Axisymmetric and spherical sources of Majumdar-Papapetrou type spacetimes

Gonzalo García-Reyes

Departamento de Física
Universidad Tecnológica de Pereira, Colombia

ggarcia@utp.edu.co

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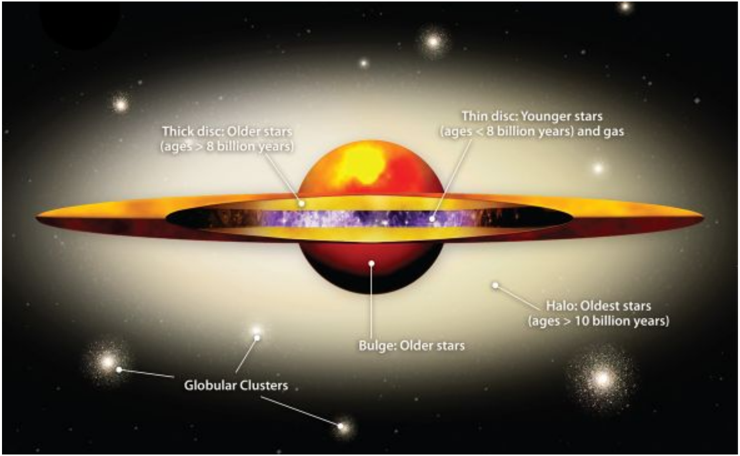
Introduction



We consider a conformastatic spacetime in spherical coordinates (t, r, θ, φ) and in the particular form

$$ds^2 = -f dt^2 + f^{-1}(dr^2 + r^2 d\Omega), \tag{1}$$

where $f(r)$ and $d\Omega = d\theta^2 + \sin^2\theta d\varphi^2$.
Making $f = (1 - \phi/2)^{-4}$, with $\phi(r)$, the same metric takes the form

$$ds^2 = -\left(1 - \frac{\phi}{2}\right)^{-4} dt^2 + \left(1 - \frac{\phi}{2}\right)^{4} (dr^2 + r^2 d\Omega). \tag{2}$$

The Einstein field equations $G_{ab} = 8\pi G T_{ab}$ yield the following non-zero components of the energy-momentum tensor

$$T^t{}_t = -\frac{\nabla^2\phi}{4\pi G\left(1 - \frac{\phi}{2}\right)^5}, \tag{3a}$$

$$T^\theta{}_\theta = T^\varphi{}_\varphi = \frac{\nabla\phi \cdot \nabla\phi}{8\pi G\left(1 - \frac{\phi}{2}\right)^6}, \tag{3b}$$

$$T^r{}_r = -\frac{\nabla\phi \cdot \nabla\phi}{8\pi G\left(1 - \frac{\phi}{2}\right)^6}. \tag{3c}$$

In terms of the orthonormal tetrad $\mathrm{e}_{(a)}{}^{b} = \{V^b, X^b, Y^b, Z^b\}$, where

$$\begin{aligned} V^a &= f^{-1/2}\delta^a_t, \quad X^a = f^{1/2}\delta^a_r, && (4a) \\ Y^a &= \frac{f^{1/2}}{r}\delta^a_\theta, \quad Z^a = \frac{f^{1/2}}{r\sin\theta}\delta^a_\varphi, && (4b) \end{aligned}$$

the energy-momentum tensor can be written as

$$T^{ab} = \rho V^a V^b + p_r X^a X^b + p_\theta (Y^a Y^b + Z^a Z^b), \qquad (5)$$

so that the energy density is given by $\rho = -T^t{}_t$, and the stresses (pressure or tensions) by $p_i = T^i{}_i$.

