



Limitations on Dispersion Relations for Generalized Parton Distributions

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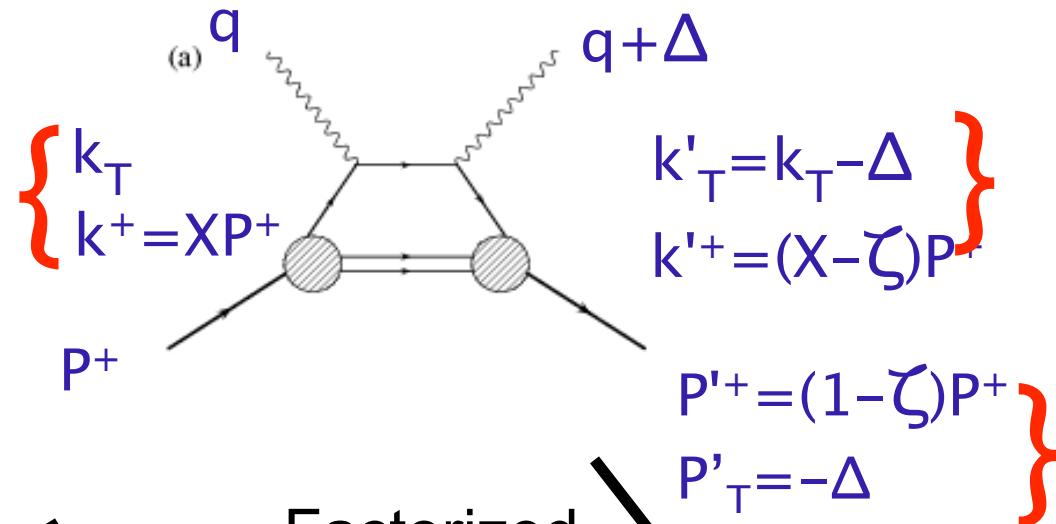
Outline of Discussion

- Exclusive lepton production of γ or mesons & GPDs
 - Factorized form of amplitudes
 - 8 helicity decomposed GPDs 4 chiral even + 4 chiral odd
 - Relation of GPDs to “Compton Form factors” & measurable reactions
 - Constraints on GPDs
 - Different models: Regge poles, Scalar diquark model (see, e.g. Brodsky-Llanes-Estrada), double distributions (Radyushkin, VGG, . . .)
- Dispersion Relations, DVCS & GPDs
 - Unitarity & analyticity
 - Threshold limits
 - Kinematic restrictions
 - Model calculations
- Conclusions

See recent: GRG, Liuti, ArXiv hep-ph/0904.3071
& Spin 2008 proceedings



DVCS $\gamma^* + P \rightarrow (\gamma \text{ or meson}) + P'$
partonic picture



$\zeta \rightarrow 0$
Regge

Factorized
“handbag”
picture

Quark-spectator
quark+diquark

Defining GPDs via quark or gluon correlation on light cone

$$\begin{aligned}
 F^q &= \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ixP^+z^-} \langle p' | \bar{q}(-\frac{1}{2}z) \gamma^+ q(\frac{1}{2}z) | p \rangle \Big|_{z^+=0, \mathbf{z}=0} \\
 &= \frac{1}{2P^+} \left[H^q(x, \xi, t) \bar{u}(p') \gamma^+ u(p) + E^q(x, \xi, t) \bar{u}(p') \frac{i\sigma^{+\alpha} \Delta_\alpha}{2m} u(p) \right] \\
 \tilde{F}^q &= \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ixP^+z^-} \langle p' | \bar{q}(-\frac{1}{2}z) \gamma^+ \gamma_5 q(\frac{1}{2}z) | p \rangle \Big|_{z^+=0, \mathbf{z}=0} \\
 &= \frac{1}{2P^+} \left[\tilde{H}^q(x, \xi, t) \bar{u}(p') \gamma^+ \gamma_5 u(p) + \tilde{E}^q(x, \xi, t) \bar{u}(p') \frac{\gamma_5 \Delta^+}{2m} u(p) \right]
 \end{aligned}$$

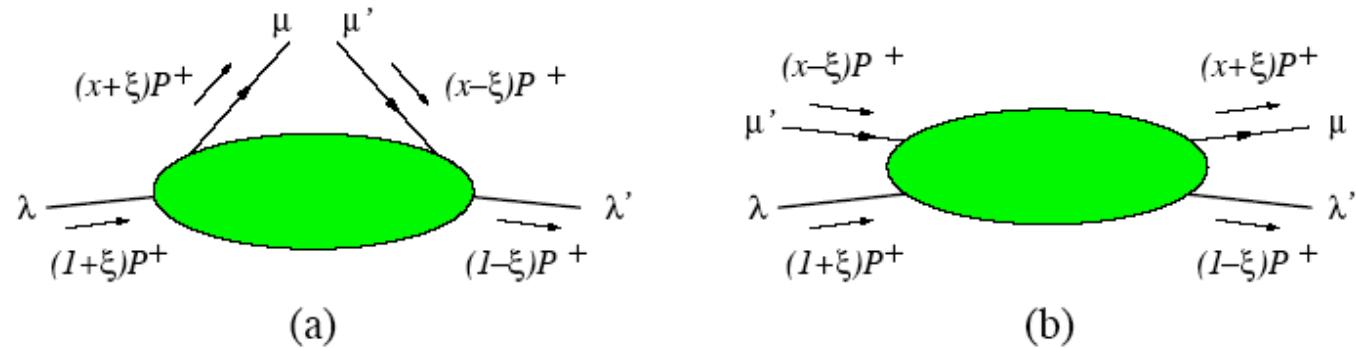
These are chiral even – no quark helicity flip
 4 other chiral odd GPDs
 Collinear factorization applied (k_T quark momenta integrated over)



Chiral odd GPDs

$$\begin{aligned}
 & \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ixP^+z^-} \langle p', \lambda' | \bar{\psi}(-\frac{1}{2}z) i\sigma^{+i} \psi(\frac{1}{2}z) | p, \lambda \rangle \Big|_{z^+=0, z_T=0} \\
 &= \frac{1}{2P^+} \bar{u}(p', \lambda') \left[H_T^q i\sigma^{+i} + \tilde{H}_T^q \frac{P^+ \Delta^i - \Delta^+ P^i}{m^2} \right. \\
 &\quad \left. + E_T^q \frac{\gamma^+ \Delta^i - \Delta^+ \gamma^i}{2m} + \tilde{E}_T^q \frac{\gamma^+ P^i - P^+ \gamma^i}{m} \right] u(p, \lambda)
 \end{aligned}$$

Eqns connecting GPD & helicity amps - M. Diehl, Eur.Phys.J.C19 (2001) 485;
Boglione & Mulders, Phys.Rev.D 60 (1999) 054007.





Additional connections

$$A_{++,+-} = \epsilon \frac{\sqrt{t_0 - t}}{2m} \left(\tilde{H}_T^q + (1 - \xi) \frac{E_T^q + \tilde{E}_T^q}{2} \right),$$

$$A_{-+,- -} = \epsilon \frac{\sqrt{t_0 - t}}{2m} \left(\tilde{H}_T^q + (1 + \xi) \frac{E_T^q - \tilde{E}_T^q}{2} \right),$$

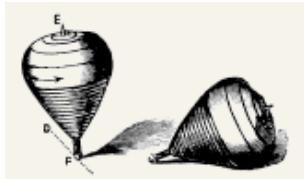
$$A_{++,- -} = \sqrt{1 - \xi^2} \left(H_T^q + \frac{t_0 - t}{4m^2} \tilde{H}_T^q - \frac{\xi^2}{1 - \xi^2} E_T^q + \frac{\xi}{1 - \xi^2} \tilde{E}_T^q \right)$$

