Ambiguities of the NLO BFKL kernel

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Contents

Introduction

Motivation:

Investigation of conformal properties of the NLO BFKL kernel and relation between this kernel and the linearized BK kernel.

The basis of the BFKL approach: gluon Reggeization.
The advantage: generality (arbitrary momentum and colour exchanges).
Original definition: momentum representation.

Alternative approach: the colour dipole picture.
The shortage: falling out of amplitudes with colour exchanges.
The advantage: simplicity of the non-linear generalization–BK equation.
Ia. Balitsky, 1996,
Yu. Kovchegov, 1999.
Formulation: impact parameter space.

Requirement: the same predictions in the common area.

Introduction

In the LO the BFKL kernel for scattering of colourless objects can be written in the Möbius invariant form

L.N. Lipatov, 1986.

The Möbius invariance can be made evident by transformation from the transverse momentum to the transverse coordinate representation.

V.F., R. Fiore, A. Papa, 2006.

The Möbius invariant form the LO BFKL kernel in the coordinate representation coincides with the kernel of the colour dipole approach

N.N. Nikolaev and B.G. Zakharov, 1994,

A. H. Mueller, 1994.

Actually this kernel can be written by Lipatov as early as 1985, if he had not restricted himself by check of the Möbius invariance in the operator form. Only by chance he had not written explicitly the kernel in the coordinate space.

Introduction

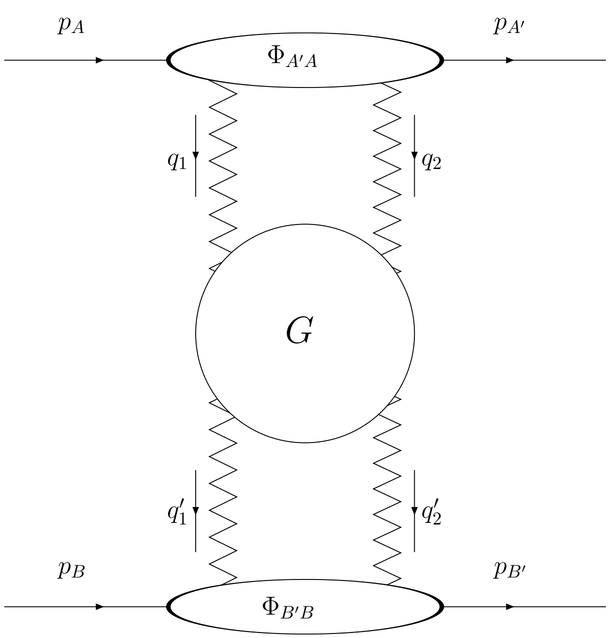
Moreover, the LO BK equation appears as a special case of the nonlinear evolution equation which sums the fan diagrams for the BFKL Green's functions in the Möbius representation
J.Bartels, L.N. Lipatov, G.P. Vacca, 2004.
In the NLO the conformal invariance is violated by renormalization. But one could expect that it is conserved in $N = 4$ SYSY Yang-Mills theories.

One could expect also coincidence of the Möbius form of the BFKL kernel and the kernel of the linearized BK equation.
However, the situation is not so simple.
The NLO kernels are not unambiguously defined.
The ambiguity of the NLO kernels is analogous to the ambiguity of the NLO anomalous dimensions. It is caused by the possibility to redistribute radiative corrections between the kernels and the impact factors.

General overview

In the BFKL approach scattering amplitudes $\mathcal{A}_{AB}^{A'B'}$ are presented in the form :

$$\Phi_{A'A} \otimes G \otimes \Phi_{B'B}.$$



The impact factors $\Phi_{A'A}$ and $\Phi_{B'B}$ describing transitions $A \to A'$ and $B \to B'$ depend on properties of scattering particles.

All energy dependence is contained in the Green's function G for two interacting Reggeized gluons.
Originally the approach was formulated in the momentum space.

The impact factors and the kernel of the BFKL equation for the Green's function are defined in the transverse momentum space.

The kernel is known now in the NLO for $t \neq 0$ and all possible t-channel colour states.

General overview

For colourless objects the impact factors in the representation

$$\delta(\vec{q}_A - \vec{q}_B) disc_s \mathcal{A}_{AB}^{A'B'} = \frac{i}{4(2\pi)^{D-2}} \langle A'\bar{A}| e^{Y\widehat{\mathcal{K}}} \frac{1}{\hat{\vec{q}}_1^{\,2} \hat{\vec{q}}_2^{\,2}} |\bar{B}'B\rangle$$

are "gauge invariant":

$$\langle A'\bar{A}|\vec{q}, 0\rangle = \langle A'\bar{A}|0, \vec{q}\rangle = 0 \ .$$

Therefore $\langle A'\bar{A}|\Psi\rangle = 0$ if $\langle \vec{r}_1, \vec{r}_2|\Psi\rangle$ does not depend either on $\vec{r}_1$ or on $\vec{r}_2$. $\langle A'\bar{A}|\widehat{\mathcal{K}}$ is "gauge invariant" as well, because $\langle \vec{q}_1, \vec{q}_2|\widehat{\mathcal{K}}_r|\vec{q}_1^{\,'}, \vec{q}_2^{\,'}\rangle$ vanishes at $\vec{q}_1^{\,'} = 0$ or $\vec{q}_2^{\,'} = 0$.
It means that we can change $|In\rangle \equiv \left(\hat{\vec{q}}_1^{\,2} \hat{\vec{q}}_2^{\,2}\right)^{-1} |\bar{B}'B\rangle$ for $|In_d\rangle$, where $|In_d\rangle$ has the "dipole " property $\langle \vec{r}, \vec{r}|In_d\rangle = 0$.
After this one can omit the terms in the kernel proportional to $\delta(\vec{r}_{1'2'})$, as well as change the terms independent either of $\vec{r}_1$ or of $\vec{r}_2$ in such a way that the resulting kernel becomes conserving the "dipole " property.

General overview

The kernel obtained in this way is called Möbius form of the BFKL kernel. It can be written as

$$\langle \vec{r}_1 \vec{r}_2 | \hat{\mathcal{K}}_M | \vec{r}_1{}' \vec{r}_2{}' \rangle = \delta(\vec{r}_{11'})\delta(\vec{r}_{22'}) \int d\vec{r}_0 \, g_0(\vec{r}_1, \vec{r}_2; \vec{r}_0)$$

$$+\delta(\vec{r}_{11'}) g_1(\vec{r}_1, \vec{r}_2; \vec{r}_2{}') + \delta(\vec{r}_{22'}) g_1(\vec{r}_2, \vec{r}_1; \vec{r}_1{}') + \frac{1}{\pi} g_2(\vec{r}_1, \vec{r}_2; \vec{r}_1{}', \vec{r}_2{}')$$

with the functions $g_{1,2}$ turning into zero when their first two arguments coincide. The first three terms contain ultraviolet singularities which cancel in their sum, as well as in the LO, with account of the “dipole” property of the “target” impact factors. The coefficient of $\delta(\vec{r}_{11'})\delta(\vec{r}_{22'})$ is written in the integral form in order to make the cancellation evident.

