PT symmetry and renormalization in a Pomeron model

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Outline

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Motivations

Strong interactions at small x (Regge kinematics) have been studied since many decades.

- About 30 years ago perturbative QCD analysis (BFKL): reggeized gluons can make non local composite states.
  The simpler, the hard QCD Pomeron, was used to investigate the leading high energy behavior of total cross sections.
  High energy factorization: decomposition in impact factors and rapidity dependent pomeron Green functions (or evolved unintegrated gluon densities), in dipole cross sections and dipole densities, etc.
- Subsequently in generalized analysis other (color singlet) composite objects appeared (Odderon, ...). But more important is the structure of interactions among them. In particular the triple Pomeron interaction (Bartels, Mueller, etc.) , phenomenologically related to diffractive physics.
- Non linear equations such as the BK equation are based on linear BFKL evolution and (non linear) interactions introduced to study onium-nucleus scattering.
  In this case one is dealing with a fan structure developing with rapidity evolution.
  Similar studies were also performed in the so called ”color glass condensate” picture (JIMWLK approach).
- Some of these investigations have been taken in the leading log and some in the next-to-leading log approximations.

Motivations

- The computation of scattering amplitudes requires to go beyond the analysis of fan structures: there are loops!
- Attempts to formulate QFT effective models to take into account loops have been made both in the BFKL framework (Braun) as well as in extensions of the JIMWLK approach (Kovner et al.)
- Let us take the point of view introduced in the BFKL approach of reggeized gluons and consider the BFKL pomeron and the triple pomeron interaction to build an effective theory based on just these two elements in 2 (transverse space) + 1 (rapidity) dimensions.
- This simplified effective field theory can be analyzed in different approximations but one should not neglect the "loop" contributions, (also addressed as fluctuation terms).
- Unfortunately this problem, already in the generalized leading log approximation, is too hard to be fully solved directly. So different approaches were considered: pure QFT, noisy stochastic systems, ... with different kinds of simplifications introduced.
- I will discuss here some developments on the study of a local approximation of the 2+1 QFT described above, known as the LRFT, introduced by Gribov before QCD.
- Our aim is to show that this problem can be studied to some extend perturbatively in an unconventional way, giving some technical tools for that.

QM and QFT local models

- The simpler model one can imagine is the reduction to zero transverse dimensions. This toy model is equivalent to a quantum mechanics of interacting pomerons in imaginary time and was intensively studied about 30 years ago and was recently reconsidered.
  It can be described by the Hamiltonian (non hermitian)

$$H = -\mu\phi^\dagger\phi + i\lambda\phi^\dagger(\phi + \phi^\dagger)\phi \quad , \quad [\phi, \phi^\dagger] = 1$$

  where ϕ is the pomeron destruction operator, μ is the intercept minus one and λ the triple pomeron coupling.

  The transition amplitudes between an initial state Ψ_i and a final state Ψ_f "after" a rapidity interval y can be written as

$$iA_{fi} = \langle\Psi_f|\Psi_i(y)\rangle = \langle\Psi_f|e^{-yH}|\Psi_i\rangle$$

  This model has been studied in full details with helps of numerical analysis.
  A recently developed perturbative analysis can be used for not too large rapidity intervals, but asymptotically essential non perturbative effects become crucial.
  In any case this model is too crude and cannot be taken seriously.

- A more realistic model, still local, is given by the LRFT for pomerons, described by the Hamiltonian density

$$\mathcal{H} = -\phi^\dagger(x)(\mu + \alpha'\nabla_x^2)\phi(x) + i\lambda\phi^\dagger(x)\Big[\phi^\dagger(x) + \phi(x)\Big]\phi(x)$$

  with α' the slope of the pomeron trajectory.