The metric function ϕ can be chosen by requiring that in the Newtonian limite $\phi \ll 1$ the expression for the relativistic energy density must reduce to Poisson's equation

$$\nabla^2 \Phi = 4\pi G \rho_N . \tag{6}$$

This condition is satisfied by taking $\phi = \Phi$.
Thus, the physical quantities associated with the matter distribution are given by

$$\rho = \frac{\rho_N}{\left(1 - \frac{\Phi}{2}\right)^5}, \tag{7a}$$

$$p_\theta = p_\varphi = -p_r = \frac{\nabla\Phi \cdot \nabla\Phi}{8\pi G \left(1 - \frac{\Phi}{2}\right)^6}, \tag{7b}$$

and the average pressure by

$$p = \frac{1}{3}(p_r + p_\theta + p_\varphi) = \frac{\nabla\Phi \cdot \nabla\Phi}{24\pi G \left(1 - \frac{\Phi}{2}\right)^6}. \tag{8}$$

In order to have a physically meaningful matter distribution the components of the energy-momentum tensor must satisfy the energy conditions. The weak energy condition requires that $\rho \geq 0$, whereas the dominant energy condition states that $|\rho| \geq |p_i|$. The strong energy condition imposes the condition $\rho_{eff} = \rho + p_r + p_\theta + p_\varphi \geq 0$, where ρ_{eff} is the "effective Newtonian density"

On the plane $\theta = \pi/2$

$$v_c^2 = \frac{v_{Nc}^2}{1 - \frac{\Phi}{2} - v_{Nc}^2}, \tag{9a}$$

$$h^2 = \frac{r^2 (1 - \frac{\Phi}{2})^4 v_{Nc}^2}{1 - \frac{\Phi}{2} - 2v_{Nc}^2}, \tag{9b}$$

where $v_{Nc}^2 = r\Phi_{N,r}$ is the Newtonian circular speed.

A simple Newtonian potential-density pair is

$$\Phi = -\frac{GM}{(r^n + a^n)^{1/n}}, \tag{10a}$$

$$\rho_N = \frac{M(n+1)a^n r^{n-2}}{4\pi(r^n + a^n)^{2+1/n}}, \tag{10b}$$

where a is a non-zero constant with the dimension of length and $n \geq 1$. For $n = 1$ the potential-density pair (10a) and (10b) reduces to Hernquist model which has a density profiles with a central cusp and it has been used to model elliptical galaxies and bulges . For $n = 2$ we have the Plummer's spherical model for globular clusters which has a mass distribution concentrated at center. This potential-density pair has also been used to model the central region of our Galaxy composed by the bulge/stellar-halo and the inner core. For $n \geq 3$ we have a shell-like matter distribution. The Newtonian circular speed is given by

$$v_{Nc}^2 = \frac{GMr^n}{(r^n + a^n)^{1+1/n}}. \tag{11}$$

In this case the metric function f is given by

$$f = \left[1 + \frac{GM}{2}\left(r^n + a^n\right)^{-1/n}\right]^{-4}, \qquad (12)$$

and the relativistic expressions for the physical quantities main associated with the matter distributions are

$$\rho = \frac{8(1+n)\tilde{a}^n\tilde{r}^{n-2}}{\pi G^3 M^2 \xi^{2-4/n}(2\xi^{1/n}+1)^5}, \qquad (13a)$$

$$p = \frac{8\tilde{r}^{2n-2}\xi^{4/n-2}}{3\pi G^3 M^2 (2\xi^{1/n}+1)^6}, \qquad (13b)$$

$$v_c^2 = \frac{2\tilde{r}^n}{2\xi^{1/n+1}+\xi-2\tilde{r}^n}, \qquad (13c)$$

$$h^2 = \frac{G^2 M^2 \tilde{r}^{2+n}(2\xi^{1/n}+1)^2}{2\xi^{2/n}(2\xi^{1/n+1}+\xi-4\tilde{r}^n)}, \qquad (13d)$$

where $\tilde{r} = r/(MG)$, $\tilde{a} = a/(MG)$, and $\xi = \tilde{r}^n + \tilde{a}^n$.

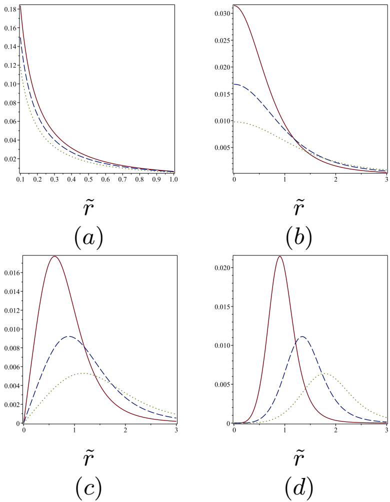


Figure : The relativistic energy density $\tilde{\rho}$.

