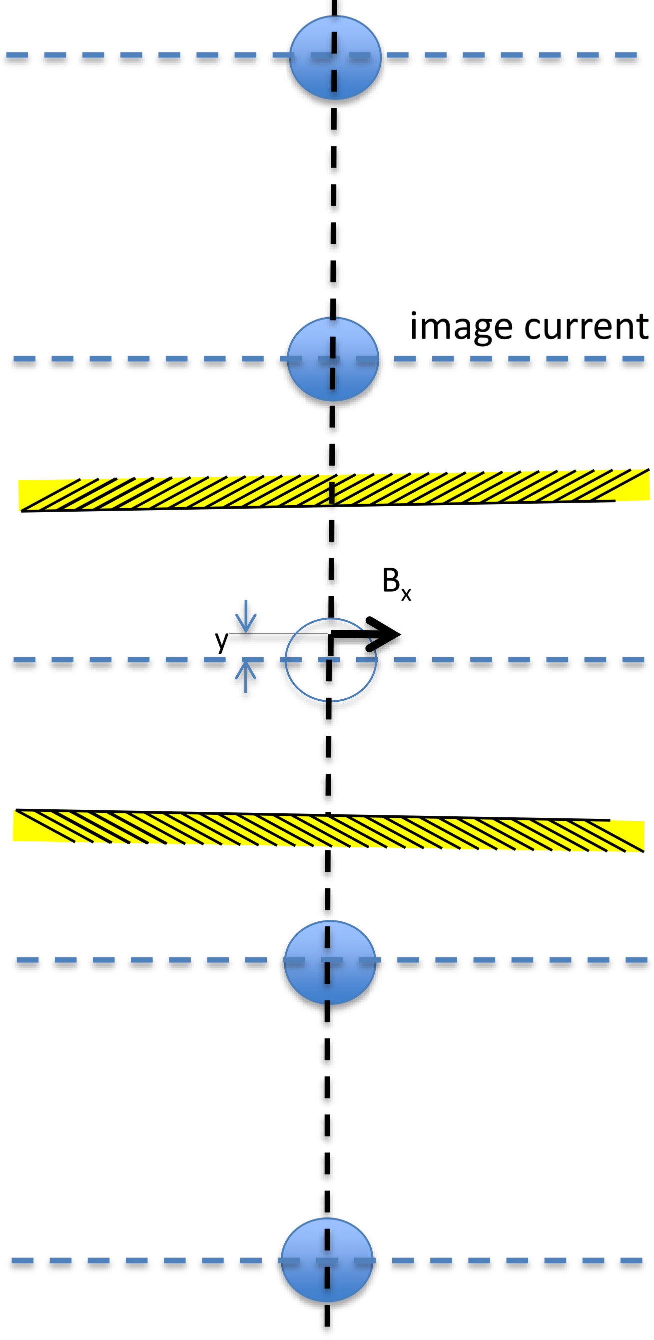
$$\frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2} \qquad B_{1,t} \simeq 0$$

Ferromagnetic Boundaries





$$B_x = \frac{\mu_0 I}{2\pi} \frac{1}{2g - y}$$



$$B_x = \frac{\mu_0 I}{2\pi} \sum_{n=1}^{\infty} \left(\frac{1}{2ng - y} - \frac{1}{2ng + y} \right)$$

for $\quad g \gg y$

$$B_x = \frac{\mu_0 I y}{4\pi g^2} \frac{\pi^2}{6}$$

In the equation of motion

$$\frac{d^2 y}{ds^2} + k_y y = \frac{2K}{Y(X+Y)} y - \frac{1}{m\gamma v_0^2} v_z B_x$$

therefore

$$\frac{d^2y}{ds^2} + k_y y = \frac{2K}{Y(X+Y)}y + K\frac{2\gamma^2\beta^2\pi^2}{24g^2}y$$

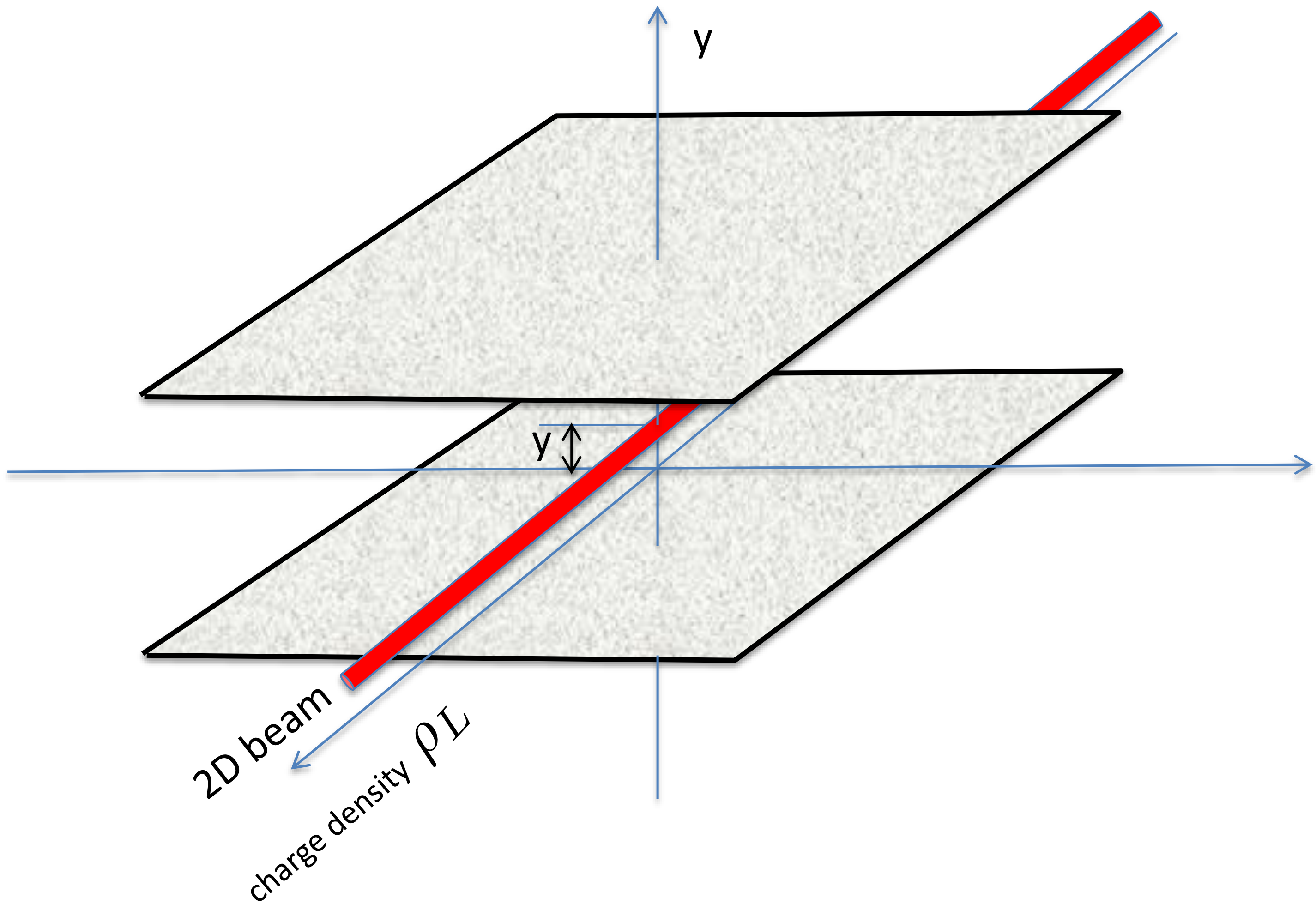
incoherent
SC

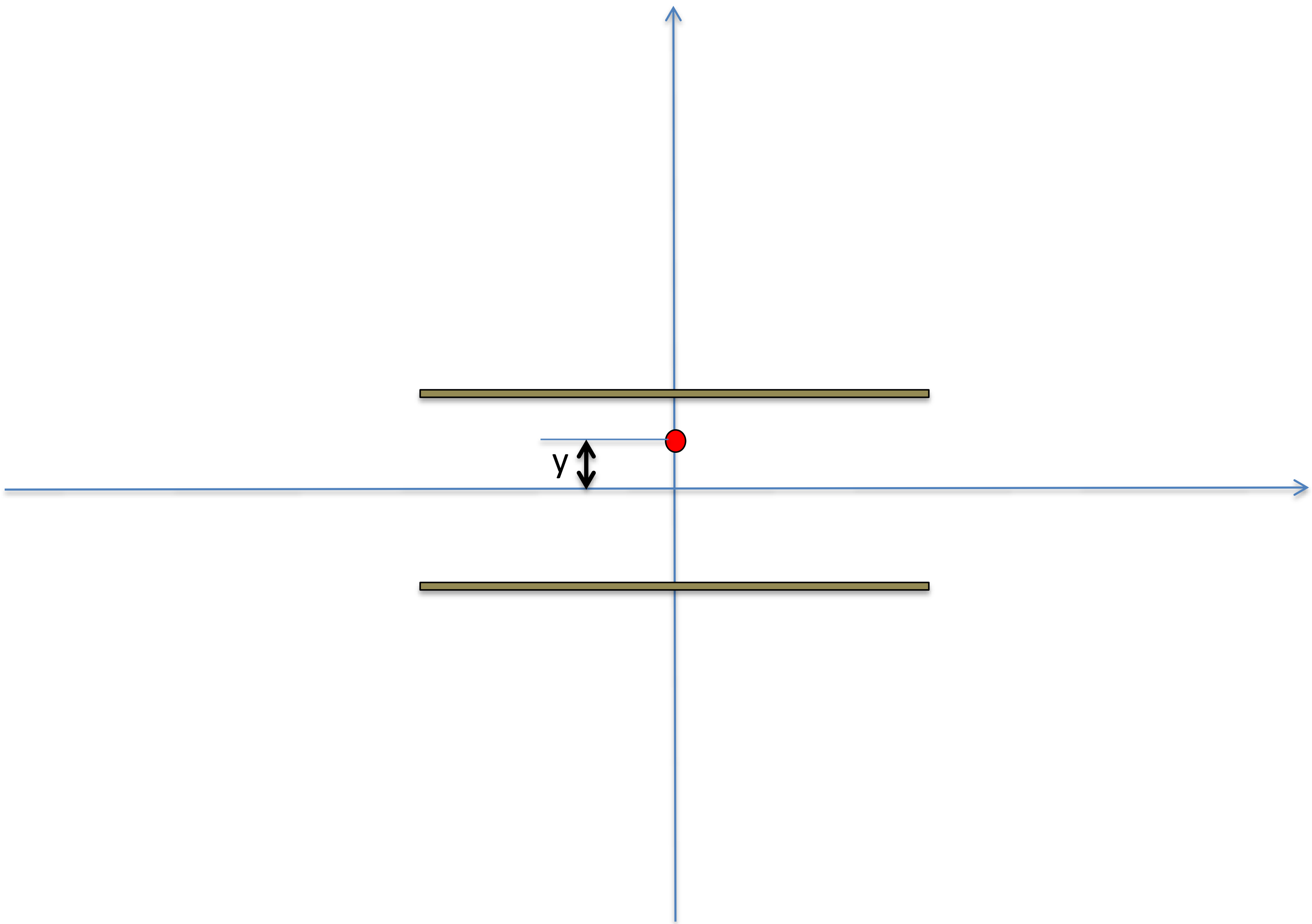
ferromagnetic
induced image
current
(coherent force)

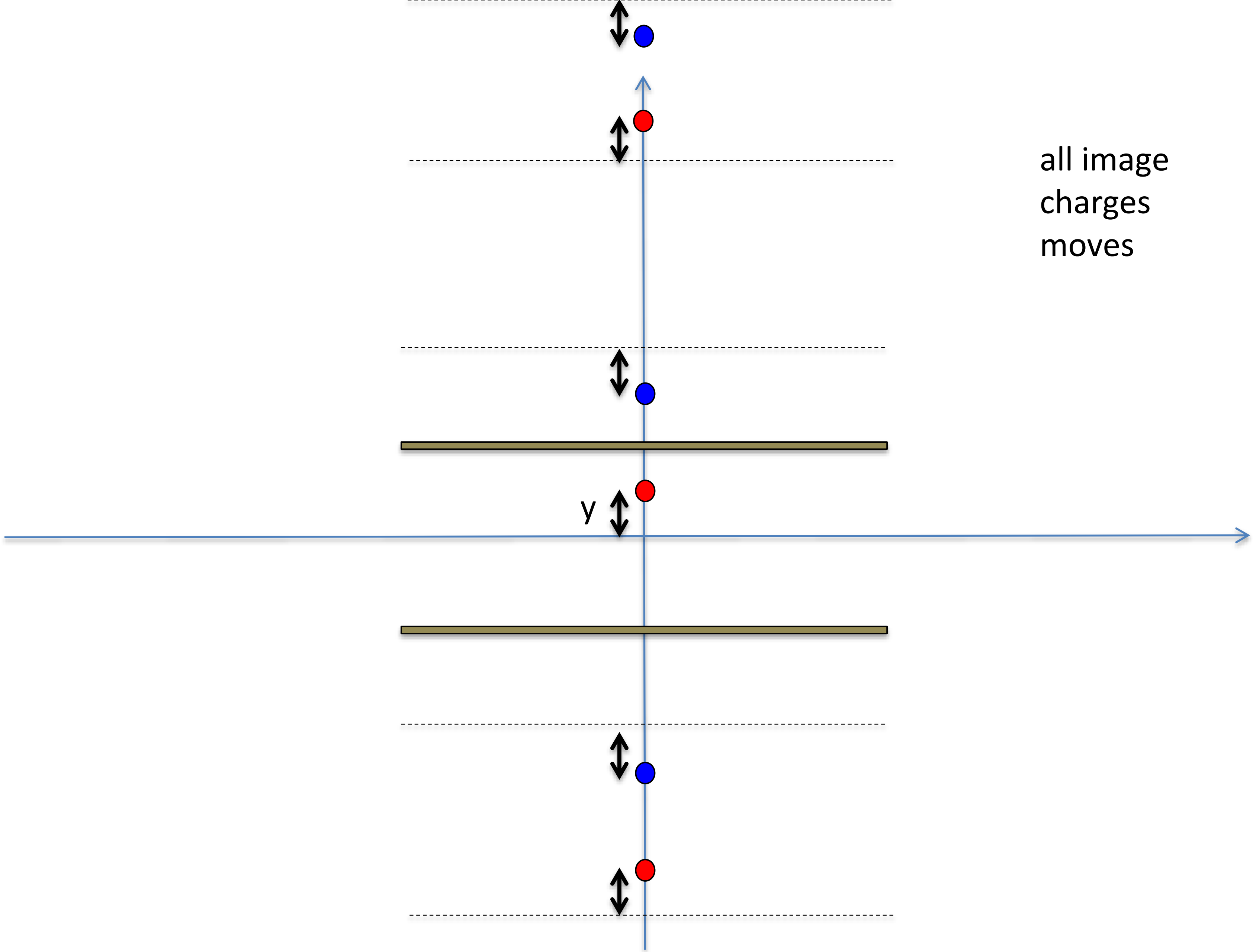
Tune-shift !

