

# M-strings and Little Strings

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1605.02591)



# Six Dimensional Superconformal Field Theories

Strong Motivation to study their dynamics:

- \* at the heart of key structures in M-theory and string theory  
(world-volume theory of M5-branes, little string theory)
- \* encode topological invariants and data of underlying string geometry
- \* connection to supersymmetric gauge theories in 4 dimensions
- \* very rich structure  
particle- and string degrees of freedom

Very involved and difficult to study using ‘traditional’ methods

- \* typically lack Lagrangian description (e.g. (2,0) theories — might not exist?)
- \* lack of perturbative description

⇒ use vast net of dualities to map the problem to a ‘tractable’ setup

Recent proposal for classification using F-theory on elliptically fibered CY-threefolds

- [Heckman, Morrison, Vafa 2013]
- [Del Zotto, Heckman, Tomasiello, Vafa 2013]
- [Heckman 2014]
- [Haghighat, Klemm, Lockhart, Vafa 2014]
- [Heckman, Morrison, Rudelius, Vafa 2015]

# Overview

Focus of this talk:

- \* Degeneracies of BPS configurations (**M- and m-strings**)
- \* Symmetries of the corresponding partition functions (modularity)
- \* Connection to **Little String Theory**
- \* Relations between different theories
- \* Relation to geometric description (Calabi-Yau threefolds)

Explicit Quantities:

- \* explicit expression of (non-perturbative) partition function
- \* BPS counting functions
- \* elliptic genera

F-theory on ell. fibered CY3  $X_N$   
 $(A_N$  fibration over  $T^2)$

Interesting opportunity: stringy tools for objects in gauge th.  
 - efficiently compute partition functions, correlators, etc.  
 - study (new) symmetries

compactification  
on  $S^1$

M-theory on CY3

configuration of M5-branes,  
M2-branes and M-waves

$\mathcal{N} = 1^*$  gauge  
theory in 5 dim.

compactification  
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type II on CY3

(p,q)-brane web in  
type IIB string th.

$\mathcal{N} = 2^*$  gauge  
theory in 4 dim.

[Aharony, Hanany, Kol 1997]

duality  
geometric  
engineering  
[Katz, Klemm, Vafa 1996]

compactification  
on  $S^1$

Main focus of this talk: string degrees of freedom

F-theory: D3-branes wrapping  $\mathbb{P}^1$  in the base

[Ganor, Hanany 1996]  
 [Seiberg, Witten 1996]  
 [Distler, Hanany 1996]

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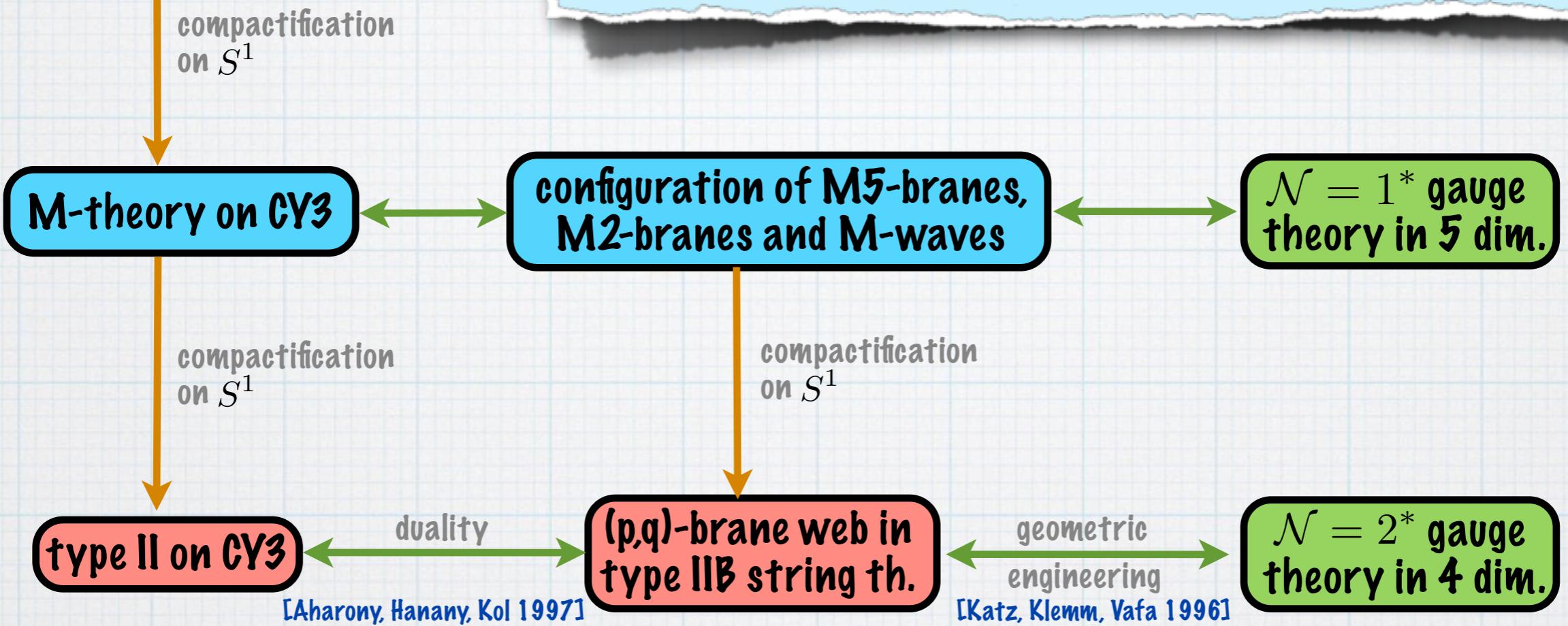
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M-theory: M-strings

[Haghighat, Iqbal, Kozcaz, Lockhart, Vafa 2013]

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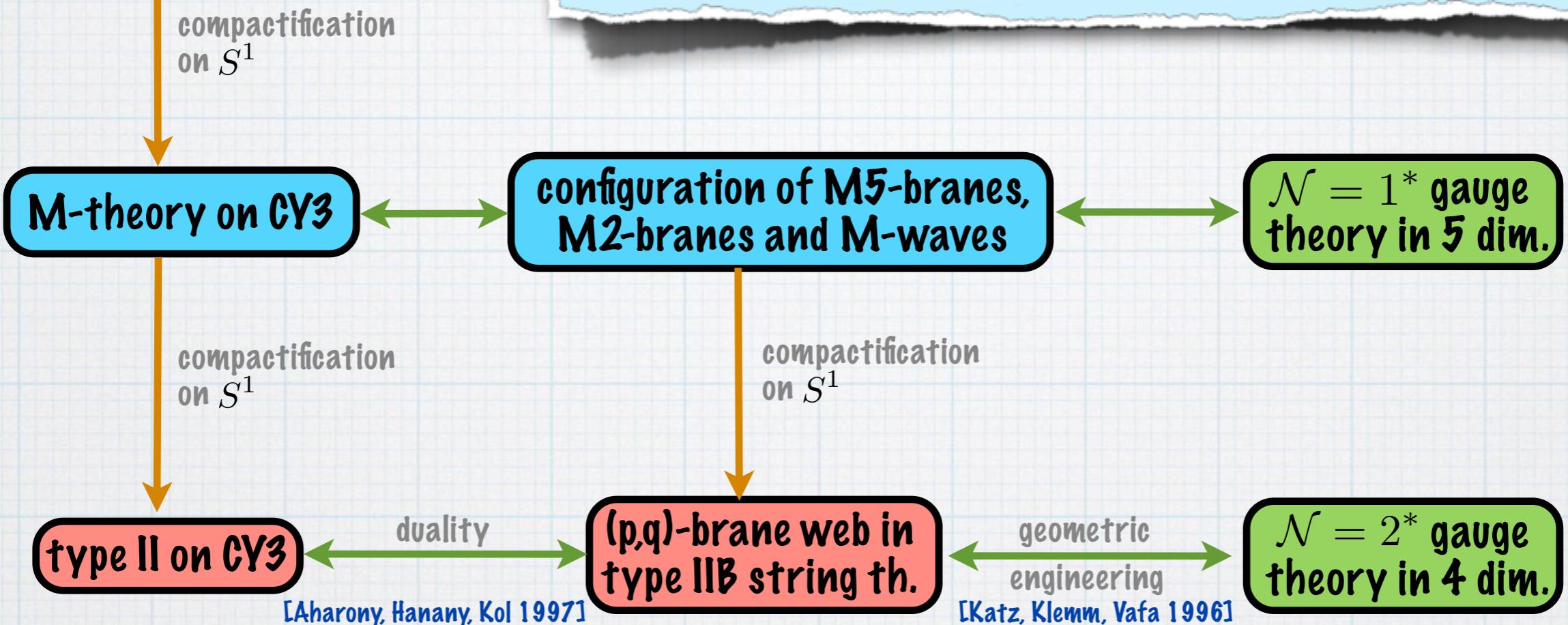
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different approaches: [Witten 1995]  
 [Aspinwall, Morrison 1997]  
 [Seiberg 1997]  
 [Intriligator 1997]  
 [Hanany, Zaffaroni 1997]  
 [Brunner, Karch 1997]