$$A_{-+,-+} = -\sqrt{1 - \xi^2} \frac{t_0 - t}{4m^2} \tilde{H}_T^q$$

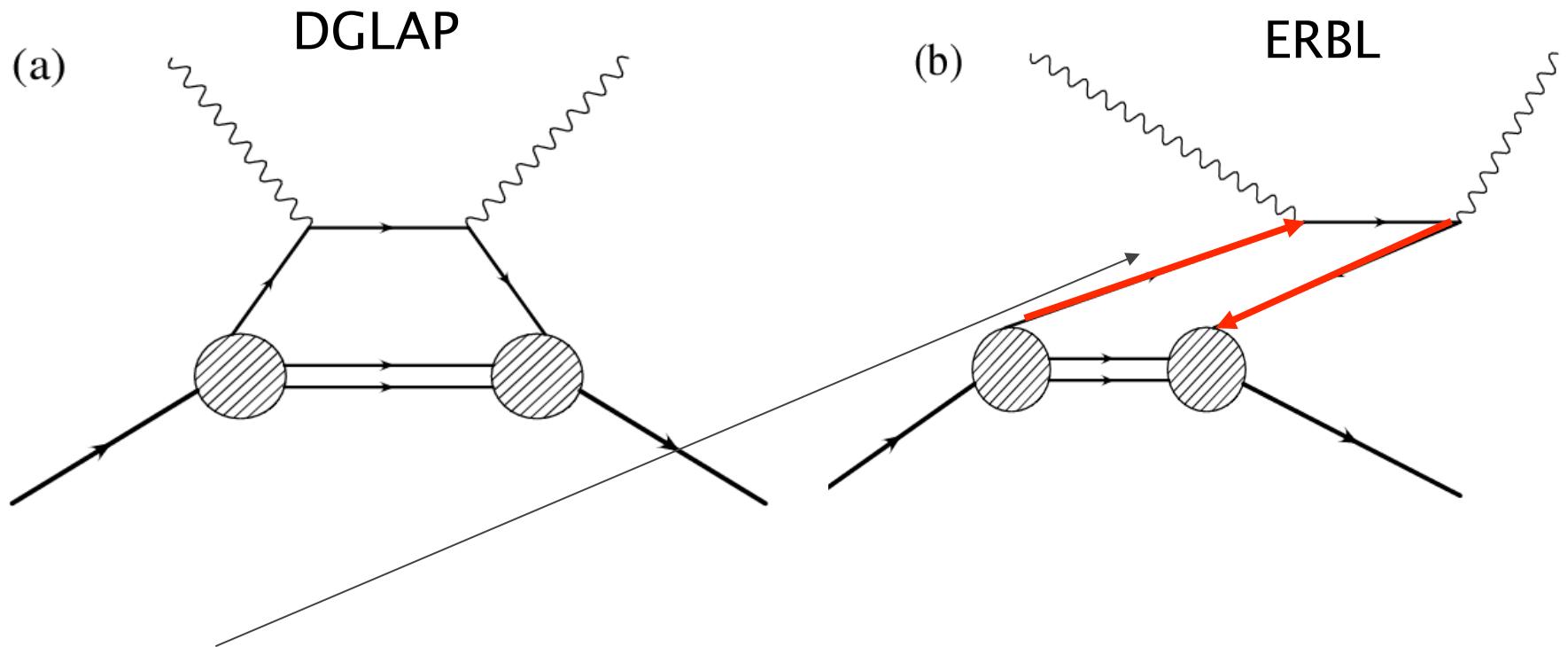
See M.Burkardt

$$\int_{\zeta=1}^1 dX [E(X, \zeta, t)]_{\zeta=t=0} = \kappa^q \quad \int_{\zeta=1}^1 dX \left[2\tilde{H}_T^q(X, \zeta, t) + E_T^q(X, \zeta, t) \right]_{\zeta=t=0} \simeq \kappa_T^q$$

$$\int d^2 k_T dX f_{1T}^{\perp q}(X, k_T) = \kappa^q \quad \int d^2 k_T dX h_1^{\perp q}(X, k_T) = \kappa_T^q \text{ times -1}$$



Two different time orderings/pole structure!



Quark anti-quark pair describes similar
physics

(dual to) Regge t-channel exchanges!!



How to determine GPDs? Constraints on GPDs

Constraints from Form Factors

$$\int_0^1 dx H(x, \xi, t) = F_1(t) \quad \text{Dirac}$$

$$\int_0^1 dxE(x, \xi, t) = F_2(t) \quad \text{Pauli}$$

Constraints from Polynomiality

Result of Lorentz invariance & causality.
Not necessarily built in to models

$$\int_{-1}^{+1} dx x^n H(x, \xi, t) = \sum_{k=0,2,\dots}^n A_{n,k}(t) \xi^k + \frac{1 - (-1)^n}{2} C_n(t) \xi^{n+1}$$

$$\int_{-1}^{+1} dx x^n E(x, \xi, t) = \sum_{k=0,2,\dots}^n B_{n,k}(t) \xi^k - \frac{1 - (-1)^n}{2} C_n(t) \xi^{n+1}$$

More constraints

Constraints from PDFs

$$H^q(x,0,0) = f_1^q(x), \quad \tilde{H}^q(x,0,0) = g_1^q(x), \quad H_T^q(x,0,0) = h_1^q(x)$$

Further Theoretical Constraints:

Sensible prediction for hadron shape at $x \sim 1$

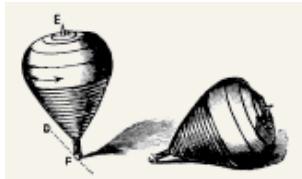
& $x \rightarrow 0$ power law

Sensible prediction for P_T' dependence $\rightarrow k_T$ dependence
(connection with TMDs)

see Ahmad, Liuti et al. (2008) & ongoing work

Dispersion Relations for DVCS amplitudes

could constrain integrated **GPDs, e.g.** $\mathcal{H}^{(+)}(v,t)$

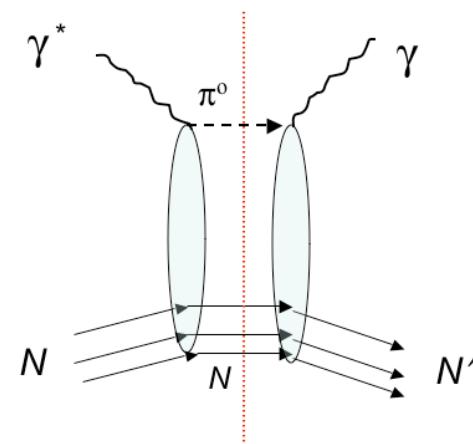
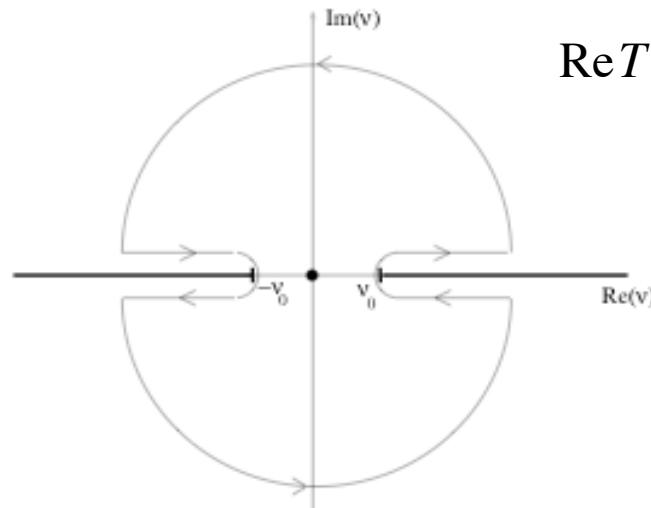


Dispersion relations: constraints on GPD modeling & experimental extraction

- Deeply Virtual Compton Scattering (DVCS) amplitudes satisfy unitarity, Lorentz covariance & analyticity
- (Generic) $T(v,t,Q^2)$ ($v=(s-u)/4M$) has v or s analytic structure determined by nucleon pole & hadronic intermediate on-shell states

$$d\sigma/dt \propto |T(v,t,Q^2)|^2$$

$$\text{Re } T(v, Q^2, t) = \frac{2v}{\pi} \int_{v_{Threshold}}^{\infty} dv' \frac{\text{Im } T(v', Q^2, t)}{v'^2 - v^2} \text{ for odd } T(v)$$



s-channel unitarity
hadron picture
Imaginary part for
on-shell
intermediate states

Dispersion relations for virtual Compton scattering & meson production

$$\text{Re}T^-(v, t, Q^2) = \frac{2v}{\pi} \int_{v_{threshold}}^{\infty} dv' \frac{\text{Im}T^-(v', t, Q^2)}{v'^2 - v^2}, \quad (1)$$

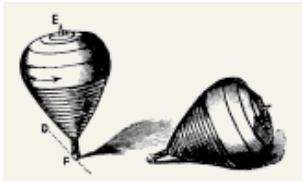
for an amplitude that is odd under the crossing symmetry $v \rightarrow -v$ (and thereby will avoid an extra subtraction for convergence) or

$$\text{Re}T^+(v, t, Q^2) = \frac{v^2}{\pi} \int_{v_{threshold}}^{\infty} dv' \frac{\text{Im}T^+(v', t, Q^2)}{v'^2(v'^2 - v^2)} + \Delta, \quad (2)$$

for an amplitude that is even under the crossing symmetry. The latter has a subtraction Δ to assure convergence in v' .

