

Random Number Generators: Design Principles and Statistical Testing

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Example: 8 possibilities for the 3 bits **???**:

000, 001, 010, 011, 100, 101, 110, 111

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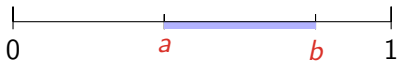
Want a probability of $1/8$ for each, independently of everything else.

For s bits, probability of $1/2^s$ for each of the 2^s possibilities.

Uniform distribution over $(0, 1)$

For simulation in general, we want (to imitate) a sequence U_0, U_1, U_2, \dots of independent random variables uniformly distributed over $(0, 1)$.

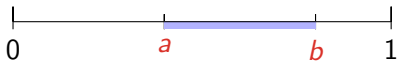
We want $\mathbb{P}[a \leq U_j \leq b] = b - a$.



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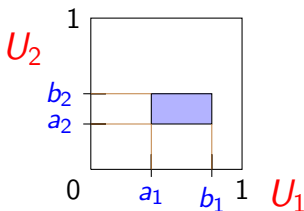
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Independence: For a random vector $\mathbf{U} = (U_1, \dots, U_s)$, we want

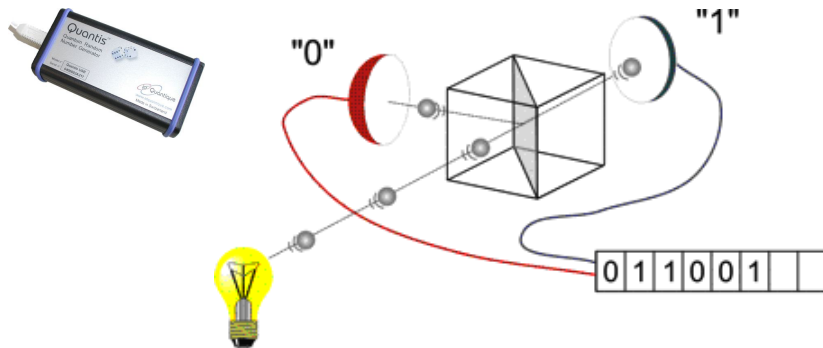
$$\mathbb{P}[a_j \leq U_j \leq b_j \text{ for } j = 1, \dots, s] = (b_1 - a_1) \cdots (b_s - a_s).$$



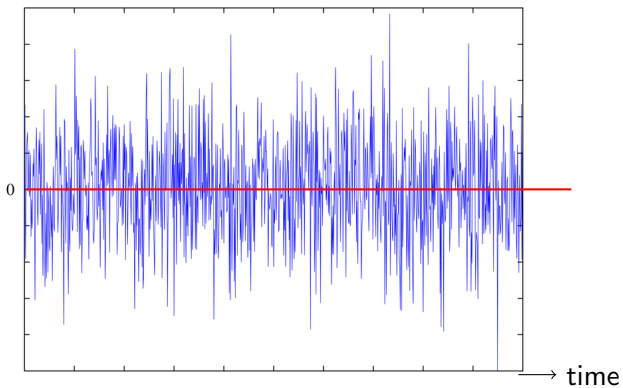
This notion of independent uniform random variables is only a **mathematical abstraction**. Perhaps it does not exist in the real world! We only wish to **imitate** it (approximately).

Physical devices for computers

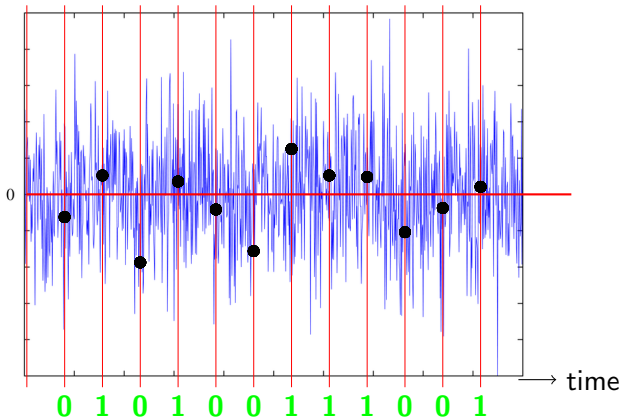
Photon trajectories (sold by **id-Quantique**):



Thermal noise in resistances of electronic circuits



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The signal is sampled periodically.

Reproducibility

Simulations are often required to be exactly replicable, and **always produce exactly the same results** on different computers and architectures, sequential or parallel.

Important for **debugging** and to **replay** exceptional events in more details, for better understanding.

Also essential when comparing systems with slightly different configurations or decision making rules, by simulating them with **common random numbers (CRNs)**. That is, to reduce the variance in comparisons, use the same random numbers at exactly the same places in all configurations of the system, as much as possible. Important for sensitivity analysis, derivative estimation, and effective stochastic **optimization**.

Algorithmic RNGs permit one to replicate without storing the random numbers, which would be required for physical devices.

Algorithmic (pseudorandom) generator

\mathcal{S} , finite state space;

$f : \mathcal{S} \rightarrow \mathcal{S}$, transition function;

$g : \mathcal{S} \rightarrow [0, 1]$, output function.

s_0 , germe (état initial);

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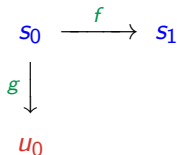
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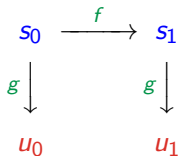
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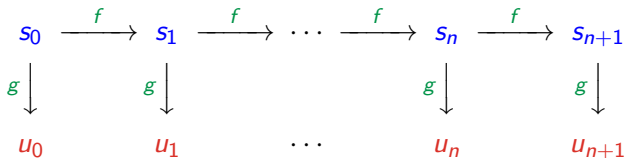
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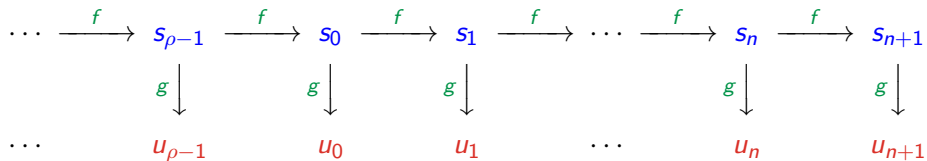


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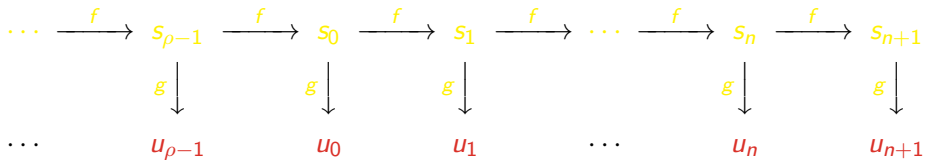
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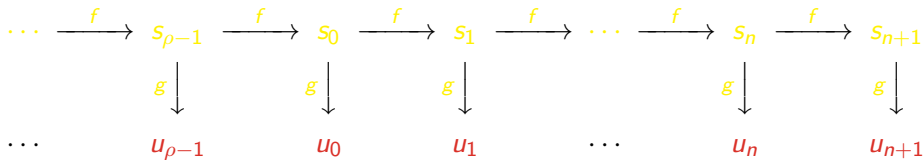
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Period of $\{s_n, n \geq 0\}$: $\rho \leq$ cardinality of \mathcal{S} .



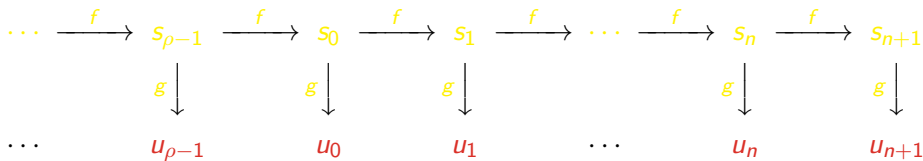
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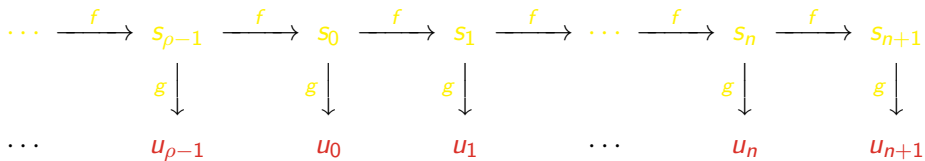
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Selecting s_0 at random and generating s random numbers means spinning the wheel and taking $\mathbf{u} = (u_0, \dots, u_{s-1})$.