The term $g(\vec{r}_1, \vec{r}_2; \vec{r}_1{}', \vec{r}_2{}')$ is absent in the LO because the LO kernel in the momentum space does not contain terms depending on all three independent momenta simultaneously.

General overview

In the LO $g_1(\vec{r}_1, \vec{r}_2; \vec{r}_0) = -\,g_0(\vec{r}_1, \vec{r}_2; \vec{r}_0) = \frac{\alpha_s N_c}{2\pi^2} \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{10}^{\,2} \vec{r}_{20}^{\,2}}$, so that the $\langle \vec{r}_1 \vec{r}_2 | \hat{\mathcal{K}}_M | \vec{r}_1{}' \vec{r}_2{}' \rangle$ coincides with the colour dipole kernel and is explicitly conformal invariant.

In QCD the NLO kernel contains quark and gluon contributions. In ones turn, the quark contribution is divided into two pieces: non-Abelian" (leading in N_c) and Abelian" (suppressed by N_c^{-2}). Their Möbius forms

V.S. F., R. Fiore, A. Papa, 2006, 2007

agrees, with account of the ambiguity of the kernel, with the results

V.V. Kovchegov, H. Weigert, 2006,

I. Balitsky, 2006,

obtained by direct calculation in the dipole picture. The Abelian part is greatly simplified in comparison with the momentum representation. Moreover, this part is conformal invariant. It could be important for the QED Pomeron.

General overview

The most important contribution to the BFKL kernel is the gluon one. In the momentum representation in the NLO for arbitrary momentum transfer it is very complicated
V.S. F, R. Fiore 2005.
The Möbius form of this contribution
V.S. F, R. Fiore, A.V. Grabovsky, A. Papa, 2007
turned out strikingly simple.

However, the conformal invariance is broken not only by the terms related to the renormalization.

Moreover, it occurred afterwards that the NLO gluon contribution to the BK kernel
I. Balitsky, G.A. Chirilli, 2008
does not agree with the Möbius form of the same contribution to the BFKL kernel.

General overview

Supersymmetric Yang-Mills theories contain gluons and Maiorana fermions in the adjoint representation of the colour group. The gluon contribution is the same as in QCD. The fermion one can be obtained by change of the group coefficients: $n_f \to n_M N_c$ for the "non-Abelian" part, and $n_f \to -n_M N_c^3$ for the "Abelian" part; n_M is the number of flavours of Maiorana quarks. For N–extended SUSY $n_M = N$.

At $N > 1$ besides quarks there are n_S scalar particles; $n_S = 2$ at $N = 2$ and $n_S = 6$ at $N = 4$. At $N = 4$ $\beta_0 = \frac{11}{3} - \frac{2}{3}n_M - \frac{1}{6}n_S = 0$ and α_s is not running. Nevertheless, the Möbius form of the NLO kernel
V.S. F, R. Fiore, 2007
is not conformal invariant.

However, the hope for conformal invariance still remains.
The reason is the ambiguity of the NLO kernel.

Ambiguity of the kernel

The BFKL kernel has an evident ambiguity connected with impact factors. The discontinuity

$$\langle A'\bar{A}|e^{Y\widehat{\mathcal{K}}}\frac{1}{\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}}|\bar{B}'B\rangle$$

remains intact under the transformation

$$\hat{\mathcal{K}} \to \hat{\mathcal{O}}^{-1}\hat{\mathcal{K}}\hat{\mathcal{O}}\ , \ \ \langle A'\bar{A}| \to \langle A'\bar{A}|\hat{\mathcal{O}}\ , \ \ \frac{1}{\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}}|\bar{B}'B\rangle \to \hat{\mathcal{O}}^{-1}\frac{1}{\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}}|\bar{B}'B\rangle.$$

If the LO kernel is fixed, one can take $\hat{\mathcal{O}} = 1 - \alpha_s\hat{U}$, and get

$$\hat{\mathcal{K}} \to \hat{\mathcal{K}} - \alpha_s[\hat{\mathcal{K}}^{(B)}, \hat{U}].$$

Secondly, there is a freedom in the energy scale s_0. At first sight, it can lead to an additional ambiguity of the NLO kernel. However, it is not so.

Ambiguity of the kernel

It was shown

V.F., 1986

that any change of the energy scale can be compensated by the corresponding redefinition of the impact factors. In the NLO dependence on s_0 of the energy factor is cancelled by the dependence of the impact factors, so that s_0 can be taken as a free parameter. This freedom can be used for optimization of perturbative results

D.Yu. Ivanov, A. Papa, 2006.

What about

V.F., L.N. Lipatov, 1998?

The scale transformation was associated with the change of the kernel there.

– Because one of the impact factors was fixed. Instead of transforming both impact factors one can compensate any change of the scale by transformation of one of the impact factors and the kernel.

Ambiguity of the kernel

A natural choice of s_0 is $s_0 = Q_A Q_B$, where Q_A and Q_B are typical virtualities for the impact factors $\Phi_{A'A}$ and $\Phi_{B'B}$ correspondingly. Let us consider the transition from such scale to the scale depending on the Reggeon momenta $\vec{q}_{Ai}$ and $\vec{q}_{Bi}, \;\; i = 1, 2$, in the impact factors $\Phi_{A'A}$ and $\Phi_{B'B}$ respectively:

$$s_0 \to f_A f_B, \;\; f_A \equiv f_A(\vec{q}_{Ai}), \;\; f_B \equiv f_B(\vec{q}_{Bi}).$$

With the NLO accuracy one can write

$$\langle \vec{q}_{A1}, \vec{q}_{A2} | \left(\frac{s}{s_0} \right)^{\hat{\mathcal{K}}} | \vec{q}_{B1}, \vec{q}_{B2} \rangle = \langle \vec{q}_{A1}, \vec{q}_{A2} | \hat{F}_A \left(\frac{s}{f_A f_B} \right)^{\hat{\mathcal{K}}} \hat{F}_B | \vec{q}_{B1}, \vec{q}_{B2} \rangle,$$

where

$$\hat{F}_A = \left(1 + \ln \left(\frac{\hat{f}_A}{s_0^{\alpha}} \right) \hat{\mathcal{K}}^{(B)} \right), \;\; \hat{F}_B = \left(1 + \hat{\mathcal{K}}^{(B)} \ln \left(\frac{\hat{f}_A}{s_0^{\beta}} \right) \right),$$

$$\alpha + \beta = 1, \;\; \hat{f}_A \equiv f_A(\hat{\vec{q}}_i), \;\; \hat{f}_B \equiv f_B(\hat{\vec{q}}_i).$$