Coherent Motion

Coherent motion

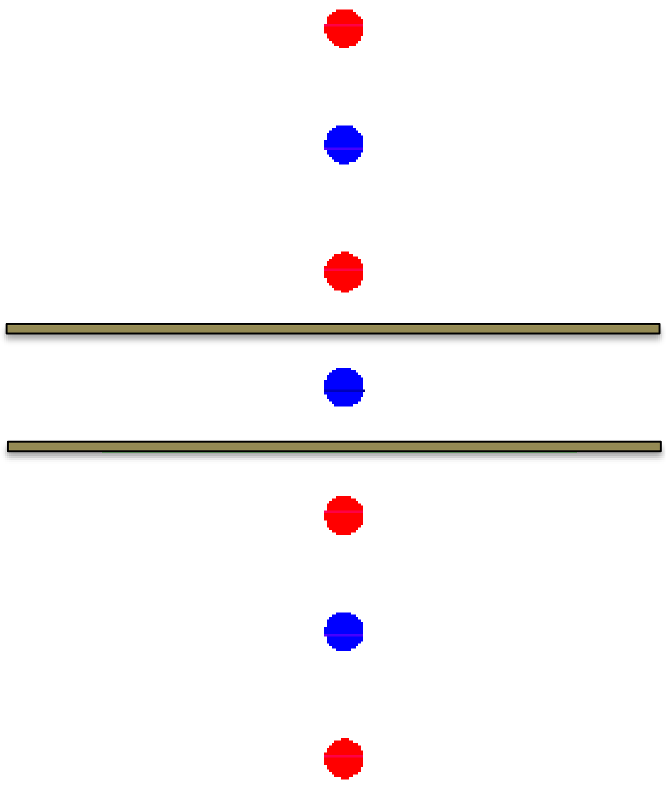


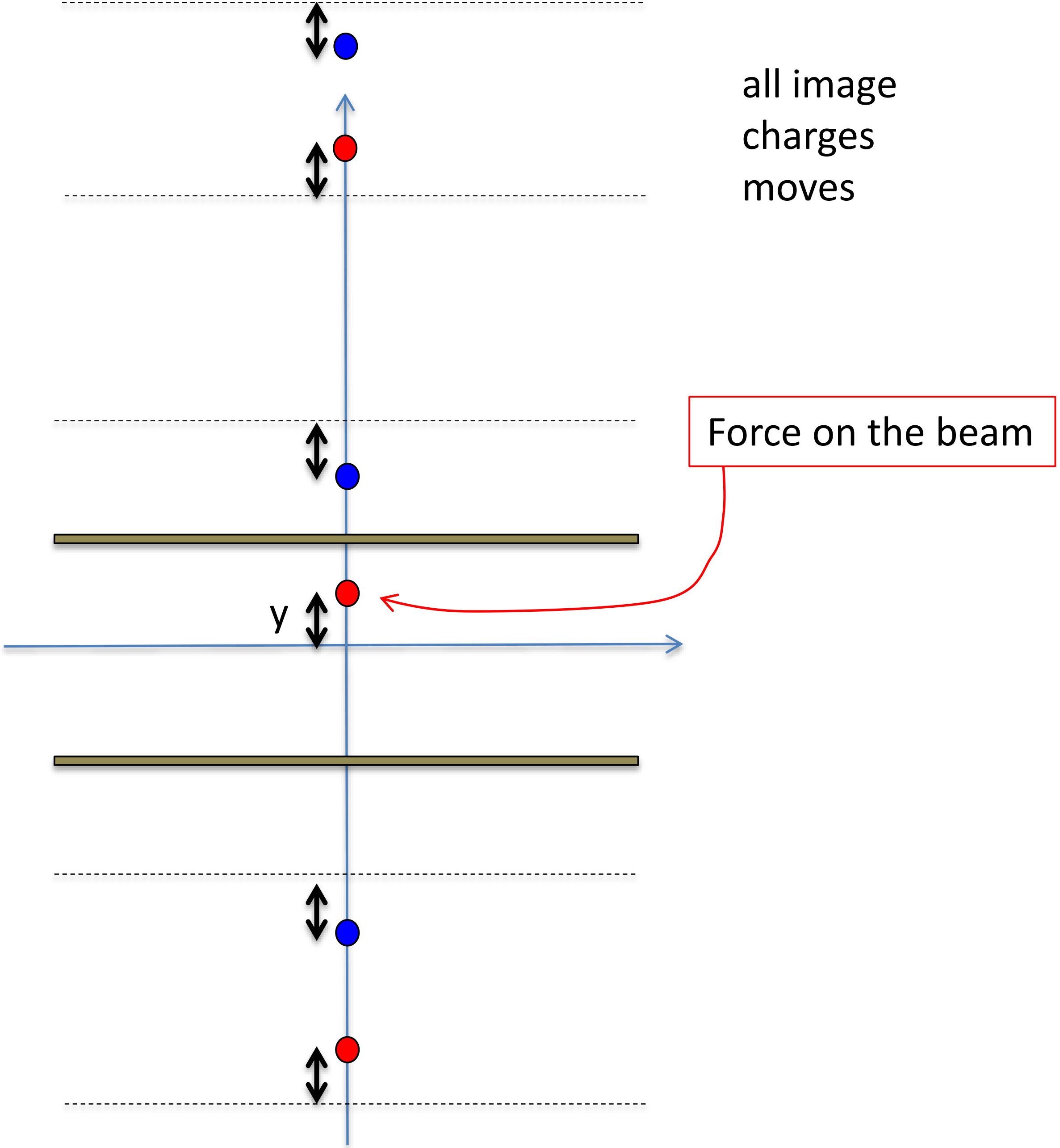

y

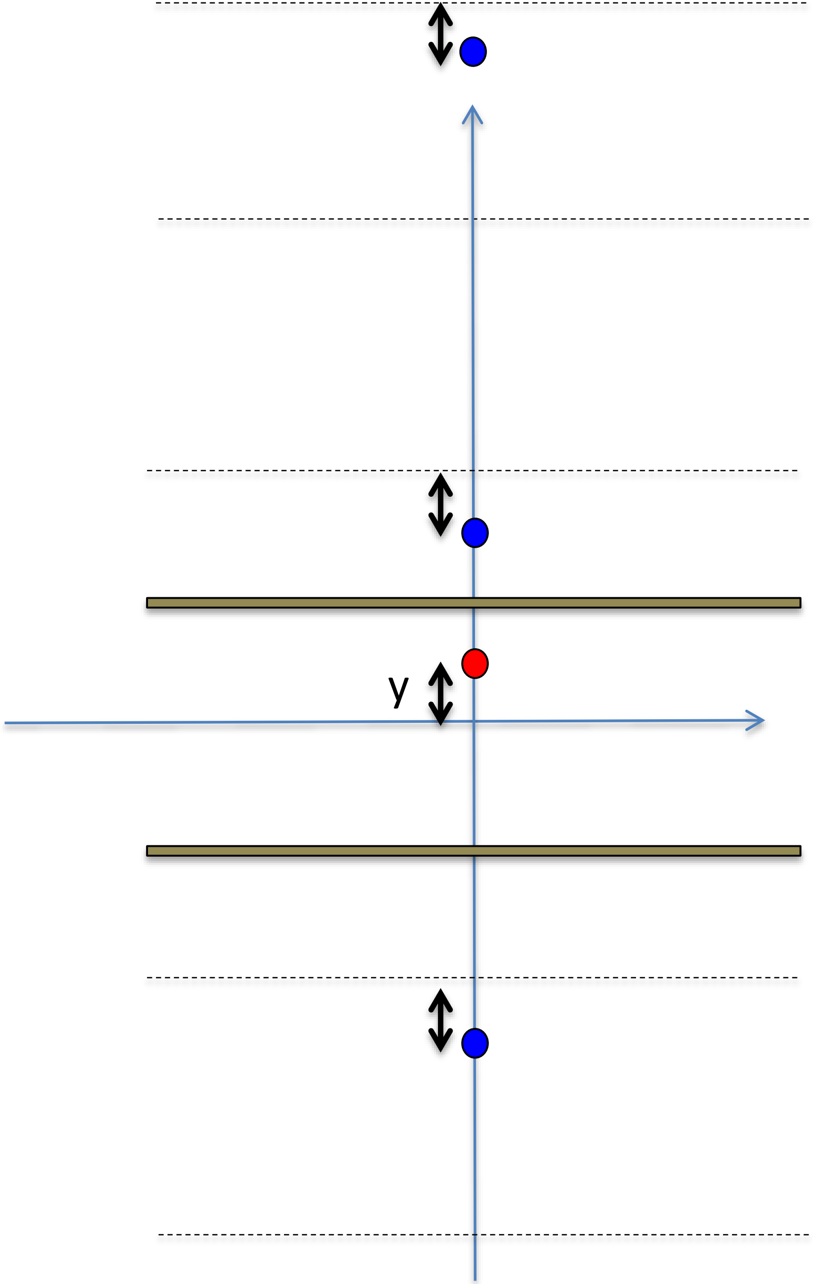


all image
charges
moves

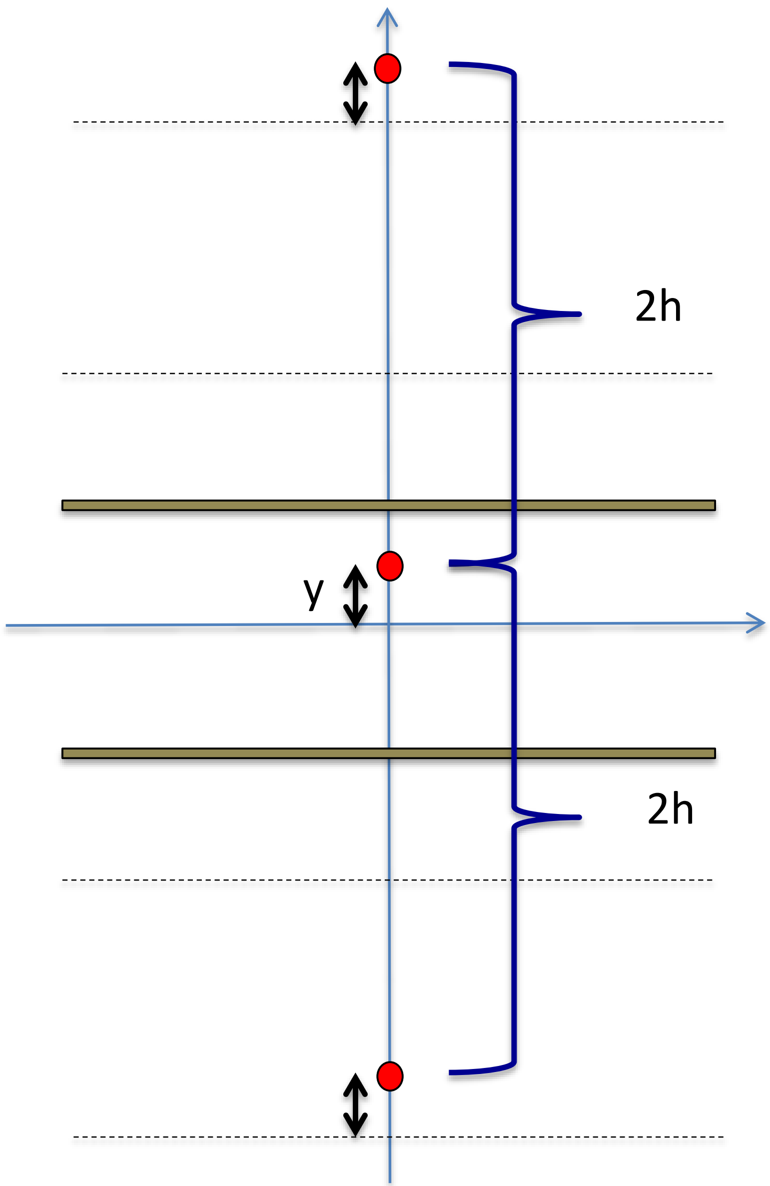
Coherent motion

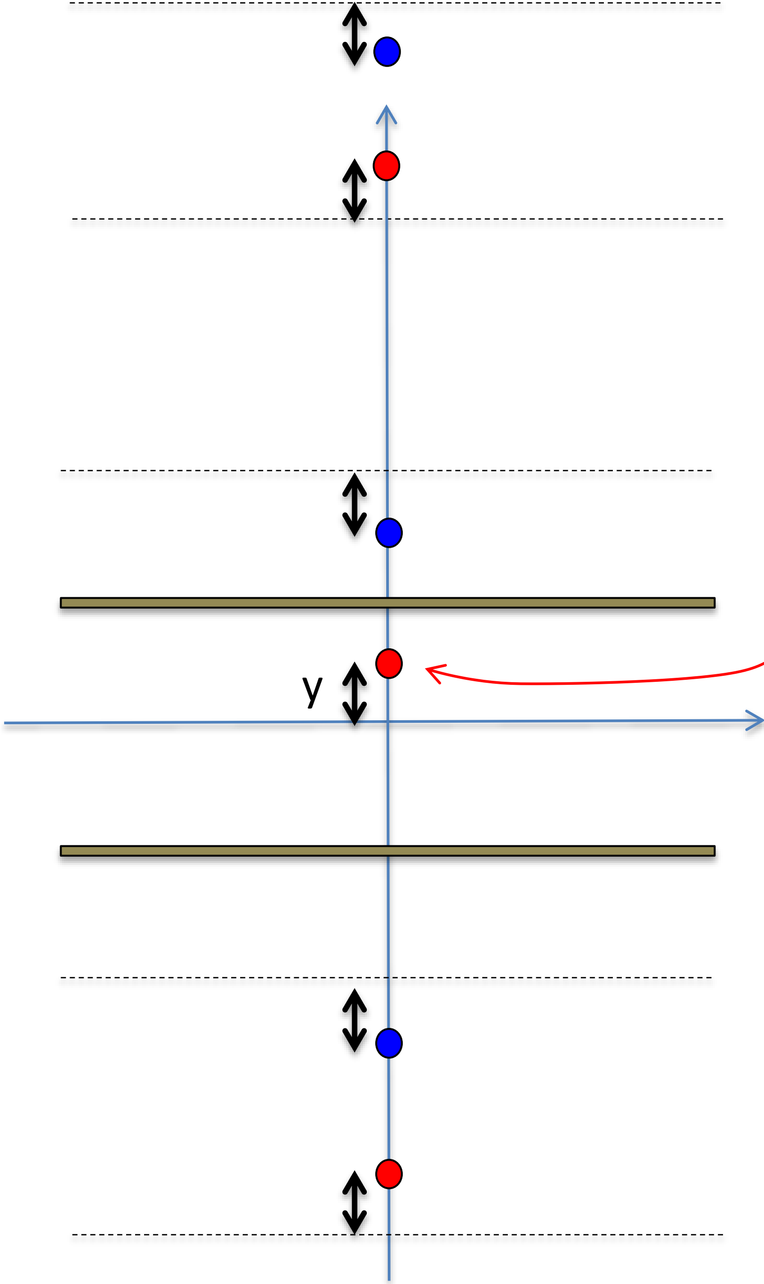



all image
charges
moves
Force on the beam
y



NO FORCE CREATED ON THE BEAM





all image
charges
moves

Force on the beam

$$E_{y,n} = -\frac{\rho_L}{2\pi\epsilon_0}\left[\frac{1}{2nh+2y} - \frac{1}{2nh-2y}\right]$$

n = 1, 3, 5, 7, ...

$$E_{y,n} = \frac{\rho_L}{2\pi\epsilon_0}\frac{4y}{(2nh)^2-(2y)^2}$$

$$E_{y,n} = \frac{\rho_L}{2\pi\epsilon_0 h^2} y \frac{1}{n^2}$$

n = 1, 3, 5, 7, ...

therefore

$$E_{y,n} = \frac{\rho_L}{4\pi\epsilon_0 h^2} y \left[\frac{1}{n^2} - \frac{(-1)^n}{n^2} \right]$$ (trick!)

with n = 1, 2, 3, 4, 5, 6, …

$$E_y = \sum_{n=1}^{\infty} E_{y,n} = \frac{\rho_L}{4\pi\epsilon_0 h^2} y \left[\frac{\pi^2}{6} + \frac{\pi^2}{12} \right] = \frac{\rho_L}{16\pi\epsilon_0 h^2} \pi^2 y$$

The electric field E_x due to coherent shift is zero on the center of mass ☺

equation of motion

$$\frac{d^2 y_c}{ds^2} + k_y y_c = \frac{e}{m\gamma v_0^2} \frac{\rho_L}{16\pi\epsilon_0 h^2} \pi^2 y_c$$

but $$I = v_z \rho_L \simeq v_0 \rho_L$$

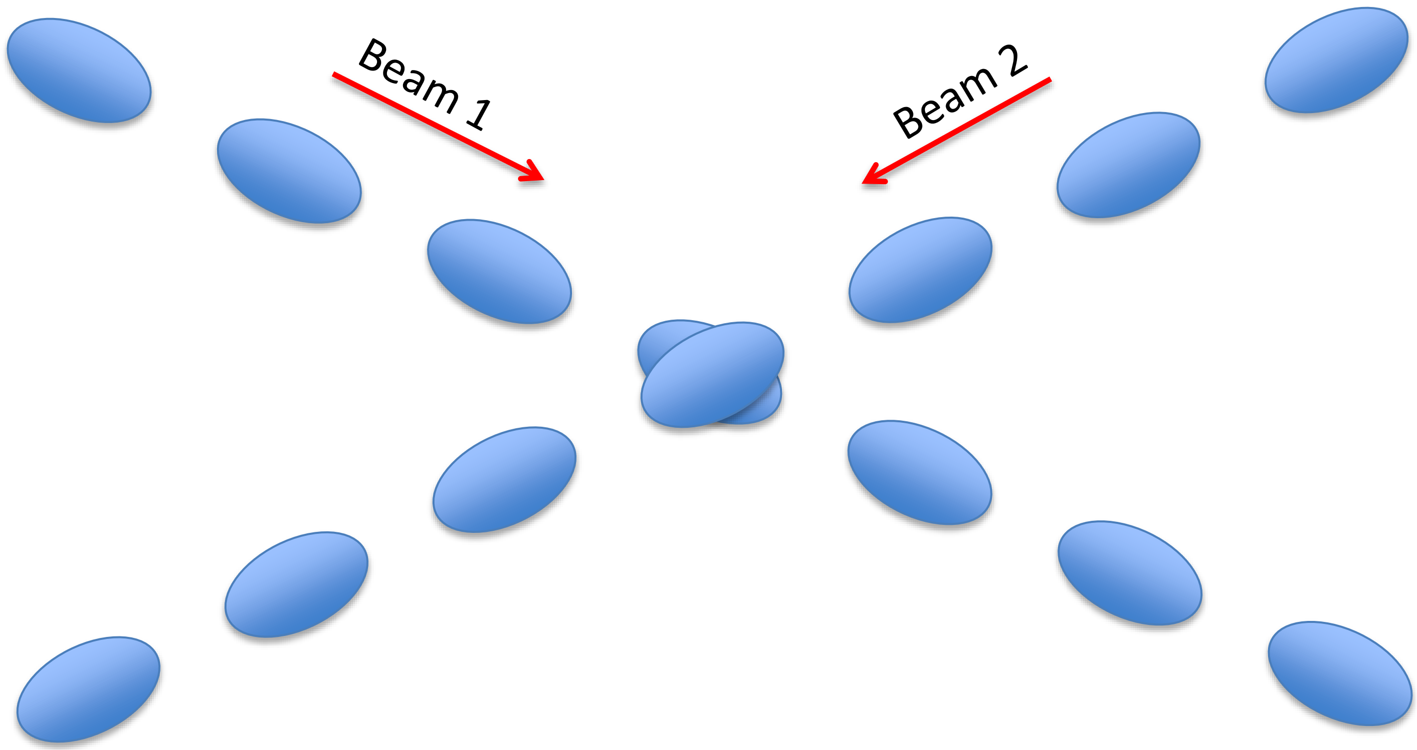
$$\frac{d^2 y_c}{ds^2} + k_y y_c = K \frac{\gamma^2 \pi^2}{8h^2} y_c$$

$$\frac{d^2 y_c}{ds^2} + \left(k_y - 2K \frac{\gamma^2 \pi^2}{16h^2} \right) y_c = 0$$

Coherent detuning

$$\Delta Q_{y,c} \simeq -\frac{R_m^2}{Q_y} K \frac{\gamma^2 \pi^2}{16 h^2}$$

Beam-beam



Bunches of the beam 1 feels the field of the other beam when it travels through the other bunch of the beam 2. The same is reverted for Beam 2.

Beam-beam tune-shift

$$\delta Q_y = \frac{r_0 N \beta_y}{2\pi B \sigma_y (\sigma_x + \sigma_y)}$$

r_0 = classical radius of particle, B=number of bunches per beam,
N = particle per beam

Usually one would expect that the acceptable tune-shift be 0.15 – 0.2

Reality requires more stringent values ~ 0.004 – 0.006

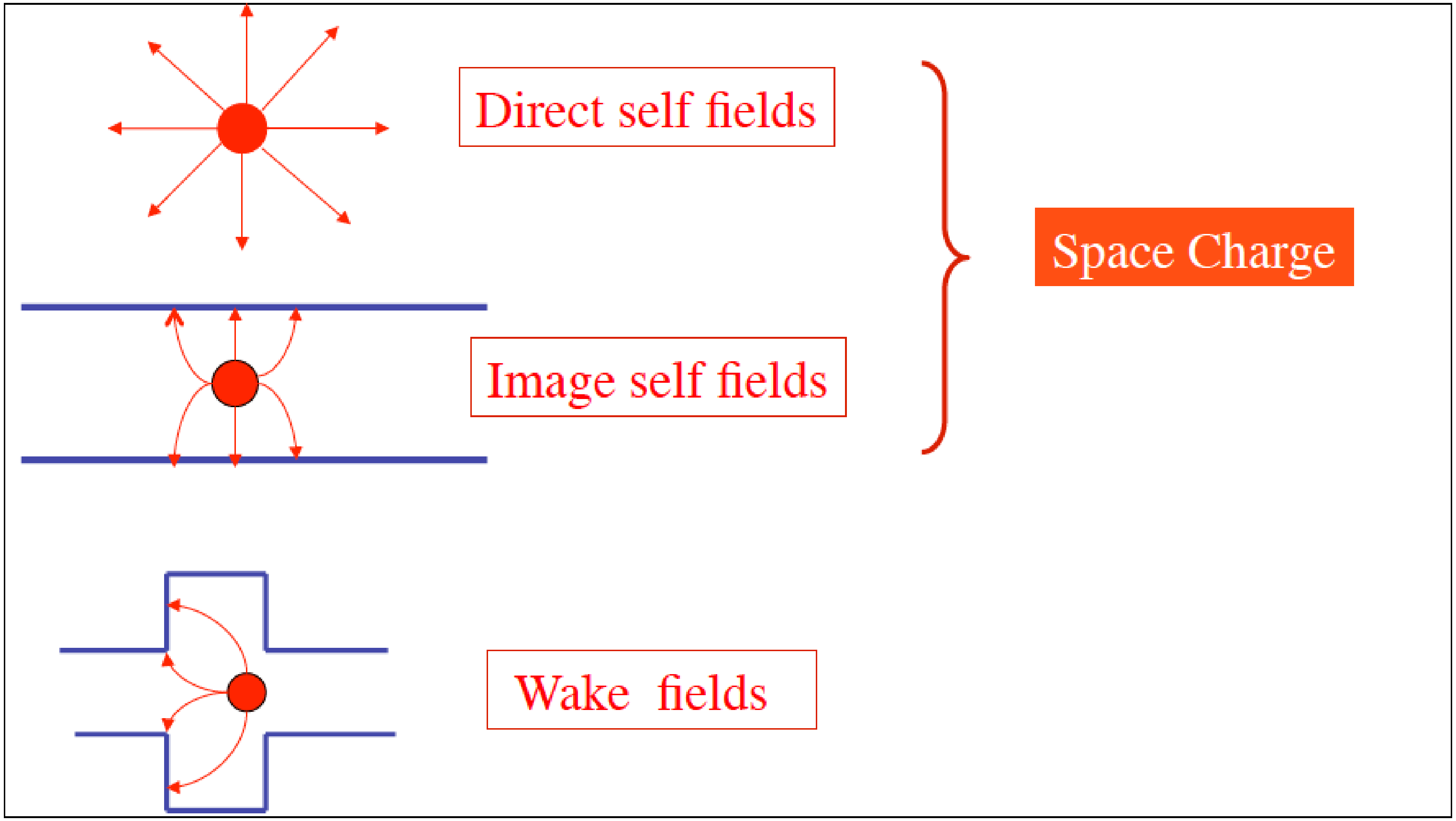
Complications: one beam feels also all the nonlinear field of the other beam!

Storage is for millions of turns, and high order resonances plays a crucial role and should be avoided.

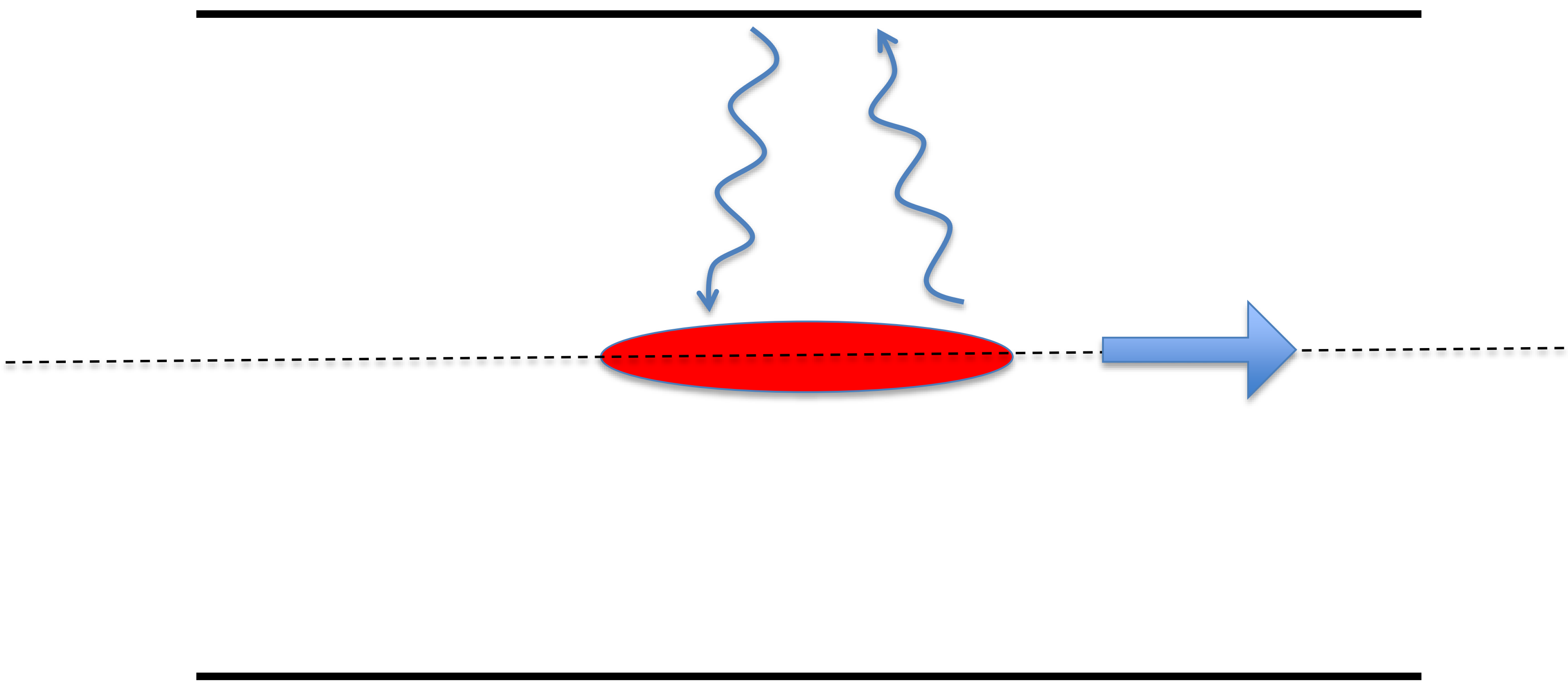
The Collective Effects

Thanks to Oliver Boine-Frankenheim, I. Hofmann, U. Niedermayer, D. Brandt

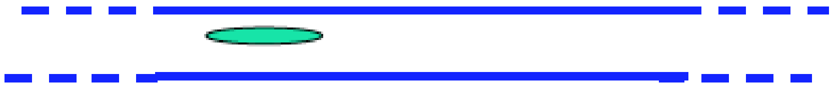
Interaction of the beam with the environment



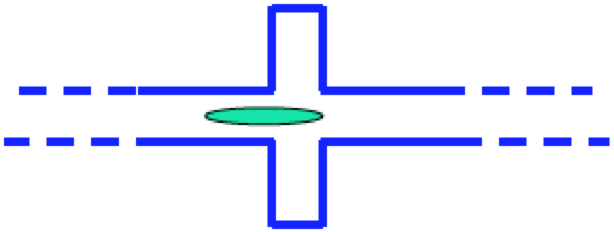
Effect on the dynamics



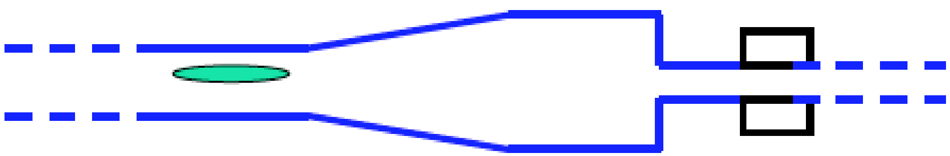
Resistive wall effect:
Finite conductivity



