INTRODUCTION TO
STOCHASTIC CALCULUS
Borun D Chowdhury
What is a stochastic process?

Series of events where a variable takes random values $X_0, X_1, \ldots, X_t, \ldots$

Causality requires $X_t$ depends only on $X_{t'}$ where $t' < t$

A Markovian process is a memory less stochastic process so that $X_t$ depends only on $X_{t'}$

Often it is useful to look at the difference process $Y_t = X_t - X_{t-1}$
Binomial Random Walk

The workhorse of stochastic processes is the Binomial Random walk.

To understand that we first look at a Bernoulli process $B(p)$, a random variable that is 1 with probability $p$ and 0 with probability $q$.

$$E[B(p)] = p, \quad Var[B(p)] = p(1 - p)$$

We then define a variable that takes values 1 with probability $p$ and -1 with probability $q$.

$$X_i = 2B(p) - 1$$

and we have

$$E[X_i] = 2p - 1, \quad Var[B(p)] = 4p(1 - p)$$

This is a model for a step taken by a drunk.
Binomial Random Walk

If the drunk takes \( n \) steps then we have a new random variable

\[
Y_n = \sum_{i}^{n} X_i
\]

Its expectation value is

\[
E[Y_n] = \sum E[X_i] = n(2p - 1)
\]

Its variance is

\[
V ar(Y_n) = \sum_{i} V ar(X_i) + \sum_{i \neq j} Cov(X_i, X_j)
\]

If the different steps are not correlated (or better yet independent)

\[
V ar(Y_n) = 4np(1 - p)
\]

Note \( Y_n \) is also a stochastic process.
Binomial Random Walk

Note that after $n$ steps the scale of fluctuations around the mean

$$\sigma_n \sim \sqrt{n}$$
Binomial Random Walk

Covariance of random walk

\[
Cov(X_n, X_m) = 4p(1 - p)\delta_{n,m}
\]

Covariance of steps

\[
Cov(Y_n, Y_m) = 4\min(n, m)p(1 - p)
\]

Covariance of paths
A glimpse of the central limit theorem

For a very large number of steps

Take the central half and zoom in

The distribution looks the same (after rescaling)
A glimpse of the central limit theorem

However, zooming in too much reveals the Bernoulli "microstructure"
A glimpse of the central limit theorem

What we are doing is the opposite of renormalisation group transformation

If we perform the RG operation of “coarse-graining”, i.e. if we start with the Bernoulli and

1. Zoom out
2. Rescale back
3. Repeat

We will converge to a stable distribution: The normal distribution

In fact this will happen independent of the microscopic distribution as long as it has a well defined mean and variance
A glimpse of the central limit theorem

Bernoulli random walk

\[ Y_n = 2\text{Binomial}(n, p) - n \]

Large n limit

\[
Y_n \sim 2\mathcal{N}(np, \sqrt{np(1-p)}) - n \\
= n(2p - 1) + 2\sqrt{np(1-p)}\mathcal{N}(0, 1) \\
= n(2p - 1) + \sqrt{4np(1-p)}\mathcal{N}(0, 1)
\]

Suppose the microscopic time scale is much shorter than times of interest

\[ \delta t \ll dt \]

Then we can write the above as

\[ dY(t) = \mu dt + \sigma \sqrt{dt} \mathcal{N}(0, 1) \]

where \( \mu = \frac{2p - 1}{\delta t} \) and \( \sigma = \sqrt{\frac{4p(1-p)}{\delta t}} \).
Wiener process

A Wiener process is the limit of a process that happens in time $dt$ and is distributed normally with variance $dt$

$$dW_t = \sqrt{dt} N(0, 1)$$

We can simulate this process and repeat the zooming in process

This is a simulation based on finite steps so still has microstructure but the theoretical Wiener process is scale independent.
Stochastic Calculus

Above we naturally came across

\[ dY(t) = \mu dt + \sigma \sqrt{dt} \mathcal{N}(0, 1) \]

And by construction we know how to solve it

In general we can have

\[ dS(t) = a(t, W_t)dt + b(t, W_t)dW_t \]

and this will involve solving

\[ \int G(t)dW_t \]

We will discuss properties of \( dW \) and how to tackle stochastic calculus next time