Collective Effect II

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Disclaimer: not all in this handouts will be presented

Type of fields

- Direct self fields
- Image self fields
- Wake fields

Collective Effects: Space Charge

Collective Effects:
Robinson Instability

It is an instability arising from the coupling of the impedance and longitudinal motion.

- The revolution frequency controls the impedance.
- Real part controls the energy loss in a bunch.
- Change particle energy.
- Change revolution frequency.
- Slower particle (if below transition).
The coupling of two effects

via the longitudinal dynamics

Energy loss due to impedance

Change of revolution frequency

because of the impedance $Z(\omega)$

Below transition

\[ \frac{\delta \omega}{\omega} = -\eta \frac{\delta E}{E} \]

$\eta < 0$
Below transition

\[ W_b = \int_0^{T_b} I(t)V(t)dt \]

Where \( V(t) \) is given by the impedance \( Z_r(\omega) \)
Below transition

energy lost per particle for non oscillating bunch

\[ U = \frac{2\pi}{I_0} \sum_{1}^{\infty} I_r^2 Z_r(p\omega) \]

In one turn energy is lost but compensated by the RF
Below transition

Energy lost $\rightarrow$ decrease $\omega$ $\rightarrow$ increase $Z_r$ $\rightarrow$ increase energy loss !!!
Below transition

$Z_r(\omega)$

$\omega_r$

Stable
Below transition

$Z_r(\omega)$

Stable

Unstable

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Below transition

Summary of the reasoning

Unstable
More complicated

No Energy Loss: RF give the same energy lost by the impedance

\[ E < E_T \]

\[ \epsilon \]

\[ \tau \]

\[ Z_T(\omega) \]

\[ \Delta \omega \]

\[ \epsilon \]

\[ \tau \]

Remember that energy lost is \( V^* I \)
Source of difficulty

\[ E < E_T \]

\[ \Delta \omega > 0 \quad \Delta \omega < 0 \]

Impedance effect → Energy gain

\[ I_K(t) = \sum_{p=-\infty}^{\infty} I_p e^{ip\omega_0 t} \]

\[ I_p(\omega) \]

\[ \omega_0 \]

Impedance effect

Energy gain

Energy lost
Still we neglect something

$E < E_T$

position of the bunch at turn "k"

$\tau_k = \hat{\tau} \cos(2\pi Q_s k)$

$Q_s$ is the synchrotron tune

$\tau_k = \hat{\tau} \cos(\omega_s t)$

$t \rightarrow t + \hat{\tau} \cos(\omega_s t)$

$I_k(t) = \sum_{p=-\infty}^{\infty} I_p e^{j p \omega_0} [t + \hat{\tau} \cos(\omega_s t)]$
**Current**

\[ I_k(t) \simeq \sum_{\omega > 0} I_p \left[ \cos(p\omega_0 t) + \frac{p\omega_0}{2} \sin((p + Q_s)\omega_0 t) + \frac{p\omega_0}{2} \sin((p - Q_s)\omega_0 t) \right] \]

The bunch current can be described by 3 components with frequency very close.

**Voltage created by the resistive impedance**

**Main component**

\[ V = 2 \sum_{\omega > 0} I_p Z_r(p\omega_0) \cos(p\omega_0 t) \]

**1st sideband**

\[ V = \sum_{\omega > 0} I_p p\omega_0 Z_r(\omega_p^+ \hat{\tau}) \sin(\omega_p^+ t) \]

**2nd sideband**

\[ V = \sum_{\omega > 0} I_p p\omega_0 Z_r(\omega_p^- \hat{\tau}) \sin(\omega_p^- t) \]
Prosthaphaeresis formulae

\[
\sin(\omega_p^+ t) = \sin(p\omega_0 t)\cos(\omega_s t) - \cos(p\omega_0 t)\sin(\omega_s t)
\]
\[
\sin(\omega_p^- t) = \sin(p\omega_0 t)\cos(\omega_s t) + \cos(p\omega_0 t)\sin(\omega_s t)
\]

But \( \tau = \hat{\tau}\cos(\omega_s t) \)

\[
\begin{align*}
\cos(\omega_s t) &= \frac{\tau}{\hat{\tau}} \\
\sin(\omega_s t) &= -\frac{\dot{\tau}}{\hat{\tau}\omega_s}
\end{align*}
\]

\[
\sin(\omega_p^+ t) = \sin(p\omega_0 t)\frac{\tau}{\hat{\tau}} - \cos(p\omega_0 t)\frac{\dot{\tau}}{\hat{\tau}\omega_s}
\]
\[
\sin(\omega_p^- t) = \sin(p\omega_0 t)\frac{\tau}{\hat{\tau}} + \cos(p\omega_0 t)\frac{\dot{\tau}}{\hat{\tau}\omega_s}
\]