Ambiguity of the kernel

It is possible to leave one of the impact factors invariable changing the kernel. Indeed,

$$\hat{F}_A \left(\frac{s}{f_A f_B} \right)^{\hat{\mathcal{K}}} \hat{F}_A^{-1} = \left(\frac{s}{f_A f_B} \right)^{\hat{\mathcal{K}}'} , \quad \hat{\mathcal{K}}' = \hat{F}_A \hat{\mathcal{K}} \hat{F}_A^{-1} \, .$$

Therefore one can take

$$\langle \vec{q}_{A1}, \vec{q}_{A2}| \to \langle \vec{q}_{A1}, \vec{q}_{A2}|, \quad \hat{\mathcal{K}} \to \hat{\mathcal{K}}', \quad \frac{1}{\hat{\vec{q}}_1^{\,2} \hat{\vec{q}}_2^{\,2}} |\bar{B}' B\rangle \to \hat{F}_A \hat{F}_B \frac{1}{\hat{\vec{q}}_1^{\,2} \hat{\vec{q}}_2^{\,2}} |\bar{B}' B\rangle,$$

where, with the NLO accuracy,

$$\hat{\mathcal{K}}' = \hat{\mathcal{K}} - \left[\hat{\mathcal{K}}^{(B)}, \ln \hat{f}_A \hat{\mathcal{K}}^{(B)} \right] \, .$$

We see that the change of the energy scale can be associated with the transformation of the kernel with the specific $\hat{U}$.

Forward scattering

In the theory with n_M Maiorana fermions and n_S scalars in the adjoint representation we have

$$g_1(\vec{r}_1, \vec{r}_2; \vec{r}_2\,') = \frac{\alpha_s(\frac{4e^{-2C}}{\vec{r}^{\,2}})N_c}{2\pi^2} \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{22'}^{\,2} \vec{r}_{12'}^{\,2}} \left[1 + \frac{\alpha_s N_c}{2\pi}\left(\frac{67}{18} - \zeta(2) - \frac{5n_M}{9} - \frac{2n_S}{9}\right.\right.$$

$$\left.\left. + \frac{\beta_0}{2N_c}\frac{\vec{r}_{12'}^{\,2} - \vec{r}_{22'}^{\,2}}{\vec{r}_{12}^{\,2}} \ln\left(\frac{\vec{r}_{22'}^{\,2}}{\vec{r}_{12'}^{\,2}}\right) - \frac{1}{2}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{22'}^{\,2}}\right)\ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{12'}^{\,2}}\right) + \frac{\vec{r}_{12'}^{\,2}}{2\vec{r}_{12}^{\,2}}\ln\left(\frac{\vec{r}_{12'}^{\,2}}{\vec{r}_{22'}^{\,2}}\right)\ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{12'}^{\,2}}\right)\right)\right].$$

Since only the integral of g^0 is fixed, it can be written in different forms. One of them is

$$g_0(\vec{r}_1, \vec{r}_2; \vec{r}_0) = -g(\vec{r}_1, \vec{r}_2; \rho) + \frac{\alpha_s^2 N_c^2}{4\pi^3}\delta(\vec{r}_0) 2\pi\zeta(3)\,.$$

The function $g_1(\vec{r}_1, \vec{r}_2; \vec{\rho})$ vanish at $\vec{r}_1 = \vec{r}_2$. Then, these functions turn into zero for $\vec{\rho}^{\,2} \to \infty$ faster than $(\vec{\rho}^{\,2})^{-1}$ to provide the infrared safety. The ultraviolet singularities of this function at $\vec{\rho} = \vec{r}_2$ and $\vec{\rho} = \vec{r}_1$ cancel with the singularities of $g^0(\vec{r}_1, \vec{r}_2; \vec{\rho})$ on account of the “dipole” property of the “target” impact factors.

Forward scattering

$$g_2(\vec r_1,\vec r_2;\vec r_1{}',\vec r_2{}') = \frac{\alpha_s^2 N_c^2}{4\pi^3}\Bigg[\frac{1}{2\vec r_{1'2'}^{\,4}}\left(\frac{\vec r_{12'}^{\,2}\,\vec r_{21'}^{\,2}}{d}\ln\left(\frac{\vec r_{12'}^{\,2}\,\vec r_{21'}^{\,2}}{\vec r_{11'}^{\,2}\vec r_{22'}^{\,2}}\right)-1\right)\left(1-n_M+\frac{n_S}{2}\right)$$

$$-\left(\frac{(4-n_M)}{4\vec r_{1'2'}^{\,4}}\frac{\vec r_{12}^{\,2}\,\vec r_{1'2'}^{\,2}}{d}-\frac{1}{4\vec r_{11'}^{\,2}\,\vec r_{22'}^{\,2}}\left(\frac{\vec r_{12}^{\,4}}{d}-\frac{\vec r_{12}^{\,2}}{\vec r_{1'2'}^{\,2}}\right)\right)\ln\left(\frac{\vec r_{12'}^{\,2}\,\vec r_{21'}^{\,2}}{\vec r_{11'}^{\,2}\vec r_{22'}^{\,2}}\right)$$

$$+\frac{\ln\left(\frac{\vec r_{12}^{\,2}}{\vec r_{1'2'}^{\,2}}\right)}{4\vec r_{11'}^{\,2}\vec r_{22'}^{\,2}}+\frac{\ln\left(\frac{\vec r_{12}^{\,2}\;\vec r_{1'2'}^{\,2}}{\vec r_{11'}^{\,2}\vec r_{22'}^{\,2}}\right)}{2\vec r_{12'}^{\,2}\vec r_{21'}^{\,2}}\left(\frac{\vec r_{12}^{\,2}}{2\vec r_{1'2'}^{\,2}}+\frac{1}{2}-\frac{\vec r_{22'}^{\,2}}{\vec r_{1'2'}^{\,2}}\right)+\frac{\vec r_{12}^{\,2}\ln\left(\frac{\vec r_{12}^{\,2}\vec r_{1'2'}^{\,2}}{\vec r_{12'}^{\,2}\;\vec r_{21'}^{\,2}}\right)}{4\vec r_{11'}^{\,2}\vec r_{22'}^{\,2}\vec r_{1'2'}^{\,2}}$$

$$+\frac{\ln\left(\frac{\vec r_{22'}^{\,2}}{\vec r_{12}^{\,2}}\right)}{2\vec r_{11'}^{\,2}\;\vec r_{12'}^{\,2}}+\frac{\ln\left(\frac{\vec r_{12}^{\,2}\vec r_{1'2'}^{\,2}}{\vec r_{12'}^{\,2}\vec r_{22'}^{\,2}}\right)}{2\vec r_{11'}^{\,2}\;\vec r_{1'2'}^{\,2}}+\frac{\ln\left(\frac{\vec r_{12}^{\,2}\vec r_{11'}^{\,2}}{\vec r_{22'}^{\,2}\;\vec r_{1'2'}^{\,2}}\right)}{2\vec r_{12'}^{\,2}\vec r_{1'2'}^{\,2}}$$