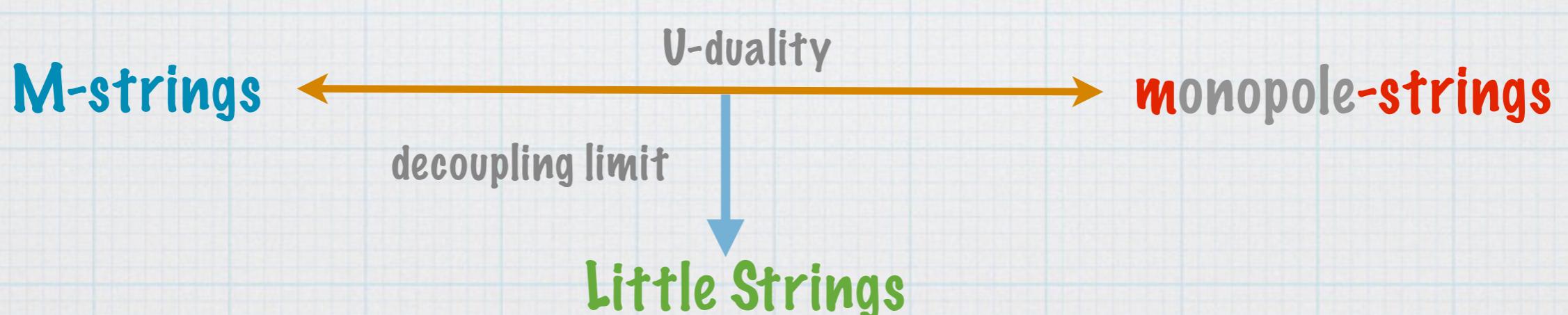
Decoupling limit: Little strings

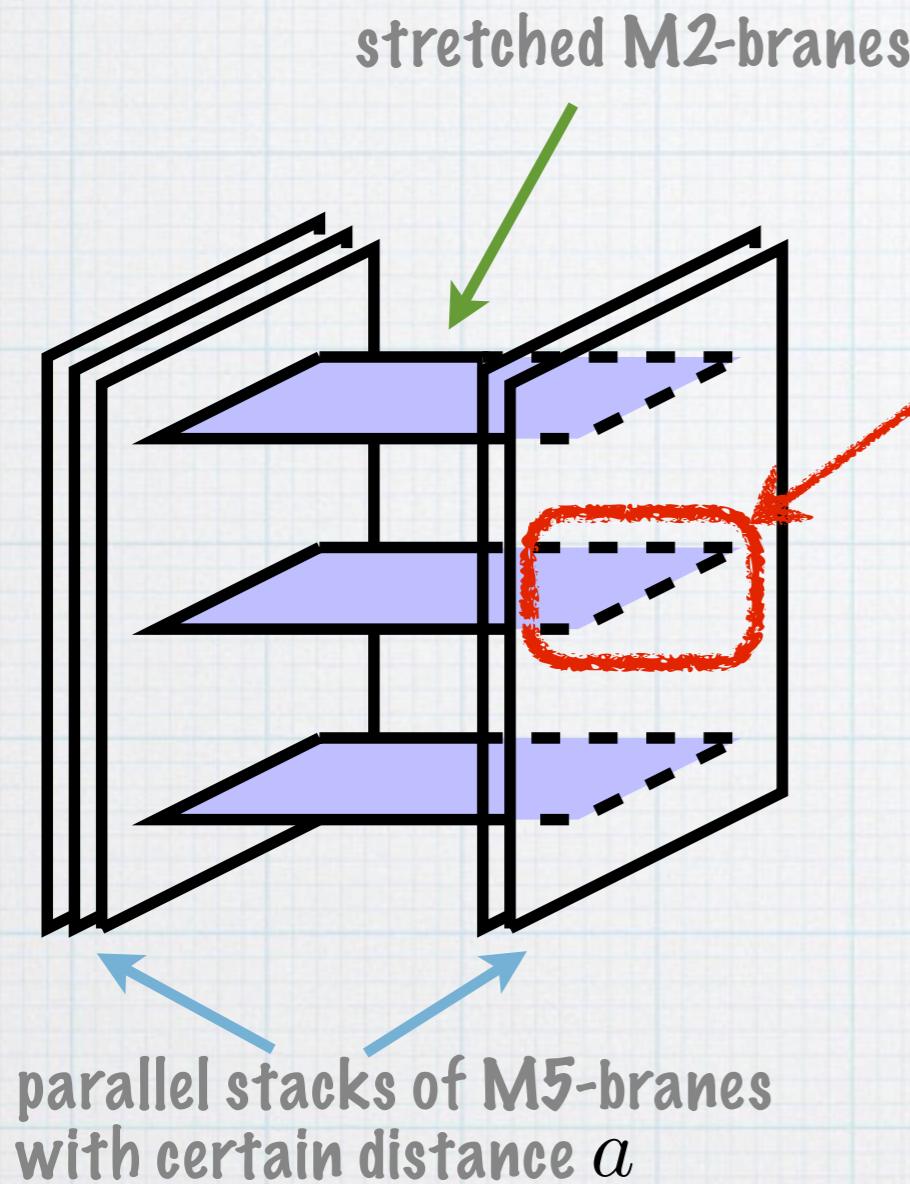
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- \* M-strings defined as one-dimensional intersection between M5- and M2-branes  
[Haghighat, Iqbal, Kozcaz, Lockhart, Vafa 2013]
- \* originally introduced to describe interacting (almost) tensionless strings in 6. dim  
 $\mathcal{N} = (2, 0)$  superconformal theories
- \* F-theory: D3-branes wrapping  $\mathbb{P}^1$  with normal bundle  $\mathcal{O}(-2)$  in the CY3
- \* replacing  $\mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \dots \times \mathbb{P}^1$   
corresponds to multiple parallel M5-branes

## Connection to Gauge Theory:

Gauge Theory partition fct. related to **elliptic genus** of M-strings

[Haghighat, Iqbal, Kozcaz, Lockhart, Vafa 2013]  
[Haghighat, Kozcaz, Lockhart, Vafa 2013]  
[SH, Iqbal 2013]

## 5-dimensional S-duality:



precise duality map and details of the BPS spectra very involved

[Douglas 2010]  
[Tachikawa 2011]

## Implication from Gauge Theory:

degeneracies of BPS m-string states can be extracted from  $\mathcal{N} = 1^*$  gauge theory partition function

→ can be connected to the elliptic genus of M-strings

## Known Example:

Taub NUT space as the moduli space of charge (1,1) monopoles in  $SU(3)$  gauge theory

elliptic genus of Taub NUT

[Harvey, Lee, Murthy 2014]  
[Bak, Gustavsson 2014]



gauge theory index

[Kim, Kim, Koh, Lee, Lee 2011]

agreement in all known instances

want to generalise to other configurations of m-strings

## 6-dimensional systems:

- gravity is decoupled
- have an intrinsic string scale
- obtained from type II string theory through the decoupling limit

$$g_{\text{st}} \rightarrow 0 \text{ while } \ell_{\text{st}} = \text{fixed}$$

## Little String Theories with 16 supercharges

### \* IIA LST of type $A_{N-1}$ with $\mathcal{N} = (1, 1)$ supersymmetry

- decoupling limit of a stack of  $N$  NS5-branes in type IIB with transverse space  $\mathbb{R}^4$
- type IIA string theory on  $A_{N-1}$  orbifold background

related by  
T-duality



### \* IIB LST of type $A_{N-1}$ with $\mathcal{N} = (2, 0)$ supersymmetry

- decoupling limit of a stack of  $N$  NS5-branes in type IIA with transverse space  $\mathbb{R}^4$
- type IIB string theory on  $A_{N-1}$  orbifold background

## M-theory Description:

alternative description of IIB LST as decoupling limit of  $N$  M5-branes with transverse space  $\mathbb{S}^1 \times \mathbb{R}^4$

BPS states from the point of view of M5-branes correspond to M2-branes (M-waves) ending on them

$\implies$  connection to M-strings

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  - have an intrinsic string scale
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## Little String Theories with 8 supercharges: particular class obtained as

- \*  $\mathbb{Z}_N$  orbifold of IIA LST of type  $A_{M-1}$  with  $\mathcal{N} = (1, 0)$  supersymmetry

- decoupling limit of a stack of  $N$  NS5-branes in type IIB with transverse space  $\mathbb{R}^4/\mathbb{Z}_N$

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$\implies$  connection to M-strings

# Brane Configurations

The most general configuration of branes in M-theory in 11 dimensions looks like

	#	0	1	2	3	4	5	6	7	8	9	10
$M5$	$N$	•	•	•	•	•	•					
$M2$	$K$	•	•					•				
$M \sim$	$M$	•	•									

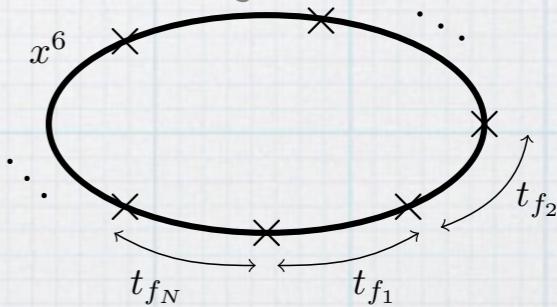
non-compact case:  $\mathbb{R}$

$\mathbb{R}^4_{||}$

$\mathbb{R}^4_{\perp}$

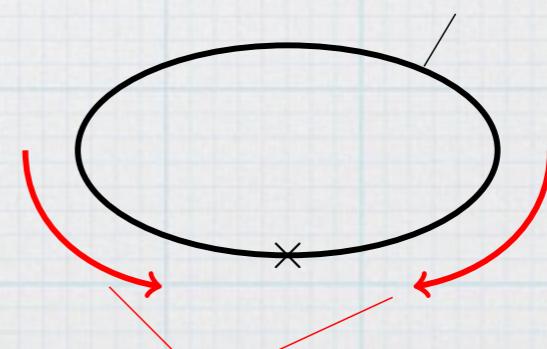
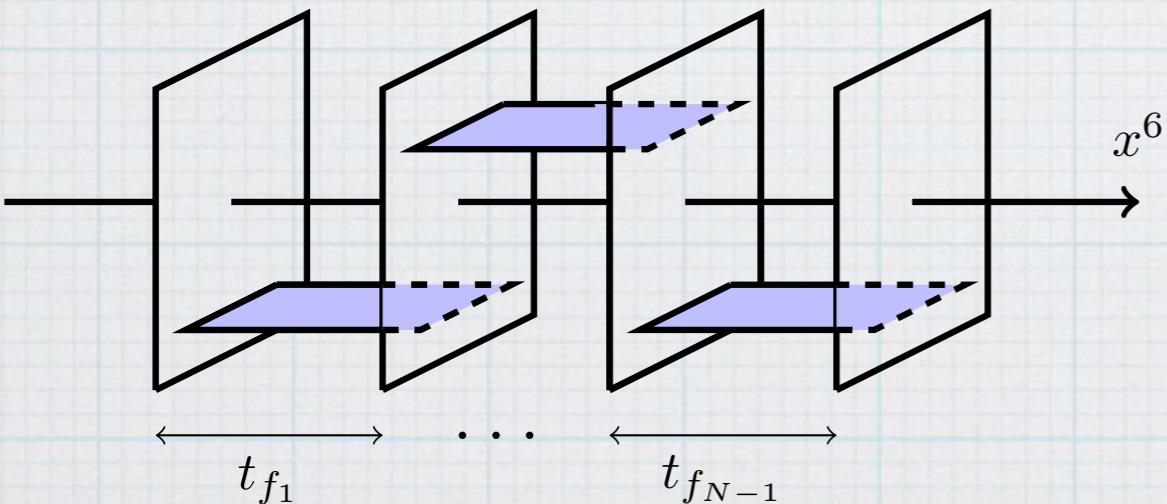
compact case:  $S^1$

M5-branes arranged on a circle



necessary for little-string interpretation

tensionful string going around  $S^1$



limit where all M5-branes form a single stack

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Symmetries:

M5-branes:  $ISO(1, 10) \longrightarrow ISO(1, 5) \times \text{Spin}_R(5) \longrightarrow ISO(1, 5) \times \text{Spin}_R(4)$

SUSY condition:  $\Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \Gamma^5 \epsilon = \epsilon$

M2-branes: distributed among  $(N - 1)/N$  intervals between the M5-branes

$ISO(1, 5) \longrightarrow ISO(1, 1) \times \text{Spin}(4)$

SUSY condition:  $\Gamma^0 \Gamma^1 \Gamma^6 \epsilon = \epsilon$

M-waves: distributed among M5-branes

preserve:  $ISO(1, 1) \times \text{Spin}(4) \times \text{Spin}(4)_R$

SUSY condition:  $\Gamma^0 \Gamma^1 \epsilon = \epsilon$

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$T^2 \sim S^1 \times S^1$ 
 $\mathbb{R}^4_{||}$ 
 $\mathbb{R}^4_{\perp}$