Voltage created by the resistive impedance

\[
V = 2\sum_{\omega > 0}^{\infty} I_p Z_r(\omega p_0) \cos(p\omega_0 t)
\]

Main component

1st sideband

\[
V = \sum_{\omega > 0}^{\infty} I_p p\omega_0 Z_r(\omega_p^+) [\sin(p\omega_0 t)\tau - \cos(p\omega_0 t)\frac{\dot{\tau}}{\omega_s}]
\]

2nd sideband

\[
V = \sum_{\omega > 0}^{\infty} I_p p\omega_0 Z_r(\omega_p^-) [\sin(p\omega_0 t)\tau + \cos(p\omega_0 t)\frac{\dot{\tau}}{\omega_s}]
\]

Therefore the induced Voltage depends on \( \tau, \dot{\tau} \)
Energy lost in one turn

\[ E_l = \int_0^{T_0} V(t)I(t)\,dt \]

energy lost per particle per turn

\[ U = \frac{2e}{I_0} \left[ \frac{I_p^2 Z_r(p\omega_0)}{2} - \frac{I_p^2 p\omega_0}{2} (Z_r(\omega_p^+)) - Z_r(\omega_p^-) \frac{1}{\omega} \right] \]

this term can give rise to a constant loss, or a constant gain of energy

In terms of the energy of a particle

\[ U = \frac{2e}{I_0} \left[ \frac{I_p^2 Z_r(p\omega_0)}{2} - \frac{I_p^2 p\omega_0}{2} (Z_r(\omega_p^+)) - Z_r(\omega_p^-) \frac{1}{\omega} \right] \]

\[ \frac{\partial U}{\partial \epsilon} = -\frac{e}{I_0} \sum_{\omega > 0} \frac{I_p^2 p\omega_0}{2} (Z_r(\omega_p^+) - Z_r(\omega_p^-)) \frac{\eta}{\omega} \]

This is a slope in the energy, and the sign of the slope depends on

\[ Z_r(\omega_p^+) - Z_r(\omega_p^-) \]

and \( \eta \)
The longitudinal motion now!

\[ \ddot{\tau} + 2\alpha_s \dot{\tau} + \omega_s^2 \tau = 0 \]

\[ \alpha_s = \frac{1}{2} \frac{\omega_0}{2\pi} \frac{\partial U}{\partial E} = \frac{\omega_0}{4\pi E} \frac{\partial U}{\partial \epsilon} = \frac{\omega_s}{2I_0 h} \frac{\sum pI_p^2 (Z_R(\omega_p^+) - Z_R(\omega_p^-))}{\cos \phi_s} \]

Robinson Instability

If \( \alpha_s > 0 \) there is a damping
If \( \alpha_s < 0 \) there is an instability

below transition
above transition

Robinson Instability

stable
unstable
stable
unstable
Longitudinal space charge and resistive wall impedance

\[ \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0} \]

\[ \oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int \vec{B} d\vec{a} \]

\[ \oint \vec{E} \cdot d\vec{l} = \int E_r(z) dr + E_w \Delta z - \int E_r(z + \Delta z) dr - E_z \Delta z \]
For a KV beam

Electric Field

\[
E_r = \begin{cases} 
\frac{\lambda(z)}{2\varepsilon_0}r & \text{if } r < r_0 \\
\frac{\lambda(z)r_0^2}{2\varepsilon_0} \left( \frac{1}{r} \right) & \text{if } r > r_0
\end{cases}
\]

\[
\int_0^{r_w} E_r(z)dr = \int_0^{r_0} \frac{\lambda(z)}{2\varepsilon_0} rdr + \int_{r_0}^{r_w} \frac{\lambda(z)r_0^2}{2\varepsilon_0} \left( \frac{1}{r} \right) dr
\]

\[
\int_0^{r_w} E_r(z)dr = \frac{\lambda(z)r_0^2}{4\varepsilon_0} \left[ 1 + 2 \ln \left( \frac{r_w}{r_0} \right) \right]
\]