$$+\frac{\vec r_{12}^{\,2}\ln\left(\frac{\vec r_{11'}^{\,2}}{\vec r_{1'2'}^{\,2}}\right)}{2\vec r_{11'}^{\,2}\vec r_{12'}^{\,2}\;\vec r_{22'}^{\,2}}+\frac{\vec r_{21'}^{\,2}\ln\left(\frac{\vec r_{21'}^{\,2}\vec r_{1'2'}^{\,2}}{\vec r_{12}^{\,2}\;\vec r_{11'}^{\,2}}\right)}{2\vec r_{11'}^{\,2}\vec r_{22'}^{\,2}\vec r_{1'2'}^{\,2}}+(1\leftrightarrow 2)\Bigg],\quad d=\vec r_{12'}^{\,2}\vec r_{21'}^{\,2}-\vec r_{11'}^{\,2}\vec r_{22'}^{\,2}.$$

Forward scattering

To get a hint on possible form of the transformation consider the forward scattering. Defining

$$\langle\vec{r}|\hat{\mathcal{K}}_M|\vec{\rho}\rangle = \int \langle\vec{r}_1\vec{r}_2|\hat{\mathcal{K}}_M|\vec{r}_1{}'\vec{r}_2{}'\rangle\delta(\vec{r}_1{}' - \vec{r}_2{}' - \vec{\rho})d^2r_1{}'d^2r_2{}' \,,$$

where $\vec{r} = \vec{r}_1 - \vec{r}_2$, we obtained

$$\langle\vec{r}|\hat{\mathcal{K}}_M^{SUSY}|\vec{r}\,'\rangle = \frac{\alpha_s(\frac{4e^{-2C}}{\vec{r}\,^2})N_c}{2\pi^2}\int\frac{d\vec{\rho}\,\vec{r}\,^2}{(\vec{r}-\vec{\rho})^2\vec{\rho}\,^2}\Big(2\delta(\vec{\rho}-\vec{r}\,') - \delta(\vec{r}-\vec{r}\,')\Big)\Bigg[1 + \frac{\alpha_s N_c}{4\pi}$$

$$\times\left(\frac{67}{9} - 2\zeta(2) - \frac{10n_M}{9} - \frac{4n_S}{9} + \beta_0\frac{\vec{\rho}\,^2 - (\vec{r}-\vec{\rho})^2}{\vec{r}\,^2}\ln\left(\frac{(\vec{r}-\vec{\rho})^2}{\vec{\rho}\,^2}\right)\right)\Bigg] + \frac{\alpha_s^2N_c^2}{4\pi^3}$$

$$\times\left[\delta(\vec{r}-\vec{r}\,')6\pi\zeta(3) + \frac{\vec{r}\,^2}{\vec{r}\,'^2}\left(f_1(\vec{r},\vec{r}\,') + f_2^{SUSY}(\vec{r},\vec{r}\,') - \frac{1}{(\vec{r}-\vec{r}\,')^2}\ln^2\left(\frac{\vec{r}\,^2}{\vec{r}\,'^2}\right)\right)\right].$$

Forward scattering

Here

$$\alpha_s(\frac{4e^{-2C}}{\vec{r}^{\,2}}) \simeq \alpha_s(\mu^2)\left(1-\frac{\alpha_s(\mu^2)}{4\pi}\beta_0\ln\left(\frac{4e^{-2C}}{\vec{r}^{\,2}\mu^2}\right)\right)\ ,$$

μ is the renormalization scale in the $\overline{MS}$-scheme,

$$\beta_0=\left(\frac{11}{3}-\frac{2n_M}{3}-\frac{n_S}{6}\right)N_c,$$

$$f_1(\vec{x},\vec{y})=\frac{(\vec{x}^{\,2}-\vec{y}^{\,2})}{(\vec{x}-\vec{y})^2(\vec{x}+\vec{y})^2}\left[\ln\left(\frac{\vec{x}^{\,2}}{\vec{y}^{\,2}}\right)\ln\left(\frac{\vec{x}^{\,2}\vec{y}^{\,2}(\vec{x}-\vec{y})^4}{(\vec{x}^{\,2}+\vec{y}^{\,2})^4}\right)+2\,\mathrm{Li}_2\left(-\frac{\vec{y}^{\,2}}{\vec{x}^{\,2}}\right)\right]$$

$$-2\,\mathrm{Li}_2\left(-\frac{\vec{x}^{\,2}}{\vec{y}^{\,2}}\right)-\left(1-\frac{(\vec{x}^{\,2}-\vec{y}^{\,2})^2}{(\vec{x}-\vec{y})^2(\vec{x}+\vec{y})^2}\right)\left[\int_0^1-\int_1^\infty\right]\frac{du}{(\vec{x}-\vec{y}u)^2}\ln\left(\frac{u^2\vec{y}^{\,2}}{\vec{x}^{\,2}}\right)\ ,$$

$$f_2^{SUSY}(\vec{r},\vec{r}\,')=(1-n_M+\frac{n_S}{2})f_2(\vec{r},\vec{r}\,')+(2n_S-3n_M)\int_0^\infty dt\frac{\ln\left|\frac{1+t}{1-t}\right|}{\vec{r}\,'^{\,2}+t^2\vec{r}^{\,2}}\ ,$$

Forward scattering

$$f_2\left(\vec{x},\vec{y}\right)=\frac{1}{8\vec{x}^{\,2}\vec{y}^{\,2}}\left\{\left(\vec{x}\,\vec{y}\right)^2\left(1-\frac{3}{2}\left(\frac{\vec{y}^{\,2}}{\vec{x}^{\,2}}+\frac{\vec{x}^{\,2}}{\vec{y}^{\,2}}\right)\right)+\left(\vec{x}^{\,2}+\vec{y}^{\,2}\right)^2-32\vec{x}^{\,2}\vec{y}^{\,2}\right\}$$

$$\times\int_0^{\infty}dt\frac{\ln\left|\frac{1+t}{1-t}\right|}{\vec{y}^{\,2}+t^2\vec{x}^{\,2}}+\frac{3\left(\vec{x}\,\vec{y}\right)^2-2\vec{x}^{\,2}\vec{y}^{\,2}}{16\vec{x}^{\,2}\vec{y}^{\,2}}\left(\ln\frac{\vec{x}^{\,2}}{\vec{y}^{\,2}}\left(\frac{1}{\vec{y}^{\,2}}-\frac{1}{\vec{x}^{\,2}}\right)+\frac{2}{\vec{x}^{\,2}}+\frac{2}{\vec{y}^{\,2}}\right)\,.$$