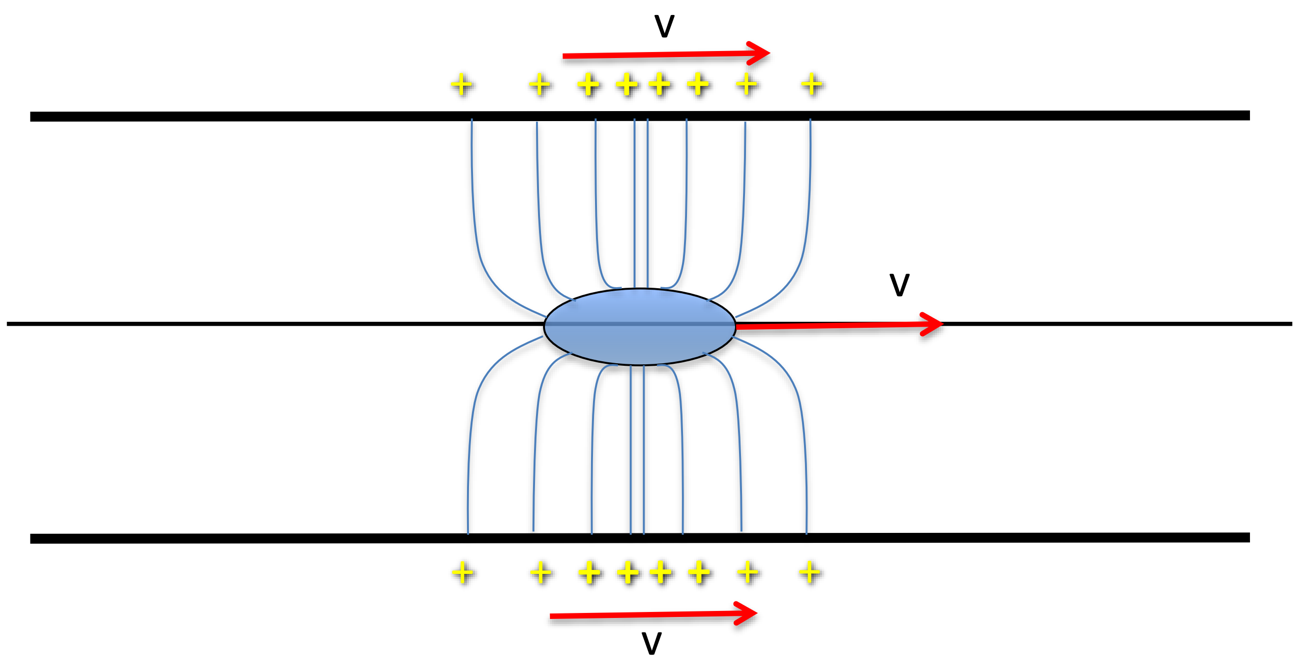
Narrow-band resonators:
Cavity-like objects



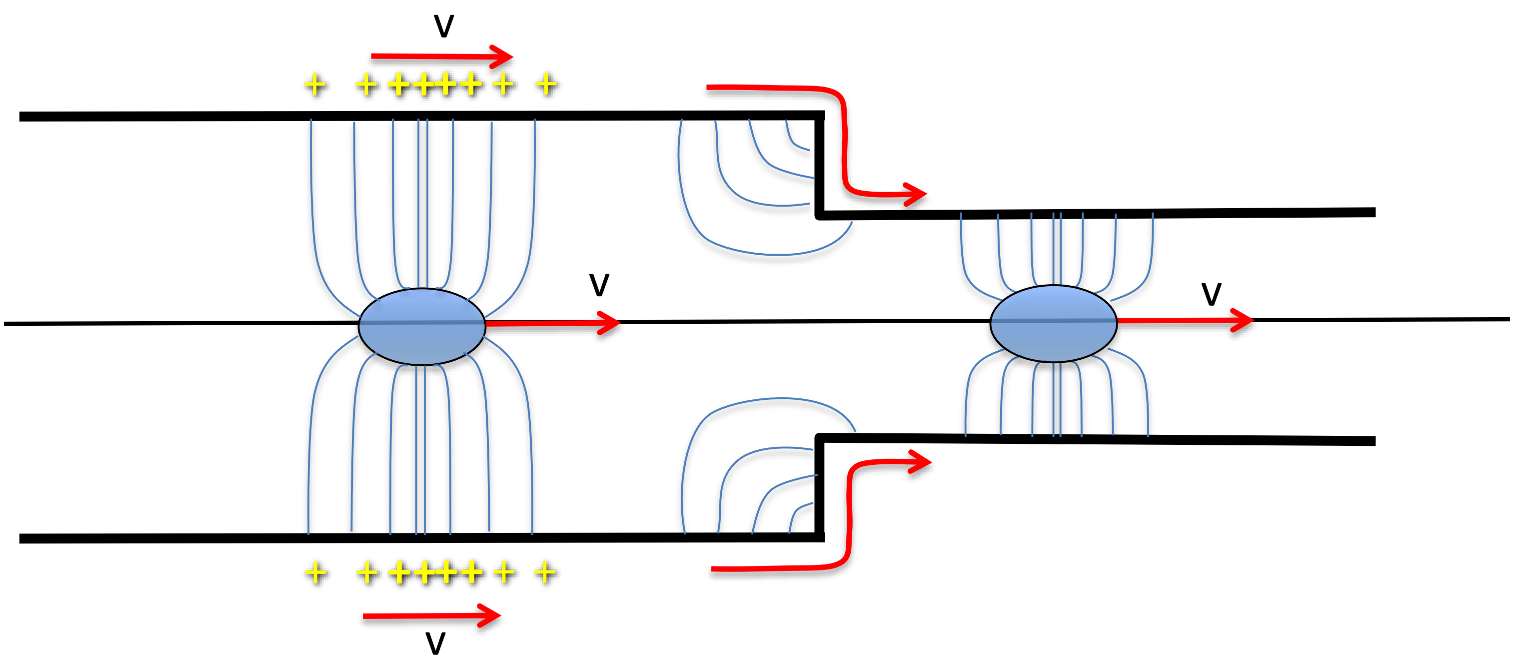
Broad-band resonators:
Tapers, other non-resonant structures



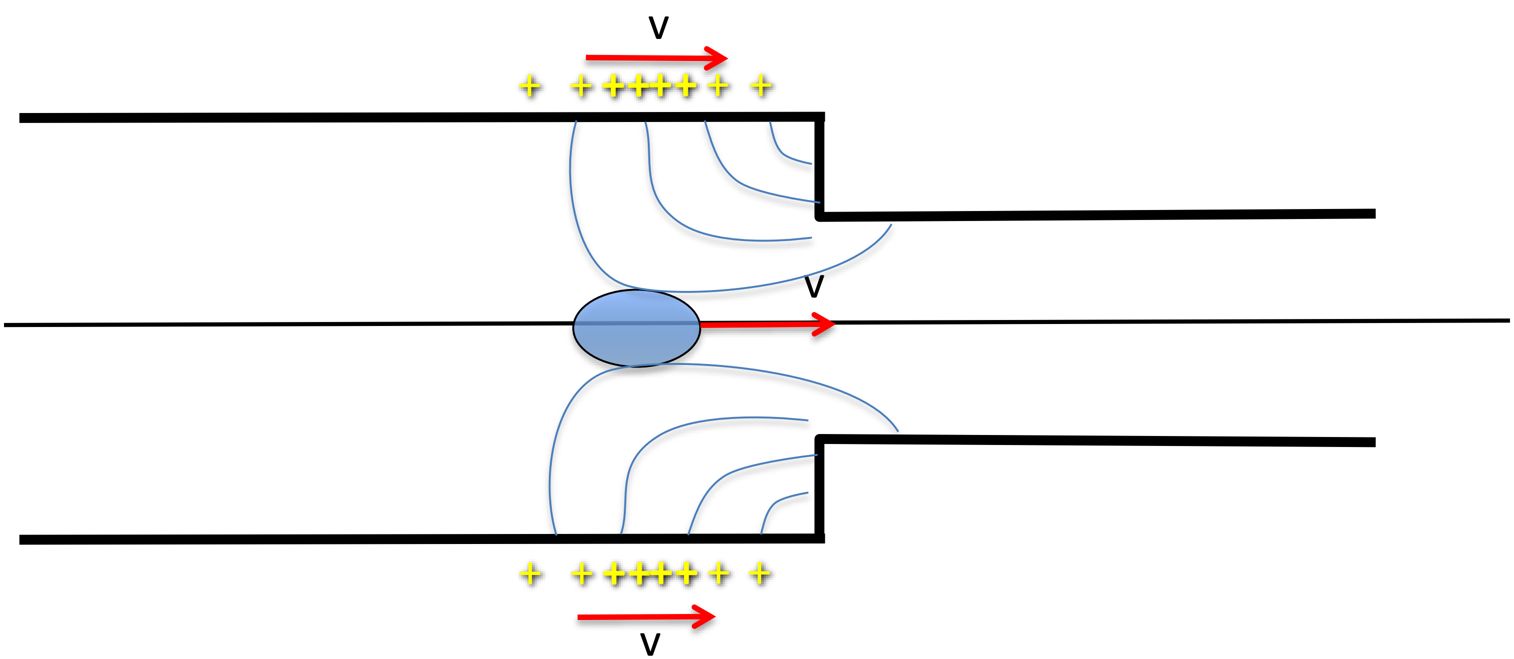
Bunch in a conducting pipe



Bunch in a conducting pipe with sudden change



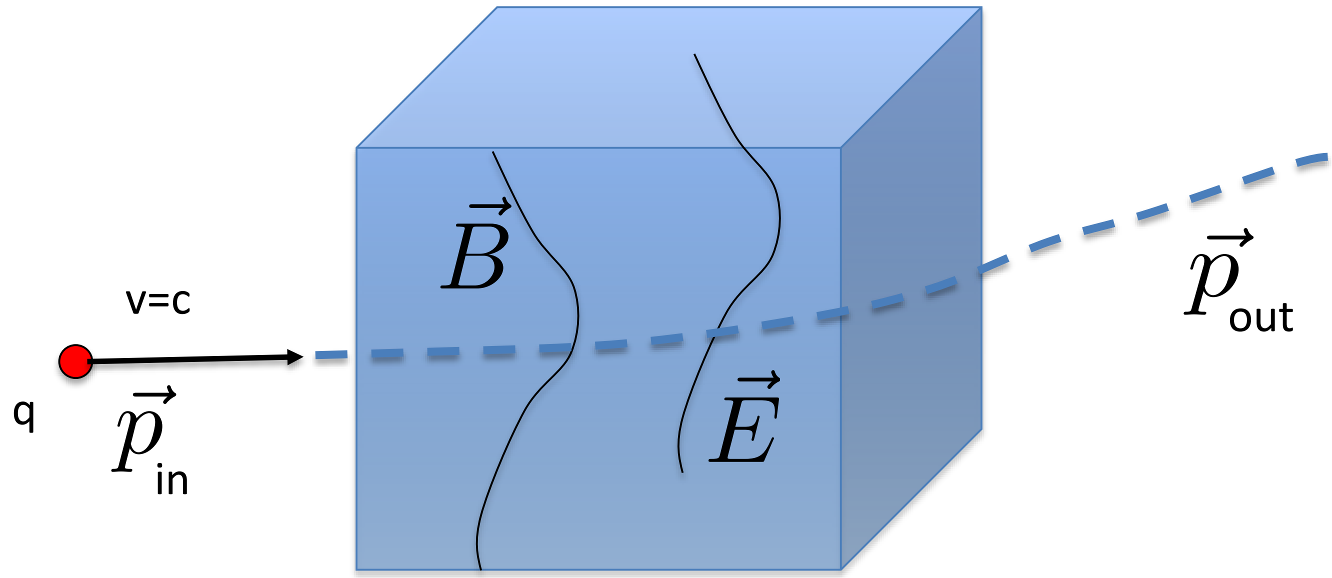
Situation



The electric field from the self-field has a delay

Energy gained by one unit charge $V(\omega) = Z(\omega)I(\omega)$

Panofsky theorem



Can we say something about $\Delta \vec{p} = \vec{p}_{out} - \vec{p}_{in}$?

Panofsky theorem

$$\vec{\nabla} \times \Delta \vec{p} = 0$$

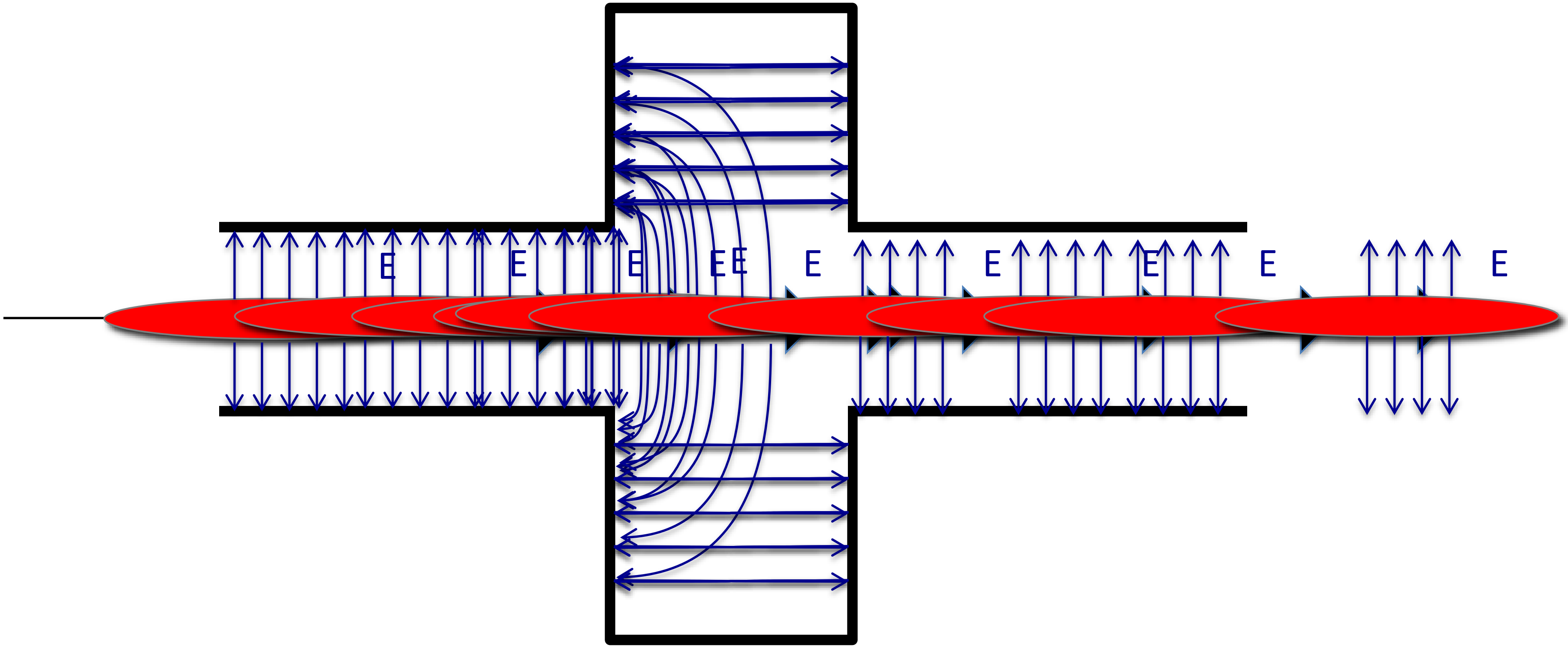
$$\frac{\mathrm{d}\boldsymbol{p}_\perp}{\mathrm{d}t} = -e\nabla_\perp \int_{-\infty}^{\infty} E_{||}\,\mathrm{d}z$$

$$\boxed{\frac{\partial}{\partial z}\boldsymbol{F}_\perp = -\nabla_\perp \boldsymbol{F}_{||}}$$

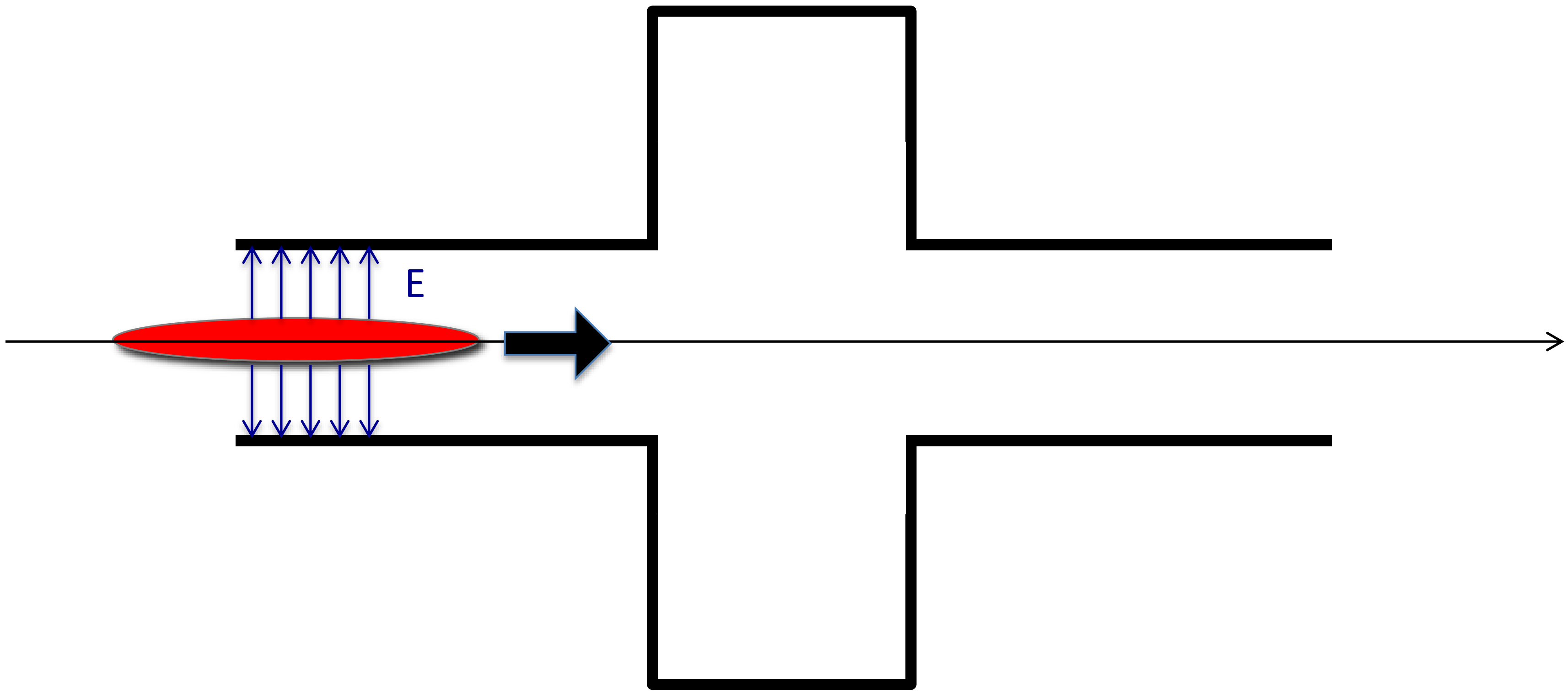
We find a constrain to the forces created by electromagnetic fields
Without actually knowing almost anything !