Therefore

\[
\int E_r(z)dr - \int E_r(z + \Delta z)dr = -\frac{r_0^2}{4\varepsilon_0} \left[ 1 + 2 \ln \left( \frac{r_w}{r_0} \right) \right] \frac{\partial\lambda(z)}{\partial z} \Delta z
\]

\[
\oint \vec{E} \cdot d\vec{l} = (E_w - E_z)\Delta z - \frac{r_0^2}{4\varepsilon_0} \left[ 1 + 2 \ln \left( \frac{r_w}{r_0} \right) \right] \frac{\partial\lambda(z)}{\partial z} \Delta z
\]
Magnetic Field

\[
B_\perp = \begin{cases} 
\frac{\mu_0 v_z \lambda(z)}{2} r & \text{if } r < r_0 \\
\frac{\mu_0 v_z \lambda(z) r_0^2}{2} \frac{1}{r} & \text{if } r > r_0 
\end{cases}
\]

\[
\int B_\perp da = \int_0^{r_0} dr \int_z^{z+\Delta z} \frac{\mu_0 v_z \lambda(z)}{2} r + \int_{r_0}^{r_w} dr \int_z^{z+\Delta z} \frac{\mu_0 v_z \lambda(z) r_0^2}{2} \frac{1}{r}
\]

\[
\int B_\perp da = \frac{\mu_0 v_z r_0^2 \lambda \Delta z}{4} \left[ 1 + 2 \ln \left( \frac{r_w}{r_0} \right) \right]
\]

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Maxwell-Faraday Law

\[
\int \mathbf{E} \cdot d\mathbf{r} = -\frac{\partial}{\partial t} \int \mathbf{B} da
\]

\[
(E_w - E_z) \Delta z - \frac{r_0^2}{4\epsilon_0} \left[ 1 + 2 \ln \left( \frac{r_w}{r_0} \right) \right] \frac{\partial \lambda(z)}{\partial z} \Delta z = \frac{\mu_0 v_z r_0^2 \Delta z}{4} \left[ 1 + 2 \ln \left( \frac{r_w}{r_0} \right) \right] \frac{\partial \lambda}{\partial t}
\]

from the equation of continuity \( \frac{\partial \lambda}{\partial t} + v_z \frac{\partial \lambda}{\partial z} = 0 \)

\[
E_z = E_w - \frac{r_0^2}{4\epsilon_0} \left[ 1 + 2 \ln \left( \frac{r_w}{r_0} \right) \right] \frac{1}{\gamma^2} \frac{\partial \lambda}{\partial z}
\]

again we find the factor \( 1/\gamma^2 \) !
**Space charge impedance**

\[ \lambda(\theta, t) = \sum_n \lambda_n e^{i(n\theta - \omega_n t)} \quad \theta = \frac{2\pi z}{L} \quad \omega_n = n\omega_0 \]

Local density

\[
V_{z0} = 2\pi RE_{zw} - i \sum_n \frac{I_n}{4\pi \epsilon_0} \frac{2\pi n}{\beta c \gamma^2} \left[ 1 + 2 \ln \left( \frac{r_w}{r_0} \right) \right] e^{i(n\theta - \omega_n t)}
\]

Perfect vacuum chamber \( E_{zw} = 0 \)

\[ I = I_n e^{i(n\theta - \omega_n t)} \quad V = -i \frac{I_n}{4\pi \epsilon_0} \frac{2\pi n}{\beta c \gamma^2} \left[ 1 + 2 \ln \left( \frac{r_w}{r_0} \right) \right] e^{i(n\theta - \omega_n t)} \]

\[ Z_{||sc} = \frac{\hat{V}}{\hat{I}} \quad Z_{||sc} = -i \frac{1}{\epsilon_0 c} \frac{n}{2\beta \gamma^2} \left[ 1 + 2 \ln \left( \frac{r_w}{r_0} \right) \right] \]
Resistive Wall impedance

Do not take into account B

$E_w = E_z$

Beam on axis

Wall currents are related to the electric field by Ohm’s law

$E_w = \sigma^{-1} J_w$

The thickness of the wall currents is called skin depth

$\delta_w = \sqrt{\frac{2}{\mu_0 \sigma_w \omega}}$
Impedance of the surface (pipe)

\[ Z_{\text{surf}} = \frac{1 + i}{\sigma \delta} \]

Longitudinal impedance (beam)

\[ Z_{||} = \frac{2\pi R}{2\pi r_p} \frac{1 + i}{\sigma \delta} \]

Transverse impedance
Origin

Beam passing through a cavity on axis

Beam passing through a cavity off-axis
But the field transform it-self!
Effect on the dynamics

The dynamics is much more affected by $B$, than $E$ because

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

this speed is high

The beam creates its own dipolar magnetic field!