**Compactification:** Compactify (0,1) to  $T^2 \sim S^1 \times S^1$  with radii  $R_0$  and  $R_1 =: \frac{\tau}{2\pi i}$

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$M5$	$N$	●	●	●	●	●	●					
$M2$	$K$	●	●					●				
$M \sim$	$M$	●	●									
$\epsilon_1$				○	○				○	○	○	○
$\epsilon_2$						○	○		○	○	○	○
$m$									○	○	○	○

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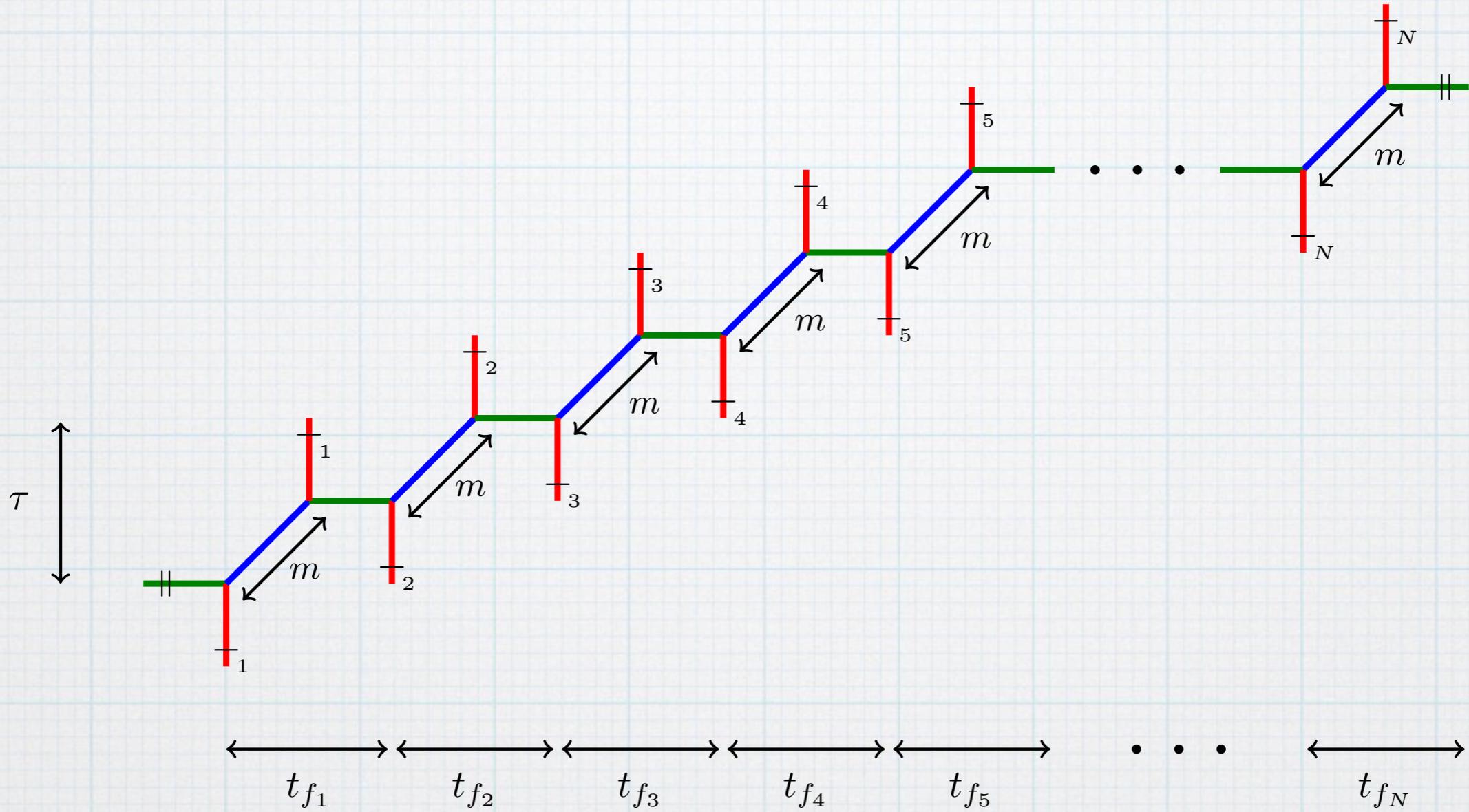
**Deformations:** there are two types of deformations with respect to the compactified (0,1)-directions introducing complex coordinates  $(z_1, z_2) = (x_2 + ix_3, x_4 + ix_5)$  and  $(w_1, w_2) = (x_7 + ix_8, x_9 + ix_{10})$

**(0)-direct.:**  $U(1)_{\epsilon_1} \times U(1)_{\epsilon_2}$ :  $(z_1, z_2) \rightarrow (e^{2\pi i \epsilon_1} z_1, e^{2\pi i \epsilon_2} z_2)$  and  $(w_1, w_2) \rightarrow (e^{-i\pi(\epsilon_1 + \epsilon_2)} w_1, e^{-i\pi(\epsilon_1 + \epsilon_2)} w_2)$

**(1)-direct.:**  $U(1)_m$ :  $(w_1, w_2) \rightarrow (e^{2\pi i m} w_1, e^{-2\pi i m} w_2)$

# Calabi-Yau Geometry

The M-brane configuration can be dualised into a toric Calabi-Yau threefold  $X_N$  with web diagram



**Compact case ( $t_{f_N} < \infty$ ):**  $\rho = i \sum_{a=1}^N t_{f_a} = 2\pi i R_6$

**double elliptic fibration:** elliptic fibration over affine  $\widehat{A}_{N-1}$  which itself is an elliptic fibration over  $\mathbb{C}$

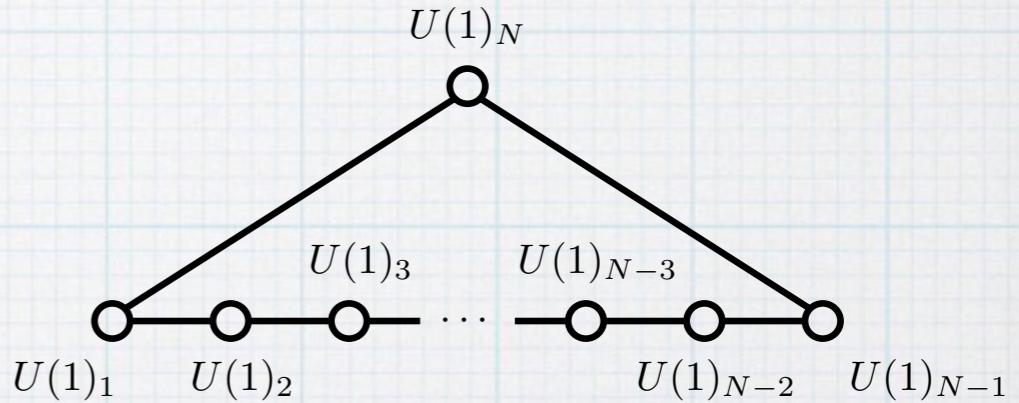
**fiber-base duality:** dual brane setup with 1 M5 brane and N M2-branes starting and ending on it

# Gauge Theory

The compact brane configuration can be related to two (dual) gauge theories

- \* **Gauge Theory 1:**  $U(N)$  gauge theory that reduces to  $\mathcal{N} = 2^*$  supersym. gauge th. in 4dim
- \* **Gauge Theory 2:**  $[U(1)]^N$  circular quiver gauge theory

**non-compact case: linear quiver**



**Identification of Parameters:**

pm.	brane configuration	Calabi-Yau	gauge theory 1	gauge theory 2
$\tau$	size of $\mathbb{S}^1$ parallel to M5-branes	Kähler moduli of elliptic base	coupling constant	compact Coulomb branch parameter
$\rho$	size of $\mathbb{S}^1$ transverse to M5 branes	Kähler moduli of affine $A_{N-1}$ -fiber	compact Coulomb branch parameter	overall coupling constant
$t_{f_a}$	separations between adjacent M5-branes	Kähler moduli of affine $A_{N-1}$ -fiber	compact Coulomb branch parameter	coupling constants $a = 1, \dots, N - 1$

# M-Strings BPS States and Topological String

**Free Energy:** Counts number of M-string BPS configurations

Computed by topological free energy  $F_{X_N} = \ln \mathcal{Z}_N$  on  $X_N$  in the dual picture

[Haghighat, Iqbal, Kozcaz, Lockhart, Vafa 2013]

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**Topological free Energy:** sum over massive particles, i.e. M2-branes wrapping holomorphic curves on  $X_N$  (classified by little group  $SU(2)_L \times SU(2)_R$ )

$$F_{X_N}(\omega, \epsilon_1, \epsilon_2) = \sum_{\beta \in H_2(X_N, \mathbb{Z})} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{j_L, j_R} \frac{e^{-n \int_{\beta} \omega} N_{\beta}^{j_L, j_R} (-1)^{2j_L + 2j_R} \text{Tr}_{j_L}(\sqrt{qt})^{n j_L, 3} \text{Tr}_{j_R}(\sqrt{q/t})^{n j_R, 3}}{(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})}$$

complexified Kähler class of  $X_N$

sum over representations of  $SU(2)_L \times SU(2)_R$

degeneracies of BPS states

$$q = e^{2\pi i \epsilon_1}$$

$$t = e^{-2\pi i \epsilon_2}$$

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$F_{X_N}$  counts single- and multi-particle states  $\rightarrow$  extract single particle bound states

$$F_{X_N} = \sum_{n=1}^{\infty} \frac{\Omega(n\omega, n\epsilon_1, n\epsilon_2)}{n}$$



$$\begin{aligned} \Omega(\omega, \epsilon_1, \epsilon_2) &= \text{PLog } \mathcal{Z}_{X_N}(\omega, \epsilon_1, \epsilon_2) && \text{M\"obius function} \\ &= \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \ln \mathcal{Z}_{X_N}(k\omega, k\epsilon_1, k\epsilon_2) \end{aligned}$$

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**Topological Vertex Calculation:** two distinct expansion of the partition function

**vertical:**  $\mathcal{Z}_{X_N} = Z_1(m, t_{f_1}, \dots, t_{f_N}, \epsilon_{1,2}) \sum_{k \geq 0} Q_{\tau}^k C_{N,k}(m, t_{f_1}, \dots, t_{f_N}, \epsilon_{1,2})$

**horizontal:**  $\mathcal{Z}_{X_N} = Z_2(N, \tau, m, \epsilon_{1,2}) \sum_{\nu_1, \dots, \nu_N} \left( \prod_{a=1}^N (-Q_{f_a})^{|\nu_a|} \right) Z_{\nu_1 \nu_2 \dots \nu_N}(\tau, m, \epsilon_1, \epsilon_2)$