Even or odd **crossing** determines J even or odd t-channel exchanges
 At large v & small $|t|$ $T^+ \propto v_P^{\alpha_{Pomeron}(t)-1}$ with $\alpha_{Pomeron}(0)=1+\delta$ leading behavior
 and $T^- \propto v_R^{\alpha_{Regge}(t)-1}$ with $\alpha_{Regge}(0) \approx 1/2$
 T^+ integral needs one subtraction, T^- does not

Recall that in elastic forward scattering $\text{Im}T(v) \propto \sigma(v)$ Optical theorem



Hadron to parton variables:

$$s = (P+q)^2 = M^2 - Q^2 + 2Mv_{\text{Lab}}$$

$$v = (s-u)/4M = (2s+t+Q^2-2M^2)/4M$$

$$= v_{\text{Lab}} + (t - Q^2)/4M$$

$$x_{\text{BJ}} = Q^2/2Mv_{\text{Lab}}, \quad X = k \cdot P / q \cdot P, \quad \zeta = q \cdot P' / q \cdot P$$

$$\xi = \zeta / (2 - \zeta), \quad x = (2X - \zeta) / (2 - \zeta)$$

$$\xi = Q^2/4Mv \quad \text{Integration variable } x = Q^2/4Mv'$$

[Anikin & Teryaev; Ivanov & Diehl; Vanderhaeghen, et al.](#);

[Brodsky, Close & Gunion \(1970's\)](#) [Brodsky, Llanes-Estrada & Szczeplaniak; Mueller, et al.](#)

$$T^{\mu\nu}(v, Q^2, t) = \frac{1}{2} g^{\mu\nu} \bar{u}(p') \hat{n} u(p) \sum_{\text{flavors}} e_f^2 \mathcal{H}_f(\xi, t)$$

$$\mathcal{H}_f(\xi, t) = \int_{-1}^{+1} dx \frac{H_f(x, \xi, t)}{x - \xi + i\varepsilon}$$

Note difference
between DR &
Direct real part
with x & ξ in GPD

$$\text{Im } \mathcal{H}_f(\xi, t) = H_f(\xi, \xi, t)$$

$$\text{Re } \mathcal{H}_f(\xi, t) = \frac{1}{\pi} PV \int_{-1}^{+1} dx \frac{H_f(x, \xi, t)}{x - \xi}$$

$$\text{Dispersion Relation: } \text{Re } \mathcal{H}_f(\xi, t) = \frac{1}{\pi} PV \int_{-1}^{\xi^{(\text{Max})} ?} dx \frac{H_f(x, x, t)}{x - \xi}$$

See R. Jaffe (NPB229 (1983) 205)
explores twist expansion in DIS & DRs

$\alpha \rightarrow x$

$x \rightarrow \xi$ and the sum may be performed,

$$\frac{p \cdot q}{M^2} t_2^{T=2}(x, q^2) = 2x \int_{-1}^1 d\alpha H(\alpha) \left\{ \frac{1}{x - \alpha} - \frac{1}{x + \alpha} \right\}. \quad (3.12)$$

$t_2(x, q^2)$ may now be analytically continued to the physical region, $x \rightarrow x - i\varepsilon$ with x real and $0 < x \leq 1$.

$$\frac{p \cdot q}{M^2} t_2^{T=2}(x, q^2) = 2x \int_{-1}^1 d\alpha H(\alpha) \left\{ \frac{1}{x - \alpha - i\varepsilon} - \frac{1}{x + \alpha - i\varepsilon} \right\}. \quad (3.13)$$

Finally we take the imaginary part of $t_2(x, q^2)$ for x real to obtain the structure function $F_2(x)$,

$$F_2(x) = x(H(x) - H(-x)). \quad (3.14)$$

Bridge to Compton Form Factor parton DR

$$\int_{-1}^{+1} dx \frac{H(x, \xi, t)}{x - \xi} = -\frac{1}{\xi} \int_{-1}^{+1} dx \frac{H(x, \xi, t)}{1 - \frac{x}{\xi}} = -\sum_{n=0}^{\infty} \xi^{-n-1} \int_{-1}^{+1} dx x^n H(x, \xi, t)$$

For $|\xi| > 1$ this converges because integrals are Mellin moments & thereby polynomials in ξ of lower powers with t -dependent coefficients

What about fixed t DRs? Lehmann ellipse & analytic continuation

Crossing symmetry

Analyticity in the energy variable $v=(s-u)/4M \propto 1/x$ requires crossing symmetric amplitudes.

At GPD level need:

$$H^{(+)}(x, \xi, t) = H(x, \xi, t) - H(-x, \xi, t)$$

where $H^{(\text{anti-}q)}(x, \xi, t) = -H^q(-x, \xi, t)$

$$H^{(-)}(x, \xi, t) = H(x, \xi, t) + H(-x, \xi, t)$$

Each is multiplied by hard part from $\gamma^* q \rightarrow \gamma q'$

$$C^\pm(x/\xi) = 1/(-X + \zeta - i\epsilon) \mp 1/(X - i\epsilon)$$

or $1/(\xi - x - i\epsilon) \mp 1/(\xi + x - i\epsilon)$

$$\& 1/(x-\xi+i\epsilon) = P/(x-\xi) + \delta(x-\xi)/\pi$$

$$\mathcal{H}^{(+)}(v, t) \propto v^{\alpha_P(t)-1} \text{ where } \alpha_{\text{Pomeron}}(0)=1+\delta$$

$$\mathcal{H}^{(-)}(v, t) \propto v^{\alpha_R(t)-1} \text{ where } \alpha_{\text{Regge}}(0) \approx 1/2$$



Where is threshold?

- Imaginary part from discontinuity across unitarity branch cut - from ν threshold for production of on-shell states - e.g. πN , $\pi\pi N$, $\pi\pi\Delta$, etc.
- Continuum starts at $s = (M + m_\pi)^2 \Rightarrow$ lowest hadronic threshold.
- Physical region for **non-zero** Q^2 and t differs from this.
- How to fill the gap?

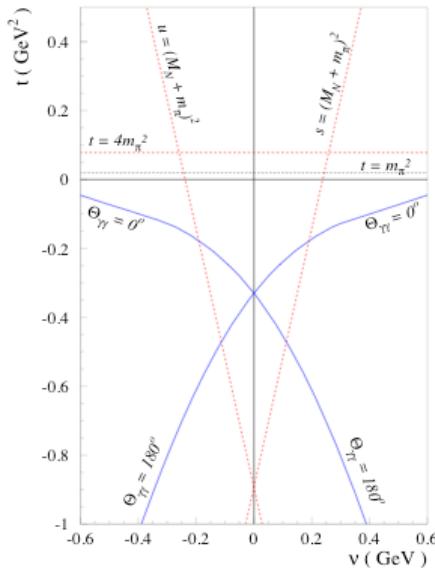
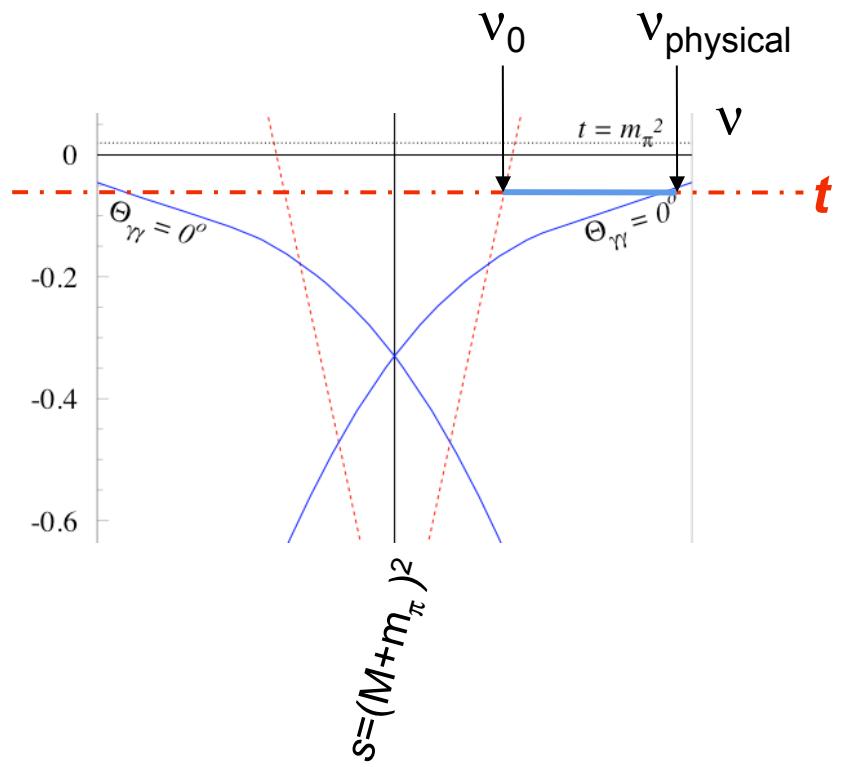


FIG. 2. The Mandelstam plane for virtual Compton scattering at $Q^2 = 0.33$ GeV 2 . The boundaries of the physical s -channel region are $\Theta_{\gamma\gamma} = 0^\circ$ and $\Theta_{\gamma\gamma} = 180^\circ$ for $\nu > 0$, the u -channel region is obtained by crossing, $\nu \rightarrow -\nu$. The curves for $\Theta_{\gamma\gamma} = 0^\circ$ and $\Theta_{\gamma\gamma} = 180^\circ$ intersect at $\nu = 0$, $t = -Q^2$, which is the point where the generalized polarizabilities are defined.