(dipolar errors create integer resonances…. we expect the same...)
**Transverse impedance**

**Definition of longitudinal impedance (classical)**

\[ I = \hat{I}e^{i\omega t} \quad \rightarrow \quad \text{System} \quad \rightarrow \quad V = \hat{V}e^{i\omega t} \]

Impedance

\[ Z(\omega) = \frac{\hat{V}}{\hat{I}} \]

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**For a displaced beam**

Source of the effect

\[ Ix_0 \]

Effect

\[ \vec{E} + \vec{v} \times \vec{B} \]

This field acts on a single particle

It means that in the equation of motion we have to add this effect

\[ \frac{d^2 x}{ds^2} + k_x x = \frac{q}{m\gamma v_0^2} \left[ E_x + (\vec{v} \times \vec{E})_x \right] \]
In the time domain

\[
\frac{d^2 x}{dt^2} + \left( \frac{Q_x}{R} \right)^2 x = \frac{q}{m \gamma v_0^2} \frac{1}{2 \pi R} \int_0^{2 \pi R} \left[ E_x + (\vec{v} \times \vec{E})_x \right] ds
\]

Therefore for a weak effect or distributed we find

\[
\frac{d^2 x}{ds^2} + \left( \frac{Q_x}{R} \right)^2 x = \frac{q}{m v_0^2} \frac{1}{2 \pi R} \int_0^{2 \pi R} \left[ E_x + (\vec{v} \times \vec{E})_x \right] ds
\]

But \( \int_0^{2 \pi R} \left[ E_x + (\vec{v} \times \vec{E})_x \right] ds \) is like a “strange” voltage

\[
V = - \int_0^{2 \pi R} \left( \vec{E} + \vec{v} \times \vec{B} \right)_\perp ds
\]

Now the situation is the following:

\[I x_0 \quad \rightarrow \quad \text{System} \quad \rightarrow \quad V = - \int_0^{2 \pi R} \left[ \vec{E} + \vec{v} \times \vec{B} \right]_\perp ds\]
Transverse beam coupling impedance

\[ Z_\perp(\omega) = i^2 \int_0^{2\pi R} \left[ \mathbf{E} + \vec{v} \times \mathbf{B} \right]_\perp ds \]

now the question is what is \( \omega \)?

What is it \( \omega \)?

It is given by the fractional tune, as this is the frequency seen in a cavity

Example: \( Q = 2.23 \) fractional tune \( q = 0.23 \)
B-field induced by beam displacement

From \( \frac{\partial E_z}{\partial x} = k I x_0 \) \( \Rightarrow \) \( E_z = k I x_0 x \)

electric field at the position of beam \( x_0 \) is
\[
E_z(x_0) = k I x_0^2
\]

Longitudinal impedance
\[
Z_\parallel = -\frac{E_z(x_0) l}{I} = -k x_0^2 l
\]

The magnetic field comes from Maxwell
\[
\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0}
\]

\[
\frac{\partial B_y}{\partial t} \bigg|_{x_0} = k I x_0
\]

taking \( I x_0 = I \hat{x} e^{i \omega t} \)
\[
B_y = \frac{k I \hat{x} e^{i \omega t}}{i \omega} = \frac{k I x_0}{i \omega}
\]

Transverse impedance
\[
Z_\perp = i \int_0^l [\vec{v} \times \vec{B}] || ds \quad \text{or} \quad Z_\perp = \frac{v_z k l}{\omega}
\]
\[
Z_\perp = \frac{v_z}{2 \omega} \frac{d^2 Z_\parallel(\omega)}{dx^2}
\]
Therefore the field on the beam is

\[ E_x = \frac{1}{2\pi \varepsilon_0} \frac{\lambda}{r_p^2} x_b \]

(for small \( x_b / r_p \))
\[
\frac{\partial E}{\partial x} \propto I x_0
\]

Transverse resistive Wall impedance

\[
Z(\omega_n)_\perp = \frac{2R}{r_p^2} \frac{Z_{||}(\omega_n)}{n} \bigg|_{res}
\]
Transverse instability

Coasting beam instability

Force due to the impedance (in the complex notation)

\[ F_\perp = i \frac{q Z \perp I_0}{2\pi R} x_b \]

Equation of motion of one particle for a beam on axis

\[ \ddot{x} + Q^2 \omega_0^2 x = 0 \]

Equation of motion of a beam particle when the beam is off-axis

\[ \ddot{x} + Q^2 \omega_0^2 x = -i \frac{q Z \perp I_0}{2\pi R m \gamma} x_b \]
Collective motion