Wake Field

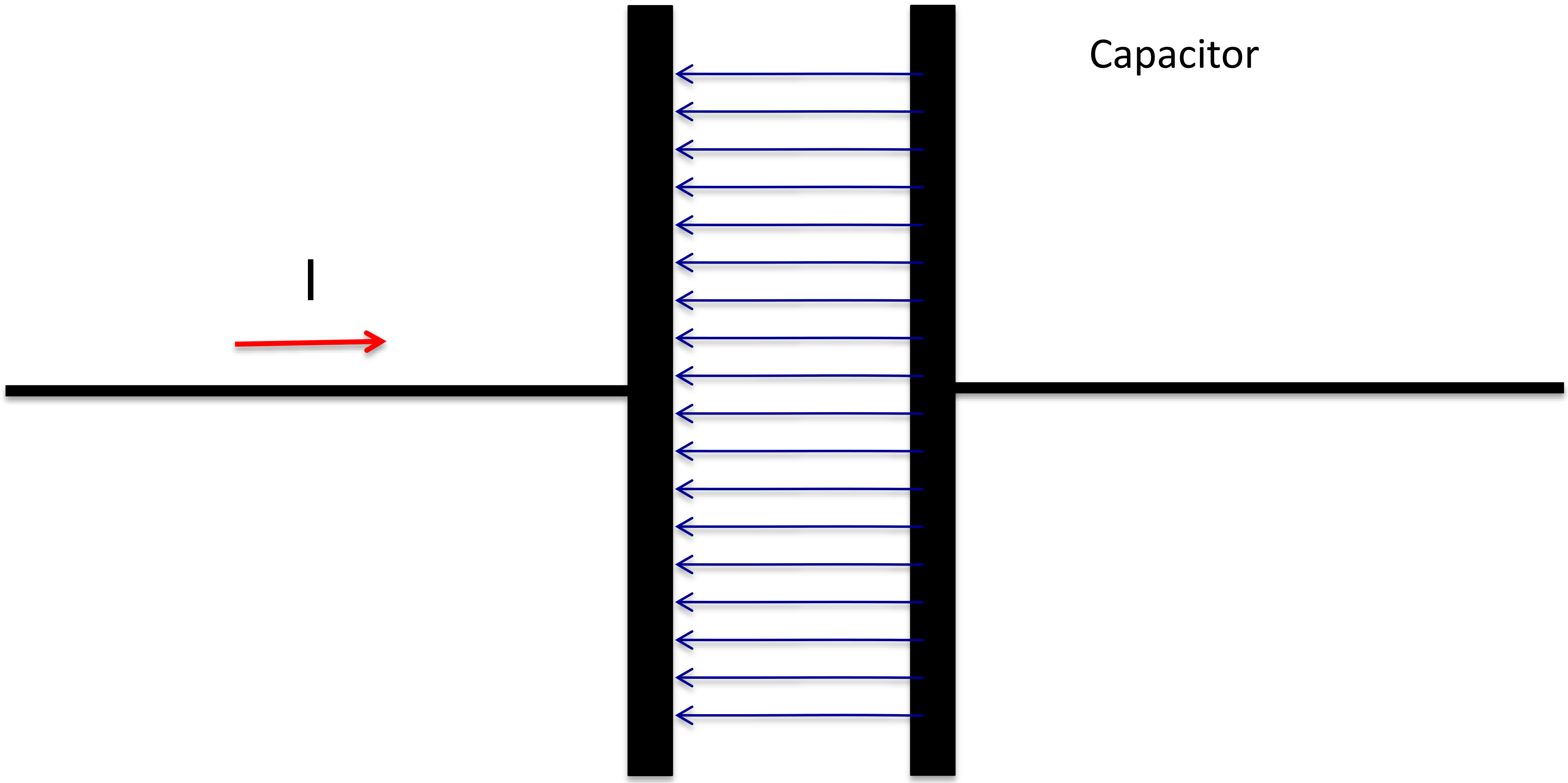
Cavities



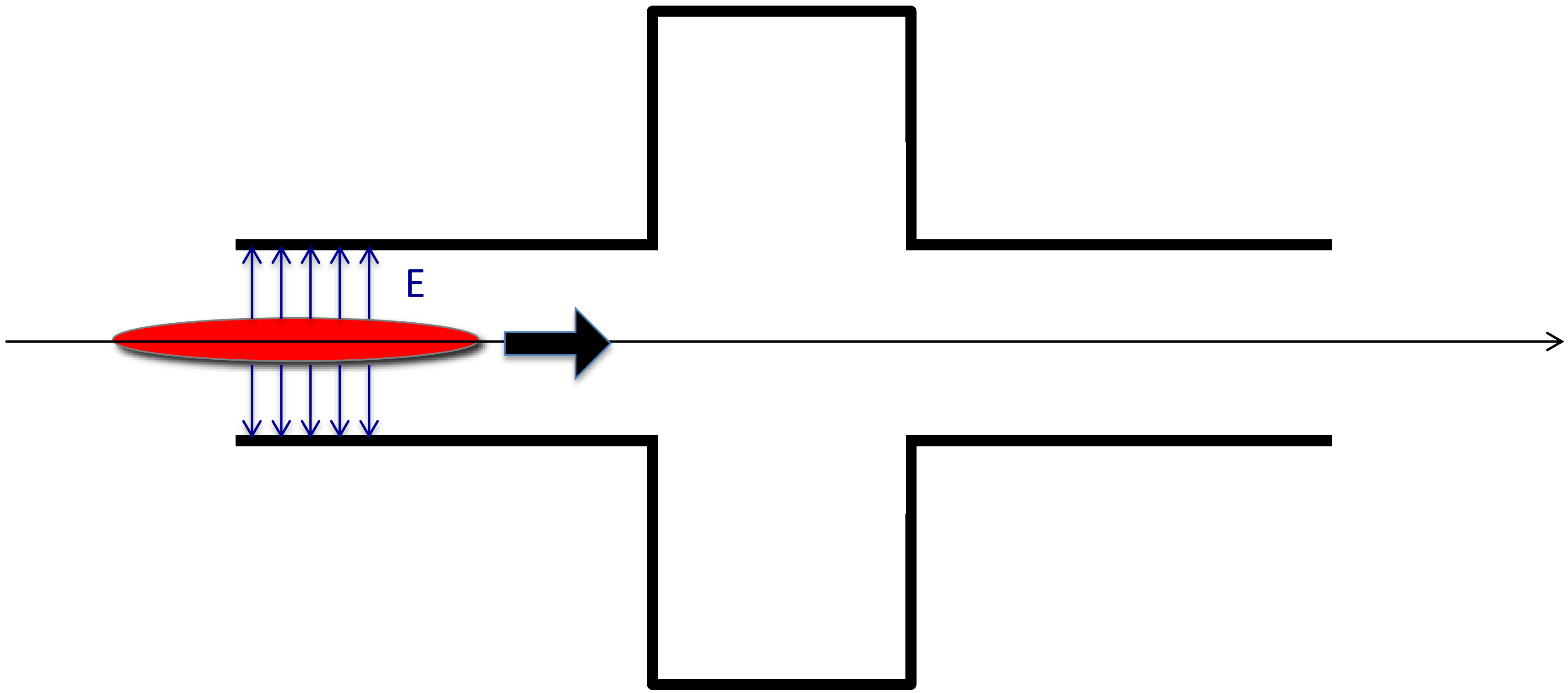
MODEL



MODEL

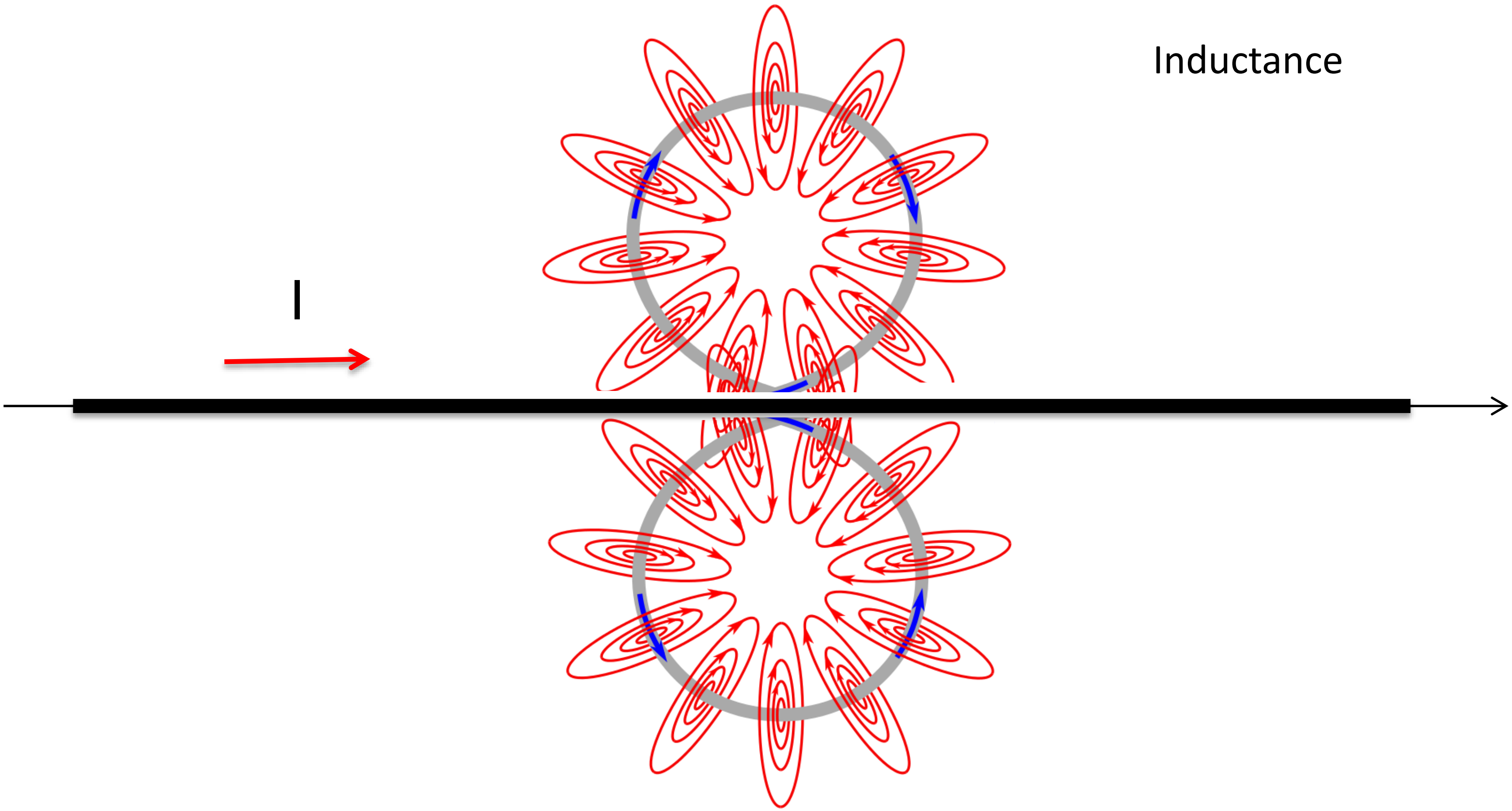


MODEL

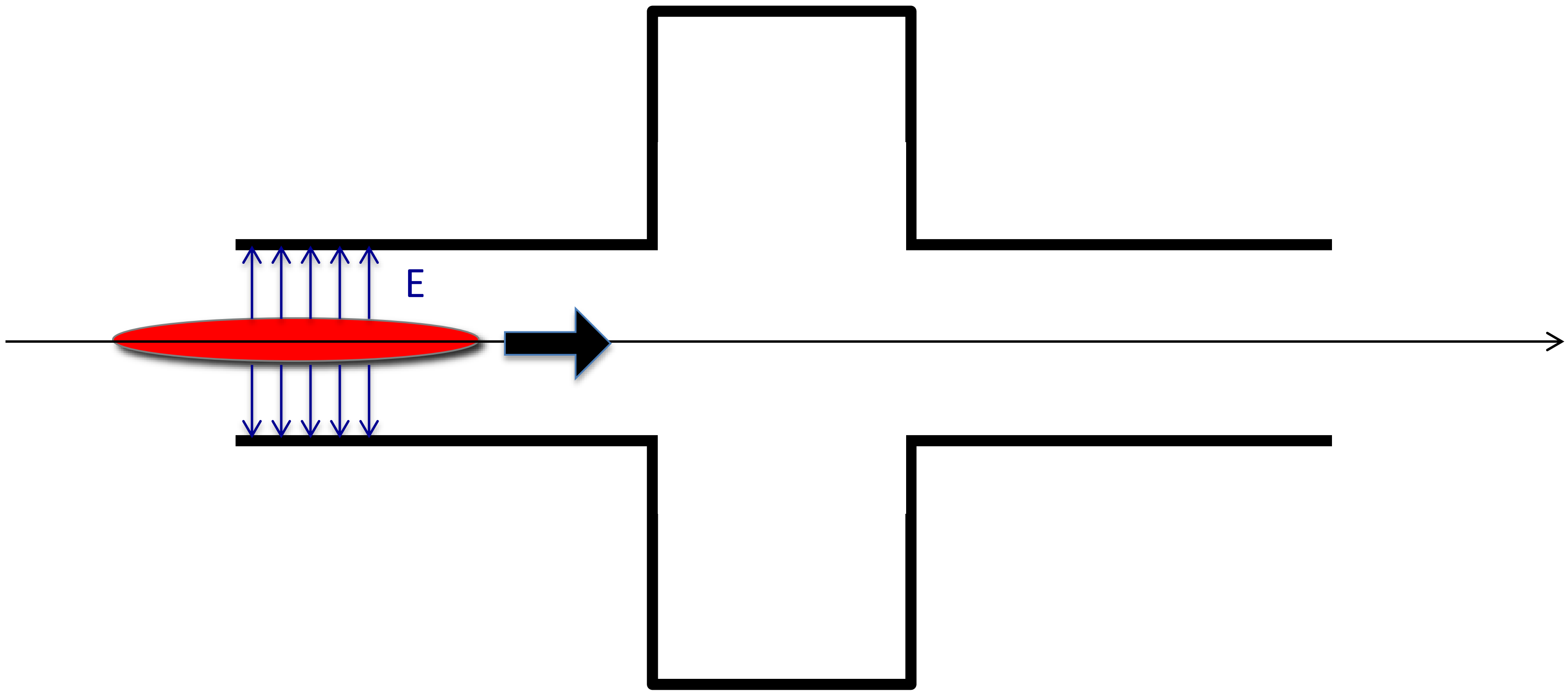


MODEL

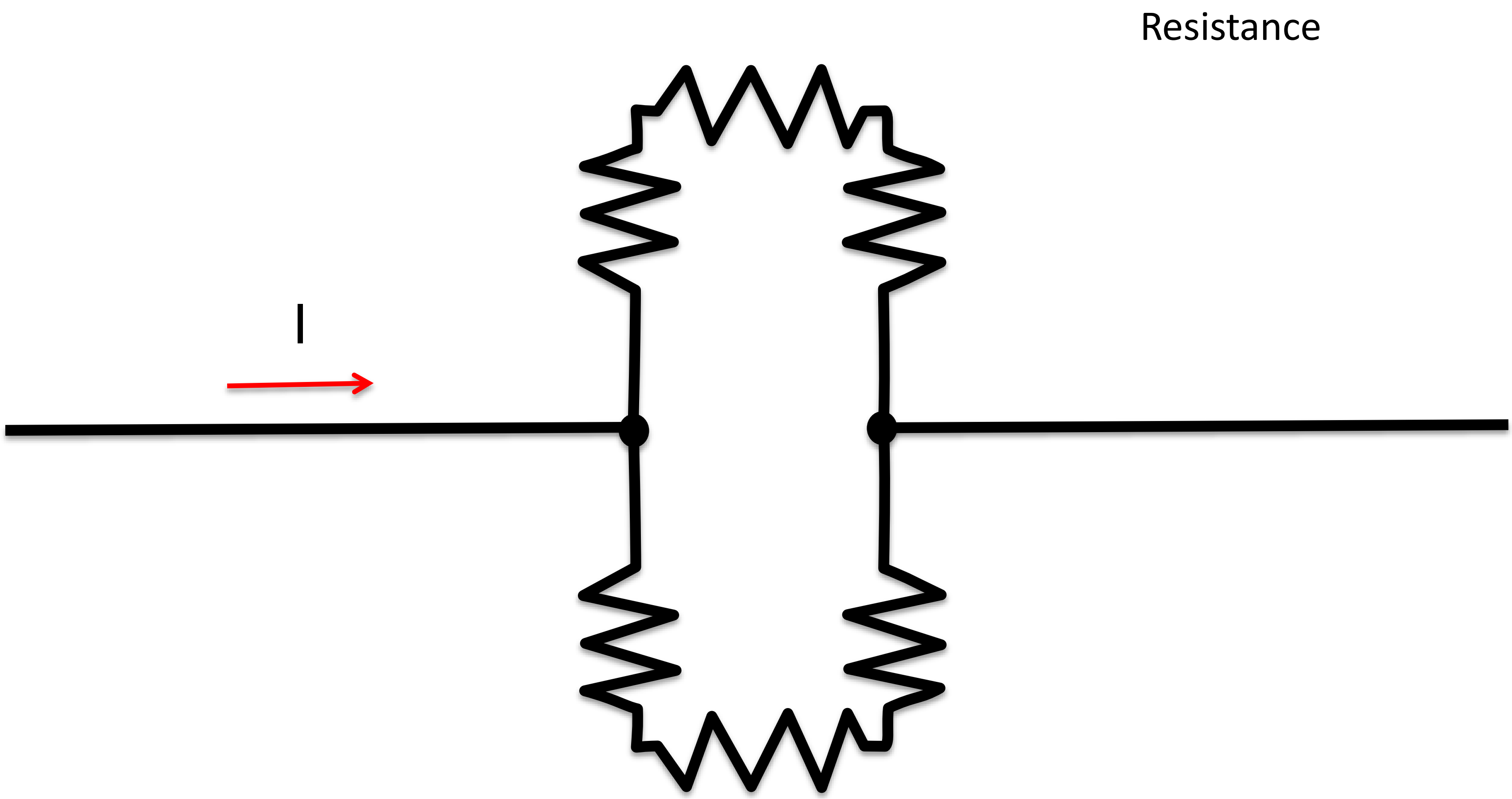
Inductance



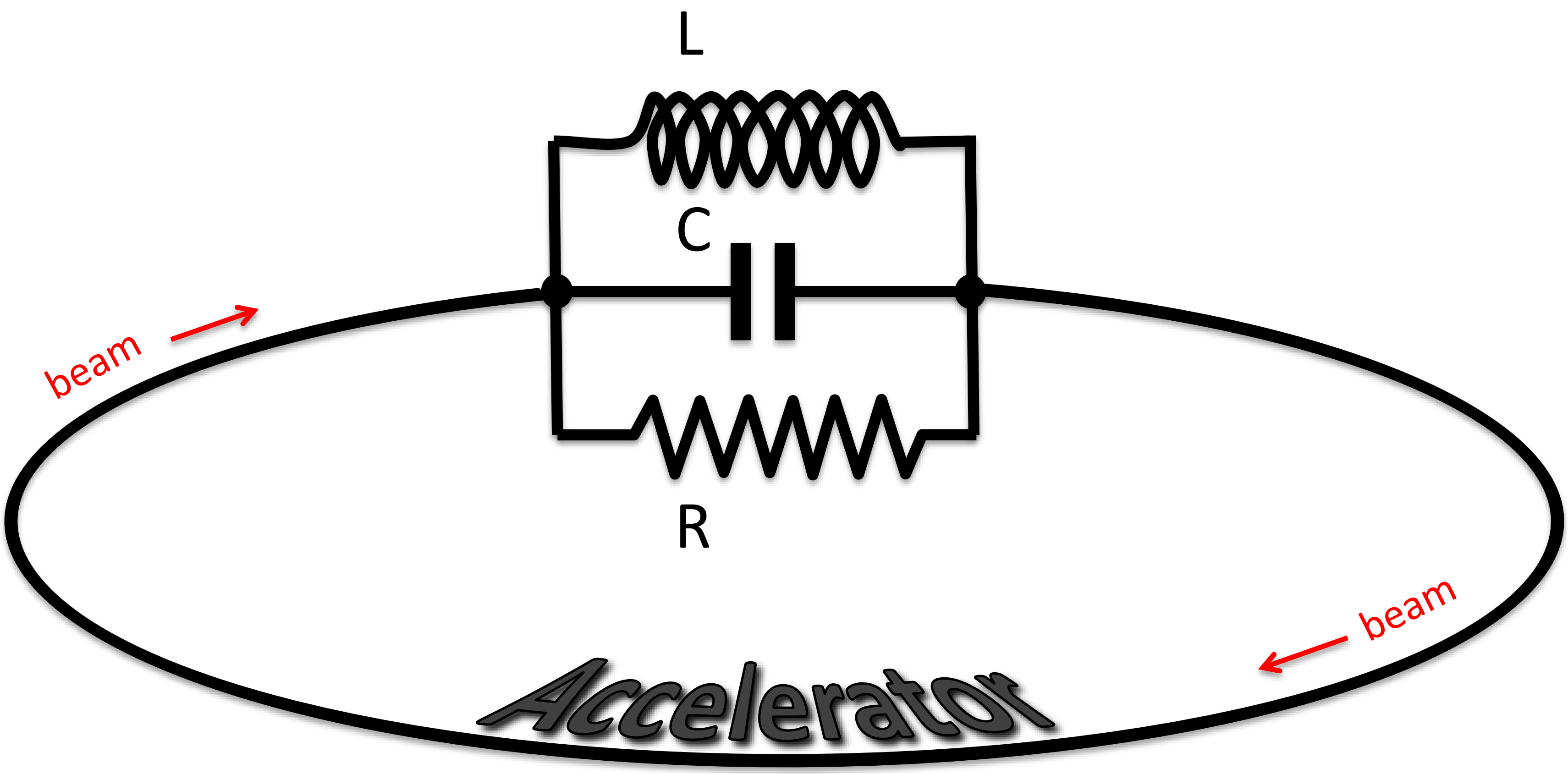
MODEL



MODEL

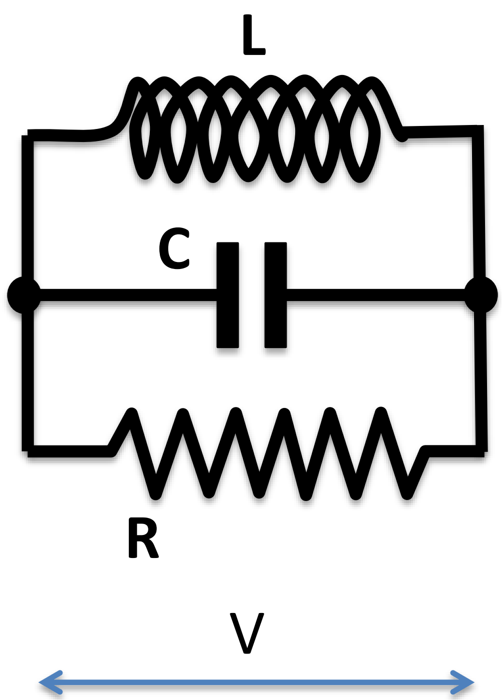


All together



RLC Features

Isolated RLC

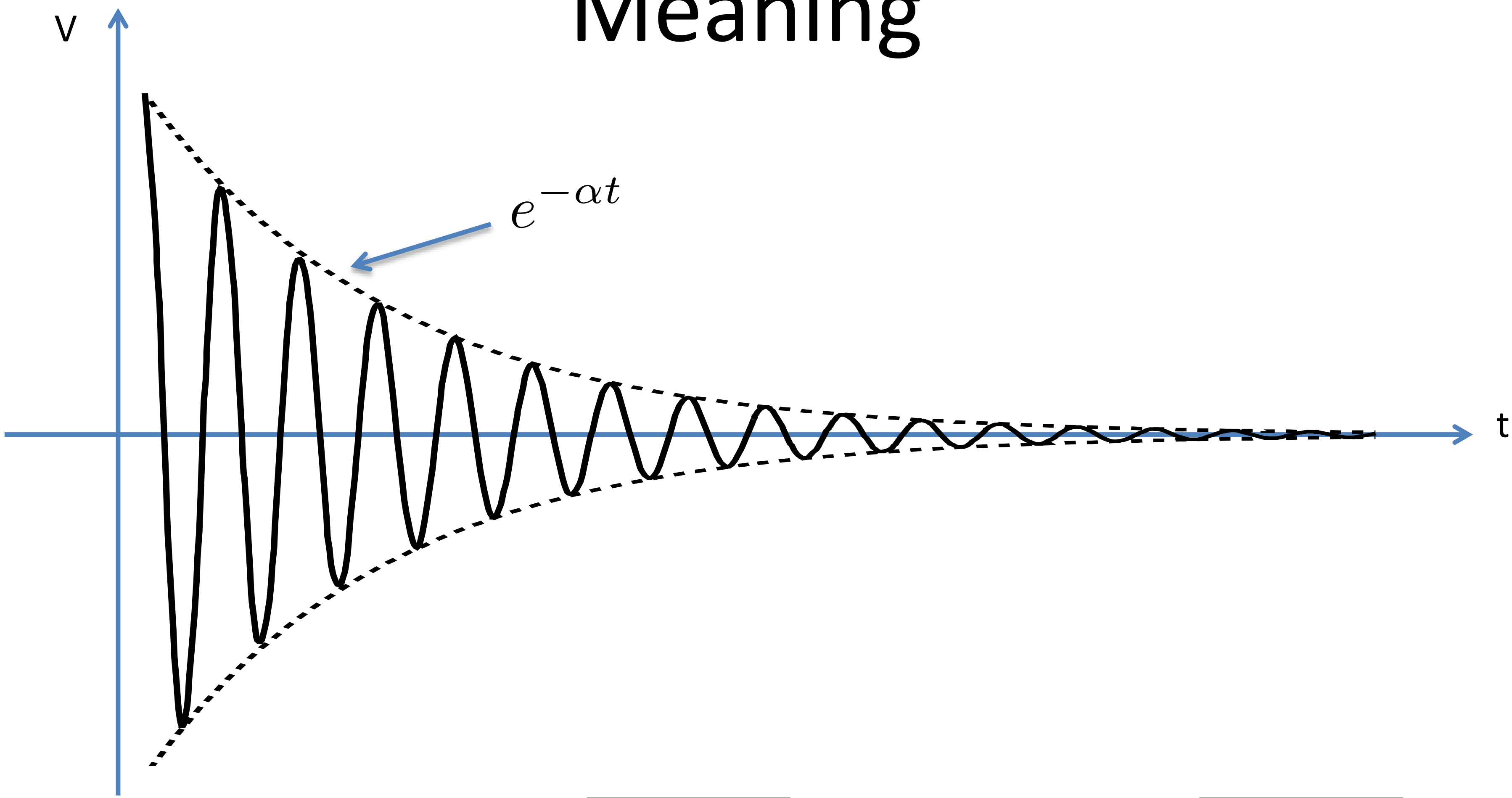


Resonance frequency $$\omega_r = \frac{1}{\sqrt{LC}}$$

Quality factor $$Q = R\sqrt{\frac{C}{L}}$$

Damping rate $$\alpha = \frac{\omega_r}{2Q}$$

Meaning



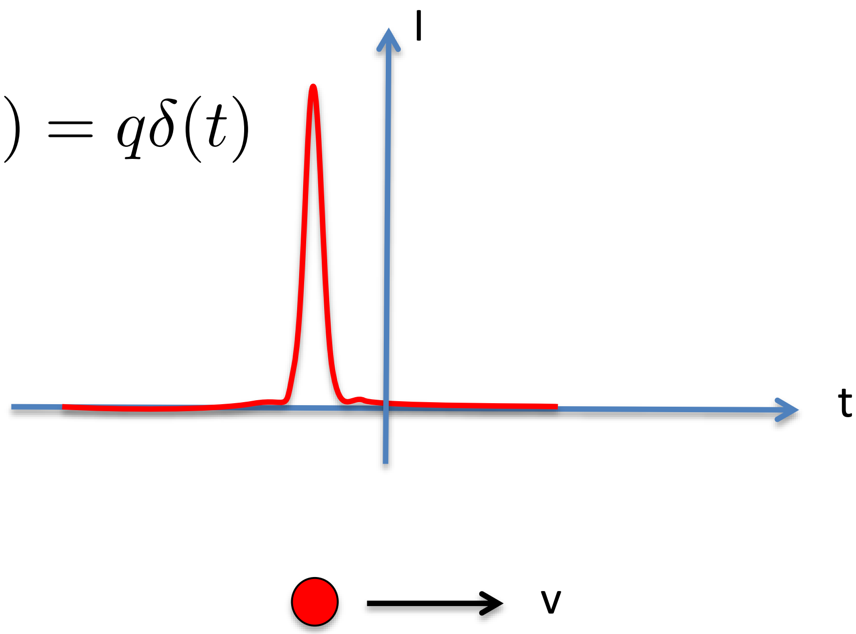
$$V(t) = e^{-\alpha t}\left[A\cos\left(\omega_r\sqrt{1-\frac{1}{4Q^2}}t\right) + B\sin\left(\omega_r\sqrt{1-\frac{1}{4Q^2}}t\right)\right]$$

Response to one particle

What happen when one particle goes through the cavity ?