The BC result for the gluon contribution in the forward case is

$$\langle\vec{r}|\hat{\mathcal{K}}_{BC}|\vec{r}\,'\rangle=\frac{\alpha_s(\frac{1}{\vec{r}^{\,2}})N_c}{2\pi^2}\int\frac{d\vec{\rho}\,\vec{r}^{\,2}}{(\vec{r}-\vec{\rho})^2\vec{\rho}^{\,2}}\Big(2\delta(\vec{\rho}-\vec{r}\,')-\delta(\vec{r}-\vec{r}\,')\Big)\left[1+\frac{\alpha_sN_c}{4\pi}\right.$$

$$\left.\times\left(\frac{67}{9}-2\zeta(2)+\frac{11}{3}\frac{\vec{\rho}^{\,2}-(\vec{r}-\vec{\rho})^2}{\vec{r}^{\,2}}\ln\left(\frac{(\vec{r}-\vec{\rho})^2}{\vec{\rho}^{\,2}}\right)-2\ln\left(\frac{(\vec{r}-\vec{\rho})^2}{\vec{r}^{\,2}}\right)\ln\left(\frac{\vec{\rho}^{\,2}}{\vec{r}^{\,2}}\right)\right)\right]$$

$$+\frac{\alpha_s^2N_c^2}{4\pi^3}\frac{\vec{r}^{\,2}}{\vec{r}^{\,\prime 2}}\left(f_1(\vec{r},\vec{r}\,')+f_2(\vec{r},\vec{r}\,')\right)\,.$$

Forward scattering

Thus, for the difference of the forward kernels one has

$$\langle \vec{r}|\hat{\mathcal{K}}_M - \hat{\mathcal{K}}_{BC}|\vec{r}\,'\rangle = \frac{\alpha_s^2 N_c^2}{4\pi^3}\left[\frac{\vec{r}^{\,2}}{(\vec{r}-\vec{r}\,')^2\vec{r}\,'^2}\ln\left(\frac{\vec{r}^{\,2}}{\vec{r}\,'^2}\right)\ln\left(\frac{\vec{r}^{\,2}\vec{r}\,'^2}{(\vec{r}-\vec{r}\,')^4}\right)\right.$$

$$\left.+\delta(\vec{r}-\vec{r}\,')2\pi\zeta(3)\right] + \frac{\alpha_s N_c}{4\pi}\frac{11}{3}\left(C - \ln 2\right)\langle \vec{r}|\hat{\mathcal{K}}_M^{(B)}|\vec{r}\,'\rangle\ ,$$

$$\langle \vec{r}|\hat{\mathcal{K}}_M^{(B)}|\vec{r}\,'\rangle = \frac{\alpha_s N_c}{2\pi^2}\int \frac{d\vec{\rho}\,\vec{r}^{\,2}}{(\vec{r}-\vec{\rho})^2\vec{\rho}^{\,2}}\Big(2\delta(\vec{\rho}-\vec{r}\,') - \delta(\vec{r}-\vec{r}\,')\Big)\ .$$

The term proportional to $11/3$ is related to renormalization. In our opinion, this term arose because the renormalization scheme used by BC is not equivalent to conventional $\overline{MS}$-scheme. The term with logarithms can be eliminated by the transformation

$$\hat{\mathcal{K}} \to \hat{\mathcal{K}} + \frac{1}{2}\left[\hat{\mathcal{K}}^{(B)},\, \ln(\hat{\vec{q}}^{\,2})\,\hat{\mathcal{K}}^{(B)}\right]$$

applied to the forward BFKL kernel. Indeed, the direct calculation shows that

$$\langle \vec{r}| \left[\hat{\mathcal{K}}^{(B)}, \, \ln(\hat{\vec{q}}^{\,2}) \, \hat{\mathcal{K}}^{(B)}\right]_M |\vec{r}^{\,\prime}\rangle = -\frac{\alpha_s^2 N_c^2}{2\pi^3} \frac{\vec{r}^{\,2}}{(\vec{r}-\vec{r}^{\,\prime})^2 \vec{r}^{\,\prime 2}} \ln\left(\frac{\vec{r}^{\,2}}{\vec{r}^{\,\prime 2}}\right) \ln\left(\frac{\vec{r}^{\,2}\vec{r}^{\,\prime 2}}{(\vec{r}-\vec{r}^{\,\prime})^4}\right).$$

This transformation corresponds to the change of the energy scale at fixed value of one of the impact factors. Actually, it is of the same type as used in
V.F., L. Lipatov, 1998:

$$\mathcal{K}(\vec{q},\vec{q}^{\,\prime}) \to \mathcal{K}(\vec{q},\vec{q}^{\,\prime}) + \frac{1}{2}\int d\vec{p} \; \mathcal{K}^{(B)}(\vec{q},\vec{p}) \, \ln\frac{\vec{p}^{\,2}}{\vec{q}^{\,2}} \mathcal{K}^{(B)}(\vec{p},\vec{q}^{\,\prime}).$$

One can come to this transformation from another side. The difference $\hat{\mathcal{K}}_M - \hat{\mathcal{K}}_{BC}$ has the same eigenfunctions

$$\langle \vec{r}^{\,\prime}|n,\gamma\rangle \sim e^{in\phi_{\vec{r}^{\,\prime}}} \left(\vec{r}^{\,\prime 2}\right)^{\gamma} \;,$$

Forward scattering

as the LO dipole kernel. The eigenvalues of the LO dipole kernel coincide with the eigenvalues of the LO BFKL kernel:

$$\omega_B(n,\gamma) = \frac{\alpha_s N_c}{\pi}\chi(n,\gamma)\ , \quad \chi(n,\gamma) = 2\psi(1) - \psi(\gamma + \frac{n}{2}) - \psi(1-\gamma+\frac{n}{2})\ .$$

It becomes evident if we write the FBKL kernel for the forward scattering as

$$\langle \vec{q}|\hat{\mathcal{K}}^{(B)}|\vec{q}\,'\rangle = \frac{\alpha_s N_c}{2\pi^2}\left[\frac{2\vec{q}\,'^2}{(\vec{q}-\vec{q}\,')^2\,\vec{q}^{\,2}} - \delta(\vec{q}-\vec{q}\,')\int \frac{d\vec{l}\;\vec{q}^{\,2}}{(\vec{q}-\vec{l})^2\vec{l}^{\,2}}\right]$$

and compare it with the dipole kernel

$$\langle \vec{r}|\hat{\mathcal{K}}_d|\vec{r}\,'\rangle = \frac{\alpha_s N_c}{2\pi^2}\left[\frac{2\vec{r}^{\,2}}{(\vec{r}-\vec{r}\,')^2\,\vec{r}\,'^2} - \delta(\vec{r}-\vec{r}\,')\int \frac{d\vec{\rho}\;\vec{r}^{\,2}}{(\vec{r}-\vec{\rho})^2\vec{\rho}^{\,2}}\right]\ .$$