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Uniform distribution over $[0, 1]^s$.

If we choose s_0 randomly in \mathcal{S} and we generate s numbers, this corresponds to choosing a random point in the finite set

$$\Psi_s = \{\mathbf{u} = (u_0, \dots, u_{s-1}) = (g(s_0), \dots, g(s_{s-1})), s_0 \in \mathcal{S}\}.$$

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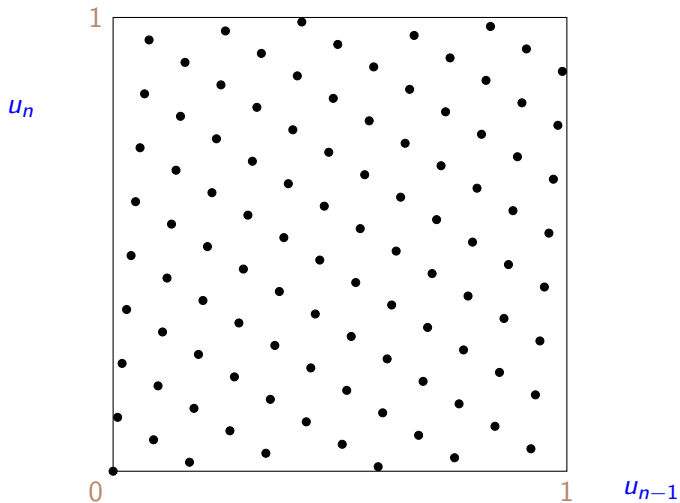
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Design and analysis:

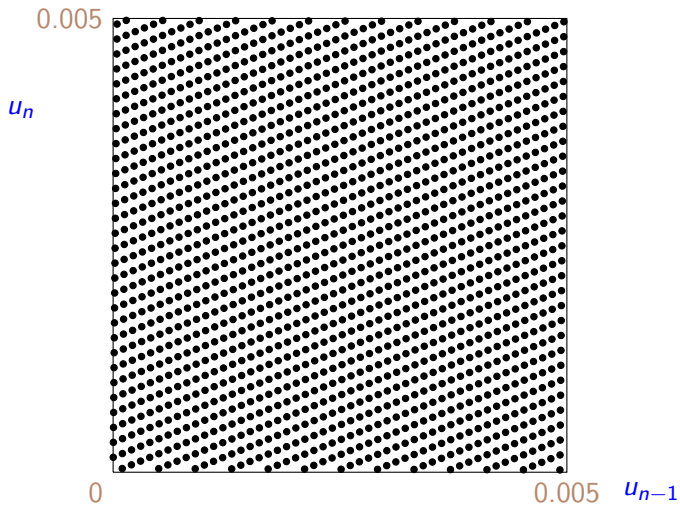
1. Define a uniformity measure for Ψ_s , computable without generating the points explicitly. Linear RNGs.
2. Choose a parameterized family (fast, long period, etc.) and search for parameters that “optimize” this measure (e.g., the worst case) over a given range of values of s .

Baby example:



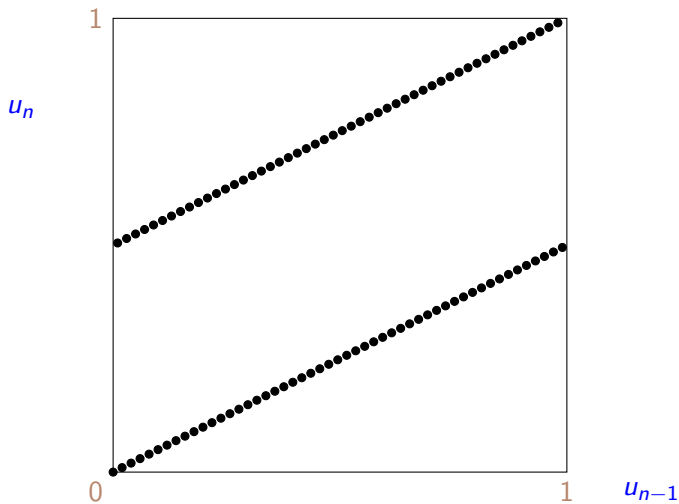
$$x_n = 12 x_{n-1} \bmod 101; \quad u_n = x_n / 101$$

Baby example:



$$x_n = 4809922 x_{n-1} \bmod 60466169 \text{ and } u_n = x_n / 60466169$$

Baby example:



$$x_n = 51 x_{n-1} \bmod 101; \quad u_n = x_n/101.$$

Good uniformity in one dimension, but not in two!

Myth: I use a fast RNG with period length $> 2^{1000}$, so it is certainly excellent!

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No.

Example: $u_n = (n/2^{1000}) \bmod 1$ for $n = 0, 1, 2, \dots$

Other examples: Subtract-with-borrow, lagged-Fibonacci, xorwow, etc.
Were designed to be very fast: simple and very few operations.
They have bad uniformity in higher dimensions.

A single RNG does not suffice.

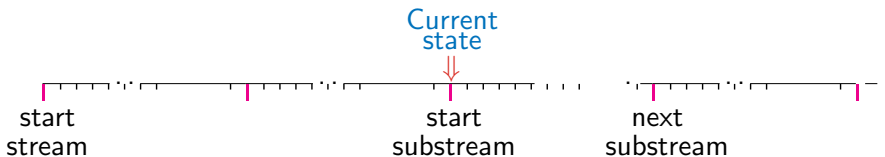
One often needs several **independent streams** of random numbers, e.g., to:

- ▶ Run a simulation on **parallel processors**.
- ▶ Compare systems with well synchronized **common random numbers** (CRNs). Can be complicated to implement and manage when different configurations do not need the same number of U_j 's.

An existing solution: RNG with **multiple streams and substreams**.

Can create **RandomStream** objects at will, behave as “independent” streams viewed as virtual RNGs. Can be further partitioned in **substreams**.

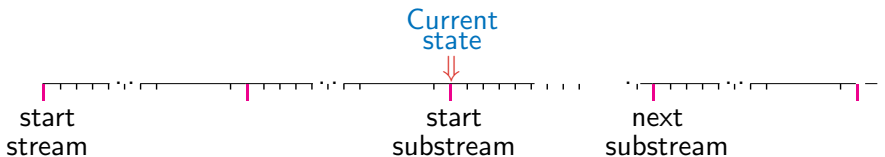
Example: With **MRG32k3a** generator, streams start 2^{127} values apart, and each stream is partitioned into 2^{51} substreams of length 2^{76} . **One stream:**



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Example: With `MRG32k3a` generator, streams start 2^{127} values apart, and each stream is partitioned into 2^{51} substreams of length 2^{76} . **One stream:**



```
RandomStream mystream1 = createStream ();
double u = randomU01 (mystream1);
double z = inverseCDF (normalDist, randomU01(mystream1));
...
rewindSubstream (mystream1);
forwardToNextSubstream (mystream1);
rewindStream (mystream1);
```

Comparing systems with CRNs: a simple inventory example

X_j = **inventory level** in morning of day j ;

D_j = **demand** on day j , uniform over $\{0, 1, \dots, L\}$;

$\min(D_j, X_j)$ **sales** on day j ;

$Y_j = \max(0, X_j - D_j)$ **inventory** at end of day j ;

Orders follow a **(s, S) policy**: If $Y_j < s$, order $S - Y_j$ items.

Each order **arrives** for next morning with probability p .

Revenue for day j : sales – inventory costs – order costs

$$= c \cdot \min(D_j, X_j) - h \cdot Y_j - (K + k \cdot (S - Y_j)) \cdot \mathbb{I}[\text{an order arrives}].$$

Number of calls to RNG for order arrivals is random!

Two streams of random numbers, one substream for each run.

Same streams and substreams for all policies (s, S) .