Before

$$I(t) = q\delta(t)$$

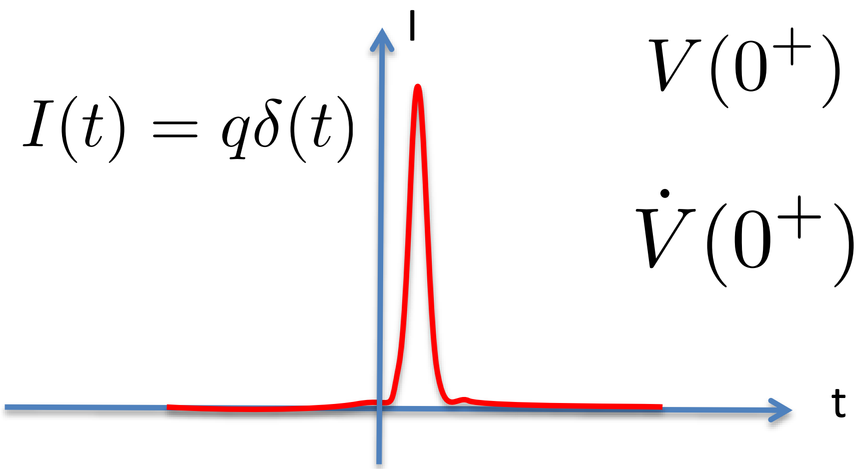


$$V(0^-) = 0$$

$$\dot{V}(0^-) = 0$$

Response to one particle

After



$$I(t) = q\delta(t)$$

$$V(0^+) = \frac{q}{C}$$

$$\dot{V}(0^+) = -\frac{2\omega_r k_{pm}}{Q} q$$

Pulse Response

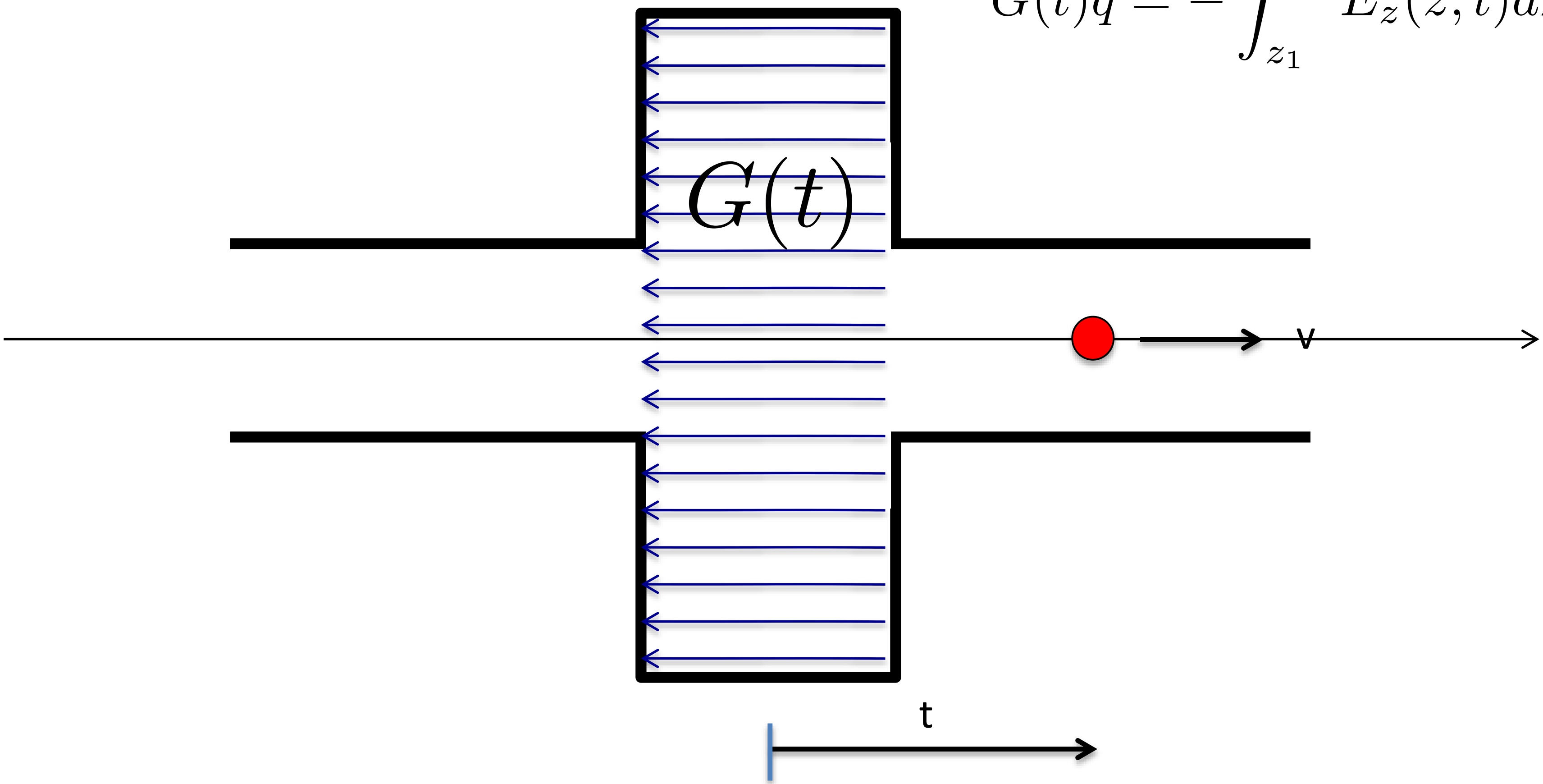
$$V(t) = 2qk_{pm}e^{-\alpha t}\left[\cos\left(\omega_r\sqrt{1-\frac{1}{4Q^2}}t\right) - \frac{\sin\left(\omega_r\sqrt{1-\frac{1}{4Q^2}}t\right)}{2Q\sqrt{1-\frac{1}{4Q^2}}}\right]$$

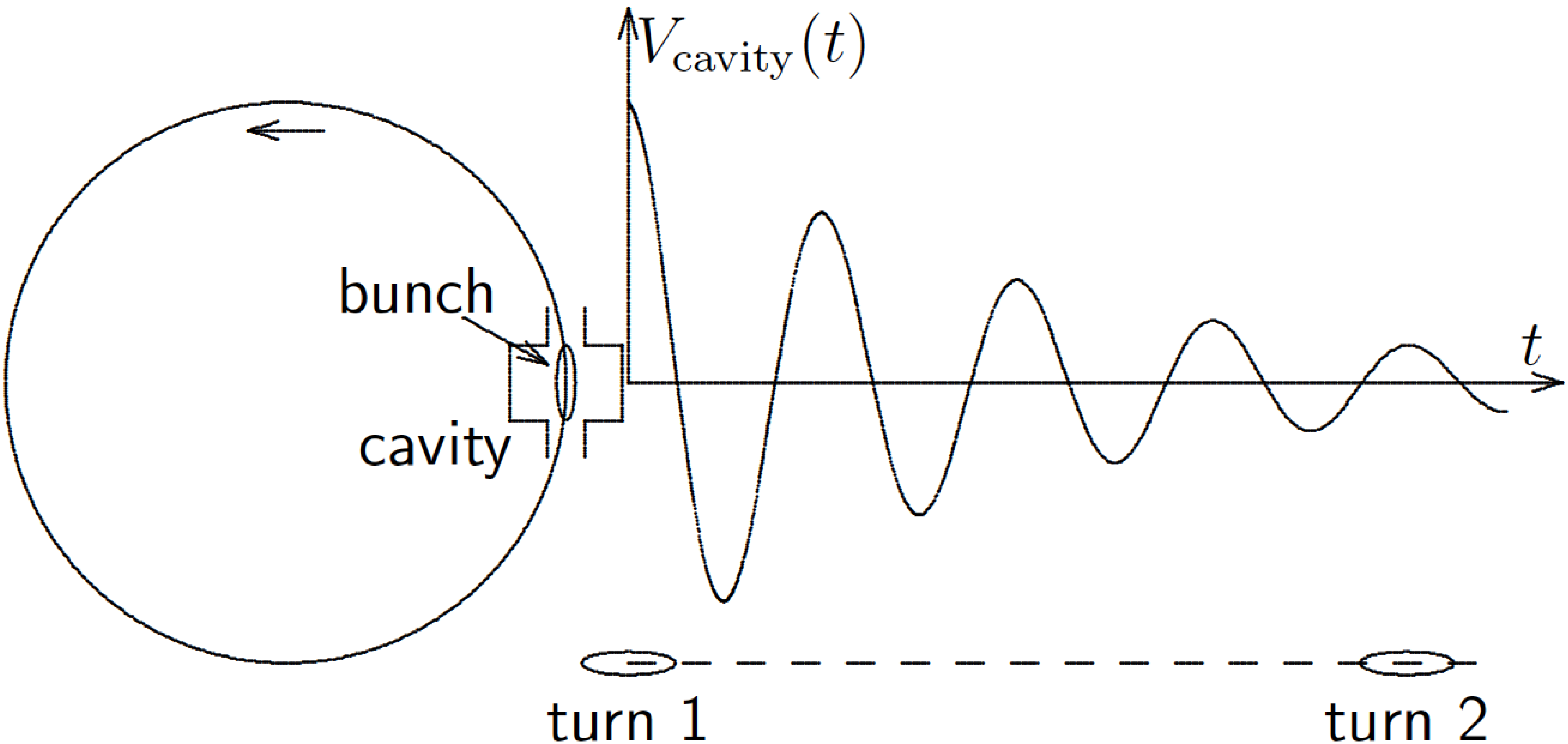
This is the potential in the cavity

Green or wake function

$$G(t) = \frac{V(t)}{q}$$

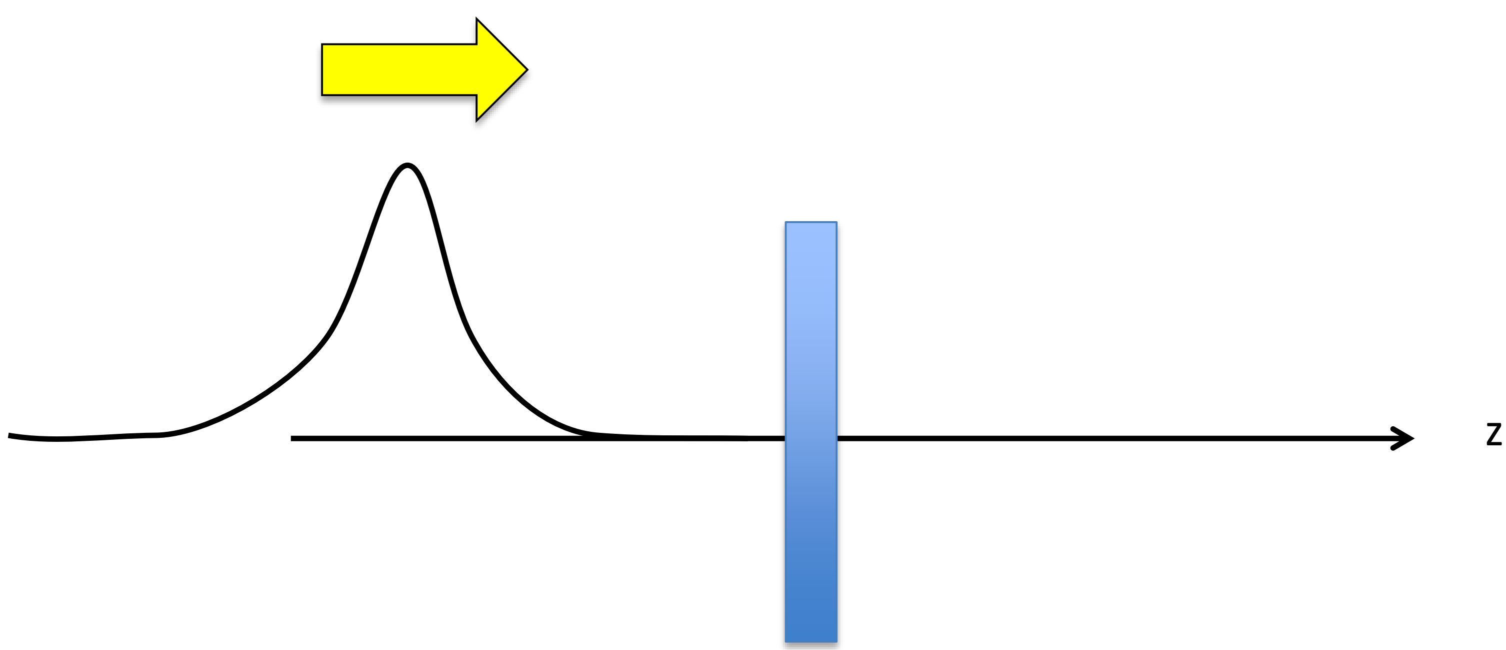
$$G(t)q = -\int_{z_1}^{z_2} E_z(z,t)dz$$




$V_{\mathrm{cavity}}(t)$
bunch
cavity
t
turn 1
turn 2

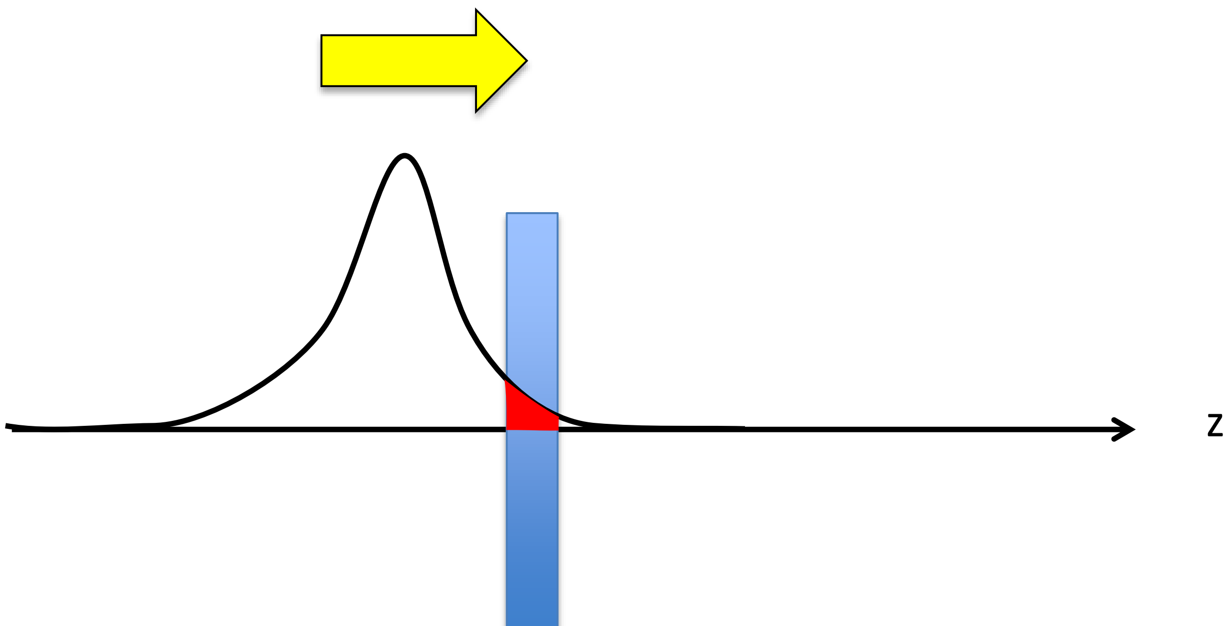
Summary

The wake function tells us what is the longitudinal field experienced by another particle passing through the cavity later



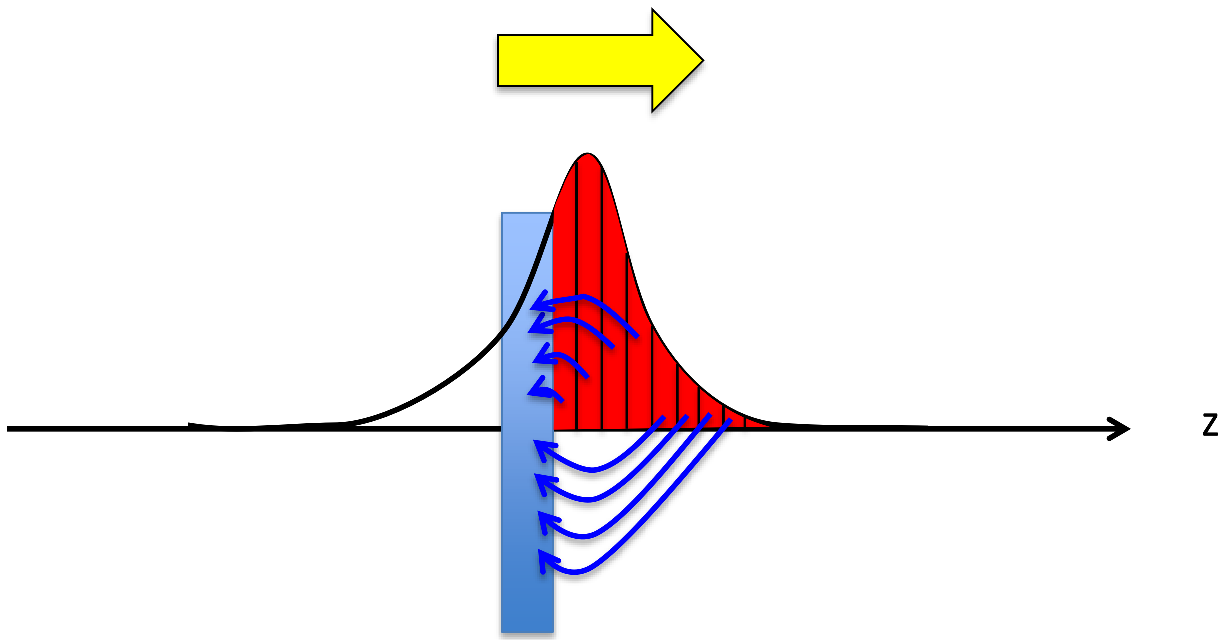
Summary

The wake function tells us what is the longitudinal field experienced by another particle passing through the cavity later



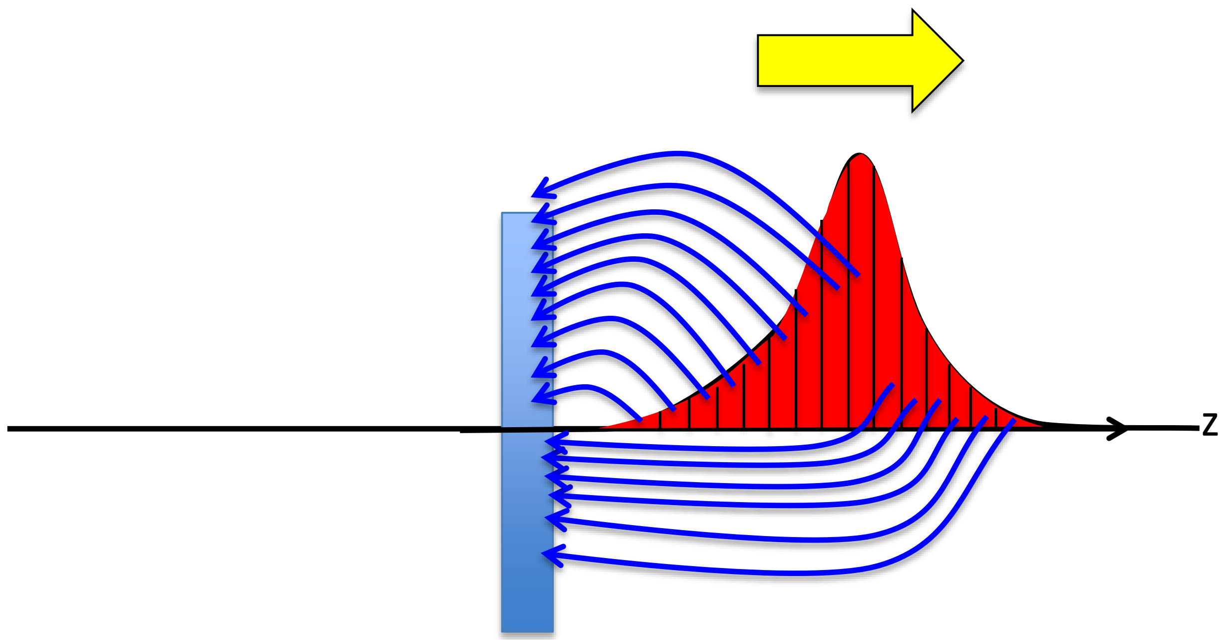
Summary

The wake function tells us what is the longitudinal field experienced by another particle passing through the cavity later