It is worthwhile to note here that the kernel used here differs from the usually used symmetric kernel;

Forward scattering

the former is obtained from the latter by the transformation $\hat{\mathcal{K}} \rightarrow \hat{\vec{q}}^{\,-2}\hat{\mathcal{K}}\hat{\vec{q}}^{\,2}$. For the non-forward case the corresponding transformation is $\hat{\mathcal{K}} \rightarrow (\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2})^{-1/2}\hat{\mathcal{K}}(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2})^{1/2}$. Let us stress that just the transformed kernel can be written in the Möbius form $\langle\vec{r}_1\vec{r}_2|\hat{\mathcal{K}}_M^{(B)}|\vec{r}_1^{\,\prime}\vec{r}_2^{\,\prime}\rangle$, which is invariant in regard to the conformal transformations of the transverse coordinates and coincides with the kernel of the colour dipole model $\langle\vec{r}_1\vec{r}_2|\hat{\mathcal{K}}_d|\vec{r}_1^{\,\prime}\vec{r}_2^{\,\prime}\rangle$. In the forward case it means the functional identity of the LO BFKL kernel in the momentum and Möbius coordinate representations: $\vec{q}^{\,2}\langle\vec{q}|\hat{\mathcal{K}}^{(B)}|\vec{q}^{\,\prime}\rangle/\vec{q}^{\,\prime 2}$ is represented by the same function as $\vec{r}^{\,\prime 2}\langle\vec{r}|\hat{\mathcal{K}}_M^{(B)}|\vec{r}^{\,\prime}\rangle/\vec{r}^{\,2}$.

The eigenvalues $\omega_B(n,\gamma)$ are associated usually with the eigenfunctions $e^{in\phi_{\vec{q}^{\,\prime}}}\left(\vec{q}^{\,\prime 2}\right)^{\gamma-2}$ in the momentum space, i.e. $e^{in\phi_{\vec{r}^{\,\prime}}}\left(\vec{r}^{\,\prime 2}\right)^{1-\gamma}$. From the functional identity of the kernels it is clear that the eigenvalues must be the same as for $e^{in\phi_{\vec{r}^{\,\prime}}}\left(\vec{r}^{\,\prime 2}\right)^{\gamma}$, i.e. $\omega_B(n,1-\gamma)$. Both requirements are fulfilled because $\omega_B(n,\gamma)=\omega_B(n,1-\gamma)$.

Forward scattering

By the direct calculation we obtain

$$\omega_M(n,\gamma)-\omega_{BC}(n,\gamma)=\frac{\alpha_s^2(\mu^2)N_c^2}{2\pi^2}\left[\chi'(n,\gamma)\,\chi(n,\gamma)+\frac{11}{3}(C-\ln 2)\,\chi(n,\gamma)+\zeta(\right.$$

where $\omega_M(n,\gamma)-\omega_{BC}(n,\gamma)$ is the eigenvalue of the difference $\hat{\mathcal{K}}_M-\hat{\mathcal{K}}_{BC}$ corresponding to the eigenfunction $e^{in\phi_{\vec{r}\,'}}\left(\vec{r}\,'^2\right)^{\gamma}$. The first term here can be written as

$$\frac{1}{2}\omega_B'(n,\gamma)\,\omega_B(n,\gamma)=-\frac{1}{2}[\omega_B,\frac{\partial}{\partial\gamma}\,\omega_B].$$

In the space of the eigenfunctions $e^{in\phi_{\vec{r}\,'}}\left(\vec{r}\,'^2\right)^{\gamma}$ we have $\hat{\mathcal{K}}^{(B)}=\omega_B(n,\gamma)$ and $\ln(\hat{\vec{q}}^{\,2})=-\partial/\partial\gamma$, so that for the forward scattering we obtain

$$\hat{\mathcal{K}}_M-\hat{\mathcal{K}}_{BC}=\frac{1}{2}[\hat{\mathcal{K}}^{(B)},\ln(\hat{\vec{q}}^{\,2})\hat{\mathcal{K}}^{(B)}]+\hat{\mathcal{K}}^{(B)}\frac{11}{3}\frac{\alpha_s(\mu^2)N_c}{2\pi}(C-\ln 2)+\frac{\alpha_s^2(\mu^2)N_c^2}{2\pi^2}\zeta(3$$

Forward scattering

Thus, the logarithmic term in the difference is eliminated by the kernel transformation. The second one, as it was already pointed out, in our opinion is related to the difference of the renormalization scheme used by BC with conventional $\overline{MS}$-scheme and can be eliminated by change of the scheme. We have to note that in fact this term is present in the difference between the eigenvalues of the NLO BFKL kernel and the linearized forward kernel in I. Balitsky, G.A. Chirilli, 2008. In the calculation of this difference presented there this term is erroneously omitted at the transition from Eq. (120) to Eq. (122).

Unfortunately, we cannot find the transformation suitable to eliminate the third term. We have to add that in the BFKL approach the term with $\zeta(3)$ passed through a great number of verifications. In particular, this term is necessary for the fulfillment of the bootstrap relations. Besides, it is confirmed by the calculation of the three-loop anomalous dimensions.

Functional identity

In $N=4$ SUSY, because of the scale invariance, one can expect the same functional identity of the NLO BFKL kernel in the momentum and Möbius representations as in the LO. At $N=4$ the coupling α_s does not run, so that $\beta_0 = 0$. Moreover, $f_2^{SUSY} = 0$ in this case. In the renormalization scheme which preserves the supersymmetry we have to change

$$\alpha_s \longrightarrow \alpha_s \left(1 - \frac{\alpha_s N_c}{12\pi}\right).$$

Finally, the kernel simplifies to

$$\langle \vec{r}|\hat{K}_M^{N=4}|\vec{r}\,'\rangle = \frac{\alpha_s N_c}{2\pi^2}\int \frac{d\vec{\rho}\,\vec{r}^{\,2}}{(\vec{r}-\vec{\rho})^2\vec{\rho}^{\,2}}\left(2\delta(\vec{\rho}-\vec{r}\,')-\delta(\vec{r}-\vec{r}\,')\right)\left[1-\frac{\alpha_s N_c}{2\pi}\zeta(2)\right]$$

$$+\frac{\alpha_s^2 N_c^2}{4\pi^3}\left[6\pi\zeta(3)\delta(\vec{r}-\vec{r}\,')+\frac{\vec{r}^{\,2}}{\vec{r}\,'^2}\left(f_1(\vec{r},\vec{r}\,')-\frac{1}{(\vec{r}-\vec{r}\,')^2}\ln^2\left(\frac{\vec{r}^{\,2}}{\vec{r}\,'^2}\right)\right)\right],$$