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$$\Omega(\omega, \epsilon_1, \epsilon_2) = \text{PLog } \mathcal{Z}_{V, \cdot}(\omega, \epsilon_1, \epsilon_2)$$

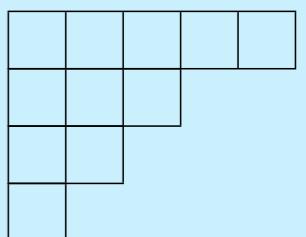
$$= \sum_{k=1}^{\infty} \frac{\mu}{\nu_a}$$

**Notation:**

$$Q_{\tau} = e^{2\pi i \tau} \text{ and } Q_{f_a} = e^{2\pi i t_{f_a}}$$

$\nu_a$  integer partitions of length  $\ell(\nu_a)$

$$|\nu_a| = \sum_{r=1}^{\ell(\nu_a)} \nu_{a,r}$$



**Topological Vertex Calculation:** two distinct expansion of the partition

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**vertical:**  $\mathcal{Z}_{X_N} = Z_1(m, t_{f_1}, \dots, t_{f_N}, \epsilon_{1,2}) \sum_{k \geq 0} Q_{\tau}^k C_{N,k}(m, t_{f_1}, \dots, t_{f_N}, \epsilon_{1,2})$

instanton expansion in gauge theory 1

**horizontal:**  $\mathcal{Z}_{X_N} = Z_2(N, \tau, m, \epsilon_{1,2}) \sum_{\nu_1, \dots, \nu_N} \left( \prod_{a=1}^N (-Q_{f_a})^{|\nu_a|} \right) Z_{\nu_1 \nu_2 \dots \nu_N}(\tau, m, \epsilon_1, \epsilon_2)$

instanton expansion in gauge theory 2

vertical:

$$\mathcal{Z}_{X_N} = Z_1(m, t_{f_1}, \dots, t_{f_N}, \epsilon_{1,2}) \sum_{k \geq 0} Q_\tau^k C_{N,k}(m, t_{f_1}, \dots, t_{f_N}, \epsilon_{1,2})$$

instanton contribution can be written as sum over integer partitions  $(\alpha_1, \dots, \alpha_N)$

$$\tilde{\mathcal{Z}}_N^{(1)} = \sum_{k \geq 0} Q_\tau^k C_{N,k}(m, t_{f_1}, \dots, t_{f_N}, \epsilon_{1,2})$$

$$= \sum_{\alpha_1 \dots \alpha_N} Q_\tau^{|\alpha_1| + \dots + |\alpha_N|} \prod_{a=1}^N \frac{\vartheta_{\alpha_a \alpha_a}(Q_m)}{\vartheta_{\alpha_a \alpha_a}(\sqrt{\frac{t}{q}})} \prod_{1 \leq a < b \leq N} \frac{\vartheta_{\alpha_a \alpha_b}(Q_{ab} Q_m^{-1}) \vartheta_{\alpha_a \alpha_b}(Q_{ab} Q_m)}{\vartheta_{\alpha_a \alpha_b}(Q_{ab} \sqrt{\frac{t}{q}}) \vartheta_{\alpha_a \alpha_b}(Q_{ab} \sqrt{\frac{q}{t}})}$$

**notation:**

$$Q_m = e^{2\pi i m}, \quad Q_\tau = e^{2\pi i \tau}, \quad q = e^{i\epsilon_1}, \quad t = e^{-i\epsilon_2}, \quad Q_{ab} = \prod_{k=a}^{b-1} Q_{f_k}$$

**theta functions:**

$$\vartheta_{\mu\nu}(x) = \prod_{(i,j) \in \mu} \theta_1(\rho; x^{-1} t^{-\nu_j^t + i - \frac{1}{2}} q^{-\mu_i + j - \frac{1}{2}}) \prod_{(i,j) \in \nu} \theta_1(\rho; x^{-1} t^{\mu_j^t - i + \frac{1}{2}} q^{\nu_i - j + \frac{1}{2}})$$

$$\theta_1(\tau, m) = -i Q_\tau^{1/8} Q_m^{1/2} \prod_{n=1}^{\infty} (1 - Q_\tau^n) (1 - Q_m Q_\tau^n) (1 - Q_m^{-1} Q_\tau^{n-1})$$

vertical:

$$\mathcal{Z}_{X_N} = Z_1(m, t_{f_1}, \dots, t_{f_N}, \epsilon_{1,2}) \sum_{k \geq 0} Q_\tau^k C_{N,k}(m, t_{f_1}, \dots, t_{f_N}, \epsilon_{1,2})$$

instanton contribution can be written as sum over integer partitions  $(\alpha_1, \dots, \alpha_N)$

$$\begin{aligned} \tilde{\mathcal{Z}}_N^{(1)} &= \sum_{k \geq 0} Q_\tau^k C_{N,k}(m, t_{f_1}, \dots, t_{f_N}, \epsilon_{1,2}) \\ &= \sum_{\alpha_1 \dots \alpha_N} Q_\tau^{|\alpha_1| + \dots + |\alpha_N|} \prod_{a=1}^N \frac{\vartheta_{\alpha_a \alpha_a}(Q_m)}{\vartheta_{\alpha_a \alpha_a}(\sqrt{\frac{t}{q}})} \prod_{1 \leq a < b \leq N} \frac{\vartheta_{\alpha_a \alpha_b}(Q_{ab} Q_m^{-1}) \vartheta_{\alpha_a \alpha_b}(Q_{ab} Q_m)}{\vartheta_{\alpha_a \alpha_b}(Q_{ab} \sqrt{\frac{t}{q}}) \vartheta_{\alpha_a \alpha_b}(Q_{ab} \sqrt{\frac{q}{t}})} \end{aligned}$$

Free Energy:

$$\Sigma_N(\tau, \rho, m, t_{f_1}, \dots, t_{f_{N-1}}, \epsilon_1, \epsilon_2) = \text{PLog } \tilde{\mathcal{Z}}_N^{(1)}(\tau, \rho, m, t_{f_1}, \dots, t_{f_{N-1}}, \epsilon_1, \epsilon_2)$$

Expansion in powers of  $Q_\tau$ :

$$\Sigma_N(\tau, \rho, m, t_{f_1}, \dots, t_{f_{N-1}}, \epsilon_1, \epsilon_2) = \sum_{k=0}^{\infty} Q_\tau^k \Sigma_{N,k}(\rho, m, t_{f_1}, \dots, t_{f_{N-1}}, \epsilon_1, \epsilon_2)$$

Further expansion in powers of  $Q_{f_a}$ :

$$\Sigma_{N,k}(\rho, m, t_{f_1}, \dots, t_{f_{N-1}}, \epsilon_1, \epsilon_2) = \sum_{k_1, \dots, k_{N-1}} Q_{f_1}^{k_1} \dots Q_{f_{N-1}}^{k_{N-1}} \Sigma_{N,k}^{(k_1, \dots, k_{N-1})}(\rho, m, \epsilon_1, \epsilon_2)$$

horizontal:

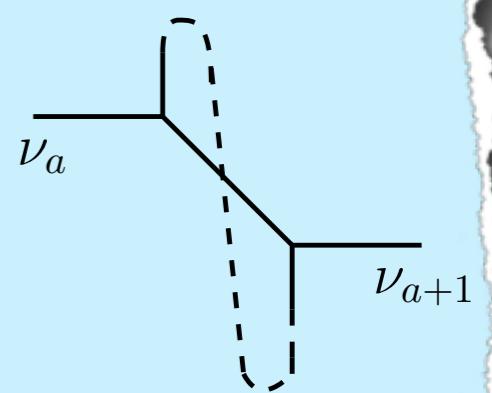
$$\mathcal{Z}_{X_N} = Z_2(N, \tau, m, \epsilon_{1,2}) \sum_{\nu_1, \dots, \nu_N} \left( \prod_{a=1}^N (-Q_{f_a})^{|\nu_a|} \right) Z_{\nu_1 \nu_2 \dots \nu_N}(\tau, m, \epsilon_1, \epsilon_2)$$

instanton contribution constructed from a single building block:

$$Z_{\nu_1 \nu_2 \dots \nu_N}(\tau, m, \epsilon_1, \epsilon_2) = D_{\nu_1 \nu_2^t}(t, q) D_{\nu_2 \nu_3^t}(t, q) D_{\nu_3 \nu_4^t}(t, q) \cdots D_{\nu_N^t \nu_1}(t, q)$$

**building block:**

$$D_{\nu_a^t \nu_{a+1}}(\tau, m, t, q) = \left[ t^{-\frac{\|\nu_{a+1}^t\|^2}{2}} q^{-\frac{\|\nu_a\|^2}{2}} Q_m^{-\frac{|\nu_a| + |\nu_{a+1}|}{2}} \right] \\ \times \prod_{k=1}^{\infty} \left[ \prod_{(i,j) \in \nu_a} \frac{(1 - Q_\tau^k Q_m^{-1} q^{-\nu_{a,i} + j - \frac{1}{2}} t^{-\nu_{a+1,j}^t + i - \frac{1}{2}})(1 - Q_\tau^{k-1} Q_m q^{\nu_{a,i} - j + \frac{1}{2}} t^{\nu_{a+1,j}^t - i + \frac{1}{2}})}{(1 - Q_\tau^k q^{\nu_{a,i} - j} t^{\nu_{a,j}^t - i + 1})(1 - Q_\tau^{k-1} q^{-\nu_{a,i} + j - 1} t^{-\nu_{a,j}^t + i})} \right. \\ \left. \times \prod_{(i,j) \in \nu_{a+1}} \frac{(1 - Q_\tau^k Q_m^{-1} q^{\nu_{a+1,i} - j + \frac{1}{2}} t^{\nu_{a,j}^t - i + \frac{1}{2}})(1 - Q_\tau^{k-1} Q_m q^{-\nu_{a+1,i} + j - \frac{1}{2}} t^{-\nu_{a+1,j}^t + i - \frac{1}{2}})}{(1 - Q_\tau^k q^{\nu_{a+1,i} - j + 1} t^{\nu_{a+1,j}^t - i})(1 - Q_\tau^{k-1} q^{-\nu_{a+1,i} + j} t^{-\nu_{a+1,j}^t + i - 1})} \right]$$



horizontal:

$$\mathcal{Z}_{X_N} = Z_2(N, \tau, m, \epsilon_{1,2}) \sum_{\nu_1, \dots, \nu_N} \left( \prod_{a=1}^N (-Q_{f_a})^{|\nu_a|} \right) Z_{\nu_1 \nu_2 \dots \nu_N}(\tau, m, \epsilon_1, \epsilon_2)$$

instanton contribution constructed from a single building block:

$$Z_{\nu_1 \nu_2 \dots \nu_N}(\tau, m, \epsilon_1, \epsilon_2) = D_{\nu_1 \nu_2^t}(t, q) D_{\nu_2 \nu_3^t}(t, q) D_{\nu_3 \nu_4^t}(t, q) \cdots D_{\nu_N^t \nu_1}(t, q)$$

leading to the non-perturbative partition function:

$$\tilde{\mathcal{Z}}_N^{(2)}(\tau, m, t_{f_1}, \dots, t_{f_N}, \epsilon_1, \epsilon_2) = \sum_{\nu_1, \dots, \nu_N} \left( \prod_{a=1}^N (-Q_{f_a})^{|\nu_a|} \right) \prod_{a=1}^N \prod_{(i,j) \in \nu_a} \frac{\theta_1(\tau; z_{ij}^a) \theta_1(\tau; v_{ij}^a)}{\theta_1(\tau; w_{ij}^a) \theta_1(\tau; u_{ij}^a)}$$