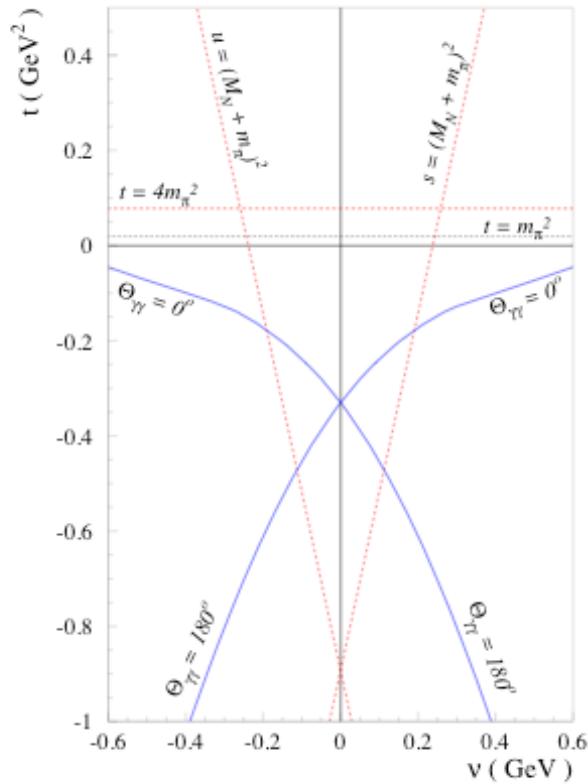
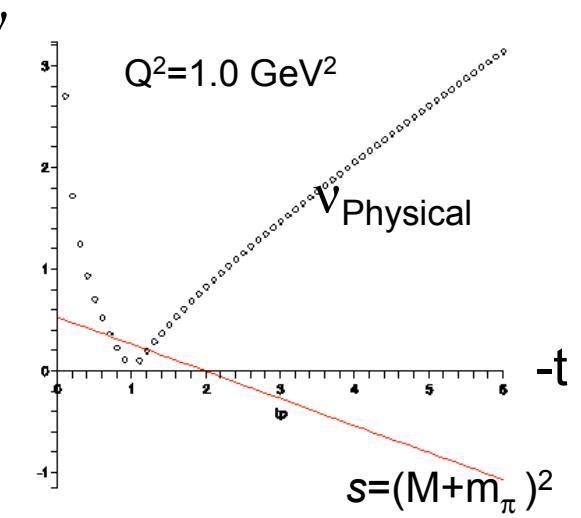
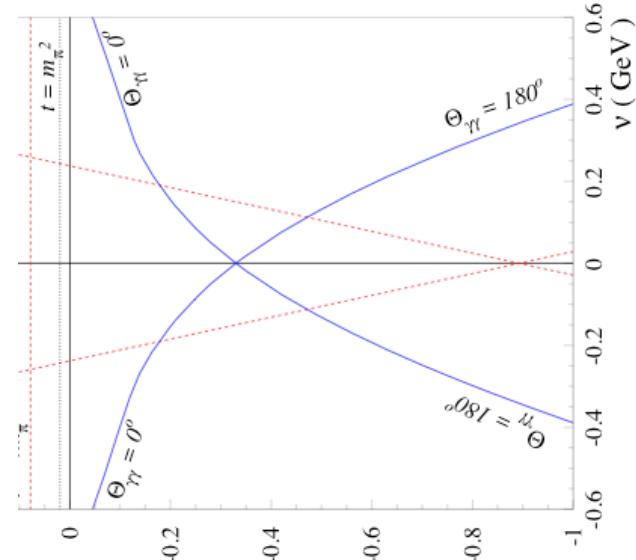


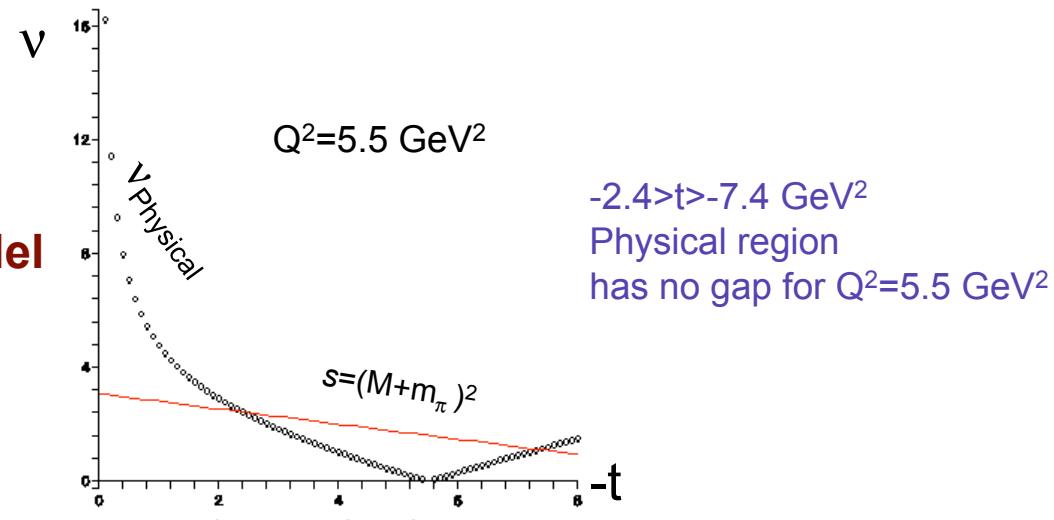
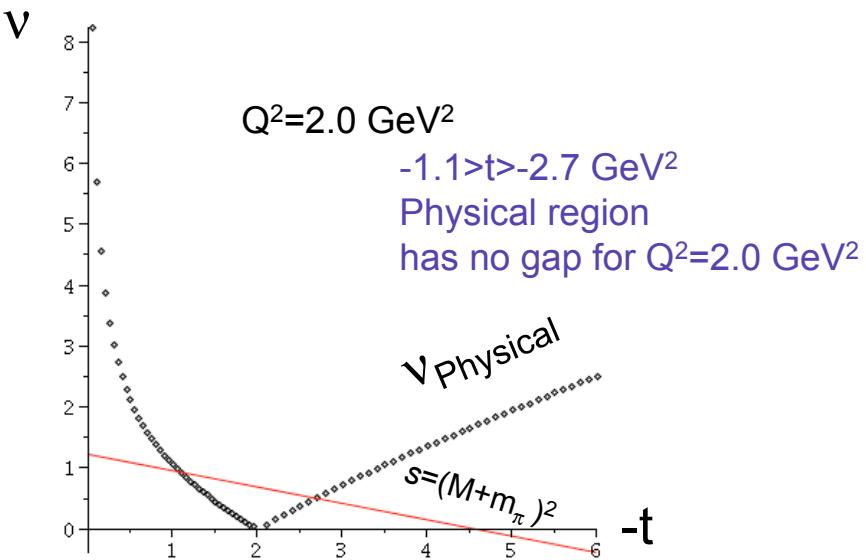
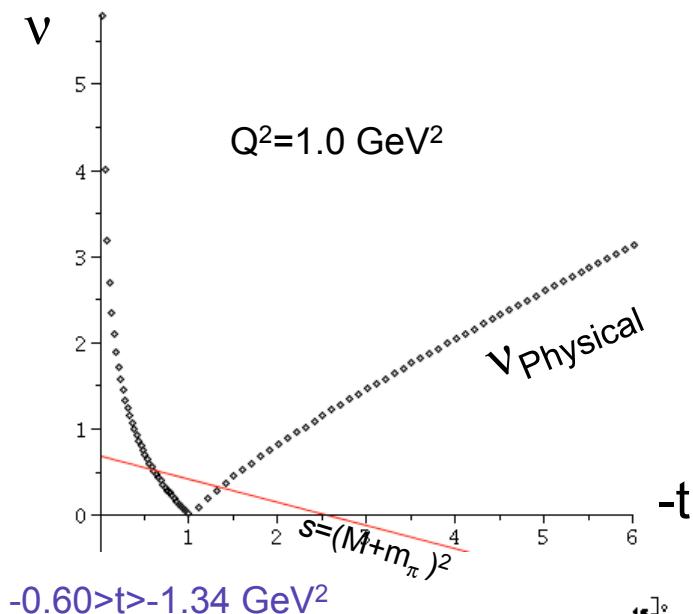
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B.Pasquini, et al., Eur.Phys.J.A11,185 (2001)