PT symmetry in a nutshell

- There are non Hermitian Hamiltonian having real spectrum bounded from below when Hilbert space is properly defined (b. c. in Sturm-Liuville problem).
  See Bender et al. (1998-) , Mostafazadeh, etc.
- A class of such systems is PT symmetric ($[H, PT] = 0$) and a norm conserved in time is obtained from the scalar product $(f, g) = \int dx [PT\, f]^t(x) g(x)$.
  Unfortunately such a norm is not positive!!!
  Nevertheless the sign of this norm can be associated to the eigenvalues of an operator C such that $C^2 = 1$, $[C, PT] = 0$ and $[C, H] = 0$.
- In order to construct a physically consistent fundamental theory in such a framework one then must use the scalar product $\langle f|g\rangle = \int dx [CPT\, f]^t(x) g(x)$ leading to a conserved positive norm and with observables such that $O^t = CPTOCPT$ ($C = P$ in hermitian QM).
- Remark: in an effective theory non hermiticity may be not fundamental (e.g. see the opening of decaying channels) and such a requirement does not rise. One should keep the normal physically motivated scalar product.
  The same happens for LRFT, where evolution is in rapidity.
- Set up of the analysis. Start from
  $$H = H_0 + \lambda H_I$$
  with H_0 Hermitian and H_I anti-Hermitian and define the P operator such that $[H_0, P] = 0$, $\{H_I, P\} = 0$, $P^2 = 1$ so that $PHP = H^\dagger$.

Perturbative analysis

- Construct the operator $C = e^Q P$ such that $[C, PT] = 0$ and $[C, H] = 0$ which requires

$$2\lambda\, e^Q H_I = [e^Q, H]$$

- Solution:

$$[H_0, Q_1] = -2H_I, \quad [H_0, Q_3] = -\frac{1}{6}\Big[[H_I, Q_1]Q_1\Big]$$

gives a perturbative solution for $Q = \lambda Q_1 + \lambda^3 Q_3 + ...$

- The operator Q is such that $e^{-Q} H e^Q = H^\dagger$ so that one has by similarity transformation

$$\boxed{h = e^{-Q/2} H e^{Q/2} = e^{Q/2} H^\dagger e^{-Q/2} = h^\dagger}$$

which, being Hermitian, is suitable to extract the real spectrum of H.

- A perturbative expression of $h = h^{(0)} + \lambda^2 h^{(2)} + \lambda^4 h^{(4)} + ...$ is given by

$$h^{(0)} = H_0 \ , \ h^{(2)} = \frac{1}{4}[H_I, Q_1] \ , \ h^{(4)} = \frac{1}{4}[H_I, Q_3] + \frac{1}{32}\Big[[H_0, Q_3], Q_1\Big]$$

- For a transition amplitude we shall write

$$\boxed{iA_{fi}(y_2 - y_1) = \langle \Psi_f(y_2) | e^{-H(y_2 - y_1)} | \Psi_i(y_1) \rangle = \langle e^{Q/2} \Psi_f(y_2) | e^{-h(y_2 - y_1)} | e^{-Q/2} \Psi_i(y_1) \rangle}$$

Outline

generalities

- For LRFT one has

$$\boxed{H_0 = \int d^2x(-\mu\phi^\dagger(x)\phi(x) + \alpha'\nabla\phi^\dagger(x)\nabla\phi(x))} \quad , \quad \boxed{H_I = i\int d^2x\,\phi^\dagger(x)\Big[\phi^\dagger(x) + \phi(x)\Big]\phi(x)}$$

- The parity operator P is such that $\phi(x) \to -\phi(-x)$ and $\phi^\dagger(x) \to -\phi^\dagger(-x)$ and the operator T perfoms complex conjugation. So $H = H_0 + \lambda H_I$ is PT symmetric.
- Starting from the c.r. $[\phi(x), \phi^\dagger(x')] = \delta^2(x - x')$ any eigenstate of H can be written as $F(i\phi^\dagger)|\Psi_0\rangle$, with F a real function since $[H, PT] = 0$ and $\phi|\Psi_0\rangle = 0$. Then one can see that the eigenvalues of H are real.
- In order to construct any transition amplitude we need to write the similarity transformation defined by Q. We compute Q to the first non trivial order in λ imposing the condition $[H_0, Q_1] = -2H_I$ on

$$Q_1 = -2\frac{i}{\mu}\int d^2x_1 d^2x_2 d^2x_3 \Big(f_1(x_1, x_2, x_3)\phi_1^\dagger\phi_2\phi_3 - h.c.\Big)$$

