A novel mathematical approach to positioning systems: Deriving an analytical algorithm for the localisation of signal sources in orb webs and other net geometries

Sophie Atzpodien, Münster, Germany
Conventional positioning systems (i.e. GPS)

Necessary information:
- point in time at which the signal was emitted
- point in time at which the signal was received
- phase speed $c$ of the signal

Sophie Atzpodien, sophie.atzpodien@yahoo.de
Novel concept

Necessary information:
locations $D_1$, $D_2$, geometry of the net phase speed $c$ of the signal
time difference $\Delta t$ between the first detections of the signal at $D_1$ and $D_2$

Key:
- Sensor locations $D_1$, $D_2$
- Location $E$ of the signal source
Mathematical abstraction

**Premise:** calculation of the resulting time difference $\Delta t$ for any choice of $D_1$, $D_2$, $E$

Assuming linear signal propagation:

$$t = \frac{c}{\min(w(P_n, D_1), w(P_n, D_2))}$$

(where $w_{\min}(P_1, P_2)$: shortest path between two arbitrary (junction) points on the net, $w_{\min}(P_1, P_2): (P_1, P_2) \rightarrow \mathbb{R}^+$)

→ analytical expression for $w_{\min}(P_1, P_2)$ must be derived (dependent on net geometry)
Deriving a novel analytical solution for $w_{\min}$

1. Evolution of the net geometries’ complexity:

- Quadratic grid
- Quadratic grid with diagonal axes
- Quadratic net
- Circular net
- Chord net
- Logarithmic spiral net

Sophie Atzpodien, sophie.atzpodien@yahoo.de
Deriving a novel analytical solution for $w_{\text{min}}$

2. $\Delta R$-$\Delta B$-theorem

general result simplifying the derivation of a case-differentiating formula for $w_{\text{min}}(P_a, P_b)$:

$$w_{\text{min}}(P_a, P_b) = \begin{cases} \left| R_a - R_b \right| + B_{R_b} & \text{if } \Delta R \geq \Delta B \\ \left| R_a - R_b \right| + 2R_b & \text{if } \Delta R \leq \Delta B \end{cases}$$

(with $B_{R_b}$: length of the series of links needed to cover at radius $R_b$)

**geometric condition:** possibility to divide the net in radial links covering only radial and angular links covering only angular distance → applicable to the circular and chord net
Deriving a novel analytical solution for $w_{\text{min}}$

2. $\Delta R - \Delta B$-theorem

polar coordinates:
\[ P_a (R_a, \theta) \quad P_b (R_b, \theta) \]

radial difference $|R_a - R_b|$

angular difference

Increase in radial path component $\Delta R$

Decrease in angular path component $\Delta B$

Sophie Atzpodien, sophie.atzpodien@yahoo.de
Deriving a novel analytical solution for $w_{\text{min}}$

3. Final formulae for $w_{\text{min}}$ (logarithmic spiral web)

Using modified $\Delta R-\Delta B$-conditions, the following result was obtained:

$$w_{\text{min}} \text{ without spiral} (P_a, P_b) = \begin{cases} w_s(P_a, P_b) & \text{if } w_s(P_a, P_b) \leq w_{\text{min}} \text{ without spiral} (P_a, P_b) \\ w_{\text{min}} \text{ without spiral} (P_a, P_b) & \text{if } w_s(P_a, P_b) \geq w_{\text{min}} \text{ without spiral} (P_a, P_b) \end{cases}$$

(2)

(where “wenn“ = “if“)

$$w_{\text{min}} (P_a, P_b) = \begin{cases} w_s(P_a, P_b) & \text{if } w_s(P_a, P_b) \leq w_{\text{min}} \text{ without spiral} (P_a, P_b) \\ w_{\text{min}} \text{ without spiral} (P_a, P_b) & \text{if } w_s(P_a, P_b) \geq w_{\text{min}} \text{ without spiral} (P_a, P_b) \end{cases}$$

(3)

$$w_{\text{min}} (P_a, P_b) = \begin{cases} w_s(P_a, P_b) & \text{if } w_s(P_a, P_b) \leq w_{\text{min}} \text{ without spiral} (P_a, P_b) \\ w_{\text{min}} \text{ without spiral} (P_a, P_b) & \text{if } w_s(P_a, P_b) \geq w_{\text{min}} \text{ without spiral} (P_a, P_b) \end{cases}$$

(4)

$$w_s(P_a, P_b) = R_0 k_2^b - k_1^b \left( \frac{360}{q_b - q_a - 1} \right)$$

(5)

$$w_{\text{min}} \text{ without spiral} (P_a, P_b) = \begin{cases} R_0 k_2^b k_1^b & \text{if } w_s(P_a, P_b) \leq \text{wenn} \\ \text{wmin without spiral} (P_a, P_b) & \text{if } w_s(P_a, P_b) \geq \text{wmin without spiral} (P_a, P_b) \end{cases}$$