Arguments of theta-functions:

$$e^{2\pi i z_{ij}^a} = Q_m^{-1} q^{\nu_{a,i} - j + \frac{1}{2}} t^{\nu_{a+1,j}^t - i + \frac{1}{2}},$$

$$e^{2\pi i v_{ij}^a} = Q_m^{-1} t^{-\nu_{a-1,j}^t + i - \frac{1}{2}} q^{-\nu_{a,i} + j - \frac{1}{2}},$$

$$e^{2\pi i w_{ij}^a} = q^{\nu_{a,i} - j + 1} t^{\nu_{a,j}^t - i},$$

$$e^{2\pi i u_{ij}^a} = q^{\nu_{a,i} - j} t^{\nu_{a,j}^t - i + 1}.$$

horizontal:

$$\mathcal{Z}_{X_N} = Z_2(N, \tau, m, \epsilon_{1,2}) \sum_{\nu_1, \dots, \nu_N} \left( \prod_{a=1}^N (-Q_{f_a})^{|\nu_a|} \right) Z_{\nu_1 \nu_2 \dots \nu_N}(\tau, m, \epsilon_1, \epsilon_2)$$

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Free Energy:

$$\Omega_N(\tau, m, t_{f_1}, \dots, t_{f_N}, \epsilon_1, \epsilon_2) = \text{PLog } \tilde{\mathcal{Z}}_N^{(2)}(\tau, m, t_{f_1}, \dots, t_{f_N}, \epsilon_1, \epsilon_2)$$

Further expansion in powers of  $Q_{f_a}$ :

$$\Omega_N(\tau, m, t_{f_1}, \dots, t_{f_N}, \epsilon_1, \epsilon_2) = \sum_{k_1, \dots, k_N \geq 0} Q_{f_1}^{k_1} \cdots Q_{f_N}^{k_N} G^{(k_1, \dots, k_N)}(\tau, m, \epsilon_1, \epsilon_2)$$

**non-compact:**

starting from the horizontal description

$$\tilde{\mathcal{Z}}_N^{(2)}(\tau, m, t_{f_1}, \dots, t_{f_N}, \epsilon_1, \epsilon_2) = \sum_{\nu_1, \dots, \nu_N} \left( \prod_{a=1}^N (-Q_{f_a})^{|\nu_a|} \right) \prod_{a=1}^N \prod_{(i,j) \in \nu_a} \frac{\theta_1(\tau; z_{ij}^a) \theta_1(\tau; v_{ij}^a)}{\theta_1(\tau; w_{ij}^a) \theta_1(\tau; u_{ij}^a)}$$

the non-compact partition function is obtained through the limit  $Q_{f_N} \rightarrow 0$  which limits  $\nu_N = \emptyset$

$$\tilde{\mathcal{Z}}_{X_N}^{\text{line}}(\tau, m, t_{f_1}, \dots, t_{f_{N-1}}, \epsilon_1, \epsilon_2) = \sum_{\substack{\nu_1, \dots, \nu_{N-1} \\ \nu_0 = \nu_N = \emptyset}} \left( \prod_{a=1}^N (-Q_{f_a})^{|\nu_a|} \right) \prod_{a=1}^N \prod_{(i,j) \in \nu_a} \frac{\theta_1(\tau; z_{ij}^a) \theta_1(\tau; v_{ij}^a)}{\theta_1(\tau; w_{ij}^a) \theta_1(\tau; u_{ij}^a)}$$

**Free Energy:**

$$\Omega_N^{\text{line}}(\tau, m, t_{f_1}, \dots, t_{f_{N-1}}, \epsilon_1, \epsilon_2) = \text{PLog } \tilde{\mathcal{Z}}_N^{\text{line}}(\tau, m, t_{f_1}, \dots, t_{f_{N-1}}, \epsilon_1, \epsilon_2)$$

Expansion in powers of  $Q_{f_a}$ :

$$\Omega_N^{\text{line}}(\tau, m, t_{f_1}, \dots, t_{f_{N-1}}, \epsilon_1, \epsilon_2) = \sum_{k_1, \dots, k_{N-1} \geq 0} Q_{f_1}^{k_1} \cdots Q_{f_{N-1}}^{k_{N-1}} F^{(k_1, \dots, k_{N-1})}(\tau, m, \epsilon_1, \epsilon_2)$$

# Compact versus Non-Compact Free Energies

Compact free energies can be expressed in terms of non-compact ones:

$$\lim_{\epsilon_2 \rightarrow 0} \epsilon_2 G^{(\{k_i\})}(\tau, m, \epsilon_1, \epsilon_2) = \lim_{\epsilon_2 \rightarrow 0} \epsilon_2 d_{(\{k_i\})} \sum_{\sum m_i = K} a_{(\{m_i\})} F^{(\{m_i\})}(\tau, m, \epsilon_1, \epsilon_2)$$

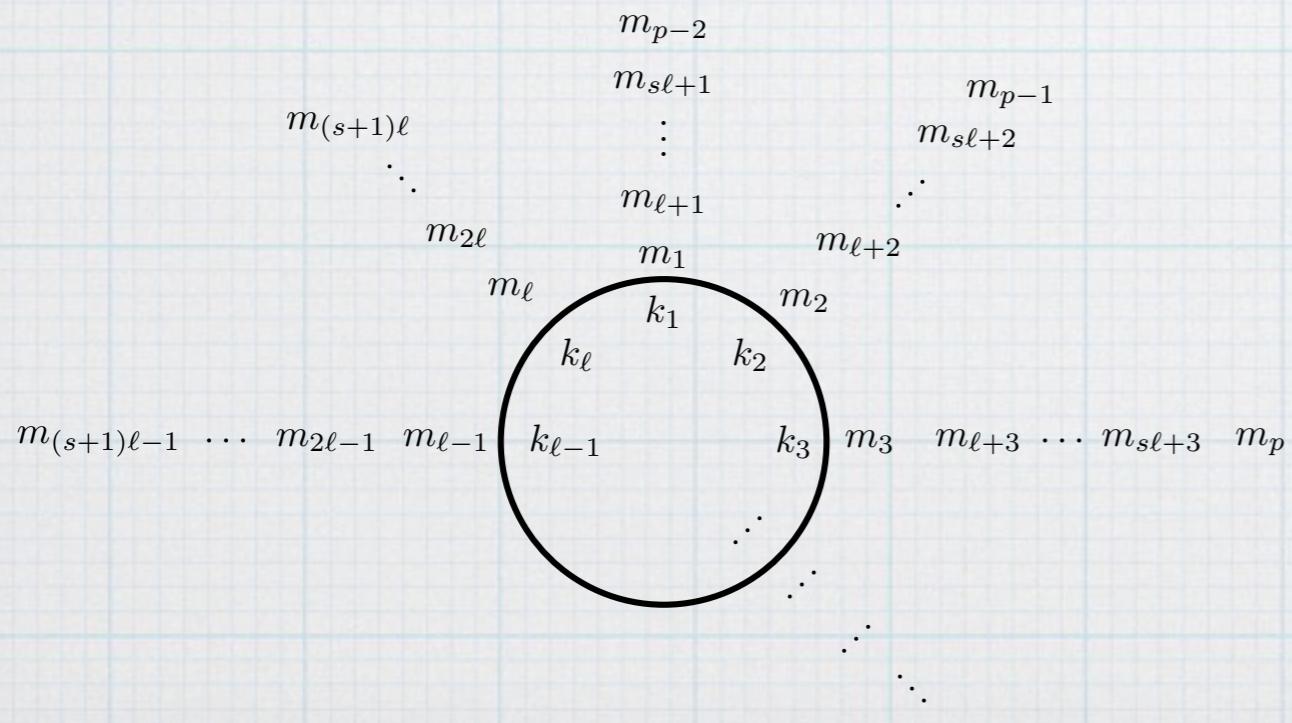
[SH, Iqbal, Rey 2015]

where the numerical coefficients are defined as:

$$d_{(\{k_i\})} = \begin{cases} n = \frac{\ell}{m} & \text{if } (\{k_i\}) = (\underbrace{\{k_j\}_m, \dots, \{k_j\}_m}_{n \text{ times}}) \\ 1 & \text{else} \end{cases}$$

$$a_{(\{m_i\})} = \begin{cases} 1 & \text{if } k_i = \sum_{r=0}^{\infty} m_{i+r\ell} \\ 0 & \text{else} \end{cases} \quad (i = 1, \dots, \ell)$$