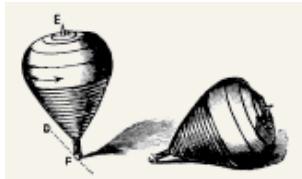




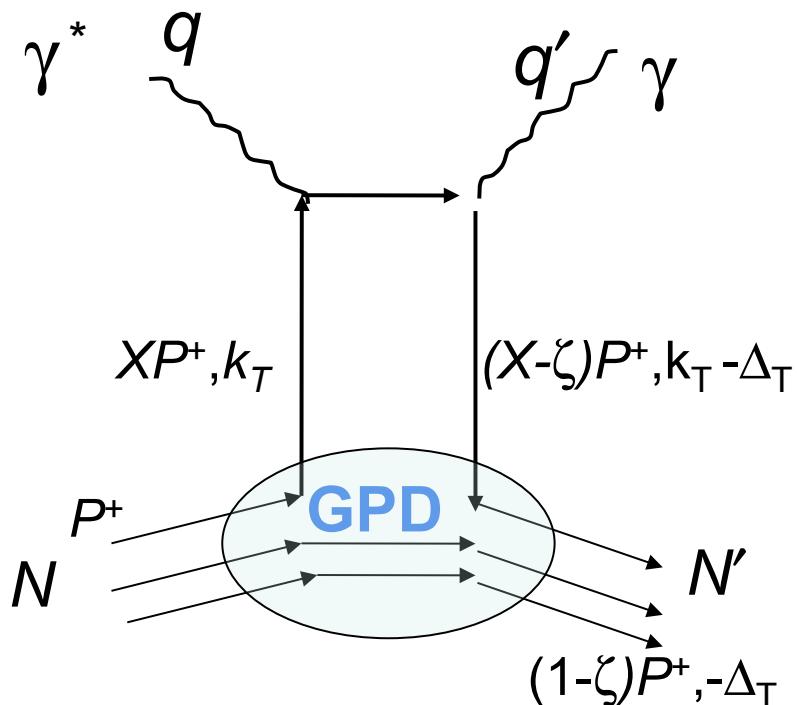
Gaps in dispersion integrals



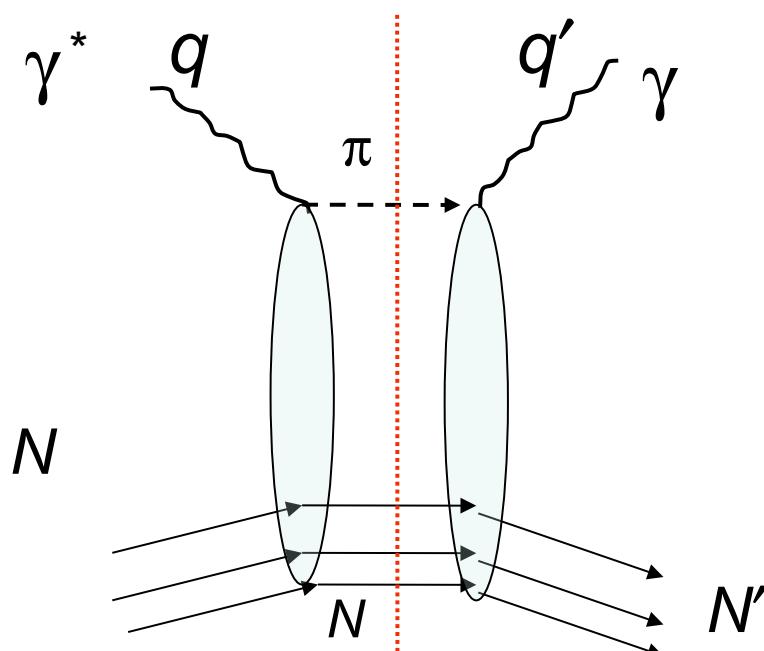
**M+2m_π next threshold
... M+nm_π, etc. Parallel
lines of increasing
intercept
– approaching jets**



Connection to GPDs? Consider Virtual Compton scattering.
Collinear factorization involves hard γ^* +quark interaction
& soft quark+nucleon amplitude - GPD



Factorized “handbag”
parton picture
Convert kinematics



s-channel unitarity hadron picture
Imaginary part for on-shell
intermediate states



Examples of DVCS dispersion integrals & threshold dependences

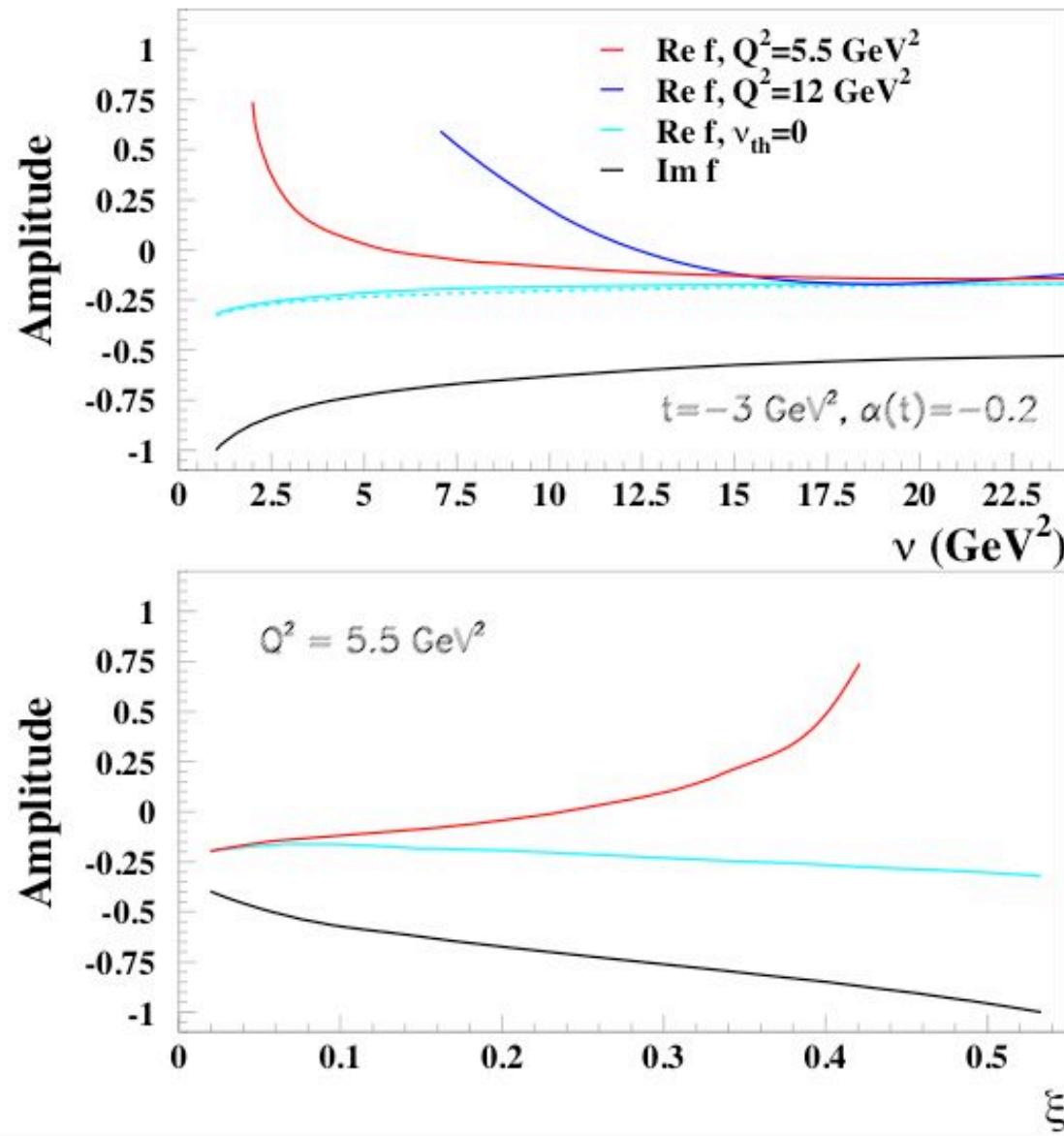
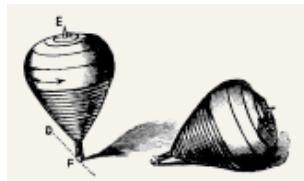
Regge form for $H(X, \zeta, t)$ or $H^R(v, Q^2, t) = \beta(t, Q^2)(1 - e^{-i\pi\alpha(t)})(v/v_0)^{\alpha(t)}$
So $\text{Re}H^R(v, Q^2, t) = \tan(\pi\alpha(t)/2) \text{Im}H(v, Q^2, t)$

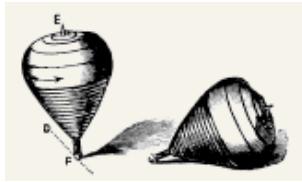
Dispersion:

$$\text{Re}H(v, Q^2, t) = \frac{2v}{\pi} \int_{v_{\text{Threshold}}}^{\infty} dv' \frac{\text{Im} H(v', Q^2, t)}{v'^2 - v^2}$$

This is exact for $v_{\text{Threshold}}=0$.