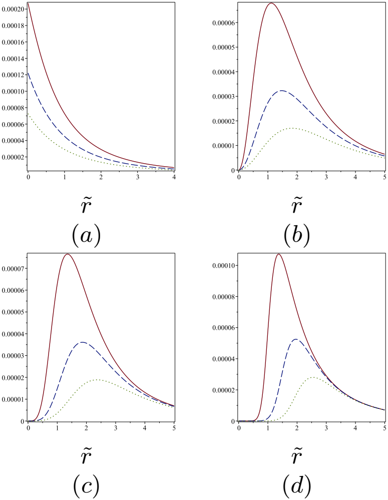


Figure : The average pressure $\tilde{p}$.

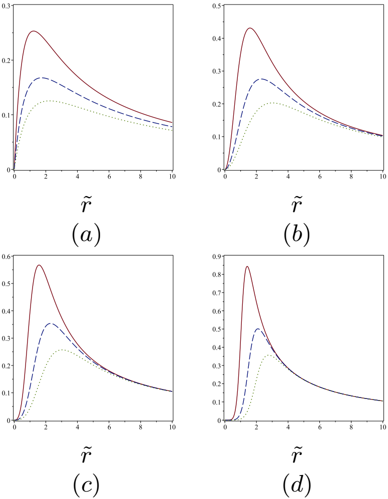


Figure : The rotation curves v_c^2.

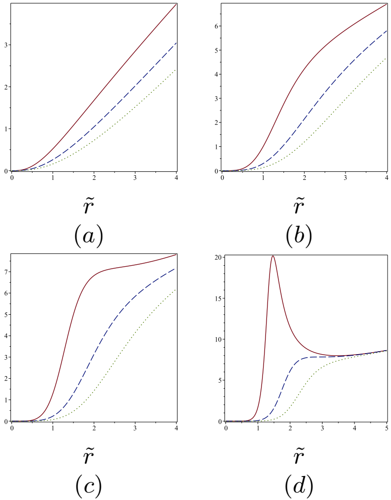


Figure : The specific angular momentum $\tilde{h}^2$.

We consider a conformastatic axisymmetric spacetime in cylindrical coordinates (t, φ, R, z) and in the particular form

$$ds^2 = -\left(1 - \frac{\phi}{2}\right)^{-4} dt^2 + \left(1 - \frac{\phi}{2}\right)^{4} (R^2 d\varphi^2 + dR^2 + dz^2), \tag{14}$$

where ϕ is function of R and z only.
The Einstein field equations $G_{ab} = 8\pi G T_{ab}$ yield the following non-zero components of the energy-momentum tensor

$$T^t{}_t = -\frac{\nabla^2\phi}{4\pi G\left(1 - \frac{\phi}{2}\right)^5}, \tag{15a}$$

$$T^\varphi{}_\varphi = \frac{\phi_{,R}^2 + \phi_{,z}^2}{8\pi G\left(1 - \frac{\phi}{2}\right)^6}, \tag{15b}$$

$$T^z{}_z = -T^R{}_R = \frac{\phi_{,R}^2 - \phi_{,z}^2}{8\pi G\left(1 - \frac{\phi}{2}\right)^6}. \tag{15c}$$

$$T^R{}_z = -\frac{\phi_{,R}\phi_{,z}}{4\pi G\left(1 - \frac{\phi}{2}\right)^6}. \tag{15d}$$

Now, in order to analyze the matter content of the disks is necessary to compute the eigenvalues and eigenvectors of the energy-momentum tensor. The eigenvalue problem for the energy-momentum tensor (15a) - (15d) has the solutions

$$\lambda_0 = T^t{}_t, \tag{16a}$$
$$\lambda_1 = T^\varphi{}_\varphi, \tag{16b}$$
$$\lambda_2 = -\lambda_3 = \sqrt{(T^R{}_R)^2 + (T^R{}_z)^2}. \tag{16c}$$