where we denote $\phi_1 \equiv \phi(x_1)$ etc. We find for the Fourier transform of f_1:

$$\tilde{f}_1(k_1, k_2, k_3) = \mu\frac{(2\pi)^2\delta(k_1 + k_2 + k_3)}{\mu - \alpha'(k_2^2 + k_3^2 - k_1^2)}$$

similar Hamiltonian : $h^{(2)}$

- The contribution of order λ^2 to h can be computed using $h^{(2)} = \frac{1}{4}[H_I, Q_1]$ and gives

$$\boxed{h^{(2)} = h^{(2)}_{single} + h^{(2)}_{pair} + h^{(2)}_{NC}}$$

- $h^{(2)}_{NC}$ is not conserving the pomeron number and contributes to order λ^4 to the spectrum, which goes beyond our approximation. We can neglect it.
- $h^{(2)}_{single}$ has a structure associated to single pomeron propagation. It gives λ^2 corrections to its energy.
- $h^{(2)}_{pair}$ has a more complicated structure associated to the interaction of two pomerons. The interaction potential is not local and is also degenerate since some terms depend only on the incoming or outgoing momenta. We have investigated both the *scattering states*, not leading to change in the spectrum, and the possible presence of *bound states* which instead could deeply affect the Hamiltonian spectrum.
- Up to λ^2 we can therefore look at the structure of the transition amplitude as a sum of contributions in the single and double pomeron channel:

$$\begin{aligned} iA_{fi}(y_2 - y_1) = \quad & \langle e^{\lambda Q_1)/2}\Psi_f(y_2)|1\rangle e^{-(h^{(0)}+\lambda^2 h^{(2)}_{single})(y_2-y_1)}\langle 1|e^{-\lambda Q_1/2}\Psi_i(y_1)\rangle + \\ & \langle e^{\lambda Q_1)/2}\Psi_f(y_2)|2\rangle e^{-\lambda^2 h^{(2)}_{pair}(y_2-y_1)}\langle 2|e^{-\lambda Q_1/2}\Psi_i(y_1)\rangle \end{aligned}$$

one pomeron contribution

- Term associated to corrections to the single pomeron propagation:

$$h^{(2)}_{single} = \int d^2k \phi^\dagger(k)\phi(k)\Delta^{(2)}\epsilon(k) \;\; , \;\; \Delta^{(2)}\epsilon(k) = -\frac{2}{(2\pi)^2}\mathrm{Re}\int \frac{d^2k_2 d^2k_3 \delta^2(k_2+k_3-k)}{\mu - \alpha'(k_2^2+k_3^2-k^2)}$$

This term gives a correction to the energy:

$$\boxed{\epsilon(k) = -\mu + \alpha' k^2 + \lambda^2 \Delta^{(2)}\epsilon(k)}$$

This correction (infinite) needs to be renormalized.
Choosing a regularization and imposing that at $k = 0$ the energy is $\epsilon(0) = -\mu$, one finds for small values of k: $\Delta^{(2)}\epsilon_{reg}(k) = -\frac{1}{8\pi\mu}k^2$
This leads to a renormalized slope

$$\boxed{\alpha' \to \alpha'_{ren} = \alpha' - \lambda^2 \frac{1}{8\pi\mu}}$$

double pomeron channel

- The two pomeron interaction term is given by

$$h^{(2)}_{pair} = \int d^2k_1 d^2k_2 d^2q_1 d^2q_2 \delta^2(q_1+q_2-k_1-k_2)\, V^{(2)}(q_1,q_2|k_1,k_2)\phi^\dagger(q_1)\phi^\dagger(q_2)\phi(k_1)\phi(k_2)$$

with the potential

$$\begin{aligned} V^{(2)}(q_1,q_2|k_1,k_2) &= -\frac{1}{2\pi^2}\frac{1}{\mu-\alpha'(k_1^2+(k_2-q_1)^2-q_2^2)} - \frac{1}{2\pi^2}\frac{1}{\mu-\alpha'(q_1^2+(q_2-k_1)^2-k_2^2)} \\ &+\frac{1}{8\pi^2}\frac{1}{\mu-\alpha'(k_1^2+k_2^2-(k_1+k_2)^2)} + \frac{1}{8\pi^2}\frac{1}{\mu-\alpha'(q_1^2+q_2^2-(q_1+q_2)^2)} \end{aligned}$$