On the other hand the beam center is

\[ x_b = \int x \, n(x, y, s) \, dx \, dy \]

with \( \int \tilde{n} \, dV = 1 \)

therefore

\[ \int \tilde{x} \tilde{n} \, dV + \int Q^2 \omega_0^2 x \tilde{n} \, dV = -i \frac{qZ \cdot I_0}{2\pi Rm} x_b \]

If all particles have the same frequency, i.e. each particle experience a force \( Q^2 \omega^2 x \)

then

\[ \ddot{x}_b + Q^2 \omega_0^2 x_b = -i \frac{qZ \cdot I_0}{2\pi Rm} x_b \]

We can define a coherent “detuning” because this is a linear equation

\[ Q^2 \omega_0^2 + i \frac{qZ \cdot I_0}{2\pi Rm} = (Q + \Delta Q^c)^2 \omega_0^2 \]

\[ \Delta Q^c = i \frac{1}{2Q^2 \omega_0^2} \frac{qZ \cdot I_0}{2\pi Rm} \]
\[ \ddot{x}_b + Q^2 \omega_0^2 x_b = -2Q \omega^2 \Delta Q^C x_b \]

that is

\[ \ddot{x}_b + (Q^2 \omega_0^2 + 2Q \omega_0^2 \Delta Q^C) x_b = 0 \]

But now \( \Delta Q^C \) is a complex number !!

Solution \( x_b = A \exp[-\omega_0 I_m(\Delta Q^C)t + i\omega_0[Q + Re(\Delta Q^C)]t] \)

\[ \tau_I^{-1} = \omega_0 Im(\Delta Q^C) \] is the growth rate of the transverse resistive wall instability

\[ \frac{1}{\tau} = \frac{qRe\{Z_\perp\} I_0}{4\pi Rm\gamma Q\omega_0} \]

This instability always take place \[ \rightarrow \] Landau damping

Instability suppression
An important assumption

We assumed that all particles have the same frequency so that

\[ \int Q^2 \omega_0^2 x \tilde{n} dV = \int Q^2 \omega_0^2 x \tilde{n} dV = Q^2 \omega_0^2 x_b \]

This assumption means that each particle of the beam respond in the same way to a change of particle amplitude

Coherent motion drive particle motion, which is again coherent

Chromaticity ?

What happened if the incoherent force created by the accelerator do not allow a coherent build up

Momentum spread
If each particle of the beam has different $dp/p$ then the force that the lattice exert on a particle depends on the particle!

$$F_x = \left(Q_0 + \frac{\xi \delta p}{p}\right)^2 \omega^2 x$$

**Incoherent motion damps $x_b$**

Equation of motion without impedances

$$\ddot{x} + \left(Q_0 + \frac{\xi \delta p}{p}\right)^2 \omega^2 x = 0$$

Motion of center of mass as an effect of the spread of the frequencies of oscillation

The momentum compaction also provides a spread of the betatron oscillations
Example:
N. particles = 5
dq/q = 0

Example:
N. particles = 5
dq/q = 5E-3
Example:
N. particles = 5
dq/q = 1E-2

Example:
N. particles = 5
dq/q = 2.5E-2

damping of x₀
But incoherent motion reduces $x_b$

Example: these are 5 sinusoid with amplitude linearly growth.

Example: now a spread dq/q of 1E-2 is added to the 5 curves.

The center of mass growth slower.
Example: now a spread dq/q of 1E-2 is added to the 5 curves

the spread of the particles damps the oscillations of the center of mass → the instability cannot develop

Situation

Coherent effect

Incoherent effect

Growth rate

Damping rate

\[ \tau_I \]

\[ \tau_D \]

The faster wins
instability of a single bunch

Example

beam position at the cavity

No oscillations $\Rightarrow$

$\omega = 0$

$\omega_{p}^{\pm} = (p \pm q)\omega_{0}$
behavior of the field in the cavity

\[ T_r = \text{time of oscillation of the field in the cavity} \]

Cavity tuned upper sideband
As for the Robinson Instability

\[ \alpha_s = \frac{1}{\tau} \propto \sum_p I_p^2 [Z_\perp(\omega_p^+) - Z_\perp(\omega_p^-)] \]
Negative mass instability

Above transition

Example with 2 particles!

Reference frame of synchronous particle

Gain speed resolution time longer

Loss speed resolution time shorter
Above transition

Example with 2 particles!

repulsive forces attract particles as if their mass were negative

reference frame of synchronous particle
Summary

Robinson instability
Longitudinal space charge and resistive wall impedance
Transverse impedance
Transverse instability
Landau damping
Single bunch instability
Negative mass instability