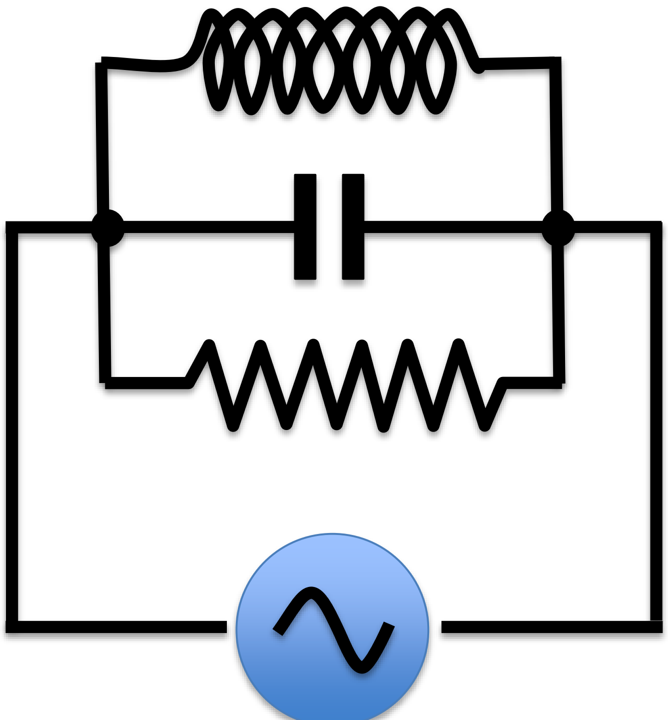
Summary

The wake function tells us what is the longitudinal field experienced by another particle passing through the cavity later



Impedance

Impedance



$$I = \hat{I}\cos(\omega t)$$

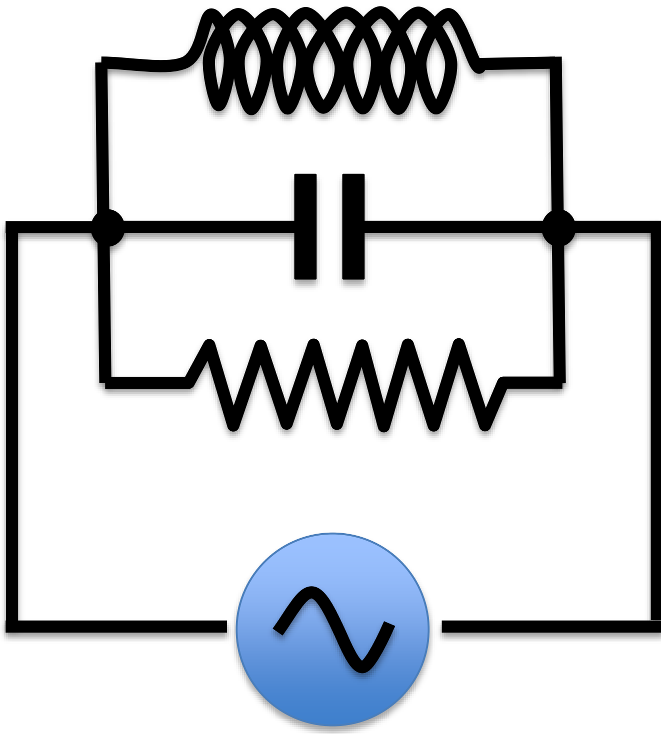
It is a quantity that relate V and I

$$\omega = 0 \quad \longrightarrow \quad V = RI$$

$$\omega > 0$$

$$V(t) = \hat{I}R\frac{\cos(\omega t) + Q\frac{\omega^2-\omega_r^2}{\omega_r\omega}\sin(\omega t)}{1+Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}$$

Impedance



$$I = \hat{I}\cos(\omega t)$$

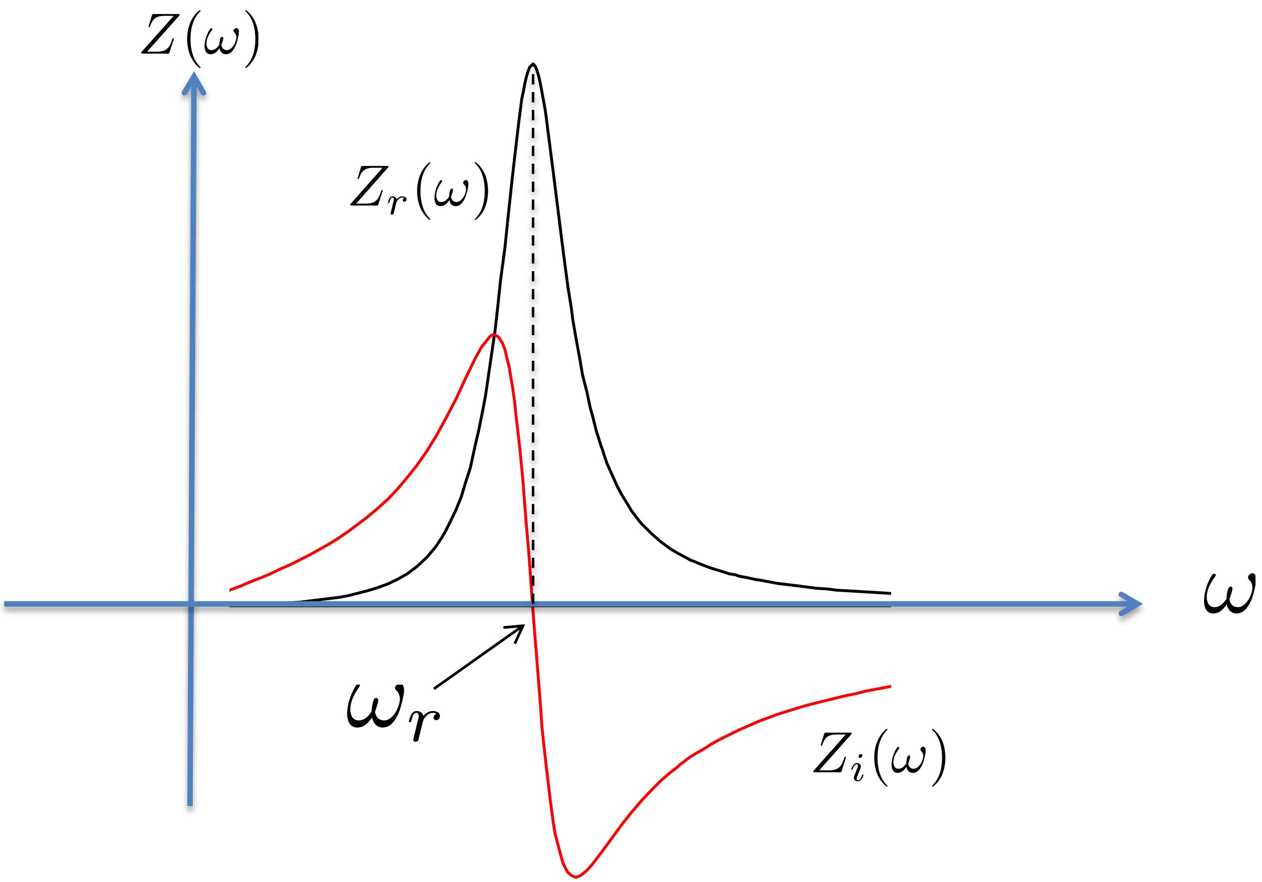
Impedance

$$V(t) = Z_r(\omega)\hat{I}\cos(\omega t) - Z_i(\omega)\hat{I}\sin(\omega t)$$

$$Z_r(\omega) = R\frac{1}{1 + Q^2\left(\frac{\omega^2 - \omega_r^2}{\omega_r \omega}\right)^2}$$

$$Z_i(\omega) = -R\frac{Q\frac{\omega^2 - \omega_r^2}{\omega_r \omega}}{1 + Q^2\left(\frac{\omega^2 - \omega_r^2}{\omega_r \omega}\right)^2}$$

Properties



Properties

At $\omega = \omega_r$ $\begin{cases} Z_i(\omega_r) & \text{is zero} \\ Z_r(\omega_r) & \text{is maximum} \end{cases}$

$$0 < \omega < \omega_r \Rightarrow Z_i(\omega) > 0 \quad \text{inductive}$$

$$\omega > \omega_r \Rightarrow Z_i(\omega) < 0 \quad \text{capacitive}$$

$$Z_r(\omega) = Z_r(-\omega) \qquad Z_i(\omega) = -Z_i(-\omega)$$

Power dissipated

$$V(t)I(t) = \hat{I}^2 R \frac{\cos^2(\omega t) + Q\frac{\omega^2-\omega_r^2}{\omega_r\omega}\sin(\omega t)\cos(\omega t)}{1+Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}$$

$$V(t)I(t) = \hat{I}^2 Z_r(\omega)\cos^2(\omega t) + \hat{I}^2 Z_i(\omega)\sin(\omega t)\cos(\omega t)$$

The power dissipated depends on the resistive impedance

$$< V(t)I(t) >_{cycle} = \frac{1}{2}\hat{I}^2 Z_r(\omega)$$

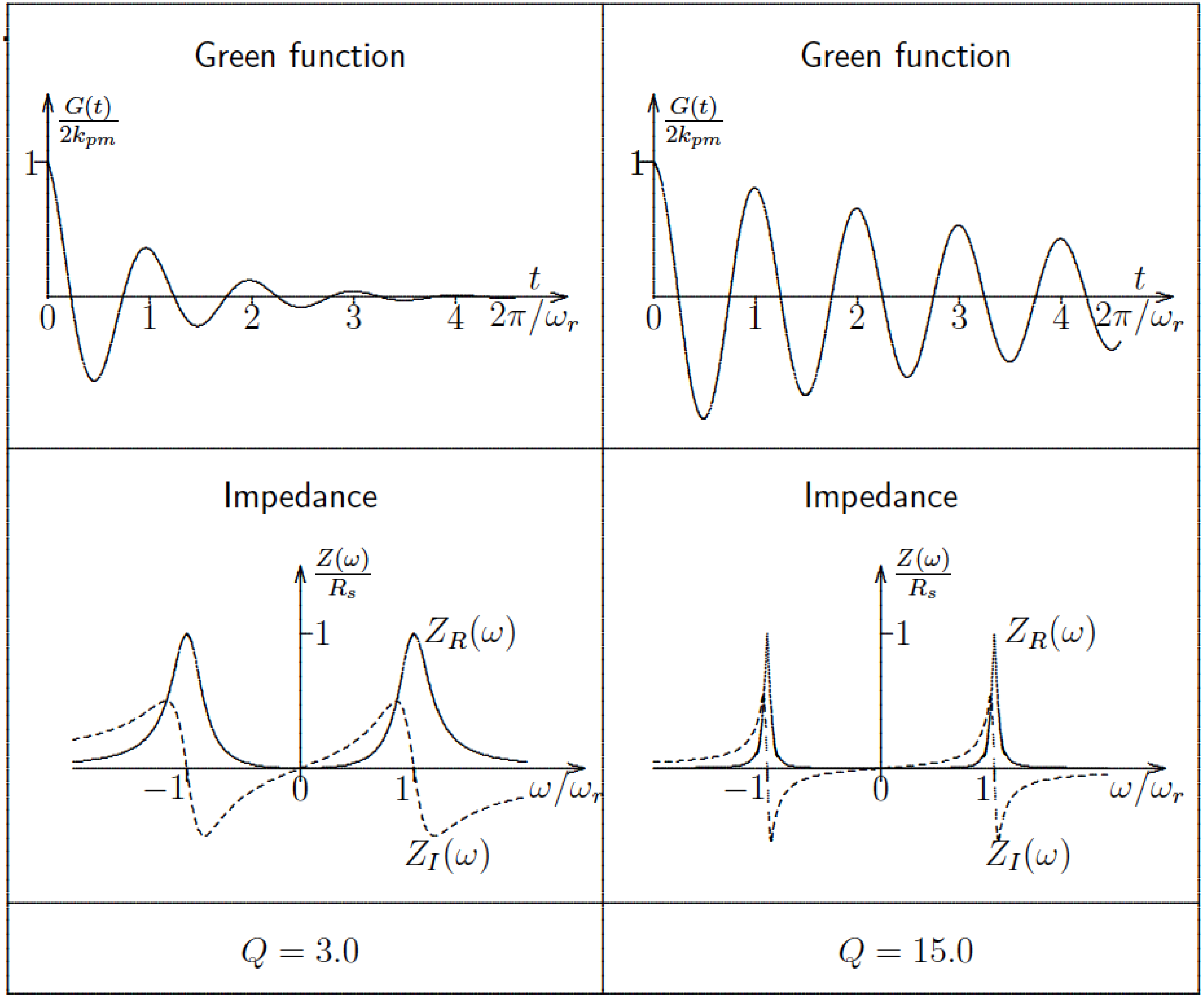
Complex notation

Complex notation $$Z(\omega) = Z_r(\omega) + iZ_i(\omega)$$

If Q is very large only for ω close to ω_r

$$\frac{\omega^2 - \omega_r^2}{\omega_r \omega} = \frac{(\omega - \omega_r)(\omega + \omega_r)}{\omega_r \omega} \simeq \frac{2\Delta\omega}{\omega_r}$$

$$Z(\omega) = R\frac{1 - iQ\frac{\omega^2 - \omega_r^2}{\omega_r \omega}}{1 + Q^2\left(\frac{\omega^2 - \omega_r^2}{\omega_r \omega}\right)^2} = R\frac{1 - i2Q\frac{\Delta\omega}{\omega_r}}{1 + 4Q^2\left(\frac{\Delta\omega}{\omega_r}\right)^2}$$


Green function
$\frac{G(t)}{2k_{pm}}$
1
0
1
2
3
4
t
$2\pi/\omega_r$
Green function
Impedance
$\frac{Z(\omega)}{R_s}$
$Z_R(\omega)$
-1
ω/ω_r
$Z_I(\omega)$
Impedance
$Q = 3.0$
$Q = 15.0$

Wake potential ←→ Impedance

Charge through the cavity at t' $\quad dq(t') = I(t')dt'$

Consider now the wake at $\quad t > t' > 0$

The wake of that charge at time t is $\quad G(t - t')$

The potential in the cavity at time *t* due to the charge passing at *t'* is

$$dq(t')G(t - t')$$

The total potential due to all charges passing through the cavity in $t > t' > 0$ is

$$V(t) = \int_0^t dq(t')G(t - t')$$

If now the current I is $$I(t') = \hat{I} e^{i\omega t'}$$

then $$V(t) = \int_0^t \hat{I} e^{i\omega t'} G(t-t') dt'$$

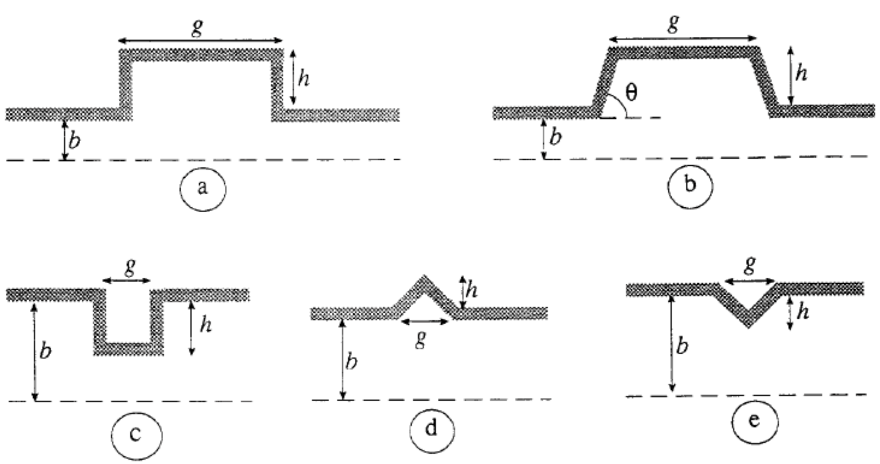
with some change of variable

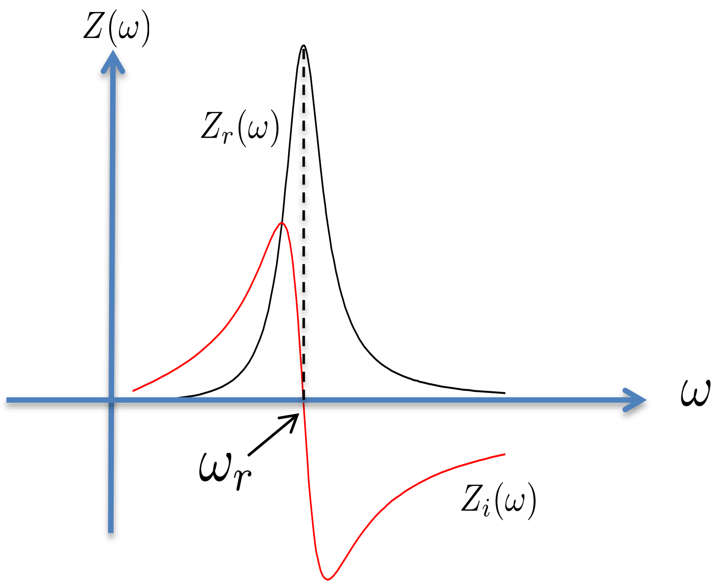
$$V(t) = I(t) \int_0^t e^{-i\omega\tau} G(\tau) d\tau$$

We wait long enough that transient effect disappears, hence

$$Z(\omega) = \frac{V(t)}{I(t)} = \int_0^\infty e^{-i\omega\tau} G(\tau) d\tau$$

Complicated geometries of the vacuum chamber give an effect on the beam which is described by the impedance Z(ω)

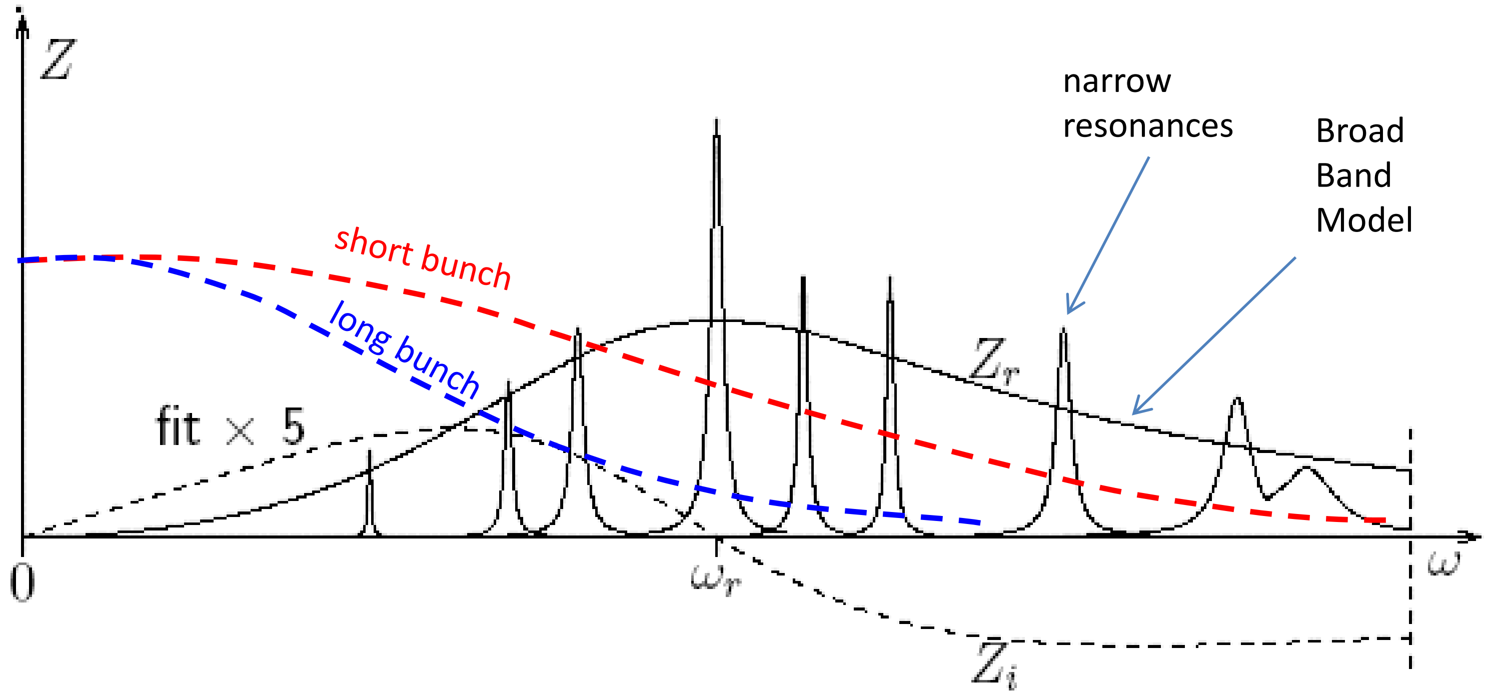




$$Z(\omega) \Longleftrightarrow G(t)$$

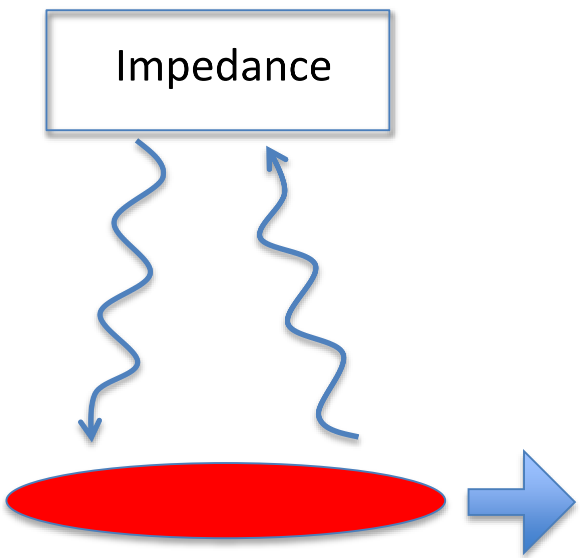
Consequences of impedances

Energy loss on pipes → heating (important if you have a superconducting machine!)