## Intuitive picture: ‘wrapping’ configurations



## examples:

$$G^{(2)} = F^{(2)} + F^{(1,1)}$$

$$G^{(1,1)} = 2 F^{(1,1)}$$

$$G^{(3)} = F^{(3)} + F^{(2,1)} + F^{(1,2)} + F^{(1,1,1)}$$

$$\equiv F^{(3)} + 2F^{(2,1)} + F^{(1,1,1)}$$

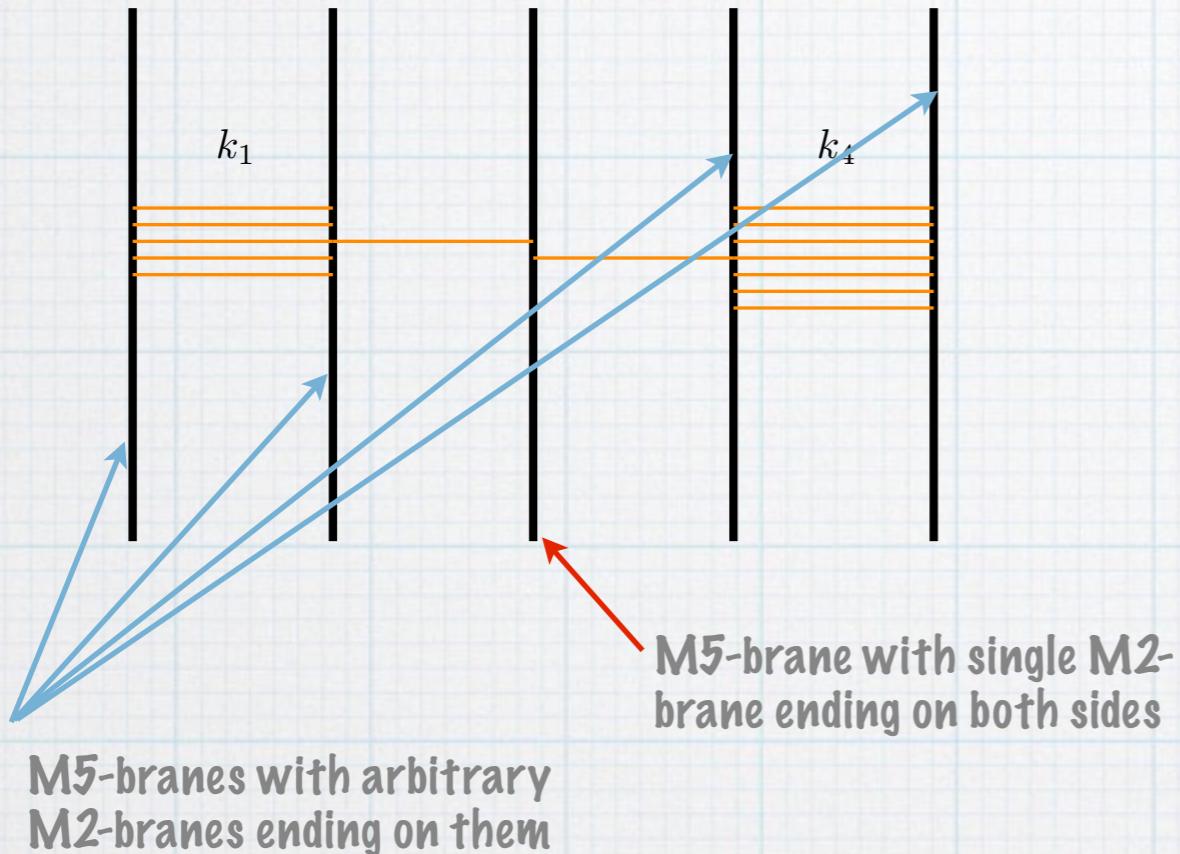
$$G^{(2,1,2,1)} = 2 \left( F^{(2,1,2,1)} + F^{(1,2,1,2)} + F^{(1,1,2,1,1)} \right)$$

$$= 2 \left( 2F^{(2,1,2,1)} + F^{(1,1,2,1,1)} \right)$$

# Relations between non-compact Free Energies

there are relations among the free energies of specific M5-M2-brane configurations

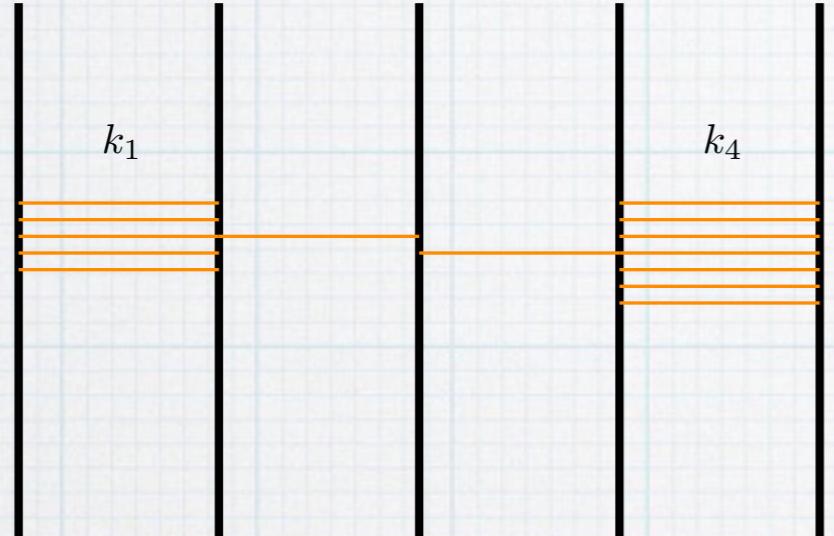
see [Bak, Gustavsson 2014] for earlier partial results



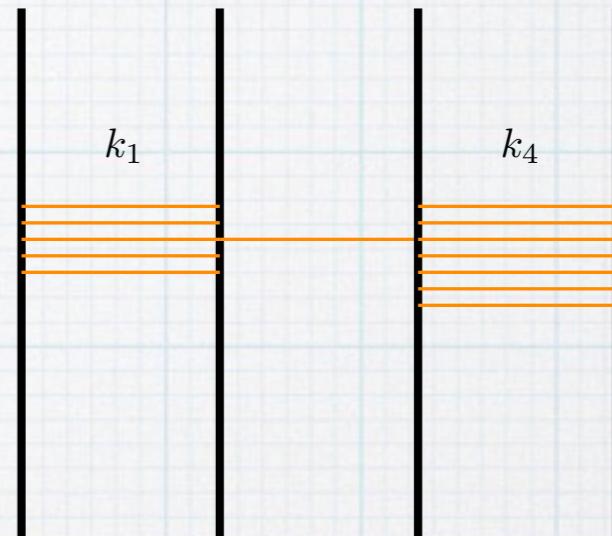
# Relations between non-compact Free Energies

there are relations among the free energies of specific M5-M2-brane configurations

see [Bak, Gustavsson 2014] for earlier partial results



$$= W(\tau, m, \epsilon_1, \epsilon_2)$$



$$F^{(k_1, \dots, k_r, 1, 1, k_{r+1}, \dots, k_{N-1})} = W(\tau, m, \epsilon_1, \epsilon_2) F^{(k_1, \dots, k_r, 1, k_{r+1}, \dots, k_{N-1})}$$

with the function

$$W(\tau, m, \epsilon_1, \epsilon_2) = \frac{\theta_1(\tau, m + \epsilon_+) \theta_1(\tau, m - \epsilon_+) - \theta_1(\tau, m + \epsilon_-) \theta_1(\tau, m - \epsilon_-)}{\theta_1(\tau, \epsilon_1) \theta_1(\tau, \epsilon_2)}$$

can be applied repeatedly to obtain

$$F^{(k_1, k_2, \dots, k_r, 1, 1, 1, \dots, 1, \dots, k_{r+s+1}, \dots, k_{N-1})} = (W(\tau, m, \epsilon_1, \epsilon_2))^{s-1} F^{(k_1, k_2, \dots, k_r, 1, k_{r+s+1}, \dots, k_{N-1})}$$

# Summary of M-string Partition Fcts. and Free

quantity	gauge theory 1	gauge theory 2	non-compact theory
variables	$\tau, \rho, m, t_{f_1}, \dots t_{f_{N-1}}, \epsilon_{1,2}$	$\tau, m, t_{f_1}, \dots t_{f_N}, \epsilon_{1,2}$	$\tau, m, t_{f_1}, \dots t_{f_{N-1}}, \epsilon_{1,2}$
partition function	$\tilde{\mathcal{Z}}_N^{(1)}(\tau, \rho, m, t_{f_a}, \epsilon_{1,2})$	$\tilde{\mathcal{Z}}_N^{(2)}(\tau, m, t_{f_a}, \epsilon_{1,2})$	$\tilde{\mathcal{Z}}_N^{\text{line}}(\tau, m, t_{f_a}, \epsilon_{1,2})$
free energy	$\Sigma_{N,k}(\rho, m, t_{f_a}, \epsilon_{1,2})$	$\Omega_N(\tau, m, t_{f_a}, \epsilon_{1,2})$	$\Omega_N^{\text{line}}(\tau, m, t_{f_a}, \epsilon_{1,2})$
counting functions	$\Sigma_{N,k}^{(\{k_i\})}(\rho, m, \epsilon_{1,2})$	$G^{(\{k_i\})}(\tau, m, \epsilon_{1,2})$	$F^{(\{k_i\})}(\tau, m, \epsilon_{1,2})$

## Modular Properties:

The partition functions have modular properties under  $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$

$$(\tau, \rho, m, t_{f_a}, \epsilon_1, \epsilon_2) \mapsto \left( \frac{a\tau + b}{c\tau + d}, \rho, \frac{m}{c\tau + d}, t_{f_a}, \frac{\epsilon_1}{c\tau + d}, \frac{\epsilon_2}{c\tau + d} \right)$$

$$(\tau, \rho, m, t_{f_a}, \epsilon_1, \epsilon_2) \mapsto \left( \tau, \frac{a\rho + b}{c\rho + d}, \frac{m}{c\rho + d}, \frac{t_{f_a}}{c\rho + d}, \frac{\epsilon_1}{c\rho + d}, \frac{\epsilon_2}{c\rho + d} \right)$$

with  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z})$

The free energies transform (almost) as Jacobi forms, but require a non-holomorphic modification

## Geometric Interpretation:

Double elliptic fibration structure with parameters  $\tau$  and  $\rho$

# Example of a Modular Transformation and NS-limit

$$Z_{k_1 \dots k_{N-1}}(\tau, m, \epsilon_{1,2}) = (-1)^{k_1 + \dots + k_{N-1}} \sum_{\nu_a, |\nu_a| = k_a} \prod_{a=1}^{N-1} \prod_{(i,j) \in \nu_a} \frac{\theta_1(\tau; z_{ij}^a) \theta_1(\tau; v_{ij}^a)}{\theta_1(\tau; w_{ij}^a) \theta_1(\tau; u_{ij}^a)}$$

**Modular Transformation:**

$$(\tau, m, \epsilon_1, \epsilon_2) \mapsto \left( \frac{a\tau + b}{c\tau + d}, \frac{m}{c\tau + d}, \frac{\epsilon_1}{c\tau + d}, \frac{\epsilon_2}{c\tau + d} \right) \quad \text{with} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z})$$

for  $r, \ell \in \mathbb{Z}$  and  $K = \sum_{a=1}^{N-1} k_a$  we have the relations

$$Z_{k_1 \dots k_{N-1}}(\tau + 1, m, \epsilon_1, \epsilon_2) = Z_{k_1 \dots k_{N-1}}(\tau, m, \epsilon_1, \epsilon_2)$$

$$Z_{k_1 \dots k_{N-1}}\left(-\frac{1}{\tau}, \frac{m}{\tau}, \frac{\epsilon_1}{\tau}, \frac{\epsilon_2}{\tau}\right) = e^{\frac{2\pi i}{\tau} f_{\vec{k}}(m, \epsilon_1, \epsilon_2)} Z_{k_1 \dots k_{N-1}}(\tau, m, \epsilon_1, \epsilon_2)$$

$$Z_{k_1 \dots k_{N-1}}(\tau, m + \ell\tau + r, \epsilon_1, \epsilon_2) = e^{-2\pi i K \ell^2 \tau + 4\pi i m K} Z_{k_1 \dots k_{N-1}}(\tau, m, \epsilon_1, \epsilon_2)$$

$Z_{k_1 \dots k_{N-1}}$  transforms as a Jacobi form with respect to  $(\tau, m)$