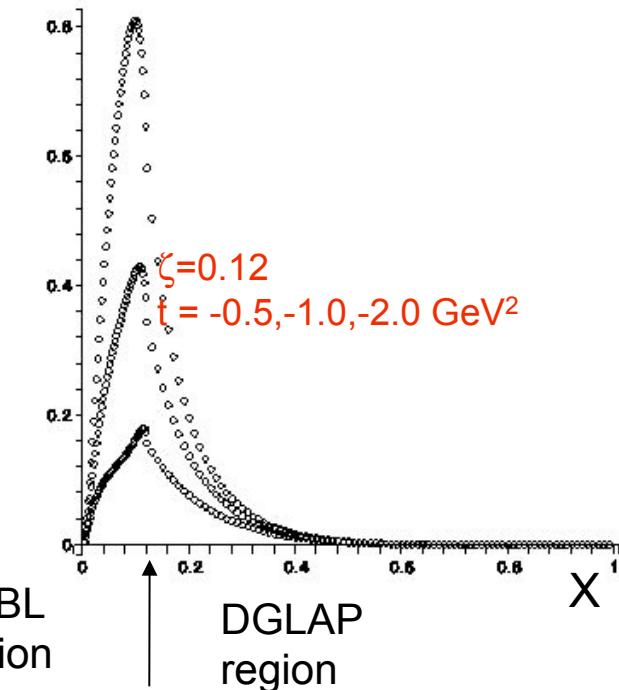
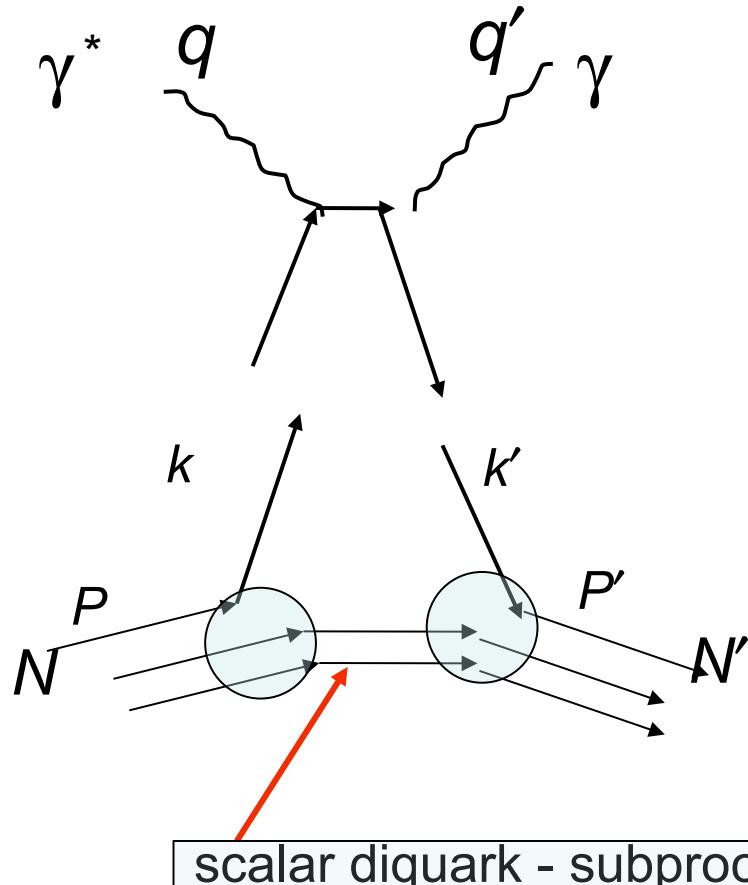
For Q^2, t have $v_{\text{Threshold}} = (2s+t+Q^2-2M^2)_{\min}/4M^2$





Diquark spectator model - simple, no spin Valence quark model

See also on Brodsky & Llanes-Estrada model



Covariant but no anti-quarks
No crossing symmetry
around $x=0$ or $X=\zeta/2$

Scalar Diquark Spectator with Scalar quarks

e.g. B-LI-E model

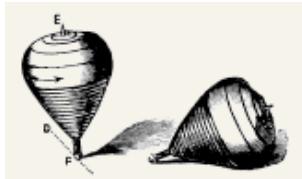
$$iT(\hat{s}, t, \hat{u}, k^2, k'^2) = ig(k^2) \frac{i}{(P - k)^2 - m_{D_{iq}}^2 + i\epsilon} ig(k'^2)$$

With $k^+ = X P^+$, $H(X, \zeta, t) = \int \frac{d\vec{k}_T dk^-}{(2\pi)^4} T(\hat{s}, t, \hat{u}, k^2, k'^2)$

T is Holomorphic in k^- with poles in \hat{u}, k^2, k'^2
 corresponding to DGLAP ($X > \zeta$ or $x > \xi$)
 & ERBL ($0 < X < \zeta$ or $-\xi < x < +\xi$) regions of X
 So $H(X < 0, \zeta, t) = 0$ No antiquarks

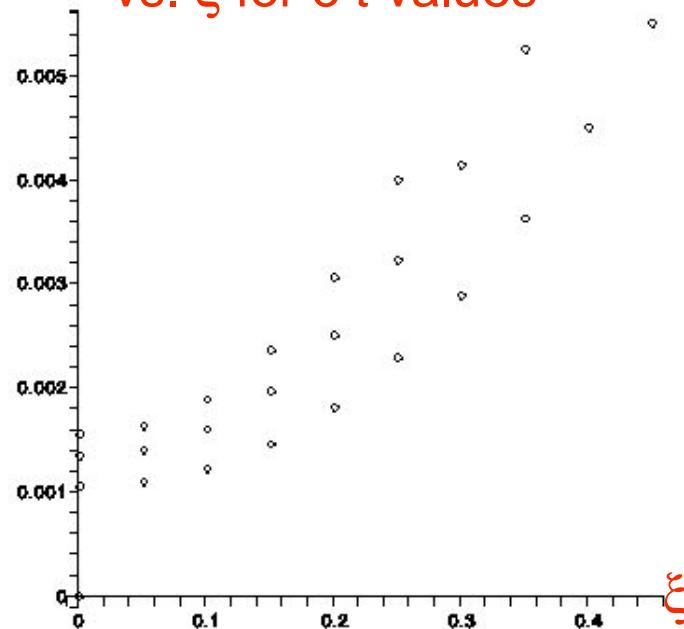
To symmetrize around $x=0$ or $X=\zeta/2$
 let $H(+)(x, \xi, t) = H(+x, \xi, t) - H(-x, \xi, t)$ for $0 < x < \xi$
 Recalling that $H(\text{anti-}q)(x, \xi, t) = -Hq(-x, \xi, t)$

Then this valence model, without anti-quarks,
 symmetrizes by adding – probability “antiquarks”