Consequences of impedances

Feed-back to the beam as a hole: collective effects



Dynamics of the
all beam is affected

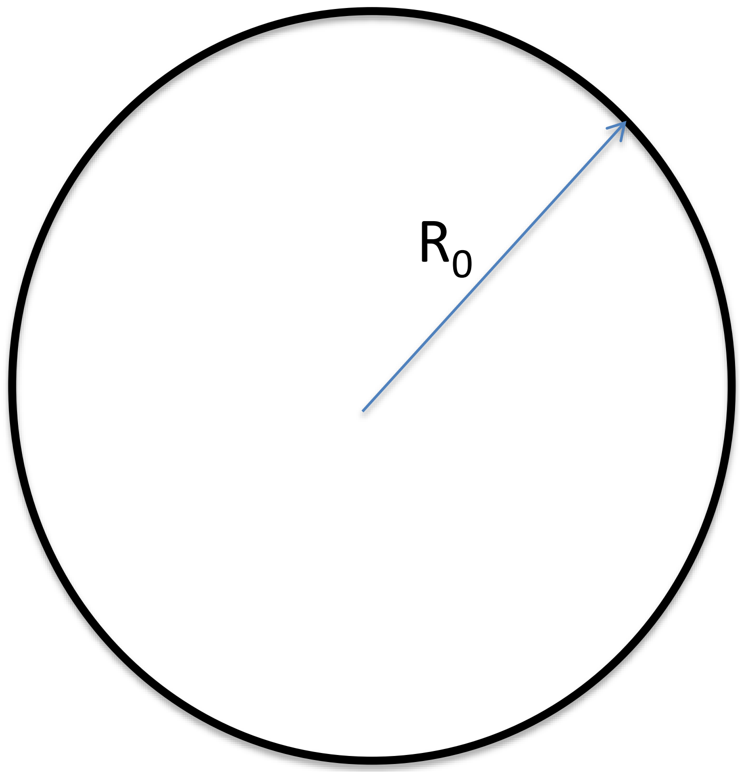
We have seen the longitudinal impedance in a cavity

More types of impedances …

Longitudinal dynamics

Longitudinal dynamics

synchronous orbit



T_0

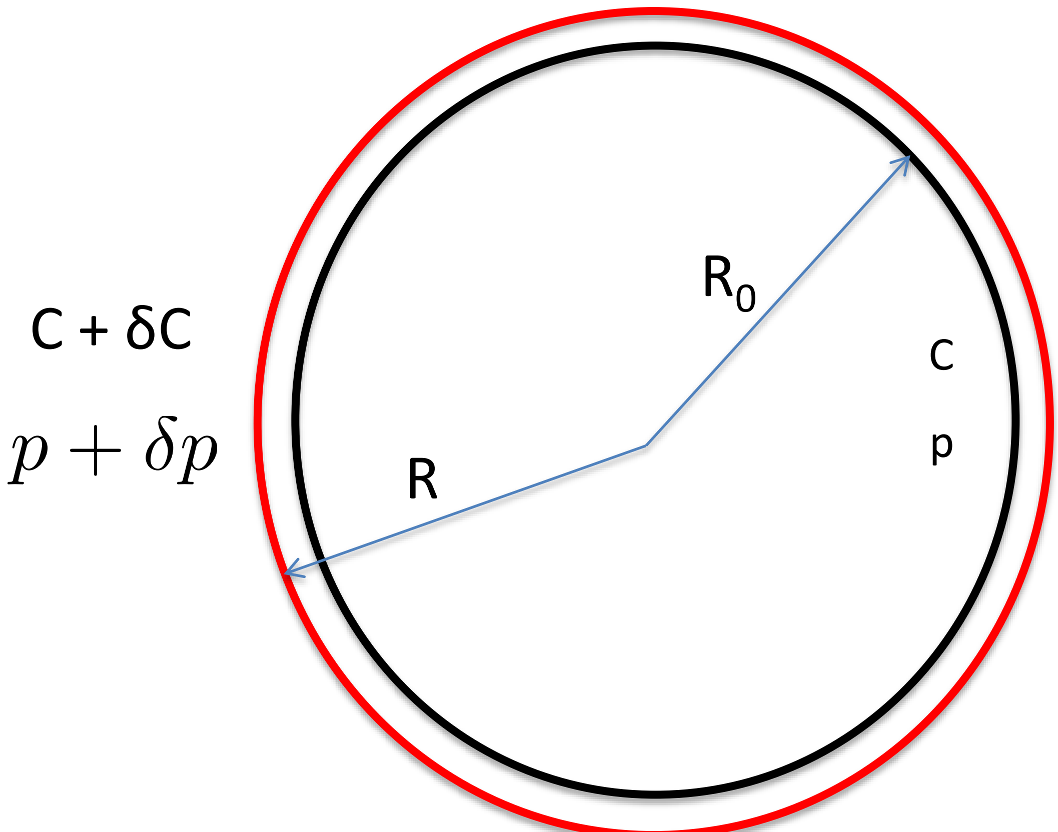
ω_0

p_0

E_0

Longitudinal dynamics

synchronous orbit

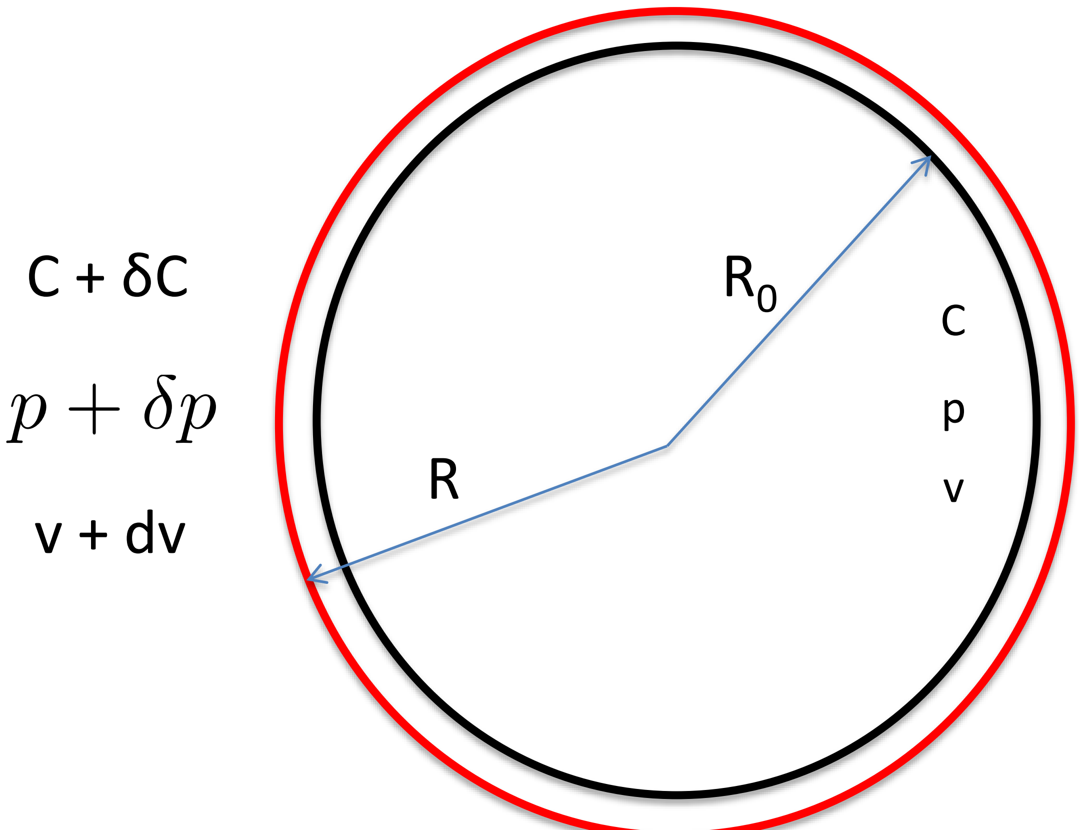


$$\frac{\delta C}{C} = \alpha_c \frac{\delta p}{p}$$

This property comes from the magnets

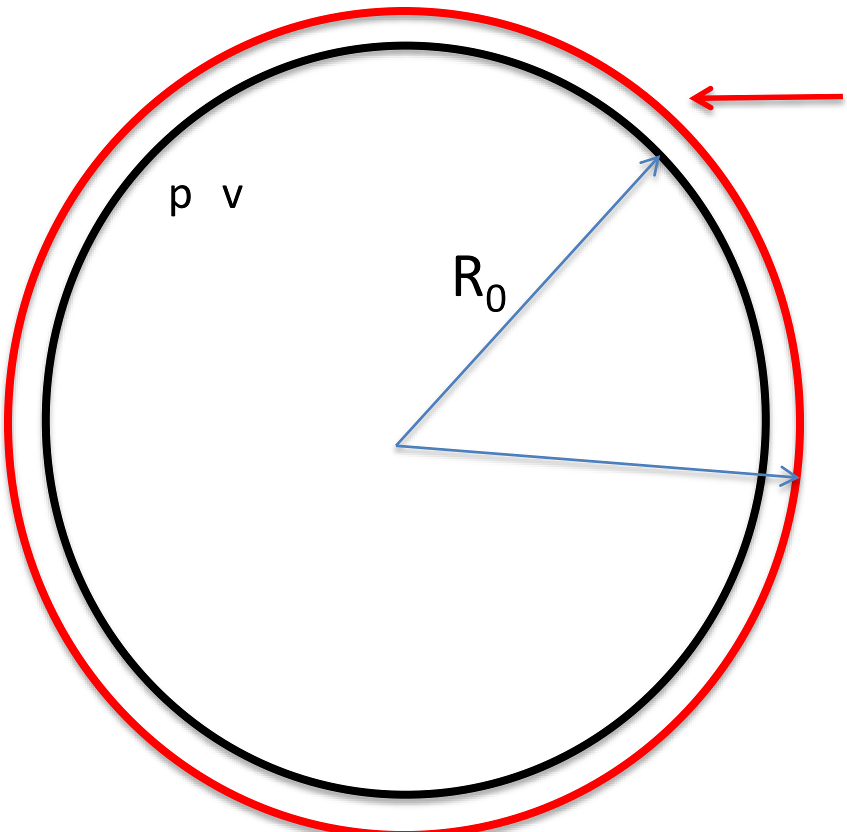
Longitudinal dynamics

revolution time



Nobody can go faster than light

revolution time



$p + \delta p$ If this is large

this velocity will always be less than "c"

Therefore at a certain point the circumference will growth, but the particle speed remains "c"

It takes longer to make one turn !

$$\frac{\delta T}{T_0} = \frac{1}{T_0}\delta\left(\frac{L}{v}\right) = \left(\alpha_c - \frac{1}{\gamma^2}\right)\frac{\delta p}{p} = \eta\frac{\delta p}{p}$$

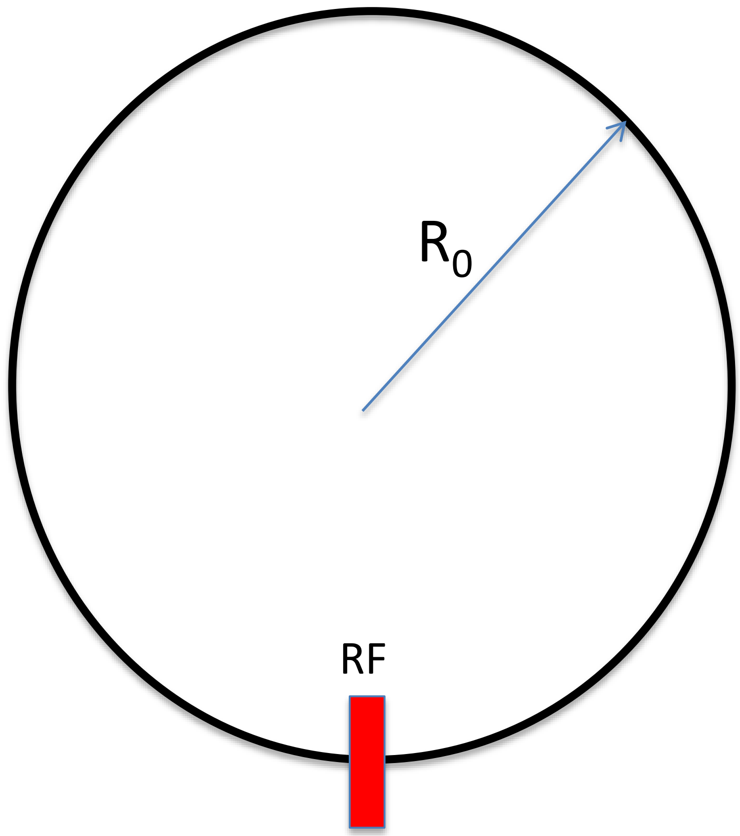
If $\alpha_c = \frac{1}{\gamma^2}$ we are at the transition energy E_T

If $E < E_T$ increasing energy $\Rightarrow$ revolution time shorter

If $E > E_T$ increasing energy $\Rightarrow$ revolution time longer !!

RF

synchronous orbit



T_0

ω_0

p_0

E_0

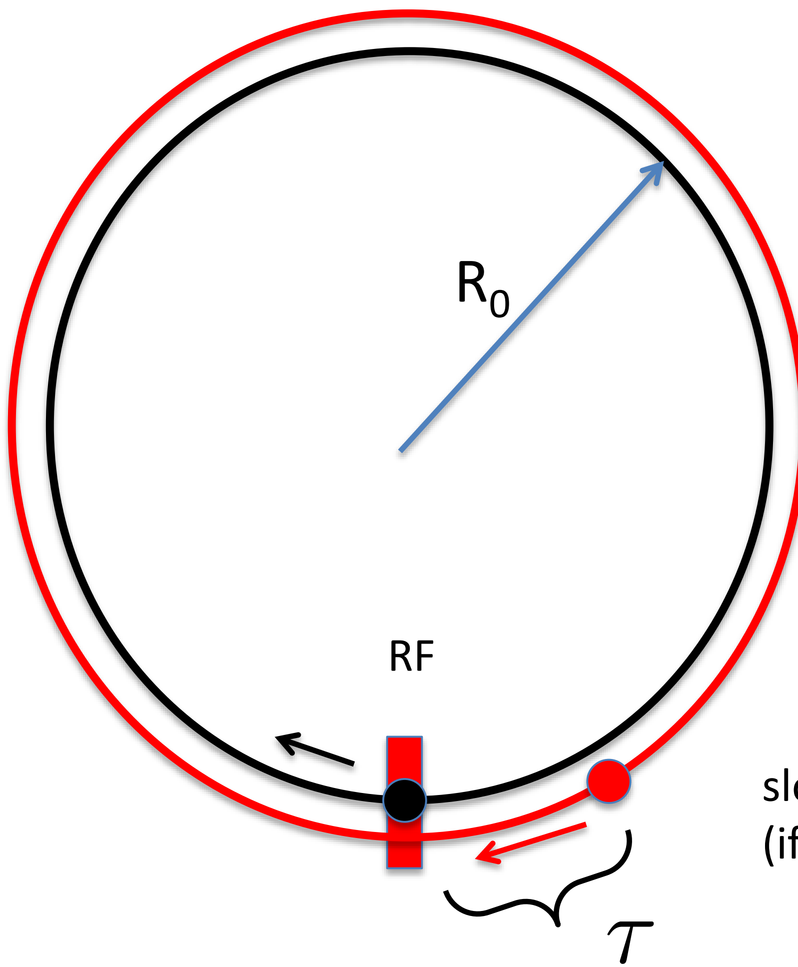
The synchronous particle has energy E and goes through the cavity at time t_s

Voltage in the cavity $V = \hat{V} \sin(h\omega_0 t_s)$

$\phi_s = h\omega_0 t_s$ this is the phase of the synchronous particle

This is a phase we know each time the particle goes through the cavity

Non synchronous particle



T_0

ω_0

p_0

E_0

slower particle
(if below transition)

Voltage on the particle

$$V = \hat{V}\sin(\phi_s + h\omega_0\tau)$$

Gain of energy $$\delta E = e\hat{V}\sin(\phi_s + h\omega_0\tau)$$

Now we include an energy loss per turn an per particle U

$$\boxed{\delta E = e\hat{V}\sin(\phi_s + h\omega_0\tau) - U}$$

Define relative energy $\epsilon = \Delta E/E_0$

$$\delta\epsilon = \frac{e\hat{V}}{E_0}\sin(\phi_s + h\omega_0\tau) - \frac{U}{E_0}$$

$$\frac{\delta\epsilon}{T_0} = \frac{e\hat{V}}{T_0 E_0}\sin(\phi_s + h\omega_0\tau) - \frac{U}{T_0 E_0}$$

If $\dfrac{\delta\epsilon}{T_0}$ is small, than this term is equal to the time derivative of ϵ

$$\dot{\epsilon} = \frac{e\hat{V}\omega_0}{2\pi E_0}\sin(\phi_s + h\omega_0\tau) - \frac{\omega_0 U}{2\pi E_0}$$

but U, depends on $\epsilon, \ \text{and}\, \tau$ ⇨ $U(\epsilon,\tau)$

If $\epsilon, \text{ and } \tau$ are small we can expand

$$\dot{\epsilon} = \frac{e\hat{V}\omega_0}{2\pi E_0}\sin(\phi_s) + \frac{e\hat{V}\omega_0}{2\pi E_0}\cos(\phi_s)h\omega_0\tau - \frac{\omega_0 U_0}{2\pi E_0} - \frac{\omega_0}{2\pi E_0}\frac{\partial U}{\partial E}\epsilon - \frac{\omega_0}{2\pi E_0}\frac{\partial U}{\partial t}\tau$$

These two terms are equal for the synchronous particle

Phase shift is used to measure U → Z

We remain with the equation

$$\dot{\epsilon} = \frac{e\hat{V}h\omega_0^2}{2\pi E_0}\cos(\phi_s)\tau - \frac{\omega_0}{2\pi E_0}\frac{\partial U}{\partial E}\epsilon - \frac{\omega_0}{2\pi E_0}\frac{\partial U}{\partial t}\tau$$

In addition at high energy $\frac{\delta T}{T} \simeq \eta \frac{\delta E}{E}$ ⇒ $\dot{\tau} = \eta\epsilon$

$$\ddot{\tau} = \eta \frac{e\hat{V} h \omega_0^2}{2\pi E_0} \cos(\phi_s)\tau - \eta \frac{\omega_0}{2\pi E_0} \frac{\partial U}{\partial E} \epsilon - \eta \frac{\omega_0}{2\pi E_0} \frac{\partial U}{\partial t} \tau$$

$$\omega_{s0}^2 = -\eta \frac{e\hat{V} h \omega_0^2}{2\pi E_0} \cos(\phi_s) \qquad \alpha_s = \frac{1}{2} \frac{\omega_0}{2\pi E} \frac{\partial U}{\partial E}$$

Final equation of motion (in tau)

$$\ddot{\tau} + 2\alpha_s \dot{\tau} + \left[\omega_{s0}^2 + \eta \frac{\omega_0}{2\pi E_0} \frac{\partial U}{\partial t} \right] \tau = 0$$

Solution

$$\tau \propto e^{\lambda t} \quad \Rightarrow \quad \lambda^2 + 2\alpha_s\lambda + \left[\omega_{s0}^2 + \eta\frac{\omega_0}{2\pi E_0}\frac{\partial U}{\partial t}\right] = 0$$

Solving for lambda: $\lambda = -\alpha_s \pm \sqrt{\alpha_s^2 - (\omega_{s0}^2 + \ldots)}$

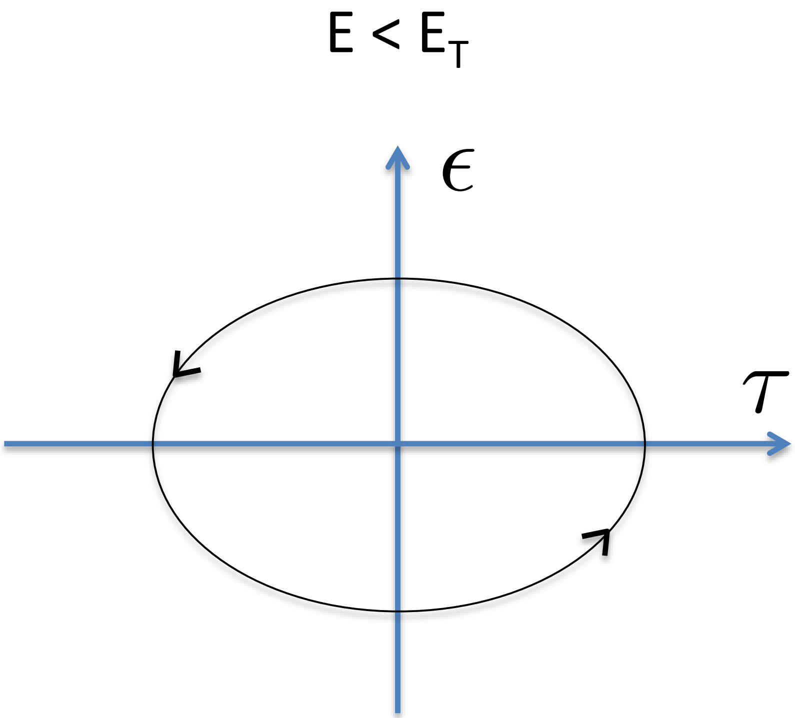
that is $\lambda = -\alpha_s \pm i\omega_{s1}$ with $\omega_{s1}^2 = \omega_{s0}^2 + \eta\frac{\omega_0}{2\pi E_0}\frac{\partial U}{\partial t} - \alpha_s^2$