The corresponding eigenvectors are

$$\mathbf{V} = (1 - \frac{\phi}{2})^2 (1, 0, 0, 0), \tag{17a}$$

$$\mathbf{X} = \frac{(1 - \frac{\phi}{2})^{-2}}{R} (0, 1, 0, 0), \tag{17b}$$

$$\mathbf{Y} = \frac{(1 - \frac{\phi}{2})^{-2} T^R{}_z}{[2D(-T^R{}_R + D)]^{1/2}} (0, 0, 1, \frac{-T^R{}_R + D}{T^R{}_z}), \tag{17c}$$

$$\mathbf{Z} = \frac{(1 - \frac{\phi}{2})^{-2} T^R{}_z}{[2D(T^R{}_R + D)]^{1/2}} (0, 0, 1, \frac{-T^R{}_R - D}{T^R{}_z}), \tag{17d}$$

where

$$D = \sqrt{(T^R{}_R)^2 + (T^R{}_z)^2}. \tag{18}$$

In terms of the orthonormal tetrad or comoving observer ${e_{(a)}}^b = \{V^b, X^b, Y^b, Z^b\}$ the energy density is given by $\rho = -\lambda_0$, and the stresses (pressure or tensions) by $p_i = \lambda_i$.

The metric function ϕ can be chosen by requiring that in the Newtonian limite $\phi \ll 1$ the expression for the relativistic energy density must reduce to Poisson's equation

$$\nabla^2 \Phi = 4\pi G \rho_N. \tag{19}$$

This condition is satisfied by taking $\phi = \Phi$.
Thus, the physical quantities associated with the matter distribution are given by

$$\rho = \frac{\rho_N}{(1 - \frac{\Phi}{2})^5}, \tag{20a}$$

$$p_\varphi = p_R = -p_z = \frac{\nabla\Phi \cdot \nabla\Phi}{8\pi G \left(1 - \frac{\Phi}{2}\right)^6}, \tag{20b}$$

and the average pressure by

$$p = \frac{1}{3}(p_\varphi + p_R + p_z) = \frac{\nabla\Phi \cdot \nabla\Phi}{24\pi G \left(1 - \frac{\Phi}{2}\right)^6}. \tag{21}$$

On galactic plane the circular speed is given by

$$v_c^2 = \frac{v_N^2}{1 - \frac{\Phi}{2} - v_N^2}, \tag{22}$$

where $v_N^2 = R\Phi_{,R}$ is the Newtonian circular speed.

$$h^2 = \frac{R^2 \left(1 - \frac{\Phi}{2}\right)^4 v_c^2}{1 - v_c^2}. \tag{23}$$

All above quantities are evaluated on the equatorial plane $z = 0$.

The first Miyamoto-Nagai potential-density pair is

$$\Phi = -\frac{GM}{\sqrt{R^2 + \left(a + \sqrt{z^2+b^2}\right)^2}}, \tag{24a}$$

$$\rho_N = \frac{b^2 M \left[aR^2 + (a + 3\sqrt{z^2+b^2})(a + \sqrt{z^2+b^2})^2\right]}{4\pi \left[R^2 + \left(a + \sqrt{z^2+b^2}\right)^2\right]^{5/2} (z^2+b^2)^{3/2}}, \tag{24b}$$

where the parameter b is related to the disc scaleheight, a to the disc scalelength, and M is the mass of the disc component. The density function (24b) describes the stratification of mass in the central bulge as well in the disk part of galaxies. The Newtonian circular speed on $z = 0$ is given by

$$v_N^2 = \frac{GMR^2}{[R^2 + (a+b)^2]^{3/2}}. \tag{25}$$

The relativistic expressions for the energy density and average pressure are

$$\rho = \frac{b^2\left[a\tilde{R}^2 + (\tilde{a}+3\zeta)(\tilde{a}+\zeta)^2\right]}{4\pi G^3 M^2\left[\sqrt{\tilde{R}^2+(\tilde{a}+\zeta)^2}+1/2\right]^5 \zeta^3}, \tag{26a}$$

$$p = \frac{(\tilde{a}+\zeta)^2\,\tilde{z}^2 + \zeta^2\tilde{R}^2}{24\pi G^3 M^2\left[\sqrt{\tilde{R}^2+(\tilde{a}+\zeta)^2}+1/2\right]^6 \zeta^2}, \tag{26b}$$

where $\tilde{R} = R/GM$, $\tilde{z} = z/GM$, $\tilde{a} = a/GM$, $\tilde{b} = b/GM$, and $\zeta = \sqrt{\tilde{z}^2 + \tilde{b}^2}$. We note that the energy density is always a positive quantity in agreement with the weak energy condition and the stresses are positive for all values of parameters, so that we have always pressure. In turn, the circular speed and the specific angular momentum on $z = 0$ are given by