Inventory example: OpenCL code to simulate m days

```

double inventorySimulateOneRun (int m, int s, int S,
    clrngStream *stream_demand, clrngStream *stream_order) {
    // Simulates inventory model for m days, with the (s,S) policy.
    int Xj = S, Yj;          // Stock Xj in morning and Yj in evening.
    double profit = 0.0;    // Cumulated profit.
    for (int j = 0; j < m; j++) {
        // Generate and subtract the demand for the day.
        Yj = Xj - clrngRandomInteger (stream_demand, 0, L);
        if (Yj < 0) Yj = 0; // Lost demand.
        profit += c * (Xj - Yj) - h * Yj;
        if ((Yj < s) && (clrngRandomU01 (stream_order) < p)) {
            // We have a successful order.
            profit -= K + k * (S - Yj); // Pay for successful order.
            Xj = S;
        } else
            Xj = Yj;          // Order not received.
    }
    return profit / m;      // Return average profit per day.
}

```

Comparing p policies with CRNs

```

// Simulate n runs with CRNs for p policies (s[k], S[k]), k=0,...,p-1.
clrngStream* stream_demand = clrngCreateStream();
clrngStream* stream_order  = clrngCreateStream();
for (int k = 0; k < p; k++) { // for each policy
    for (int i = 0; i < n; i++) { // perform n runs
        stat_profit[k, i] = inventorySimulateOneRun (m, s[k], S[k],
                                                    stream_demand, stream_order);
        // Realign starting points so they are the same for all policies
        clrngForwardToNextSubstream (stream_demand);
        clrngForwardToNextSubstream (stream_order);
    }
    clrngRewindStream (stream_demand);
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// Print and plot results ...
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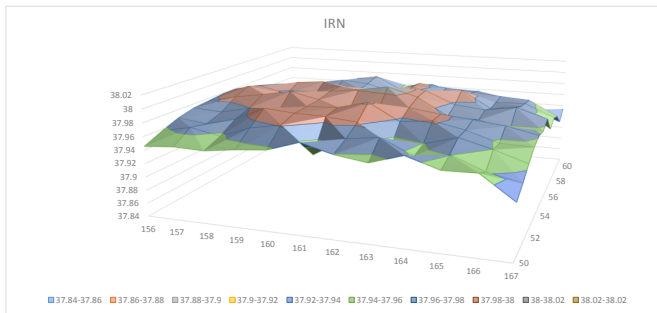
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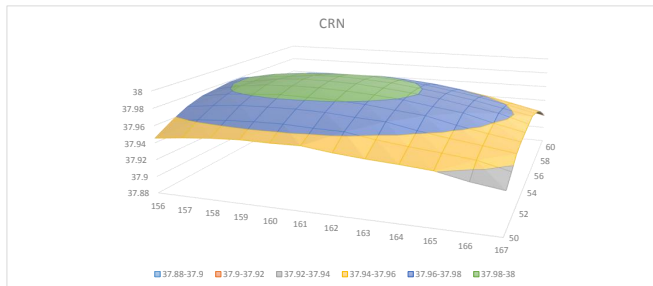
Can perform these pn simulations on thousands of [parallel processors](#) and obtain exactly the same results, using the same streams and substreams.

Comparison with independent random numbers

	156	157	158	159	160	161	162	163	164	165	166	167
50	37.94537	37.94888	37.94736	37.95314	37.95718	37.97194	37.95955	37.95281	37.96711	37.95221	37.95325	37.92063
51	37.9574	37.9665	37.95732	37.97337	37.98137	37.94273	37.96965	37.97573	37.95425	37.96074	37.94185	37.93139
52	37.96725	37.96166	37.97192	37.99236	37.98856	37.98708	37.98266	37.94671	37.95961	37.97238	37.95982	37.94465
53	37.97356	37.96999	37.97977	37.97611	37.98929	37.99089	38.00219	37.97693	37.98191	37.97217	37.95713	37.95575
54	37.97593	37.9852	37.99233	38.00043	37.99056	37.9744	37.98008	37.98817	37.98168	37.97703	37.97145	37.96138
55	37.97865	37.9946	37.97297	37.98383	37.99527	38.00068	38.00826	37.99519	37.96897	37.96675	37.9577	37.95672
56	37.97871	37.9867	37.97672	37.9744	37.9955	37.9712	37.96967	37.99717	37.97736	37.97275	37.97968	37.96523
57	37.97414	37.97797	37.98816	37.99192	37.9678	37.98415	37.97774	37.97844	37.99203	37.96531	37.97226	37.93934
58	37.96869	37.97435	37.9625	37.96581	37.97331	37.95655	37.98382	37.97144	37.97409	37.96631	37.96764	37.94759
59	37.95772	37.94725	37.9711	37.97905	37.97504	37.96237	37.98182	37.97656	37.97212	37.96762	37.96429	37.93976
60	37.94434	37.95081	37.94275	37.95515	37.98134	37.95863	37.96581	37.95548	37.96573	37.93949	37.93839	37.9203
61	37.922	37.93006	37.92656	37.93281	37.94999	37.95799	37.96368	37.94849	37.954	37.92439	37.90535	37.93375



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51	37.9574	37.96169	37.96379	37.96524	37.96546	37.96379	37.96293	37.95726	37.95295	37.94944	37.94536	37.93685
52	37.96725	37.97117	37.97402	37.97476	37.97492	37.97387	37.971	37.96879	37.96184	37.95627	37.95154	37.94626
53	37.97356	37.97852	37.98098	37.98243	37.98187	37.98079	37.97848	37.97436	37.97088	37.96268	37.95589	37.94995
54	37.97593	37.98241	37.98589	37.98692	37.98703	37.98522	37.9829	37.97931	37.97397	37.96925	37.95986	37.95186
55	37.97865	37.98235	37.9874	37.9894	37.98909	37.9879	37.98483	37.98125	37.97641	37.96992	37.96401	37.95343
56	37.97871	37.98269	37.98494	37.98857	37.98917	37.98757	37.98507	37.98073	37.97594	37.96989	37.96227	37.95519
57	37.97414	37.98035	37.98293	37.98377	37.98603	37.98528	37.98239	37.97858	37.97299	37.96703	37.95981	37.95107
58	37.96869	37.97207	37.97825	37.97944	37.97895	37.97987	37.97776	37.97358	37.96848	37.9617	37.95461	37.94622