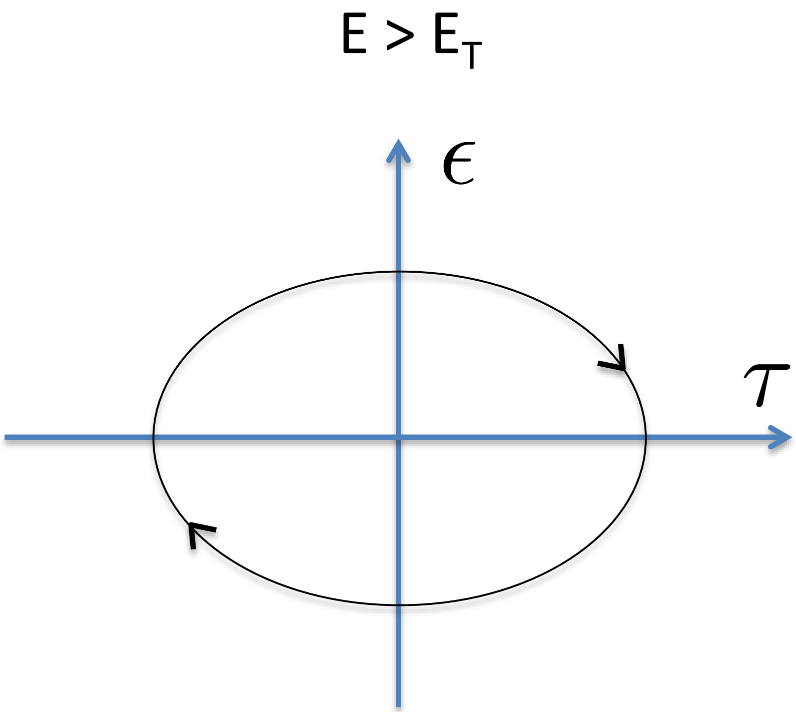
$$\boxed{\tau = \hat{\tau}e^{-\alpha_s t}\cos(\omega_{s1}t)} \quad \Rightarrow$$

if $\alpha_s > 0$ Solution stable

Interpretation

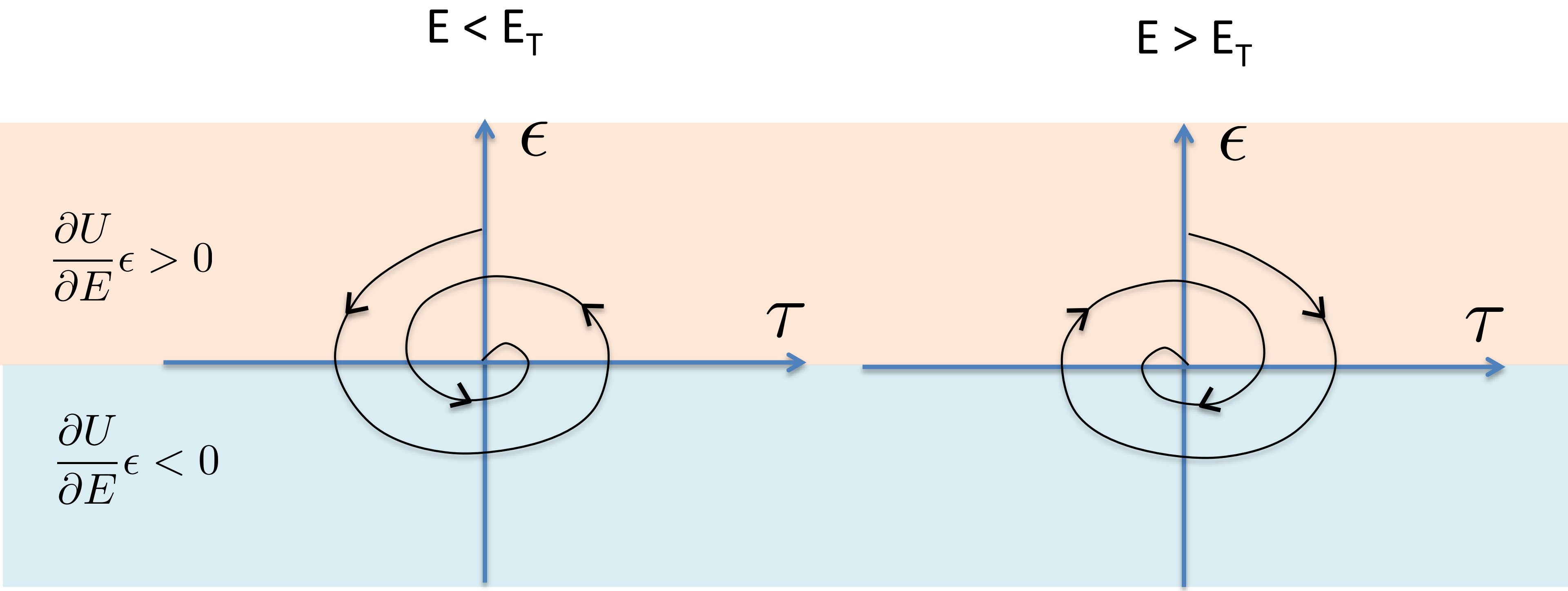
No Energy Loss





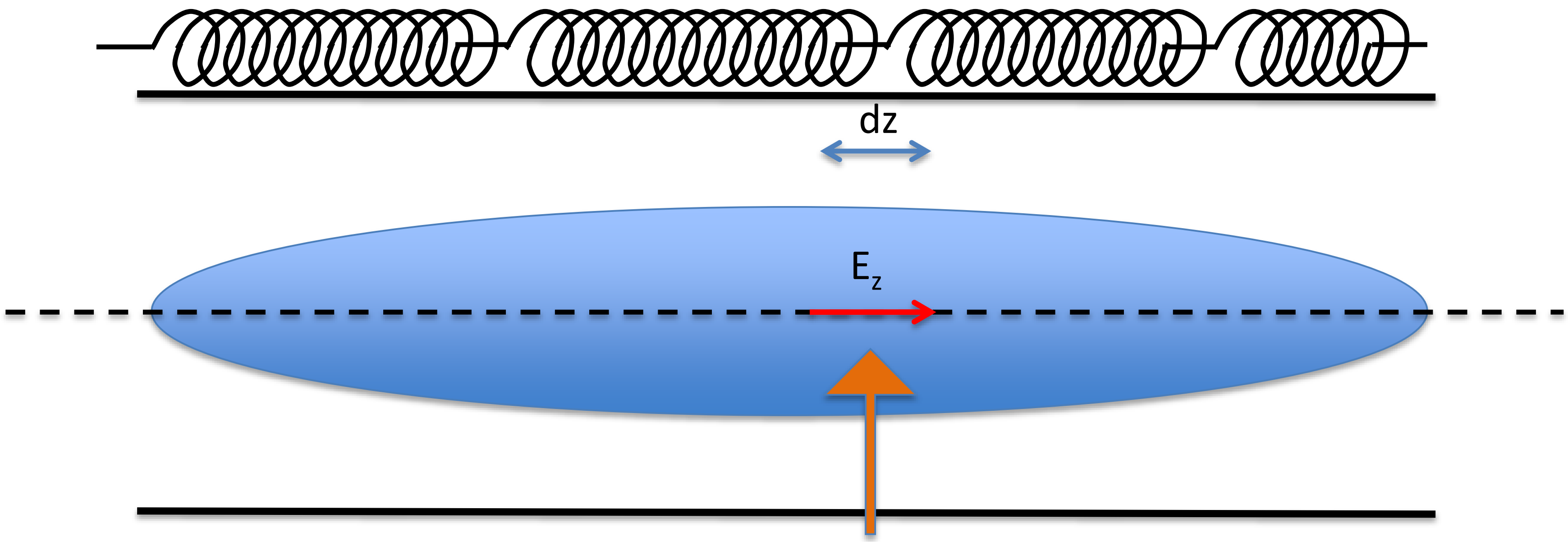
Interpretation

With Energy Loss



Bunch Lengthening

Bunch lengthening



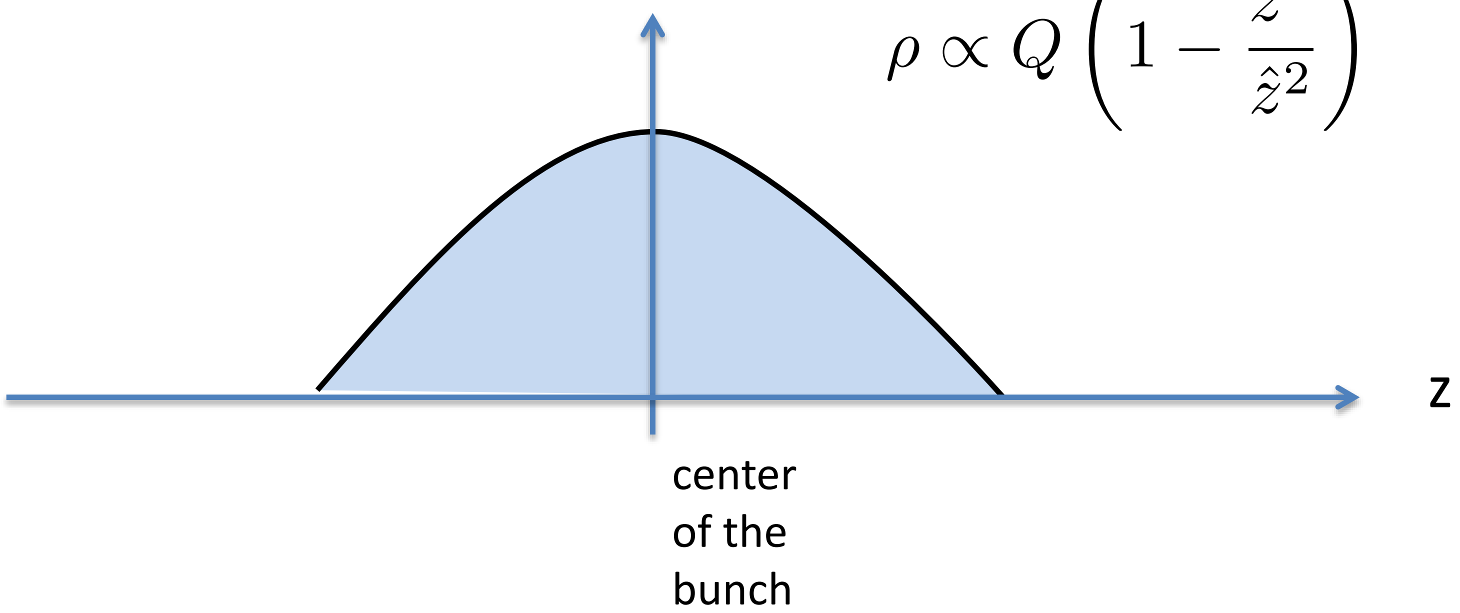
In one turn
change of energy
per charge

$$V = -L\frac{dI_b}{dt}$$

L is the integrated inductance

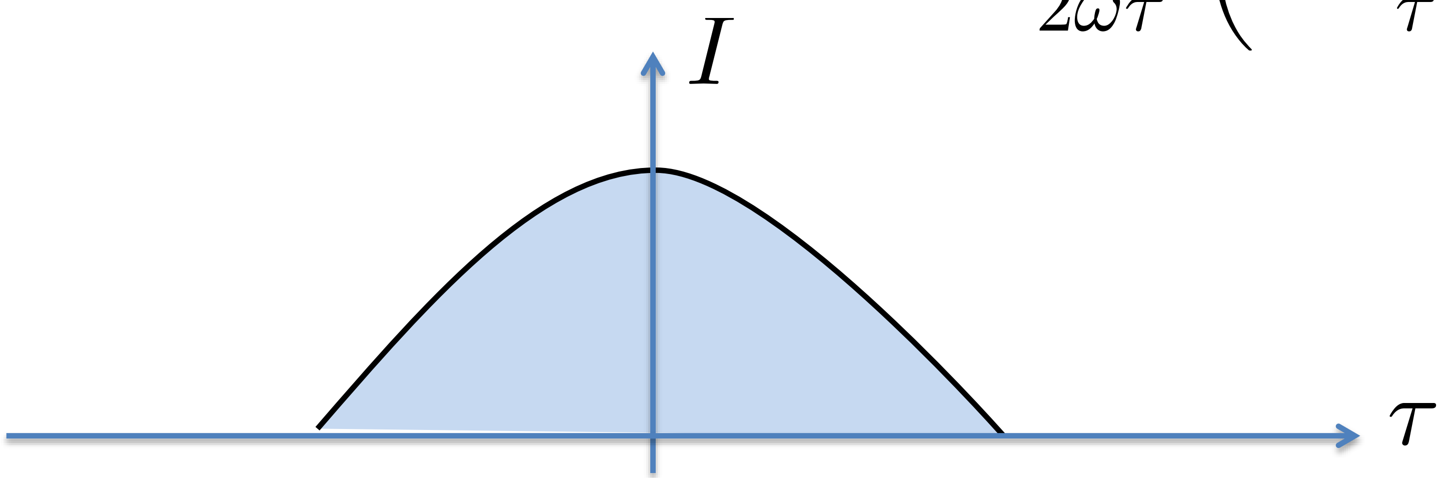
Parabolic bunch

$$\rho \propto Q \left(1 - \frac{z^2}{\hat{z}^2}\right)$$



Parabolic bunch

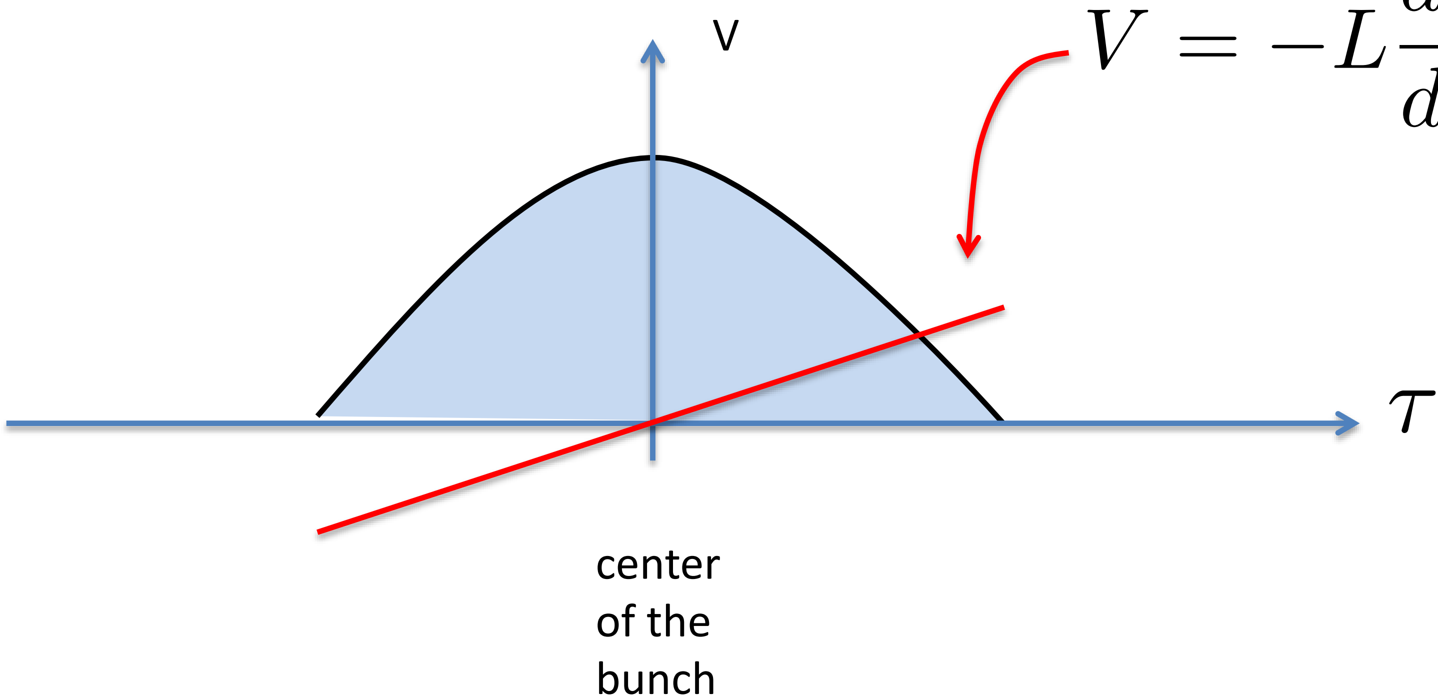
$$I = \frac{3\pi I_0}{2\omega\hat{\tau}}\left(1 - \frac{\tau^2}{\hat{\tau}^2}\right)$$



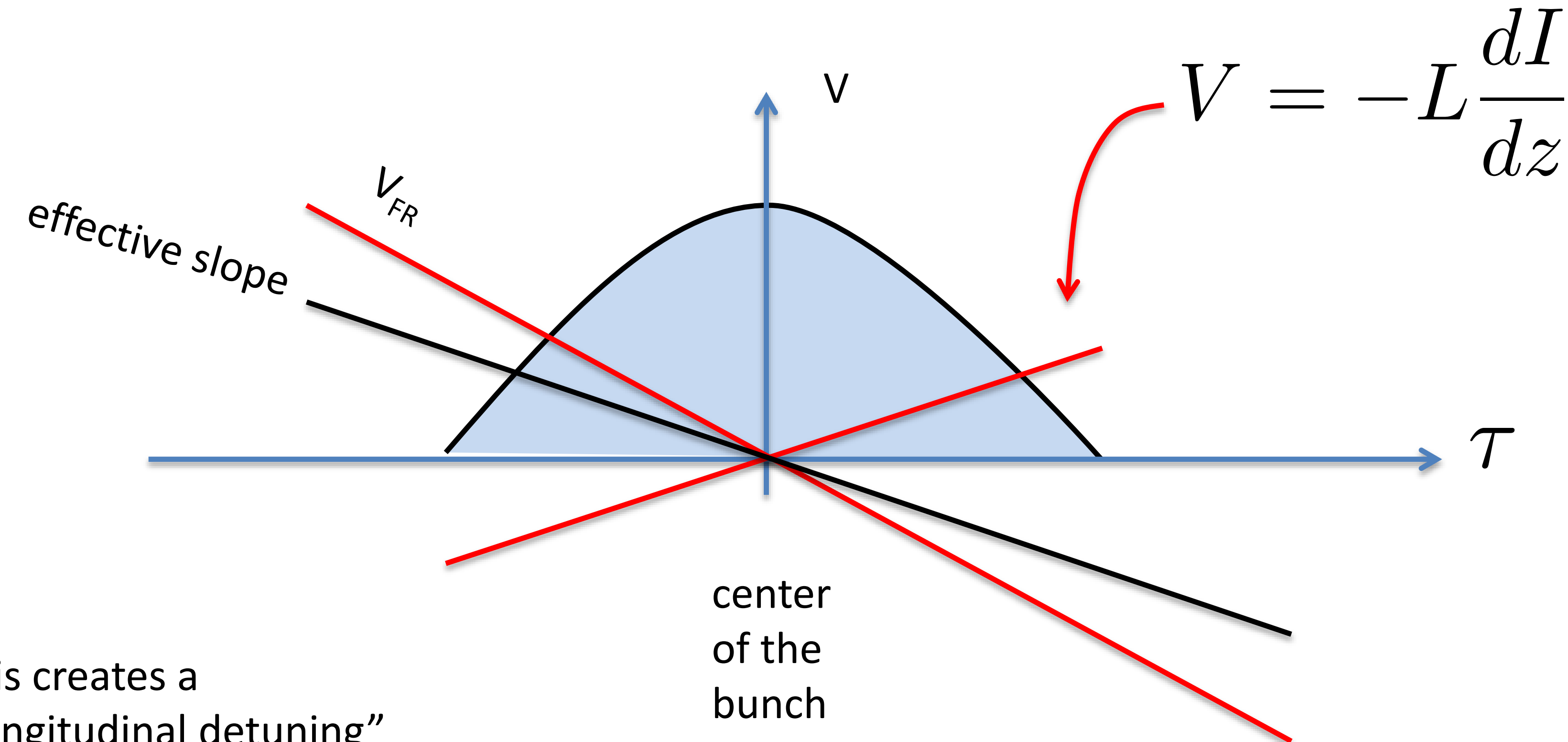
center
of the
bunch

Voltage induced

$$V = -L\frac{dI}{dz}$$



If we compare with RF



This creates a
"longitudinal detuning"

By using a bunch with the same longitudinal emittance a reduction of longitudinal focusing strength produces a bunch lengthening

The bunch becomes matched with the effective voltage slope

Effective voltage

$$V = \hat{V}\sin(\phi_s + h\omega_0\tau) + \frac{3\pi I_0 L}{\omega_0\hat{\tau}^3}\tau$$

induced voltage

Linearizing in tau

$$V = \hat{V}\sin(\phi_s) + \boxed{\hat{V}\cos(\phi_s)h\omega_0\tau} + \boxed{\frac{3\pi I_0 L}{\omega_0\hat{\tau}^3}\tau}$$

focusing from RF

defocusing from impedance

$$\dot{\epsilon} = \frac{e\hat{V}\omega_0}{2\pi E_0}\cos(\phi_s)h\omega_0\tau + e\frac{\omega_0}{2\pi E_0}\frac{3\pi I_0 L}{\omega_0\hat{\tau}^3}\tau$$

But $\dot{\tau} = \eta\epsilon$ therefore

$$\ddot{\tau} = \frac{\eta e\hat{V}\omega_0}{2\pi E_0}\cos(\phi_s)h\omega_0\tau + e\frac{\eta\omega_0}{2\pi E_0}\frac{3\pi I_0 L}{\omega_0\hat{\tau}}\tau$$

but $$\left|\frac{Z}{n}\right|_0 = L\omega_0$$

Therefore

$$\ddot{\tau} = \frac{\eta e h \hat{V}\omega_0^2}{2\pi E_0}\cos(\phi_s)\left[1 + \frac{1}{\hat{V}\cos(\phi_s)}\frac{3\pi I_0}{h\omega_0^3\hat{\tau}^3}\left|\frac{Z}{n}\right|_0\right]\tau$$

$$\omega_{s0}^2 = -\frac{\eta e h \hat{V}\omega_0^2}{2\pi E_0}\cos(\phi_s)$$

is the longitudinal strength in absence of impedance

$$\omega_s^2 = \omega_{s0}^2 \left[1 + \frac{1}{\hat{V}\cos(\phi_s)} \frac{3\pi I_0}{h\omega_0^3 \hat{\tau}^3} \left| \frac{Z}{n} \right|_0 \right]$$

Therefore the relative change in omega is

$$\frac{\Delta\omega_s}{\omega_{s0}} = \frac{1}{2} \frac{1}{\hat{V}\cos(\phi_s)} \frac{3\pi I_0}{h\omega_0^3 \hat{\tau}^3} \left| \frac{Z}{n} \right|_0$$

For protons $\hat{\tau}\hat{\epsilon} = \text{constant}$

$$\frac{\Delta\hat{\tau}}{\tau} \simeq -\frac{\Delta\omega_s}{2\omega_s}$$

Observation

The effect of the impedance is local, hence the voltage induced by impendence does not effect the center of mass (like for the space charge)

Summary

1) Wall charges creates detuning → incoherent tunes
2) Ferromagnetic material creates image currents:
   Coherent motion → coherent tunes
3) Concept of Wake field
4) Impedance of a cavity, Wake ←→ impedance
5) Energy loss
6) Longitudinal dynamics, effect of energy loss
7) Bunch lengthening

References

Lectures of Albert Hofmann, CAS
Physics of collective beam instabilities in high energy accelerators - A.W. Chao, 1993
Theory and Design of Charged Particle Beams - M. Reiser, 1994
Particle Accelerator Physics - H. Wiedemann