Functional identity

$$f_1(\vec{x},\vec{y}) = \frac{\left(\vec{x}^{\,2}-\vec{y}^{\,2}\right)}{(\vec{x}-\vec{y})^2(\vec{x}+\vec{y})^2}\left[\ln\left(\frac{\vec{x}^{\,2}}{\vec{y}^{\,2}}\right)\ln\left(\frac{\vec{x}^{\,2}\vec{y}^{\,2}(\vec{x}-\vec{y})^4}{(\vec{x}^{\,2}+\vec{y}^{\,2})^4}\right)+2\,\mathrm{Li}_2\left(-\frac{\vec{y}^{\,2}}{\vec{x}^{\,2}}\right)\right]$$

$$-2\,\mathrm{Li}_2\left(-\frac{\vec{x}^{\,2}}{\vec{y}^{\,2}}\right)-\left(1-\frac{\left(\vec{x}^{\,2}-\vec{y}^{\,2}\right)^2}{(\vec{x}-\vec{y})^2(\vec{x}+\vec{y})^2}\right)\left[\int_0^1-\int_1^\infty\right]\frac{du}{(\vec{x}-\vec{y}u)^2}\ln\left(\frac{u^2\vec{y}^{\,2}}{\vec{x}^{\,2}}\right).$$

Now turn to the momentum representation. Unfortunately, here the kernel is written in the space-time dimension $D=4+2\epsilon$ to regularize infrared divergencies. Here we solve two problems. First, we found the explicit form of the kernel for SUSY Yang-Mills with any N. Second, we perform explicitly the cancellation of the infrared divergencies and write the kernel at physical space-time dimension $D=4$. It permits us to demonstrate that the functional identity of the forward BFKL kernels in the momentum and Möbius coordinate representations exhibited in the previous section in the LO is preserved in the NLO in the N=4 SUSY case.

Functional identity

For the symmetric kernel we obtain

$$K(\vec{q},\vec{q}\,') = K_r(\vec{q},\vec{q}\,') + 2\delta(\vec{q}-\vec{q}\,')\omega\left(-\vec{q}^{\,2}\right),$$

where

$$\omega\left(-\vec{q}^{\,2}\right) = -\bar{g}^2_\mu\left(\frac{2}{\epsilon}+2\ln\frac{\vec{q}^{\,2}}{\mu^2}\right) - \bar{g}^4_\mu\left[\frac{\beta_0}{N_c}\left(\frac{1}{\epsilon^2}-\ln^2\left(\frac{\vec{q}^{\,2}}{\mu^2}\right)\right)+\left(\frac{1}{\epsilon}+2\ln\left(\frac{\vec{q}^{\,2}}{\mu^2}\right)\right.\right.$$

$$\left.\times\left(\frac{67}{9}-2\zeta(2)-\frac{10}{9}n_M-\frac{4n_S}{9}\right)-\frac{404}{27}+2\zeta(3)+\frac{56}{27}n_M+\frac{26}{27}n_S\right],$$

$$g = g_\mu\mu^{-\epsilon}\left[1+\frac{\beta_0}{N_c}\frac{\bar{g}^2_\mu}{2\epsilon}\right], \qquad \bar{g}^2_\mu = \frac{g^2_\mu N_c\Gamma(1-\epsilon)}{(4\pi)^{2+\epsilon}}, \qquad \beta_0 = \left(\frac{11}{3}-\frac{2}{3}n_M-\frac{n_S}{6}\right.$$

Functional identity

$$K_r(\vec{q},\vec{q}\,') = \frac{4\,\bar{g}_\mu^2\,\mu^{-2\epsilon}}{\pi^{1+\epsilon}\Gamma(1-\epsilon)}\frac{1}{\vec{k}^{\,2}} + \frac{4\bar{g}_\mu^4\,\mu^{-2\epsilon}}{\pi^{1+\epsilon}\Gamma(1-\epsilon)}\Bigg\{\frac{1}{\vec{k}^{\,2}}\Bigg[\frac{\beta_0}{N_c\epsilon}\Bigg(1-\left(\frac{\vec{k}^{\,2}}{\mu^2}\right)^{\epsilon}(1-\epsilon^2\frac{\pi^2}{6}$$

$$+\left(\frac{\vec{k}^{\,2}}{\mu^2}\right)^{\epsilon}\Bigg(\frac{67}{9}-\frac{\pi^2}{3}-\frac{10}{9}n_M-\frac{4n_S}{9}+\epsilon\Bigg(-\frac{404}{27}+\frac{11}{3}\zeta(2)+14\zeta(3)+\frac{56}{27}n_M$$

$$+\frac{26}{27}n_S\Bigg)-\ln^2\frac{\vec{q}^{\,2}}{\vec{q}^{\,\prime 2}}\Bigg)\Bigg]+f_1\left(\vec{q}_1,\vec{q}_1{}'\right)+f_2^{SUSY}\left(\vec{q}_1,\vec{q}_1{}'\right)\Bigg\},\qquad \vec{k}=\vec{q}-\vec{q}\,'.$$

The infrared singularities make difficulties for use of this representation. However, one can cancel the divergencies and write the kernel at physical space-time dimension $D = 4$ and using the integral representation for the trajectory

$$\omega(-\vec{q}^{\,2}) = -\frac{\bar{g}_\mu^2\,\vec{q}^{\,2}}{\pi^{1+\epsilon}\Gamma(1-\epsilon)}\int\frac{d^{2+2\epsilon}k\;\mu^{-2\epsilon}}{\vec{k}^{\,2}(\vec{k}-\vec{q})^2}\left(1+\bar{g}_\mu^2 f_\omega(\vec{k},\vec{k}-\vec{q})\right)\;,$$

Functional identity

where

$$f_\omega(\vec{k}_1,\vec{k}_2) = \frac{\beta_0}{N_c\epsilon}+\left[\frac{\beta_0}{N_c\epsilon} - \frac{67}{9} + 2\zeta\,(2) + \frac{10}{9}n_M + \frac{4n_S}{9} + \epsilon\left(\frac{404}{27} - \frac{11}{3}\zeta(2) - 6\zeta(\right.\right.$$

$$\left.\left. - \frac{56}{27}n_M - \frac{26}{27}n_S\right)\right]\left[\left(\frac{\vec{k}_{12}^{\,2}}{\mu^2}\right)^\epsilon - \left(\frac{\vec{k}_1^{\,2}}{\mu^2}\right)^\epsilon - \left(\frac{\vec{k}_2^{\,2}}{\mu^2}\right)^\epsilon\right] - \ln\left(\frac{\vec{k}_{12}^{\,2}}{\vec{k}_1^{\,2}}\right)\ln\left(\frac{\vec{k}_{12}^{\,2}}{\vec{k}_2^{\,2}}\right).$$