**Simplification: Nekrasov-Shatashvili limit**

[Nekrasov, Shatashvili 2009]  
[Mironov, Morozov 2009]

$$\epsilon_2 = 0 \longrightarrow f_{\vec{k}}(m, \epsilon_1) = f_K(m, \epsilon_1) \quad \text{depends only on the total number of M2-branes}$$

# Duality and Relation to m-strings

**Goal:** we want to map the counting of BPS M-string states in 6 dims. to the counting of BPS monopole strings (m-strings) in 5 dimensions

**Three Steps:** 1) compactify (5)-direction

[Haghighat 2015]  
[SH, Iqbal, Rey 2015]

recall brane-setup and deformations

	#	(0)	(1)	2	3	4	(5)	6	7	8	9	10
$M5$	$N$	●	●	●	●	●	●					
$M2$	$K$	●	●					●				
$M \sim$	$M$	●	●									
$\epsilon_1$				○	○				○	○	○	○
$\epsilon_2$						○	○		○	○	○	○
$m$									○	○	○	○

twisted by  $\Omega$  rotation      untwisted KK-cycle wrapped by M2-branes      S-dualise

topologically  $T^3$        $SL(3, \mathbb{Z})$  action

# Duality and Relation to m-strings

**Goal:** we want to map the counting of BPS M-string states in 6 dims. to the counting of BPS monopole strings (m-strings) in 5 dimensions

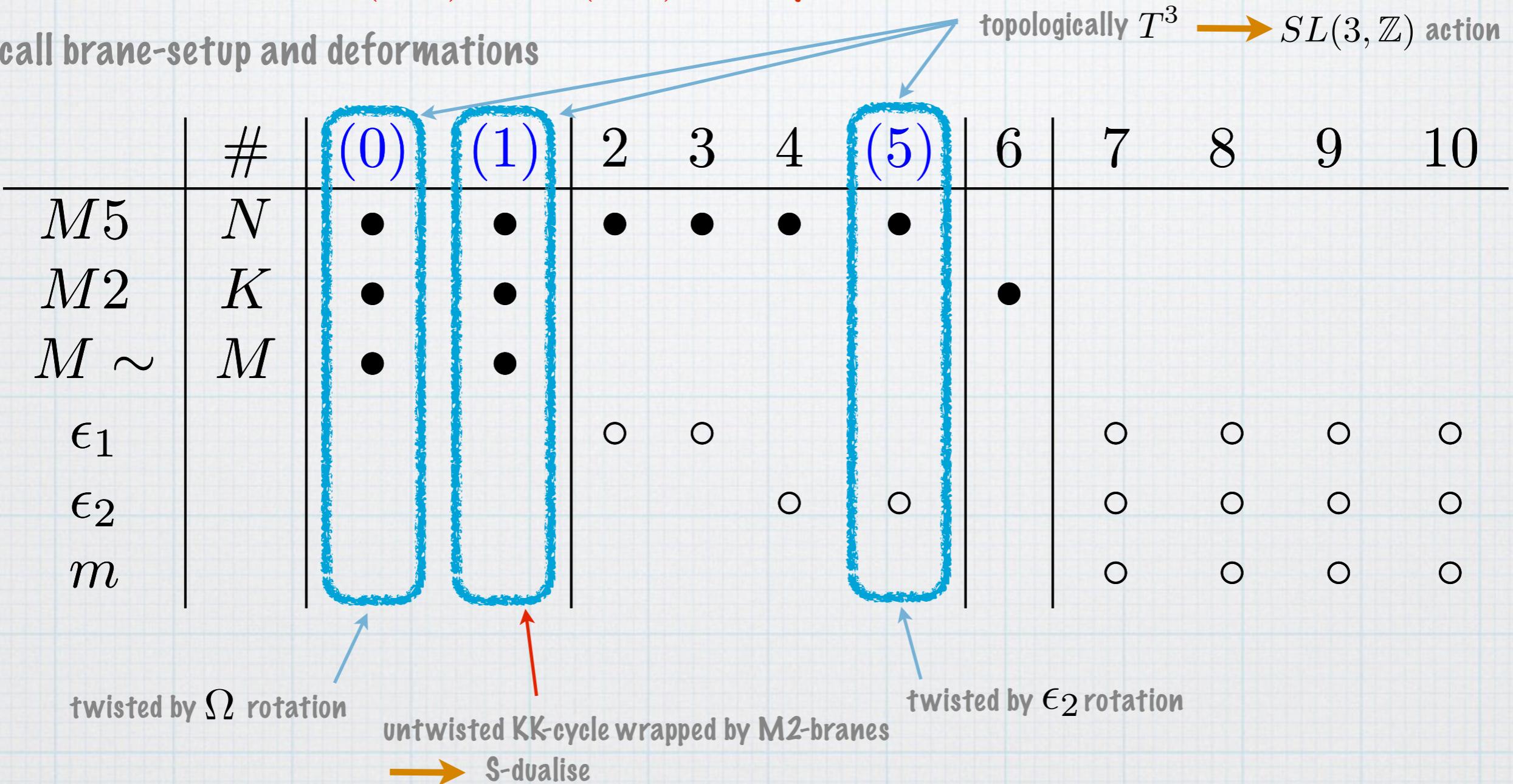
**Three Steps:** 1) compactify (5)-direction

[Haghighat 2015]  
[SH, Iqbal, Rey 2015]

2) Nekrasov-Shatashvili (NS) limit:  $\epsilon_2 \rightarrow 0$

3)  $S \in SL(2, \mathbb{Z}) \subset SL(3, \mathbb{Z})$  duality action

recall brane-setup and deformations



# Explicit Relation between M&m-string BPS Counting Functions

We are looking for relations between BPS excitations of M-strings and m-strings.

Simplest cases: 'uniform' excitations

M-strings:

$$\Sigma_{N,k}^{(0,\dots,0)}(\rho, m, \epsilon_1, \epsilon_2)$$

M-branes: summation over contrib. with same number of M2-branes between neighbouring M5, but fixed power of  $Q_\tau^k$

gauge theory:

instantons of charge  $k$   
in  $U(N)$  gauge theory

Little Strings:

little strings with momentum number  $k$  in IIA of type  $A_{N-1}$

T-duality

little strings with winding number  $k$  in IIB of type  $A_{N-1}$

[Bhardwaj, Del Zotto, Heckman, Morrison, Rudelius, Vafa 2015]  
[SH, Iqbal, Rey 2015]

geometric:

fiber-base duality of  $X_N$  acting as exchange  $\tau \longleftrightarrow \rho$

$$\lim_{\epsilon_2 \rightarrow 0} \frac{\Sigma_{N,k}^{(0,\dots,0)}(t, m, \epsilon_1, \epsilon_2)}{\Sigma_{1,1}^{(0,\dots,0)}(t, m, \epsilon_1, \epsilon_2)} = \lim_{\epsilon_2 \rightarrow 0} \frac{\overbrace{G^{(k, \dots, k)}}^{N \text{ times}}(t, m, \epsilon_1, \epsilon_2)}{G^{(1)}(t, m, \epsilon_1, \epsilon_2)}$$

monopole strings:

$$G^{(k, \dots, k)}(\tau, m, \epsilon_1, \epsilon_2)$$

$N$  M5-branes with  $k$  M2-branes stretched between each of them

monopole strings of charges  $(k, \dots, k)$  in affine  $A_{N-1}$  theory

Analysis of generalisation of this relation to other configurations is currently under way!

# Elliptic Genera of Moduli Spaces

More geometrically, we are interested in the moduli spaces of the instantons and monopoles:

**M-strings:**

**Moduli Space:**

$\mathcal{M}(N, k)$  moduli space of  $SU(N)$  instantons of charge  $k$

**monopole strings:**

$\mathcal{M}_{k_1, \dots, k_N}$  moduli space of monopole strings of charges  $(k_1, \dots, k_N)$

**Elliptic Genus:**

$$\phi_{\mathcal{M}(N, k)}(\rho, m, t_{f_1}, \dots, t_{f_{N-1}}, \epsilon_{1,2}) = C_{N,k}(m, t_{f_a}, \epsilon_{1,2})$$

defined through the expansion of the part.fct.

$$\tilde{\mathcal{Z}}_N^{(1)} = \sum_{k \geq 0} Q_\tau^k C_{N,k}(m, t_{f_1}, \dots, t_{f_N}, \epsilon_{1,2})$$

[Hollowood, Iqbal, Vafa 2003]  
[Iqbal, Kozcaz, Vafa 2007]

**proposal:**

$$\phi(\mathcal{M}_{k_1, \dots, k_N}) = \lim_{\epsilon_2 \mapsto 0} \frac{G^{(k_1, \dots, k_N)}(\tau, m, \epsilon_1, \epsilon_2)}{G^{(1)}(\tau, m, \epsilon_1, \epsilon_2)}$$

**Parameters:**

$\rho = i \sum_{a=1}^N t_{f_a}$  ... modular parameter

$(t_{f_1}, \dots, t_{f_{N-1}})$  ... fugacities of  $[U(1)]^{N-1}$

$(\epsilon_1, \epsilon_2)$  ... equivariant deformation parameters (regularisation)

**Parameters:**

$\tau$  ... modular parameter

$(\epsilon_1, \epsilon_2)$  ... equivariant deformation parameters (regularisation)

# Elliptic Genus of m-string Moduli Spaces

we re-interpret the M-strings free energies in the NS-limit by defining

$$P_{k_1, \dots, k_N}(\tau, m, \epsilon_1) := \lim_{\epsilon_2 \mapsto 0} \frac{G^{(k_1, \dots, k_N)}(\tau, m, \epsilon_1, \epsilon_2)}{G^{(1)}(\tau, m, \epsilon_1, \epsilon_2)}$$

transformation properties under  $SL(2, \mathbb{Z})$

$$P_{k_1, \dots, k_N}(\tau + 1, m, \epsilon_1) = P_{k_1, \dots, k_N}(\tau, m, \epsilon_1)$$

$$P_{k_1, \dots, k_N}\left(-\frac{1}{\tau}, \frac{m}{\tau}, \frac{\epsilon_1}{\tau}\right) = e^{\frac{2\pi i(m^2 - \epsilon_1^2)}{\tau}(K-1)} P_{k_1, \dots, k_N}(\tau, m, \epsilon_1)$$

$$P_{k_1, \dots, k_N}(\tau, m + \ell\tau + r, \epsilon_1) = e^{-2\pi i K \ell^2 \tau + 4\pi i m K} P_{k_1, \dots, k_N}(\tau, m, \epsilon_1)$$

**Genus Zero Limit:**  $\epsilon_1 \rightarrow 0$  has the following properties for  $\gcd(k_1, \dots, k_{N-1}) = 1$

- \*  $P_{k_1, \dots, k_N}(\tau, m, 0)$  has weight 0 under  $SL(2, \mathbb{Z})$  transformations
- \*  $P_{k_1, \dots, k_N}(\tau, m, 0)$  has index  $K = \sum_{a=1}^N k_a$  under  $SL(2, \mathbb{Z})$  transformations

we thus propose  $P_{k_1, \dots, k_N}(\tau, m, \epsilon_1)$  to be the **elliptic genus** of the **relative moduli space**  $\widehat{\mathcal{M}}_{\vec{k}}$  of monopoles of charge  $\vec{k} = (k_1, \dots, k_N)$  for  $\gcd(k_1, \dots, k_N)$  (note:  $K = \frac{1}{2} \dim_{\mathbb{C}} \widehat{\mathcal{M}}_{\vec{k}}$ )