(6)
Implementation as an algorithm

Input: number of points included in the simulation, distance $R_0$ of the first spiral-axis-intersection to the web's middle, geometric parameter values $(\alpha, \beta)$, sensor positions $D_{1}, D_{2}$, measured values of $c$ and measured values of $\Delta$

Calculate each point's radial coordinate $R_i$, angular coordinate $\alpha_i$, and value of $q_i$

Compute twice, first with $D_m = D_{1}$ and then with $D_m = D_{2}$:

<table>
<thead>
<tr>
<th>yes</th>
<th>Is $q_{m_1} &gt; 0$?</th>
</tr>
</thead>
<tbody>
<tr>
<td>no</td>
<td>calculate $\Delta R$ and $\Delta B$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>yes</th>
<th>Is $\Delta R \geq \Delta B$ and $q_{m_1} &gt; 1$?</th>
</tr>
</thead>
<tbody>
<tr>
<td>no</td>
<td>calculate $\Delta B_1, \Delta R_1, RB_1$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>no, and $\Delta R_1 \geq \Delta B_1$</th>
<th>Is $RB_1 = \min(\Delta R_1, \Delta B_1, RB_1)$?</th>
</tr>
</thead>
<tbody>
<tr>
<td>yes</td>
<td>set value of $w_{\text{min without group}}(P_{m}, D_{m})$ to $R_0(k_b^b - k_i^{b}\alpha^{\text{group}} + k_i^{b}\sum_{i=0}^{\infty}k_i^{b}\alpha^{\text{group}})$ to $R_0(k_i^{b} + k_i^{a})$.</td>
</tr>
</tbody>
</table>

Output $w_{\text{min without group}}(P_{m}, D_{m})$

Calculate $w_d(P_{m}, D_{m}) = \sum_{i=0}^{\infty} (R_0(k_i^{b} + k_i^{a})$)

<table>
<thead>
<tr>
<th>yes</th>
<th>Is $w_d(P_{m}, D_{m}) \geq w_{\text{min without group}}(P_{m}, D_{m})$?</th>
</tr>
</thead>
<tbody>
<tr>
<td>no</td>
<td>Set $w_{\text{min}}(P_{m}, D_{m}) = w_{\text{min without group}}(P_{m}, D_{m})$</td>
</tr>
</tbody>
</table>

Set $w_{\text{max}}(P_{m}, D_{m}) = w_d(P_{m}, D_{m})$

Output $w_{\text{max}}(P_{m}, D_{m})$

Set the maximum and minimum measured values of $\Delta$ and $c$ equal to $\Delta \text{max}(E)$ and $\Delta \text{max}(E)$ and $c_{\text{max}}$ and the average values of $\Delta$ and $c$ equal to $\Delta \text{avg}$ and $c_{\text{avg}}$

Set $\Delta_t(P_{m}) = \frac{\Delta \text{max}(P_{m}) - \Delta \text{min}(P_{m})}{c_{\text{avg}}}$, $\Delta \text{max}(P_{m}) = \frac{w_{\text{max}}(P_{m}, D_{m})}{c_{\text{max}}}$ and $\Delta \text{min}(P_{m}) = \frac{w_{\text{min}}(P_{m}, D_{m})}{c_{\text{min}}}$

Set $P_{m} = \text{E if } \Delta \text{max}(E) \geq \Delta \text{max}(P_{m}) \geq \Delta \text{min}(E) \vee \Delta \text{max}(E) \geq \Delta \text{max}(P_{m}) \geq \Delta \text{min}(E)$

Sophie Atzpodien, sophie.atzpodien@yahoo.de
Experimental study

Setup of the experiment:

Sophie Atzpodien, sophie.atzpodien@yahoo.de
Experimental study

3D-printed net mounting:

Sophie Atzpodien, sophie.atzpodien@yahoo.de
Results and discussion

Comparison of the measured values of $\Delta t$ with those predicted by the algorithm

experimentally measured and predicted values of $\Delta t$ correlate by 87.8% and show similar progression

→ mathematical model likely yields correct $\Delta t$, thus enabling the localisation of a signal source with a high certainty

Sophie Atzpodien, sophie.atzpodien@yahoo.de
Conclusion

Objective: Localization of a signal source in different net-like geometrical structures using a novel analytical algorithm

I. Derivation of a novel mathematical solution to the problem of determining the shortest path between two arbitrary points $P_1$, $P_2$ on a net for increasingly complex net geometries up to the logarithmic spiral web

II. Implementation of the resulting mathematical models as algorithms in order to automate the localisation process and the uncertainty analysis of the necessary measurements

III. Comparison of theoretical predictions of $\Delta t$ with values measured in a self-devised experimental setup $\rightarrow$ correlation of 87.8% for the most complex system (logarithmic spiral net)

$\rightarrow$ analytical algorithm based on the mathematical results succeeds in locating a signal source with a high certainty