59	37.95772	37.96302	37.9663	37.97245	37.97234	37.97055	37.9701	37.96664	37.96122	37.95487	37.94695	37.93871
60	37.94434	37.94861	37.95371	37.95691	37.96309	37.96167	37.9586	37.95678	37.95202	37.9454	37.93785	37.92875
61	37.922	37.93169	37.93591	37.94085	37.94401	37.95021	37.94751	37.94312	37.94	37.93398	37.92621	37.91742



Parallel computers

Processing elements (PEs) or “cores” are organized in a hierarchy. Many in a chip. SIMD or MIMD or mixture. Many chips per node, etc. Similar hierarchy for memory, usually more complicated and with many types of memory and access speeds.

Since about 10 years, clock speeds of processors no longer increase, but number of cores increases instead. Roughly doubles every 1.5 to 2 years.

Simulation algorithms (such as for RNGs) must adapt to this.

Some PEs, e.g., on GPUs, only have a small fast-access (private) memory and have limited instruction sets.

Streams for parallel RNGs

Why not a **single source** of random numbers (one stream) for all threads?

Bad because (1) too much overhead for transfer and (2) non reproducible.

A **different RNG** (or parameters) for each stream? **Inconvenient** and **limited**: hard to handle millions of streams.

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Random starting points: acceptable if period ρ is huge.
 For period ρ , and m streams of length ℓ ,

$$\mathbb{P}[\text{overlap somewhere}] = P_o \approx m^2 \ell / \rho.$$

Example: if $m = \ell = 2^{20}$, then $m^2 \ell = 2^{60}$.

For $\rho = 2^{128}$, $P_o \approx 2^{-68}$. For $\rho = 2^{1024}$, $P_o \approx 2^{-964}$ (negligible).

How to use streams in parallel processing?

One can use several PEs to fill rapidly a large buffer of random numbers, and use them afterwards (e.g., on host processor). Many have proposed software tools to do that. But this is rarely what we want.

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Typically, we want independent streams produced and used by the threads. E.g., simulate the inventory model on each PE.

One stream per PE? One per thread? One per subtask? **No.**

How to use streams in parallel processing?

One can use several PEs to fill rapidly a large buffer of random numbers, and use them afterwards (e.g., on host processor). Many have proposed software tools to do that. But this is rarely what we want.

Typically, we want independent streams produced and used by the threads. E.g., simulate the inventory model on each PE.

One stream per PE? One per thread? One per subtask? **No.**

For reproducibility and effective use of CRNs, streams must be assigned and used at a logical (hardware-independent) level, and it should be possible to have many distinct streams in a thread or PE at a time.

Single monitor to create all streams. Perhaps multiple creators of streams. To run on GPUs, the state should be small, say at most 256 bits. Some small robust RNGs such as LFSR113, MRG31k3p, and MRG32k3a are good for that. Also some counter-based RNGs.

Other scheme: streams that can split to create new children streams.

Linear multiple recursive generator (MRG)

$$x_n = (a_1 x_{n-1} + \cdots + a_k x_{n-k}) \bmod m, \quad u_n = x_n / m.$$

State: $s_n = (x_{n-k+1}, \dots, x_n)$. Max. period: $\rho = m^k - 1$.

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Numerous variants and implementations.

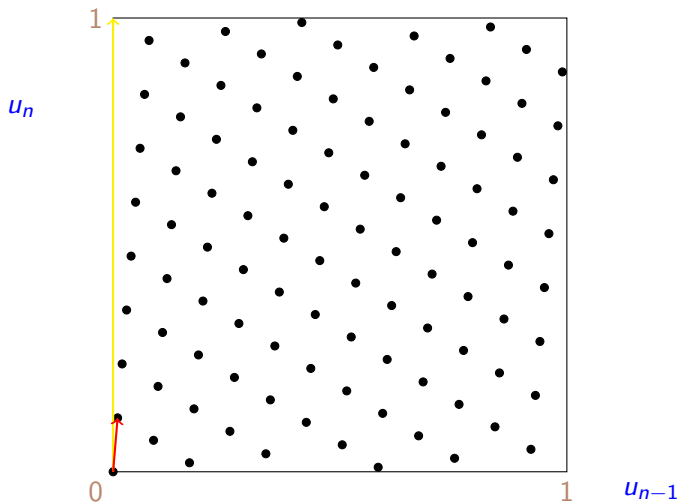
For $k = 1$: classical **linear congruential generator** (LCG).

Structure of the points Ψ_s :

x_0, \dots, x_{k-1} can take any value from 0 to $m - 1$, then x_k, x_{k+1}, \dots are determined by the linear recurrence. Thus,

$(x_0, \dots, x_{k-1}) \mapsto (x_0, \dots, x_{k-1}, x_k, \dots, x_{s-1})$ is a **linear mapping**.

It follows that Ψ_s is a linear space; it is the intersection of a lattice with the unit cube.



$$x_n = 12x_{n-1} \bmod 101; \quad u_n = x_n/101$$

Example of bad structure: lagged-Fibonacci

$$x_n = (x_{n-r} + x_{n-k}) \pmod{m}.$$

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$$x_n = (x_{n-r} + x_{n-k}) \pmod{m}.$$

Very fast, but bad. We always have $u_{n-k} + u_{n-r} - u_n = 0 \pmod{1}$.

This means: $u_{n-k} + u_{n-r} - u_n = q$ for some integer q .

If $0 < u_n < 1$ for all n , we can only have $q = 0$ or 1 .

Then **all** points (u_{n-k}, u_{n-r}, u_n) are in **only two parallel planes** in $[0, 1)^3$.

Other example: subtract-with-borrow (SWB)