In the non-compact case (for configurations (1, 1) and (2)), this proposal agrees with the known elliptic genera of the Taub-Nut and Atiyah-Hitchin space

# Orbifolds of M-brane Configurations

Consider M5-branes probing a transverse orbifold background

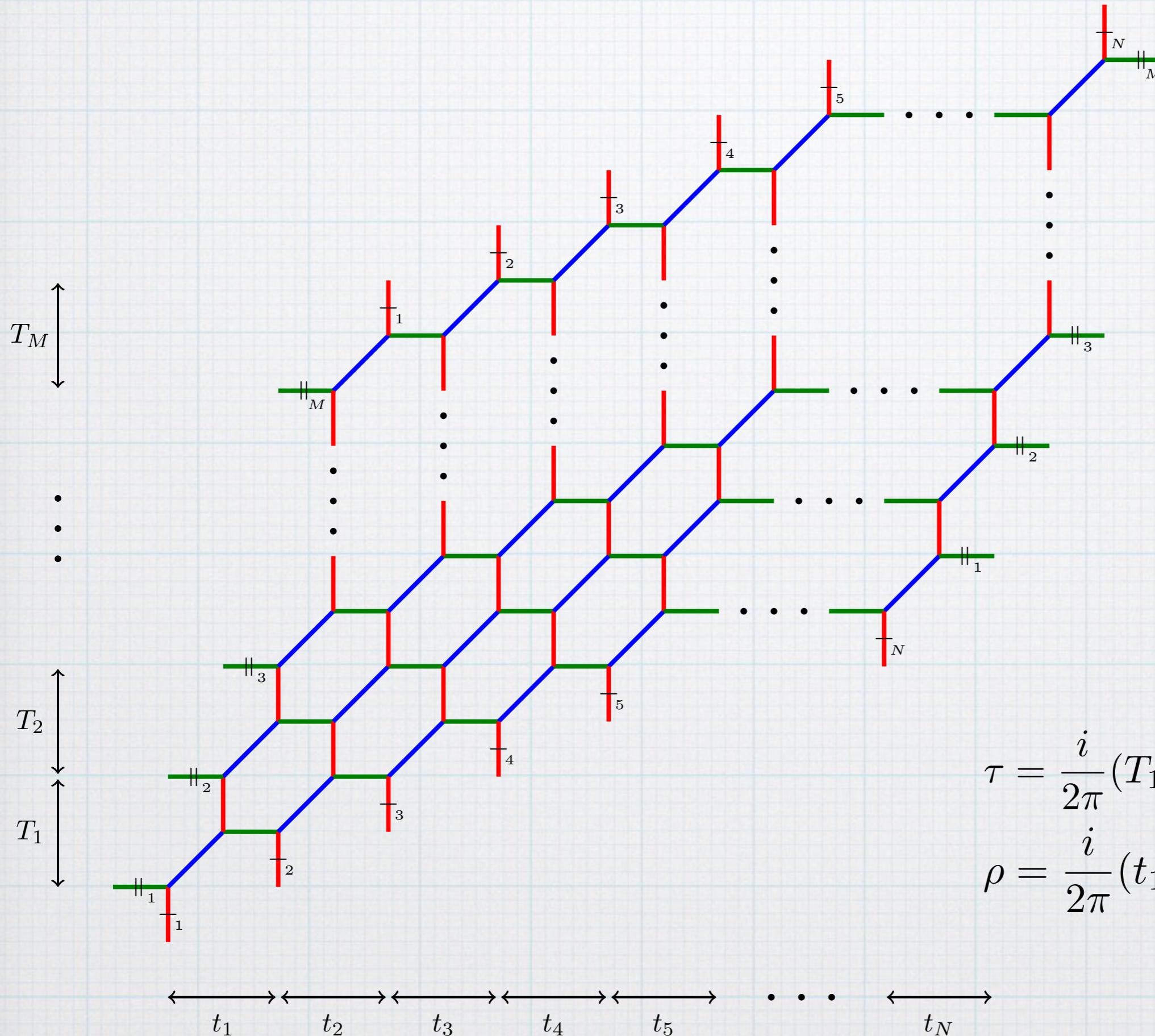
	#	0	1	2	3	4	5	6	7	8	9	10
M5	$N$	•	•	•	•	•	•					
M2	$K$	•	•					•				

$S^1_{R_6}$        $\mathbb{R}_{\perp}^4 / \mathbb{Z}_M$

Orbifold keeps key structures intact:

- \* M-string world-sheet supersymmetry is compatible with the orbifold
- \* mass and  $\Omega$ -deformation can still be introduced
- \* connection to little string theories with  $\mathcal{N} = (1, 0)$  supersymmetry
- \* Dual description in terms of toric Calabi-Yau threefold  $X_{N,M}$

**new feature:** stacks of M2-branes can be separated in the vertical direction



$$\tau = \frac{i}{2\pi} (T_1 + T_2 + \dots + T_M)$$

$$\rho = \frac{i}{2\pi} (t_1 + t_2 + \dots + t_N)$$

# Topological String Partition Function

M-string part. fct. is computed by topological string part. fct. of  $X_{N,M}$

[Haghighat, Kozcaz, Lockhart, Vafa 2013]  
[SH, Iqbal 2013]

$$\begin{aligned} \mathcal{Z}_{X_{N,M}}(\tau, \rho, t_1, \dots, t_{N-1}, T_1, \dots, T_{M-1}, m, \epsilon_{1,2}) &= W_N(\emptyset)^M \sum_{\alpha_a^{(i)}} Q_\tau^{\sum_a |\alpha_a^{(M)}|} \left( \prod_{i=1}^M \bar{Q}_i^{\sum_a (|\alpha_a^{(i)}| - |\alpha_a^{(M)}|)} \right) \\ &\times \prod_{i=1}^M \prod_{a=1}^N \frac{\vartheta_{\alpha_a^{(i+1)} \alpha_a^{(i)}}(Q_m; \rho)}{\vartheta_{\alpha_a^{(i)} \alpha_a^{(i)}}(\sqrt{t/q}; \rho)} \prod_{1 \leq a < b \leq N} \prod_{i=1}^M \frac{\vartheta_{\alpha_a^{(i)} \alpha_b^{(i+1)}}(Q_{ab} Q_m^{-1}; \rho) \vartheta_{\alpha_a^{(i+1)} \alpha_b^{(i)}}(Q_{ab} Q_m; \rho)}{\vartheta_{\alpha_a^{(i)} \alpha_b^{(i)}}(Q_{ab} \sqrt{t/q}; \rho) \vartheta_{\alpha_a^{(i)} \alpha_b^{(i)}}(Q_{ab} \sqrt{q/t}; \rho)} \end{aligned}$$

**Notation:**

$$\bar{Q}_i = e^{-T_i}, \quad \forall i = 1, \dots, M,$$

$$Q_a = e^{-t_a}, \quad \forall a = 1, \dots, N,$$

$$Q_{ab} = Q_a Q_{a+1} \dots Q_{b-1}, \quad 1 \leq a < b \leq N$$

$$W_N(\emptyset)(t_1, \dots, t_N, m, \epsilon_{1,2}) = \lim_{\tau \rightarrow i\infty} \mathcal{Z}_{X_{N,1}}(\tau, \rho, t_1, \dots, t_{N-1}, m, \epsilon_{1,2})$$

$$= \frac{1}{\prod_{n=1}^{\infty} (1 - Q_\rho^n)} \prod_{i,j,k} \prod_{a,b=1}^N \frac{1 - Q_\rho^{k-1} Q_{a,a+b} Q_m^{-1} t^{i-\frac{1}{2}} q^{j-\frac{1}{2}}}{1 - Q_\rho^{k-1} Q_{a,a+b} t^{i-1} q^j} \frac{1 - Q_\rho^{k-1} Q_{a,a+b-1} Q_m t^{i-\frac{1}{2}} q^{j-\frac{1}{2}}}{1 - Q_\rho^{k-1} Q_{a,a+b} t^i q^{j-1}}$$

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Free Energy:

$$\Sigma_{N,M}(\mathbf{t}, \mathbf{T}, m, \epsilon_{1,2}) = \text{Plog } \mathcal{Z}_{X_{N,M}}(\tau, \rho, t_1, \dots, t_{N-1}, T_1, \dots, T_{M-1}, m, \epsilon_{1,2})$$

Little String Theory Partition Function:

$$Z_{\text{IIa}}^{(N,M)}(\mathbf{T}, \mathbf{t}, m, \epsilon_{1,2}) = \mathcal{Z}_{X_{N,M}}(\tau, \rho, t_1, \dots, t_{N-1}, T_1, \dots, T_{M-1}, m, \epsilon_{1,2})$$

$$Z_{\text{IIb}}^{(M,N)}(\mathbf{T}, \mathbf{t}, m, \epsilon_{1,2}) = \mathcal{Z}_{X_{M,N}}(\rho, \tau, T_1, \dots, T_{M-1}, t_1, \dots, t_{N-1}, m, \epsilon_{1,2})$$

# Self-Similarity

Partition function drastically simplifies at the point in moduli space

$$Q_1 = Q_2 = \dots = Q_N = Q_\rho^{\frac{1}{N}} \quad \text{and} \quad \bar{Q}_1 = \bar{Q}_2 = \dots = \bar{Q}_M = Q_\tau^{\frac{1}{M}}$$

We observe the following self-similarity in the NS-limit

$$\lim_{\epsilon_2 \rightarrow 0} \epsilon_2 \Sigma_{N,M}(t, \dots, t, T, \dots, T, m, \epsilon_{1,2}) = NM \lim_{\epsilon_2 \rightarrow 0} \epsilon_2 \Sigma_{1,1}(t, T, m, \epsilon_{1,2})$$

[SH, Iqbal, Rey 2016]

Consequences:

- \* Partition Function can fully be reconstructed just from  $(N, M) = (1, 1)$
- \* symmetry in the exchange  $t \longleftrightarrow T$
- \* symmetries of (1,1) inherited by other setups
- \* self-duality of little string theories

# Conclusions

## Summary:

- \* studied string degrees of freedom in supersymmetric gauge theories
- \* duality between M-strings and monopole strings and little strings
- \* modular properties of M-string partition function
- \* proposal for elliptic genus of relative moduli space of m-strings
- \* checked for all known examples
- \* self-similarity of orbifold theories

## Outlook:

- \* more general charge configurations       $\gcd(k_1, \dots, k_{N-1}) \neq 1$
- \* analyse fractional momentum states in the little string setup
- \* study symmetries away from special point in the moduli space