$$v_c^2 = \frac{2\tilde{R}^2}{2\left(\tilde{R}^2+\tilde{d}^2\right)^{3/2} - \tilde{R}^2 + \tilde{d}^2}, \tag{27a}$$

$$h^2 = \frac{G^2M^2\tilde{R}^4\left[2\sqrt{\tilde{R}^2+\tilde{d}^2}+1\right]^4}{8\left(\tilde{R}^2+\tilde{d}^2\right)^2\left[2\left(\tilde{R}^2+\tilde{d}^2\right)^{3/2} - 3\tilde{R}^2 + \tilde{d}^2\right]}, \tag{27b}$$

where $\tilde{d} = \tilde{a} + \tilde{b}$.

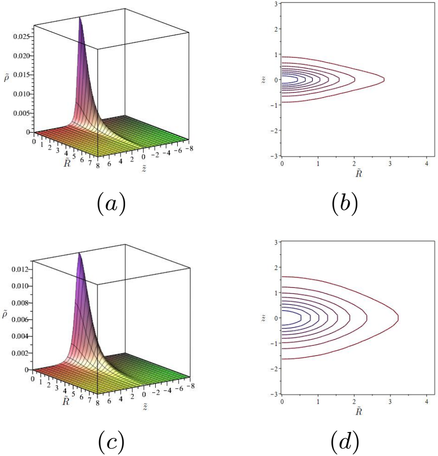


Figure : The relativistic energy density $\tilde{\rho}$ and the isodensity curves for the first model of Miyamoto-Nagai with parameters $\tilde{a} = 1$ and $\tilde{b} = 0.5$ (top figures), 1 (bottom figures) .

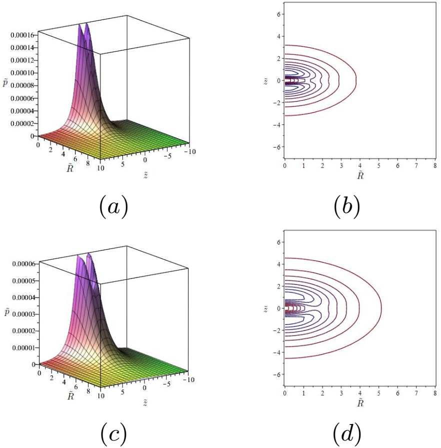


Figure : The relativistic average pressure $\tilde{p}$ and the level curves for the first model of Miyamoto-Nagai.

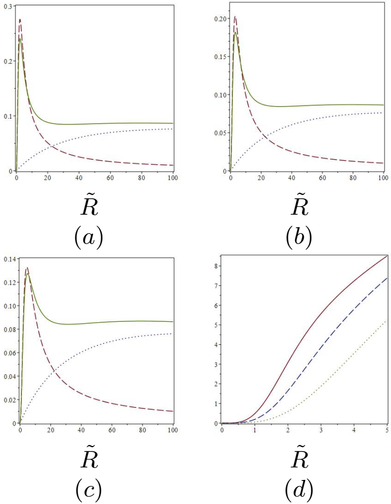


Figure : The relativistic rotation curves v_c^2 for the first model of Miyamoto-Nagai (curves dashed), matter dark halo (dotted curves) and the composite system thick disk plus dark matter halo. (d) $\tilde{h}^2$ for the first model of Miyamoto-Nagai.

NFW model

$$\Phi_H = -v_0^2 \frac{\ln(1 + r/r_s)}{r/r_s}, \tag{28}$$

where $r = \sqrt{R^2 + z^2}$, v_0 and r_s arbitrary constants. The Newtonian circular speed is given by

$$v_H^2 = v_0^2 \left[\frac{\ln(1 + r/r_s)}{r/r_s} - \frac{1}{1 + r/r_s} \right]. \tag{29}$$

References

- Gonzalo García-Reyes, Gen. Relativ. Gravit. **49**, 3, 1-13 (2017).
- Gonzalo García-Reyes and Kevin A. Hernández-Gómez, Three-dimensional axisymmetric sources for Majumdar-Papapetrou type spacetimes, arXiv:1703.08514.

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