State $(x_{n-48}, \dots, x_{n-1}, c_{n-1})$ where $x_n \in \{0, \dots, 2^{31} - 1\}$ and $c_n \in \{0, 1\}$:

$$x_n = (x_{n-8} - x_{n-48} - c_{n-1}) \bmod 2^{31},$$

$$c_n = 1 \text{ if } x_{n-8} - x_{n-48} - c_{n-1} < 0, \quad c_n = 0 \text{ otherwise,}$$

$$u_n = x_n / 2^{31},$$

Period $\rho \approx 2^{1479} \approx 1.67 \times 10^{445}$.

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Tezuka, L'Ecuyer, and Couture (1993). “On the Add-with-Carry and Subtract-with-Borrow Random Number Generators.”

Couture and L'Ecuyer (1994) “On the Lattice Structure of Certain Linear Congruential Sequences Related to AWC/SWB Generators.”

Combined MRGs.

Two [or more] MRGs in parallel:

$$\begin{aligned}x_{1,n} &= (a_{1,1}x_{1,n-1} + \cdots + a_{1,k}x_{1,n-k}) \pmod{m_1}, \\x_{2,n} &= (a_{2,1}x_{2,n-1} + \cdots + a_{2,k}x_{2,n-k}) \pmod{m_2}.\end{aligned}$$

One possible **combinaison**:

$$z_n := (x_{1,n} - x_{2,n}) \pmod{m_1}; \quad u_n := z_n/m_1;$$

L'Ecuyer (1996): the sequence $\{u_n, n \geq 0\}$ is also the output of an MRG of modulus $m = m_1 m_2$, with small added “noise”. The period can reach $(m_1^k - 1)(m_2^k - 1)/2$.

Permits one to implement efficiently an MRG with large m and several large nonzero coefficients.

Parameters: L'Ecuyer (1999); L'Ecuyer et Touzin (2000).

Implementations with multiple streams.

One popular and recommendable generator: MRG32k3a

Choose **six 32-bit integers**:

x_{-2}, x_{-1}, x_0 in $\{0, 1, \dots, 4294967086\}$ (not all 0) and

y_{-2}, y_{-1}, y_0 in $\{0, 1, \dots, 4294944442\}$ (not all 0). For $n = 1, 2, \dots$, let

$$x_n = (1403580x_{n-2} - 810728x_{n-3}) \bmod 4294967087,$$

$$y_n = (527612y_{n-1} - 1370589y_{n-3}) \bmod 4294944443,$$

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(x_{n-2}, x_{n-1}, x_n) visits each of the $4294967087^3 - 1$ possible values.

(y_{n-2}, y_{n-1}, y_n) visits each of the $4294944443^3 - 1$ possible values.

The sequence u_0, u_1, u_2, \dots is periodic, with 2 cycles of period

$$\rho \approx 2^{191} \approx 3.1 \times 10^{57}.$$

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Robust and reliable for simulation.

Used by SAS, R, MATLAB, Arena, Automod, Witness, Spielo gaming, ...

A similar (faster) one: MRG31k3p

State is six 31-bit integers:

Two cycles of period $\rho \approx 2^{185}$.

Each nonzero multiplier a_j is a sum or a difference or two powers of 2.

Recurrence is implemented via shifts, masks, and additions.

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The original MRG32k3a was designed to be implemented in (double) floating-point arithmetic, with 52-bit mantissa.

MRG31k3p was designed for 32-bit integers.

On 64-bit computers, both can be implemented using 64-bit integer arithmetic. Faster.

General linear recurrence modulo m

State (vector) \mathbf{x}_n evolves as

$$\mathbf{x}_n = \mathbf{A} \mathbf{x}_{n-1} \bmod m.$$

Jumping Ahead:

$$\mathbf{x}_{n+\nu} = (\mathbf{A}^\nu \bmod m) \mathbf{x}_n \bmod m.$$

The matrix $\mathbf{A}^\nu \bmod m$ can be precomputed for selected values of ν . This takes $\mathcal{O}(\log \nu)$ multiplications mod m .

If output function $u_n = g(\mathbf{x}_n)$ is also linear, one can study the [uniformity of each \$\Psi_s\$](#) by studying the linear mapping. Many tools for this.

RNGs based on linear recurrences modulo 2

$$\begin{aligned}
 \mathbf{x}_n &= \mathbf{A} \mathbf{x}_{n-1} \bmod 2 &= (x_{n,0}, \dots, x_{n,k-1})^t, & \text{(state, } k \text{ bits)} \\
 \mathbf{y}_n &= \mathbf{B} \mathbf{x}_n \bmod 2 &= (y_{n,0}, \dots, y_{n,w-1})^t, & \text{(} w \text{ bits)} \\
 u_n &= \sum_{j=1}^w y_{n,j-1} 2^{-j} &= .y_{n,0} y_{n,1} y_{n,2} \dots, & \text{(output)}
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Clever choice of \mathbf{A} : transition via shifts, XOR, AND, masks, etc., on blocks of bits. Very fast.

Special cases: Tausworthe, LFSR, GFSR, twisted GFSR, Mersenne twister, WELL, xorshift, etc.

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Each coordinate of \mathbf{x}_n and of \mathbf{y}_n follows the recurrence

$$x_{n,j} = (\alpha_1 x_{n-1,j} + \dots + \alpha_k x_{n-k,j}),$$

with **characteristic polynomial**

$$P(z) = z^k - \alpha_1 z^{k-1} - \dots - \alpha_{k-1} z - \alpha_k = \det(\mathbf{A} - z\mathbf{I}).$$

Max. period: $\rho = 2^k - 1$ reached iff $P(z)$ is primitive.

Example of fast RNG: operations on blocks of bits.

Example: Choose $x_0 \in \{2, \dots, 2^{32} - 1\}$ (32 bits). Evolution:

$$x_{n-1} = \quad |00010100101001101100110110100101|$$

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This implements $x_n = \mathbf{A} x_{n-1} \bmod 2$ for a certain \mathbf{A} .

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For real numbers in $(0, 1)$: $u_n = x_n / (2^{32} + 1)$.

More realistic: LFSR113

Take 4 recurrences on blocks of 32 bits, in parallel.

The periods are $2^{31} - 1$, $2^{29} - 1$, $2^{28} - 1$, $2^{25} - 1$.

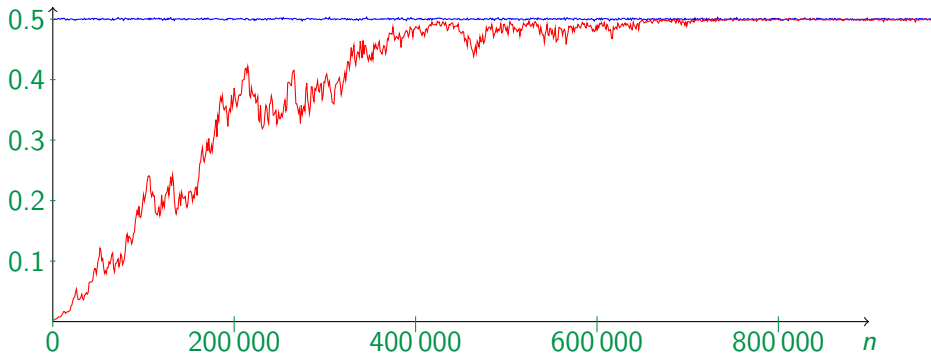
We add these 4 states by a XOR, then we divide by $2^{32} + 1$.

The output has period $\approx 2^{113} \approx 10^{34}$.

Impact of a matrix A that changes the state too slowly.

Experiment: take an initial state s_0 with a single bit at 1 and run for n steps to compute u_n . Try all k possibilities for s_0 and average the k values of u_n . Also take a moving average over 1000 iterations.

MT19937 (Mersenne twister) vs WELL19937:



Combined linear/nonlinear generators

Linear generators fail statistical tests built to detect linearity.

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To escape linearity, we may

- ▶ use a nonlinear transition f ;
- ▶ use a nonlinear output transformation g ;
- ▶ do both;
- ▶ combine RNGs of different types.

There are various proposals in this direction. Many behave well empirically.

L'Ecuyer and Granger-Picher (2003): [Large linear generator modulo 2 combined with a small nonlinear one, via XOR.](#)

Counter-Based RNGs

State at step n is just n , so $f(n) = n + 1$, and $g(n)$ is more complicated.

Advantages: trivial to jump ahead, can generate a sequence in any order.

Typically, g is a bijective block cipher **encryption algorithm**.

It has a parameter c called the encoding **key**.

One can use a different key c for each stream.

Examples: MD5, TEA, SHA, AES, ChaCha, Threefish, etc.

The encoding is often simplified to make the RNG faster.

Threefry and Philox, for example. Very fast!

$g_c : (k\text{-bit counter}) \mapsto (k\text{-bit output})$, period $\rho = 2^k$.

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Changing one bit in n should change 50% of the output bits on average.

No theoretical analysis for the point sets Ψ_S .

But some of them perform very well in empirical statistical tests.

See Salmon, Moraes, Dror, Shaw (2011), for example.

Empirical statistical Tests

Hypothesis \mathcal{H}_0 : “ $\{u_0, u_1, u_2, \dots\}$ are i.i.d. $U(0, 1)$ r.v.'s”.

We know that \mathcal{H}_0 is false, but can we detect it ?

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Test:

- Define a statistic T , function of the u_i , whose distribution under \mathcal{H}_0 is known (or approx.).
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Different tests detect different deficiencies.

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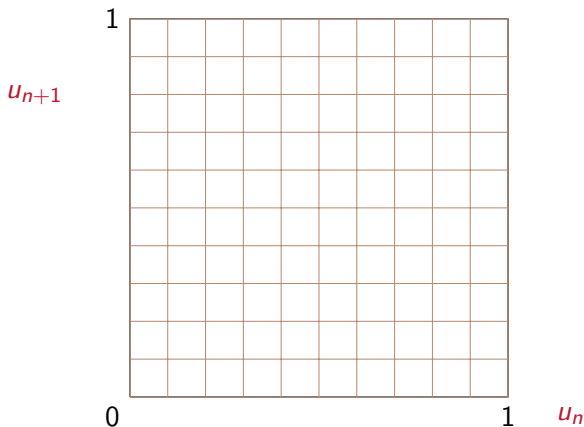
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Compromise: Build an RNG that passes most **reasonable** tests.

Tests that fail are hard to find.

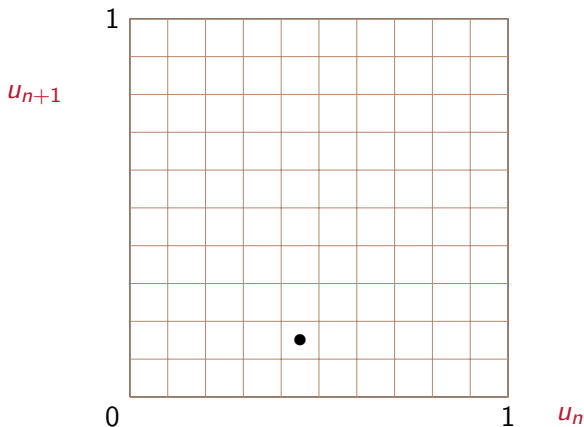
Formalization: computational complexity framework.

Example: A collision test



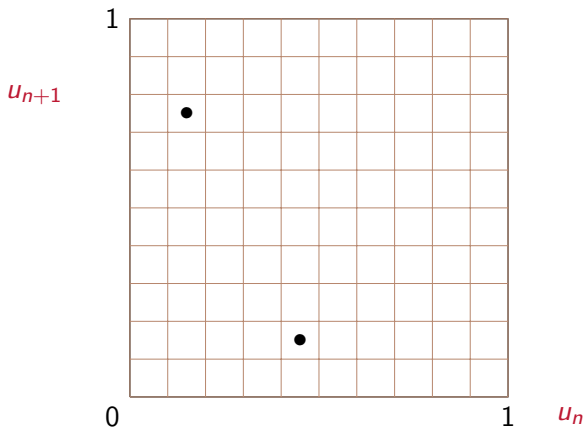
Throw $n = 10$ points in $k = 100$ boxes.

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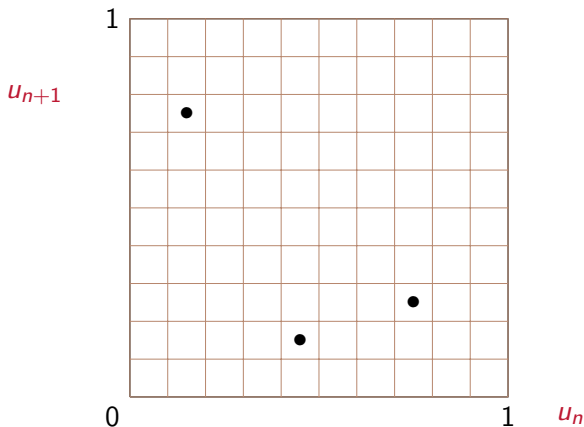
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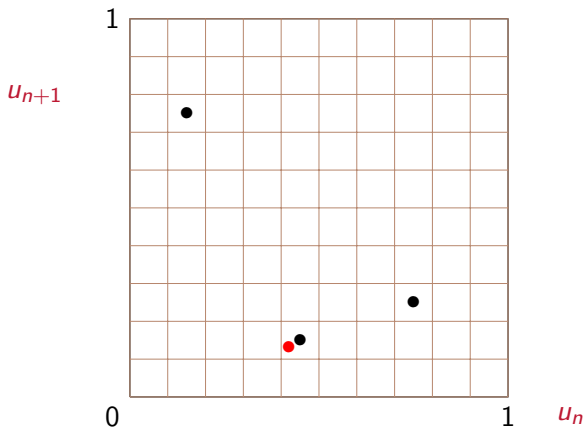
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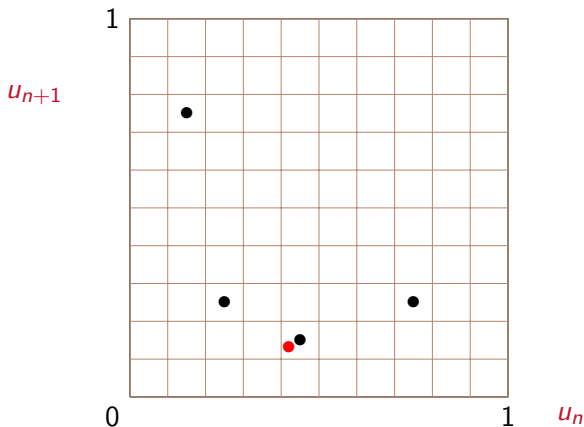
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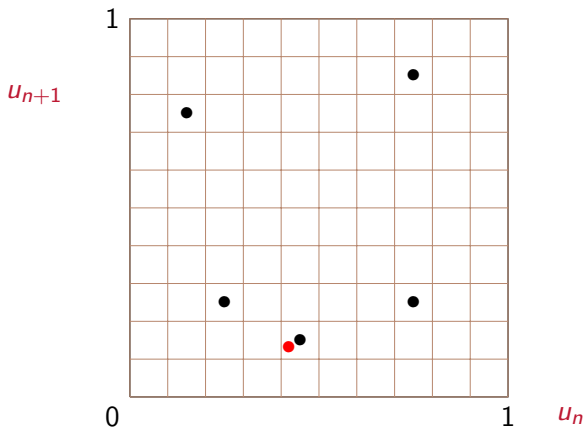
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Example: A collision test



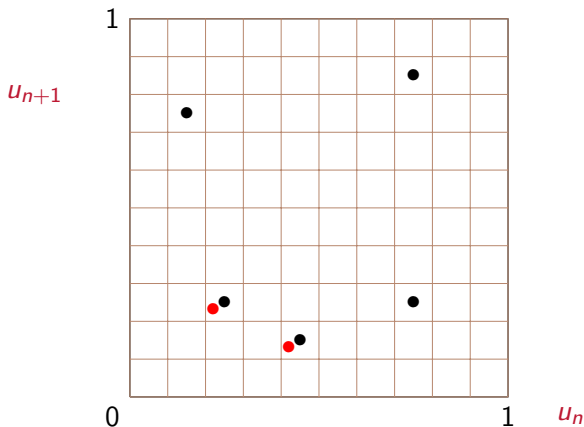
Throw $n = 10$ points in $k = 100$ boxes.

Example: A collision test



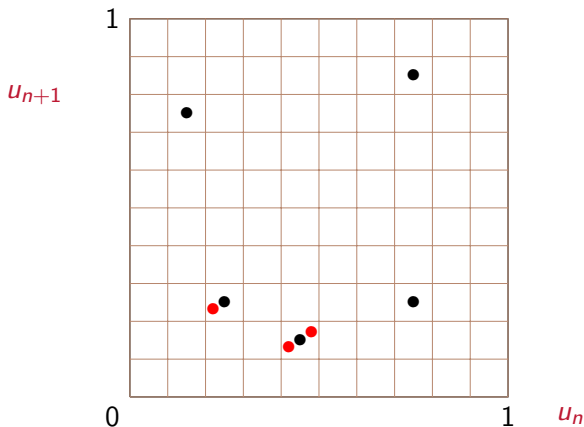
Throw $n = 10$ points in $k = 100$ boxes.

Example: A collision test



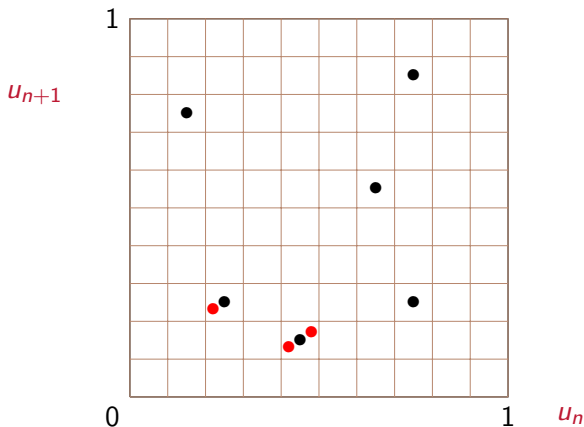
Throw $n = 10$ points in $k = 100$ boxes.

Example: A collision test



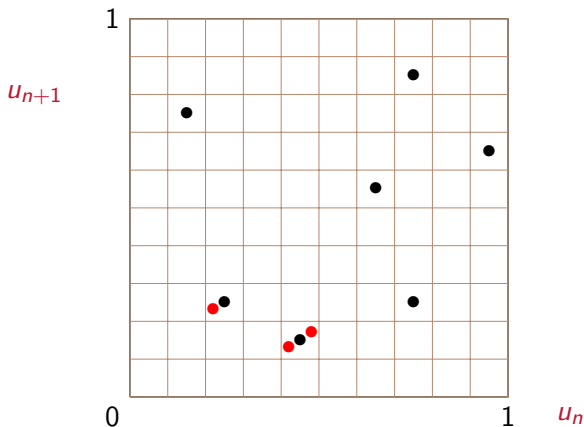
Throw $n = 10$ points in $k = 100$ boxes.

Example: A collision test



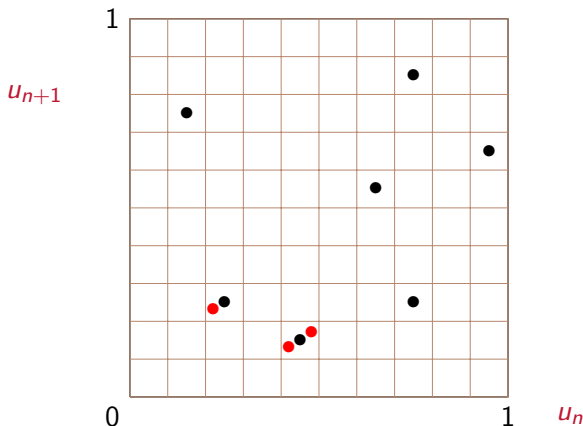
Throw $n = 10$ points in $k = 100$ boxes.

Example: A collision test



Throw $n = 10$ points in $k = 100$ boxes.

Example: A collision test



Throw $n = 10$ points in $k = 100$ boxes.