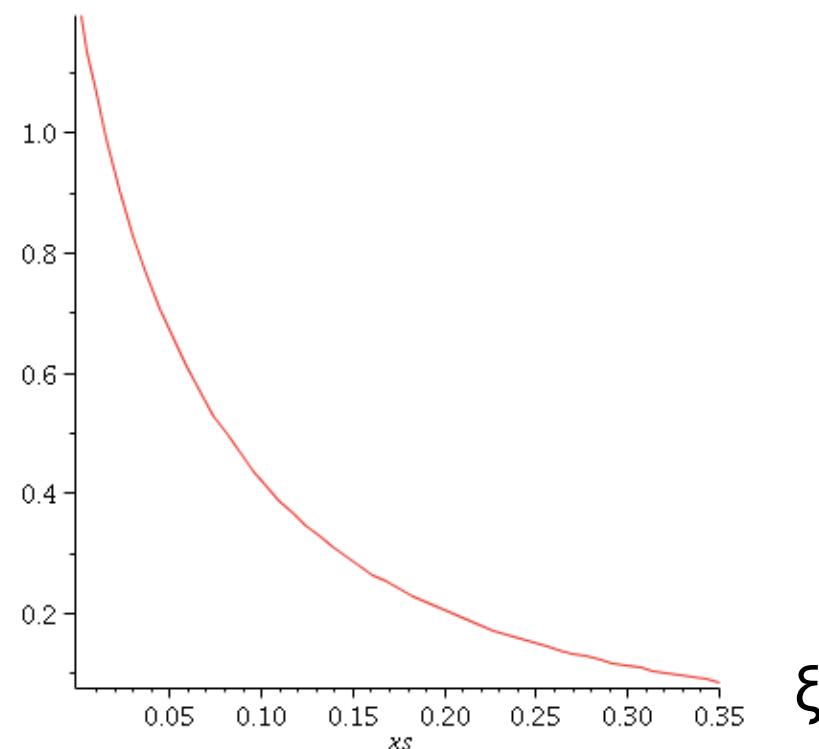
Scalar diquark, polynomiality & DR

x^2 moment of $H(x, \xi, t)$
vs. ξ for 3 t values



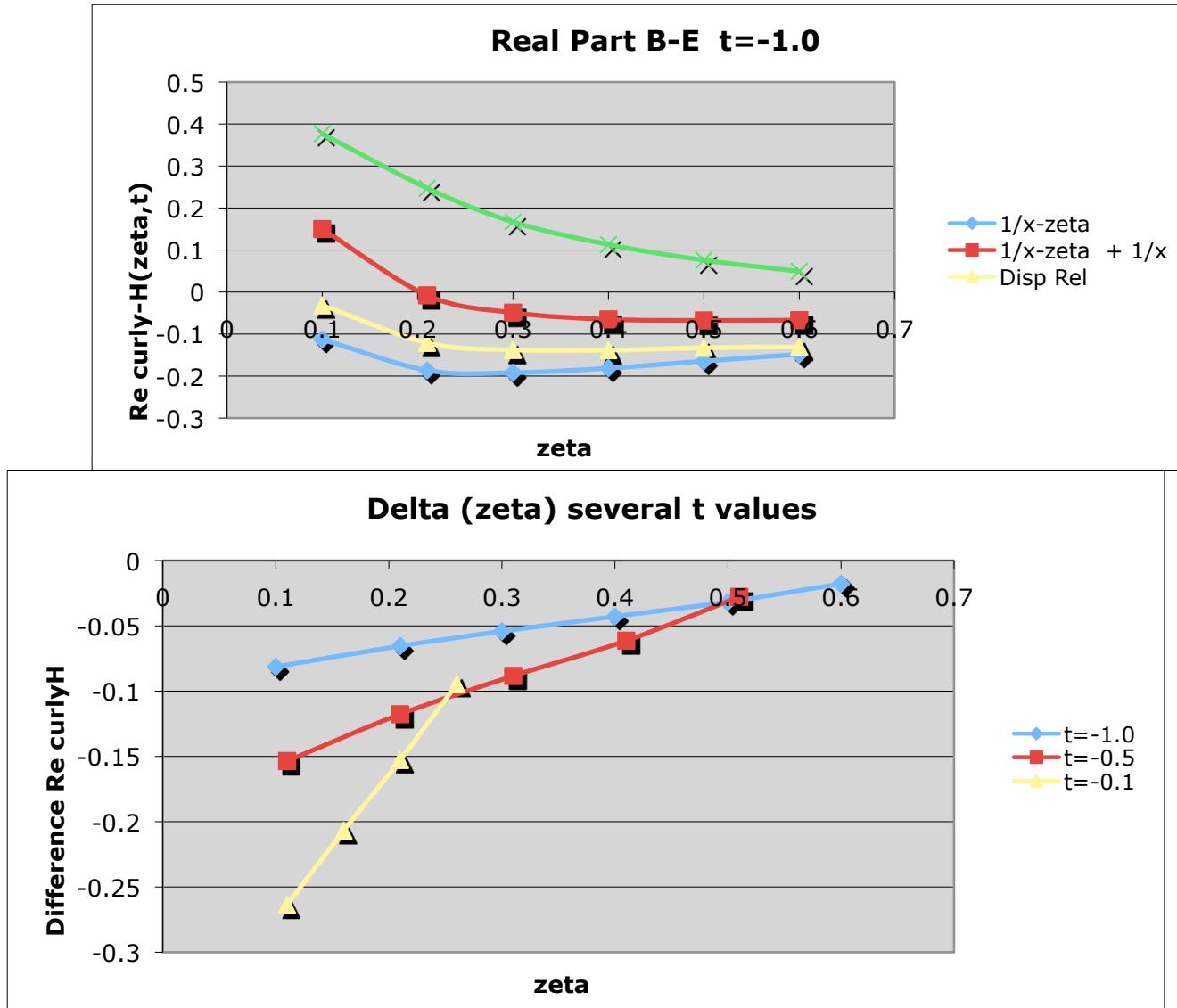
$H(x, x, t=-0.5 \text{ GeV}^2)$

$$\text{Im } \mathcal{H}_f(\xi, t) = H_f(\xi, \xi, t)$$

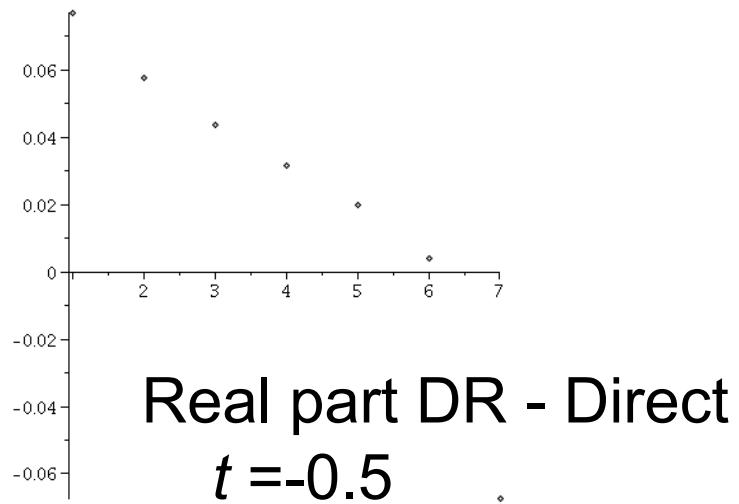
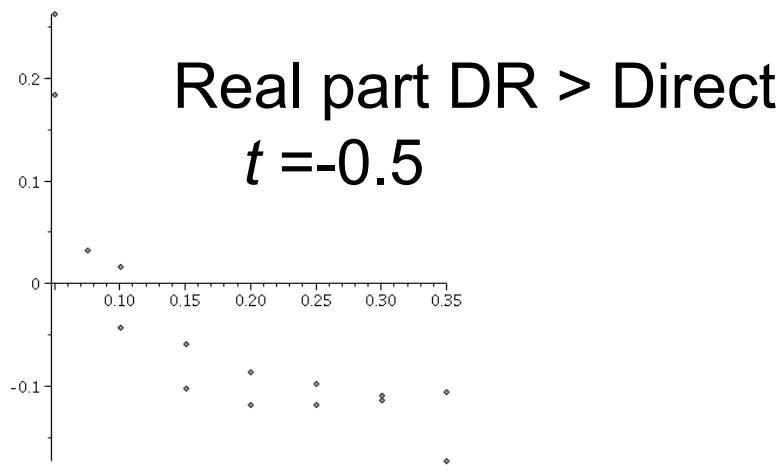


$\sim \xi^2 \Rightarrow$ polynomiality

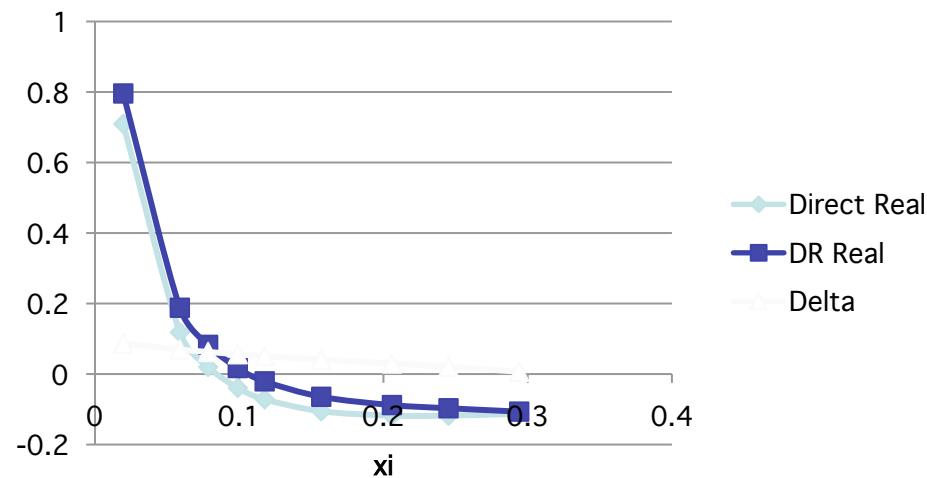
Direct real part & DR for B-E model (nonsymmetrized)