In the limit $\epsilon \to 0$ we introduce the cut-off $\lambda \to 0$ keeping $\epsilon \ln \lambda \to 0$. Then in the regions $\vec{k}^{\,2} \leq \lambda^2$ we have

$$f_\omega(\vec{k},\vec{k}-\vec{q}) = \frac{\beta_0}{N_c\epsilon} - \left(\frac{\vec{k}^{\,2}}{\mu^2}\right)^\epsilon\left[\frac{\beta_0}{N_c\epsilon} - \frac{67}{9} + 2\zeta\,(2) + \frac{10}{9}n_M + \frac{4n_S}{9}\right.$$

$$\left. +\epsilon\left(\frac{404}{27} - \frac{11}{3}\zeta(2) - 6\zeta(3) - \frac{56}{27}n_M - \frac{26}{27}n_S\right)\right],$$

Functional identity

and in the region $(\vec{k}-\vec{q})^2 \le \lambda^2$ the same expression with the substitution $\vec{k}^{\,2} \to (\vec{k}-\vec{q})^2$. Then we see that, when the kernel $K(\vec{q},\vec{q}^{\,\prime})$ acts on any function nonsingular at $\vec{q}=\vec{q}^{\,\prime}$, the contribution of the region $\vec{k}^{\,2} \le \lambda^2$ in the “real" part cancels almost completely the contributions of the regions $\vec{k}^{\,2} \le \lambda^2$ and $(\vec{k}-\vec{q})^2 \le \lambda^2$ in the doubled trajectory $\omega(-\vec{q}^{\,2})$. The only piece which remains uncancelled is

$$2\frac{\bar{g}_\mu^4}{\pi^{1+\epsilon}\Gamma(1-\epsilon)}\int\frac{d^{2+2\epsilon}k\;\mu^{-2\epsilon}}{\vec{k}^{\,2}}16\epsilon\zeta(3)\left(\frac{\vec{k}^{\,2}}{\mu^2}\right)^{\epsilon}\theta(\lambda^2-\vec{k}^{\,2}) = 2\frac{\alpha_s^2(\mu)N_c^2}{2\pi^2}\zeta(3).$$

Outside the regions $\vec{k}^{\,2} \le \lambda^2$ and $(\vec{k}-\vec{q})^2 \le \lambda^2$ one can put $\epsilon = 0$. Thus we come to the representation of the symmetric kernel which solves the second problem: presentation of the kernel in the physical space-time dimension $D=4$ with explicit cancellation of the infrared divergencies.

Functional identity

$$K(\vec{q},\vec{q}\,') = \frac{\alpha_s(\vec{q}^{\,2})N_c}{2\pi^2}\left[\frac{2}{(\vec{q}-\vec{q}\,')^2} - \delta(\vec{q}-\vec{q}\,')\int\frac{d\vec{l}\,\vec{q}^{\,2}}{(\vec{q}-\vec{l})^2\vec{l}^{\,2}}\right]$$

$$\times\left[1+\frac{\alpha_s N_c}{4\pi}\left(\frac{67}{9}-2\zeta\left(2\right)-\frac{10}{9}n_M-\frac{4n_S}{9}\right)\right]+\frac{\alpha_s^2N_c^2}{4\pi^3}\left[\frac{1}{(\vec{q}-\vec{q}\,')^2}\frac{\beta_0}{N_c}\right.$$

$$\times\ln\left(\frac{\vec{q}^{\,2}}{(\vec{q}-\vec{q}\,')^{\,2}}\right)+f_1(\vec{q},\vec{q}\,')+f_2^{SUSY}(\vec{q},\vec{q}\,')\frac{1}{(\vec{q}-\vec{q}\,')^2}\ln^2\left(\frac{\vec{q}^{\,2}}{\vec{q}\,'^2}\right)$$

$$\left.+\delta(\vec{q}-\vec{q}\,')\left(\frac{\beta_0}{2N_c}\int\frac{d\vec{l}\,\vec{q}^{\,2}}{(\vec{q}-\vec{l})^2\vec{l}^{\,2}}\ln\left(\frac{(\vec{q}-\vec{l})^2\vec{l}^{\,2}}{\vec{q}^{\,4}}\right)+6\pi\zeta(3)\right)\right]\;.$$

To compare the BFKL kernel in the Möbius representation and in the momentum representation, we have to take into account that $\langle\vec{q}|\hat{\mathcal{K}}|\vec{q}\,'\rangle = \vec{q}\,'^{\,2}K(\vec{q},\vec{q}\,')\vec{q}^{\,-2}$,

Functional identity

so that

$$\langle\vec q|\hat{\mathcal{K}}|\vec q\,'\rangle = \frac{\alpha_s(\vec q\,^2)N_c}{2\pi^2}\int\frac{d\vec l\,\vec q\,'^{\,2}}{(\vec q-\vec l)^2\vec l\,^2}\left[2\delta(\vec q-\vec l)-\delta(\vec q-\vec q\,')\right]\left[1+\frac{\alpha_s N_c}{4\pi}\left(\frac{67}{9}-2\zeta\right.\right.$$

$$\left.\left.-\frac{10}{9}n_M-\frac{4n_S}{9}\right)\right]+\frac{\alpha_s^2N_c^2}{4\pi^3}\frac{\vec q\,'^{\,2}}{\vec q\,^2}\left[\frac{1}{(\vec q-\vec q\,')^2}\frac{\beta_0}{N_c}\ln\left(\frac{\vec q\,^2}{(\vec q-\vec q\,')^{\,2}}\right)+f_1f_2^{SUSY}\right.$$

$$-\frac{1}{(\vec q-\vec q\,')^2}\ln^2\left(\frac{\vec q\,^2}{\vec q\,'^2}\right)+\delta(\vec q-\vec q\,')\left(\frac{\beta_0}{2N_c}\int\frac{d\vec l\,\vec q\,^2}{(\vec q-\vec l)^2\vec l\,^2}\ln\left(\frac{(\vec q-\vec l)^2\vec l\,^2}{\vec q\,^4}\right)+6\pi\zeta(3)\right.$$

At $\beta=0$ we have

$$\frac{\vec r\,'^{\,2}}{\vec r\,^2}\langle\vec r|\hat{\mathcal{K}}_M|\vec r\,^2\rangle|_{\beta_0=0}=\frac{\vec q\,^2}{\vec q\,'^2}\langle\vec q|\hat{\mathcal{K}}|\vec q\,'\rangle_{\beta_0=0}\bigg|_{\vec q\to\vec r,\ \vec q\,'\to\vec r\,'}.$$

Summary

- The NLO BFKL kernel is not unambiguously defined.
- The ambiguity can be used
  - for restoration of the conformal invariance of the BFKL kernel at $N = 4$ SUSY Yang-Mills theory
  - for elimination of the discrepancy between the BFKL and BK kernels.
- There are two possible sources of the ambiguity: the impact factors and the energy scale.
- For the forward scattering the discrepancy can be partly removed.
- The forward BFKL kernel has a simple form in the momentum representation at the physical space-time dimension.
- This form is the same as the Möbius for of the kernel in the coordinate representation.