- We shall restrict the analysis to the forward direction ($q_1+q_2=k_1+k_2=0$) and write

$$V^{(2)} = V(q|k) = v(q)+v(k)+V_1(q|k)$$

In momentum space the spectrum in the two pomeron subchannel is found by solving the associated Schrödinger equation:

$$\Big(\epsilon(q)-E\Big)\psi(q) = -\int d^2k\, V(q|k)\psi(k)$$

The solutions may correspond to both scattering states (continuous eigenvalues) and bound states (discrete eigenvalues).

Two pomerons: scattering states

- We start from the Lippman-Schwinger equation

$$T(q|l) = V(q|l) + \int d^2k \frac{V(q,k)T(k|l)}{\epsilon(l) - \epsilon(k) \pm i0}$$

whose solution gives

$$\psi_l(q) = \delta^2(q-l) + \frac{T(q|l)}{\epsilon(l) - \epsilon(q) \pm i0}$$

corresponding to the energy of a two pomerons scattering state $\epsilon(q) = -2\mu + 2\alpha'_{ren}q^2$

- The solution for the scattering matrix T can be found up to order λ^2 after a tedious but straightforward calculation, for any positive value of the energy.
The divergent behavior of some terms needs to be regularized and finally one finds a non trivial scattering matrix.

- Defining $I_n(l) = \int d^2k \frac{v^n(k)}{\epsilon(l)-\epsilon(k)}$ and $\chi_2(q) = \int d^2k \frac{V_1(q,k)v(k)}{\epsilon(l)-\epsilon(k)}$ one finds to order λ^2

$$\boxed{T(q|l) = V_1(q|l) - \frac{v(l)}{I_2(l)}\chi_2(q) - \frac{v(q)v(l)}{I_2(l)}}$$

Two pomerons: bound states

- Bound state solutions can be found solving

$$t_E(q) = \int d^2k \frac{V(q|k)t_E(k)}{E - \epsilon(k)} \quad , \quad \psi_E(q) = \frac{t_E(q)}{E - \epsilon(q)}$$

- The condition for the existence of bound states can be reduced to solve a secular equation for a finite algebraic problem.
  For small values of λ, i.e. in a perturbative sense, one can see that there are no solutions.
- But in general it is likely that bound states do exist for a range of finite values of λ.
- Let us note that in the perturbative approach the rapidity evolution can be investigated up to $y \approx \frac{\alpha'}{\lambda^2} \ll \frac{\alpha'^2}{\lambda^4}$

Summary

- The Local Reggeon Field Theory of pomerons is a model to study some features which are too difficult to be investigated in the QCD in the Regge limit. In particular the effects of pomeron loops induced by interactions.
- We have considered a perturbative approach starting from the PT symmetry of this QFT model in $2+1$ dimensions, which is described by a non Hermitian Hamiltonian with a real spectrum.
- The analysis is not trivial already at the first perturbative order showing the need to renormalize the energy of one pomeron states and the appearance of two pomeron states. The two pomeron states at very small perturbative coupling λ are not forming bound states but are just scattering states with a trivial continuous spectrum, but non trivial wave functions.
  This study of a Schrödinger equation with a degenerate potential can be useful in other contexts.
- Our perturbative treatment is valid up to rapidity intervals $y \sim 1/\lambda^2$.
- For finite λ bound states in the two (multi) pomeron channel are likely to appear. Their presence can affects the large rapidity asymptotic behavior which nevetheless can be analyzed reliably only non perturbatively. This fact is already a feature of the 0-transverse dimension QM toy model.
- The perturbative approach can be taken to higher orders: an analysis at NNLO (λ^4) can extend the range of applicability and make possible to better estimate the validity of the NLO results.