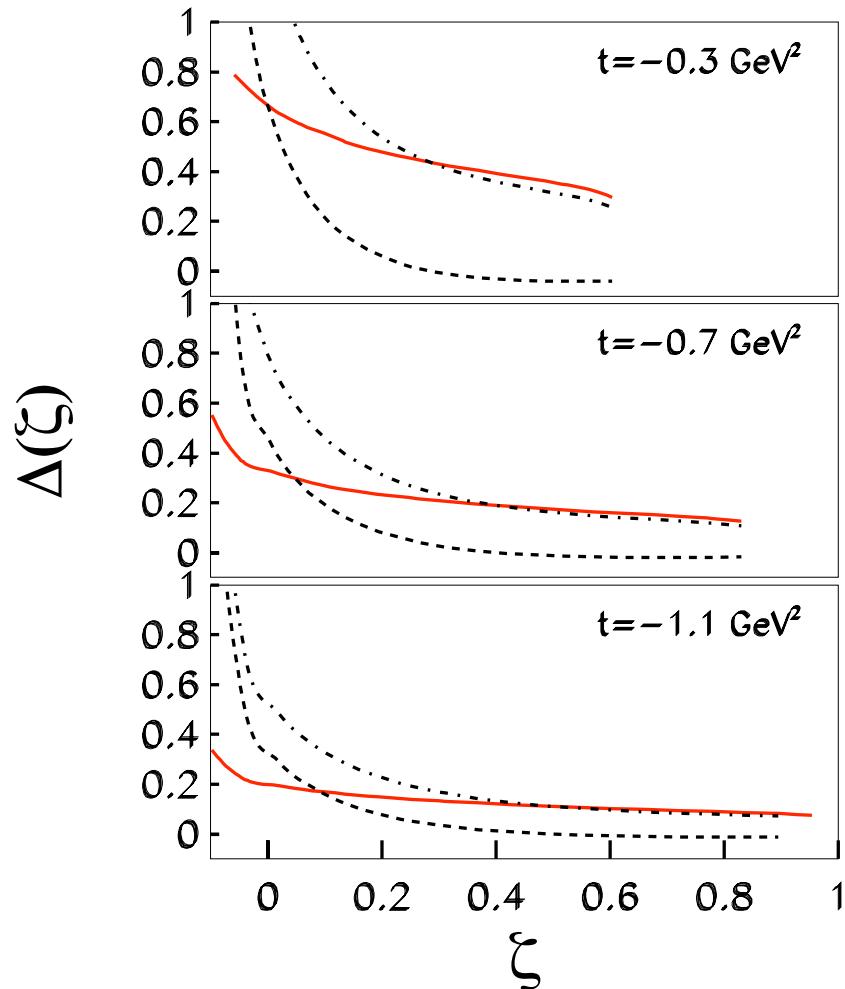
Direct real part & DR for B-E model (symmetrized)



Real $H(\xi, t=-0.5)$

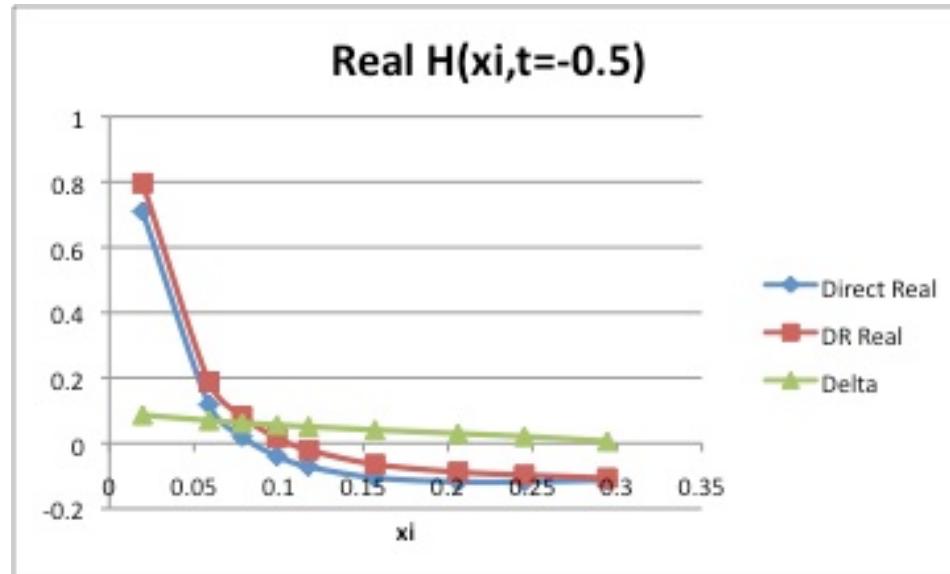


Subtraction “constant” vs. t & ζ



Upper dashed
curve is DR real.
Lower dashed curve is
direct real part.
Solid red = difference
or $\Delta(\zeta, t)$.

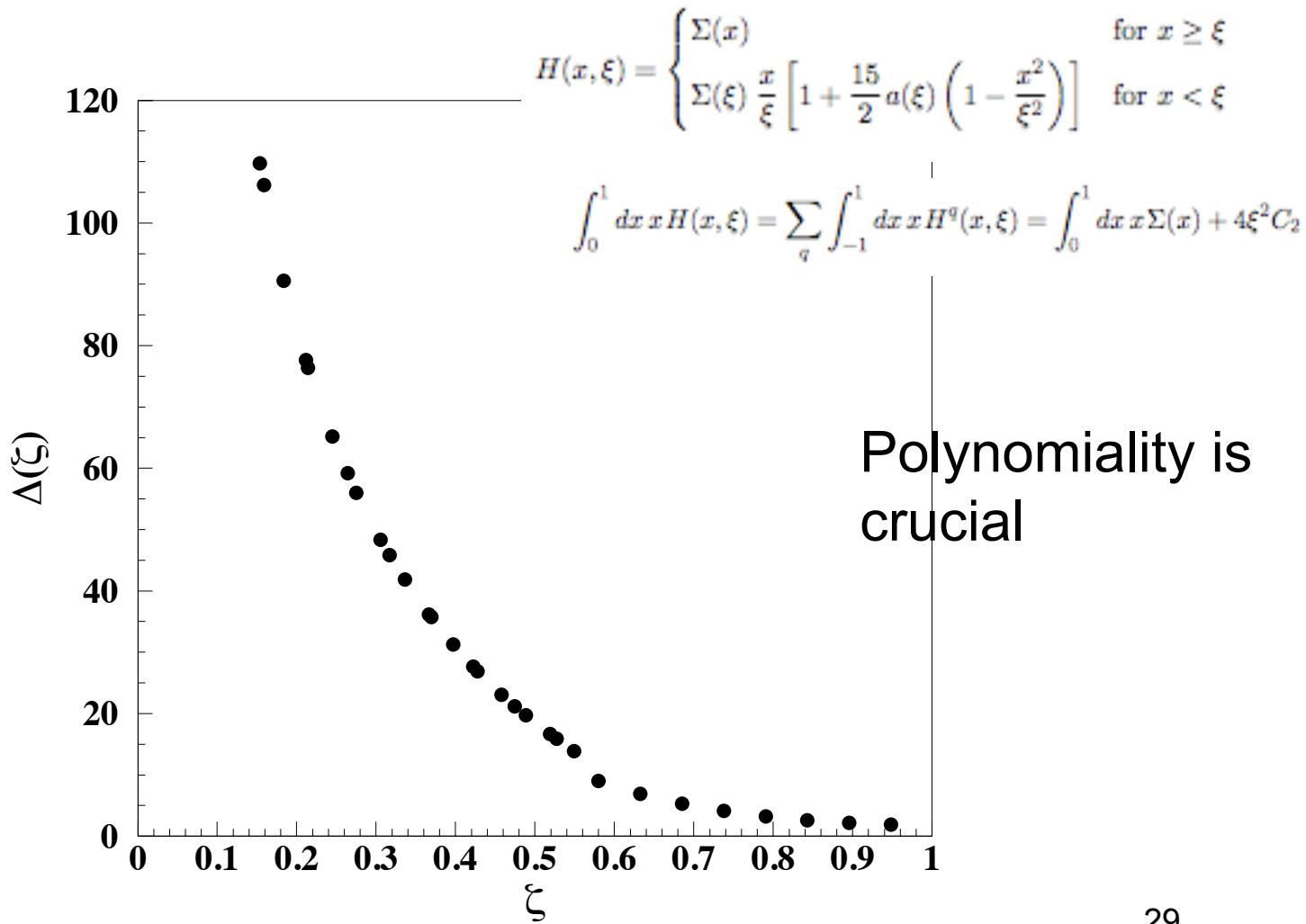
$\Delta(\xi,t)$ subtraction “constant”



Threshold “mismatch” leads to Δ dependence on ξ
Shallower for higher $|t|$

$$\zeta_{\max} = -\frac{t}{4M^2} \left(\sqrt{1 - \frac{4M^2}{t}} - 1 \right)$$

Another example: Freund, McDermott PRD65, 074008 (2002)





Lessons from examples

- Moderate reach of $E\gamma$ ($J_{lab} < 12$ GeV) or $s < 25$ $\text{GeV}^2 = (5 \text{ GeV})^2$ not far from t -dependent thresholds
- Non-forward dispersion relations require some model-dependent analytic continuation
- Difference between direct & dispersion Real $H(\zeta, t)$ depends on thresholds – is subtraction “constant” really constant?
- Antiquark distributions are critical for crossing symmetric amplitudes & GPDs
- DVCS & Bethe-Heitler interference measures complex $H(\zeta, t)$ directly – Direct Real from data complements DR constraints & necessitates determination of ERBL GPD region away from ridge $x = \xi$.