Here we observe 3 collisions. $\mathbb{P}[C \geq 3 \mid \mathcal{H}_0] \approx 0.144$.

Collision test

Partition $[0, 1)^s$ in $k = d^s$ cubic boxes of equal size.

Generate n points $(u_{i_1}, \dots, u_{i_1+s-1})$ in $[0, 1)^s$.

C = number of collisions.

Collision test

Partition $[0, 1]^s$ in $k = d^s$ cubic boxes of equal size.

Generate n points $(u_{is}, \dots, u_{is+s-1})$ in $[0, 1]^s$.

C = number of collisions.

Under \mathcal{H}_0 , $C \approx$ Poisson of mean $\lambda = n^2/(2k)$, if k is large and λ is small.

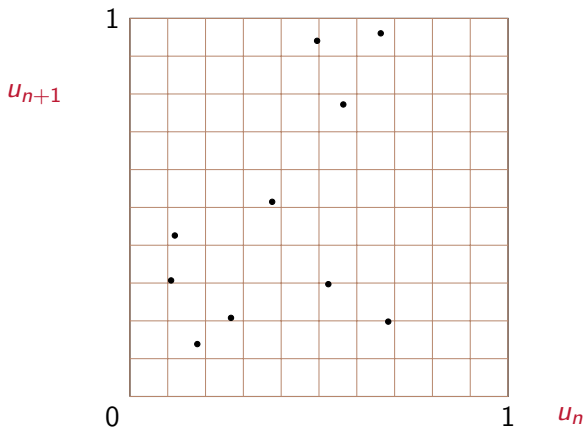
If we observe c collisions, we compute the p -values:

$$p^+(c) = \mathbb{P}[X \geq c \mid X \sim \text{Poisson}(\lambda)],$$

$$p^-(c) = \mathbb{P}[X \leq c \mid X \sim \text{Poisson}(\lambda)],$$

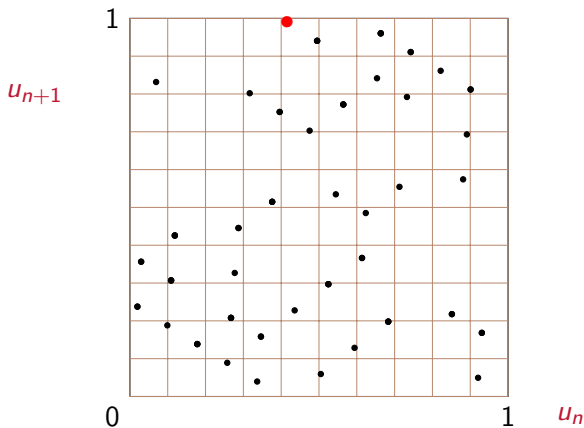
We reject \mathcal{H}_0 if $p^+(c)$ is too close to 0 (too many collisions) or $p^-(c)$ is too close to 1 (too few collisions).

Example: LCG with $m = 101$ and $a = 12$:

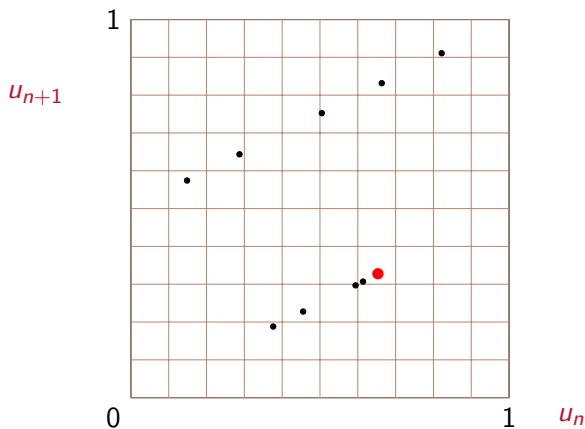


n	λ	C	$p^-(C)$
10	1/2	0	0.6281

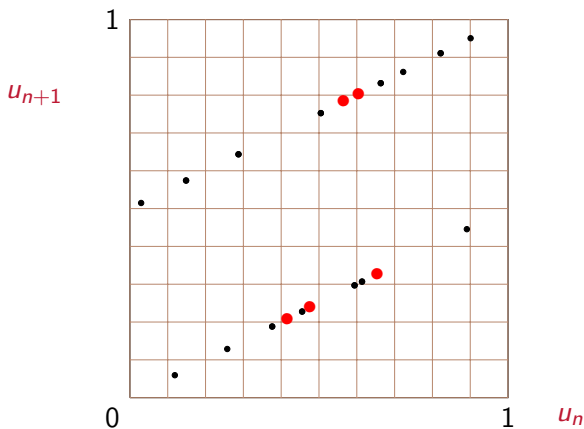
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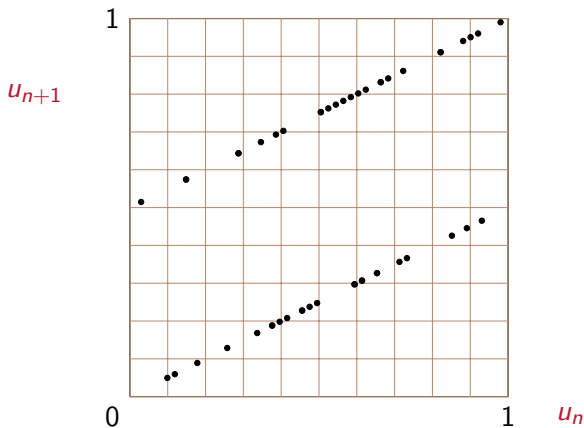
n	λ	C	$p^-(C)$
10	1/2	0	0.6281
20	2	0	0.1304
40	8	1	0.0015



n	λ	C	$p^+(C)$
10	$1/2$	1	0.3718



n	λ	C	$p^+(C)$
10	$1/2$	1	0.3718
20	2	5	0.0177



n	λ	C	$p^+(C)$
10	$1/2$	1	0.3718
20	2	5	0.0177
40	8	20	2.2×10^{-9}

SWB in (older) Mathematica

For the unit cube $[0, 1]^3$, divide each axis in $d = 100$ equal intervals. This gives $k = 100^3 = 1$ million boxes.

Generate $n = 10\,000$ vectors in 25 dimensions: (U_0, \dots, U_{24}) .

For each, note the box where (U_0, U_{20}, U_{24}) falls.

Here, $\lambda = 50$.

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Results: $C = 2070, 2137, 2100, 2104, 2127, \dots$

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Results: $C = 2070, 2137, 2100, 2104, 2127, \dots$

With MRG32k3a: $C = 41, 66, 53, 50, 54, \dots$

A Class of Multinomial Tests

Partition $[0, 1]^s$ in $k = d^s$ cubic boxes of equal size.

Generate n points $(u_{is}, \dots, u_{is+s-1})$ in $[0, 1]^s$.

Let X_j = number of points in box j .

Under \mathcal{H}_0 , (X_0, \dots, X_{k-1}) is Multinomial $(n, 1/k, \dots, 1/k)$.

Can measure the divergence from uniformity by a statistic:

$$Y = \sum_{j=0}^{k-1} f_{n,k}(X_j).$$

For example, with $\lambda = n/k$:

Y	$f_{n,k}(x)$	name
D_δ	$2x[(x/\lambda)^\delta - 1]/(\delta(1 + \delta))$	power divergence
X^2	$(x - \lambda)^2/\lambda$	Pearson
G^2	$2x \ln(x/\lambda)$	loglikelihood
$-H$	$(x/n) \log_2(x/n)$	(negative) entropy
N_b	$I[x = b]$	num. boxes with exactly b points
W_b	$I[x \geq b]$	num. boxes with at least b points
N_0	$I[x = 0]$	num. empty boxes
C	$(x - 1) I[x > 1]$	num. collisions

Distribution (multinomial case).

For well-behaved $f_{n,k}$ (finite positive variance; okay for D_δ).

(Dense). If k fixed, $n \rightarrow \infty$, $\kappa_c^2 = \text{Var}[Y]/(2(k-1))$, then

$$\frac{Y - E[Y] + (k-1)\kappa_c}{\kappa_c} \Rightarrow \chi^2(k-1).$$

(Sparse). If $k \rightarrow \infty$, $n \rightarrow \infty$, and $n/k \rightarrow \lambda_0$, $0 < \lambda_0 < \infty$, then

$$\frac{Y - E[Y]}{\sqrt{\text{Var}[Y]}} \Rightarrow N(0, 1).$$

In sparse case, $E[Y]$ and $\text{Var}[Y]$ can be computed numerically. Otherwise, use asymptotic values.

(Very sparse). Let $k \rightarrow \infty$.

If $b \geq 2$ and $n^b/(k^{b-1}b!) \rightarrow \lambda_b$, $W_b \Rightarrow N_b \Rightarrow \text{Poisson}(\lambda_b)$.

If $n^2/(2k) \rightarrow \lambda_2$, then $C \Rightarrow W_2 \Rightarrow \text{Poisson}(\lambda_2)$.

If $b = 0$ and $n/k - \ln(k) \rightarrow \gamma_0$, $N_0 \Rightarrow \text{Poisson}(e^{-\gamma_0})$.

Left and right p -values if $Y = y$:

$$p^-(y) = P[Y \leq y \mid \mathcal{H}_0] \quad \text{and} \quad p^+(y) = P[Y \geq y \mid \mathcal{H}_0].$$

Overlapping vectors.

Ex. for $s = 3$: $(u_0, u_1, u_2), (u_1, u_2, u_3), (u_2, u_3, u_4), (u_3, u_4, u_5), \dots$

$X_{s,j}^\emptyset$ = nb. of **overlapping** vectors in box j .

$$D_{\delta,(s)} = \sum_{j=0}^{k-1} \frac{2}{\delta(1+\delta)} X_{s,j}^\emptyset \left[(X_{s,j}^\emptyset / \lambda)^\delta - 1 \right].$$

Dense case: Under \mathcal{H}_0 , fixed k , $n \rightarrow \infty$, one has

$$\tilde{D}_{\delta,(s)} = D_{\delta,(s)} - D_{\delta,(s-1)} \Rightarrow \chi^2(d^s - d^{s-1}).$$

Sparse case:

$$\frac{\tilde{\chi}^2 - (d^s - d^{s-1})}{\sqrt{2(d^s - d^{s-1})}} \Rightarrow N(0, 1).$$

For **very sparse case**, Poisson approx. stills holds and is quite good for (say) $\lambda \leq 1$ if n is large.

Empirical findings: Overlapping tests are typically almost as sensitive as non-overlapping ones for same n , and use s times fewer random numbers.

What is detected? Sample size?

A. Ψ_s too regular.

Then, Y will be too small.

Extreme case: All points in different boxes.

Then, for $b \geq 2$, $p^- = P[W_b \leq 0 \mid \mathcal{H}_0] \approx \exp(-n^b / (k^{b-1} b!))$.

Need $n = O(k^{(b-1)/b})$ to reject H_0 .

Best: W_2 or C ; need $n = O(\sqrt{k})$.

$E[C] \approx n^2 / (2k)$	4	8	16	32	64
$p^- = P[C \leq 0 \mid \mathcal{H}_0]$	1.8E-2	3.3E-4	1.1E-7	1.3E-14	1.6E-28

B. Ψ_s too clustered.

Then, Y will be too large.

Ex.: Points unif. dist. over k_1 boxes, other $k - k_1$ boxes empty.

Then need $n \approx O(k_1^{(b-1)/b})$ if k_1 large enough.

If k_1 small, optimal b is $b = \max(2, \lceil n/k_1 \rceil - 1)$.

Alternatives with D_δ :

Hole: take small δ (e.g., entropy) or N_0 . Hard to detect.

Peak: take large δ (e.g., D_4) (or W_b for $b \approx n/k_1$). Easy.

Split: Use C or W_2 . Easy.

For $k \approx \rho \approx \text{card}(\Psi_s)$, cases A and B give $n = O(\sqrt{\rho})$.

Systematic tests for RNG families

For a given RNG family, seek a relationship:

$$n_0 \approx K\rho^\gamma,$$

n_0 = min. sample size for very strong rejection;

ρ is the period length;

K and γ are constants.

For example, good LCGs:

$$x_i = ax_{i-1} \pmod m; \quad u_i = x_i/m.$$

For each e , take largest prime $m < 2^e$, and choose full period LCG with excellent lattice structure in up to 32 dimensions.

Collision test: we obtain $n_0 \approx 16\rho^{1/2}$.

Examples of detailed results.

Serial test in two dim.: Divide each axis in d intervals: $k = d^2$.

Good LCGs, $s = 2$, $k \approx \rho$, $n = K\sqrt{\rho}$, nb of collisions C ,
suspect p -values $p^-(c) = P[C \leq c \mid \mathcal{H}_0]$. (ϵ means $< 10^{-15}$.)

	$n = 2\sqrt{\rho}$	$4\sqrt{\rho}$	$8\sqrt{\rho}$	$16\sqrt{\rho}$	$32\sqrt{\rho}$
e					
18			7.3E-04	$< 10^{-15}$	$< 10^{-15}$
19		3.1E-03	8.4E-11	$< 10^{-15}$	$< 10^{-15}$
20		3.4E-04	6.8E-10	$< 10^{-15}$	$< 10^{-15}$
21		3.0E-03	4.9E-05	$< 10^{-15}$	$< 10^{-15}$
22			3.0E-04	8.6E-12	$< 10^{-15}$
23		3.4E-04	4.3E-09	$< 10^{-15}$	$< 10^{-15}$
24		3.4E-04	4.3E-13	$< 10^{-15}$	$< 10^{-15}$
25			4.2E-09	$< 10^{-15}$	$< 10^{-15}$
26		3.0E-03	1.1E-07	$< 10^{-15}$	$< 10^{-15}$
27		3.0E-03	1.1E-07	$< 10^{-15}$	$< 10^{-15}$
28			2.8E-03	$< 10^{-15}$	$< 10^{-15}$
29		3.3E-04	1.3E-14	$< 10^{-15}$	$< 10^{-15}$
30			5.5E-06	$< 10^{-15}$	$< 10^{-15}$

No right p -value is suspect. C is too small: Structure too regular.

$E[C]$ vs C , $k \approx \rho$, Good LCGs. $n = K\sqrt{\rho}$.

e	$K = 2$		$K = 4$		$K = 8$		$K = 16$		$K = 32$	
18	2	0	8	4	32	15	127	45	501	214
19	2	0	8	1	32	3	127	19	505	61
20	2	0	8	0	32	4	127	14	507	74
21	2	0	8	1	32	12	128	48	508	178
22	2	0	8	3	32	14	128	59	509	220
23	2	0	8	0	32	5	128	16	510	45
24	2	0	8	0	32	1	128	15	511	37
25	2	0	8	3	32	5	128	29	511	133
26	2	2	8	1	32	7	128	24	511	71
27	2	1	8	1	32	7	128	38	512	152
28	2	0	8	3	32	17	128	41	512	193
29	2	1	8	0	32	0	128	8	512	25
30	2	0	8	3	32	10	128	29	512	189

For C , X^2 , $-H$, the p -values are almost exactly the same!

Almost all X_j 's are 0, 1, or 2.

The value of any one of C , X^2 , or $-H$ (almost) tells us the others.

What about the **dense case**? Needs much larger n .

Suspect left p -values for Good LCGs.

X^2 , $s = 2$, $n \approx K\rho^{2/3}$, $k \approx n/8$, chi-square approx.

e	$K = 1$	$K = 2$	$K = 4$	$K = 8$	$K = 16$
12				1.5E-08	
13				1.3E-15	
14				4.8E-11	4.3E-07
15				9.3E-10	$< 10^{-15}$
16				2.2E-10	$< 10^{-15}$
17				1.2E-09	$< 10^{-15}$
18				1.4E-09	$< 10^{-15}$
19			1.2E-04	1.6E-06	$< 10^{-15}$
20		3.4E-03	4.1E-03	2.1E-10	$< 10^{-15}$
21			3.3E-03	1.1E-06	$< 10^{-15}$
22				8.9E-08	$< 10^{-15}$
23			4.8E-03	1.2E-04	$< 10^{-15}$
24				2.2E-08	$< 10^{-15}$
25				3.1E-09	$< 10^{-15}$
26			5.5E-04	4.0E-07	$< 10^{-15}$
27				4.0E-06	—
28				4.3E-07	—

Collision tests for RNG families, with $k \approx 2^e$.

$K^* = \min\{K \mid n = K\rho^\gamma \text{ gives a majority of } \pm\epsilon\}$.

RNG family	γ	dim.	K^*
GoodLCG	1/2	2	16
		4	32
BadLCG2	1/2	2	2
		4	32
LFSR3, d odd	1/2	2	32
		4	64
LFSR3, d power of 2	1/2	2	16
		4	16
MRG2	1/2	2	16
		4	32
CombL2	1/2	2	32
		4	128
InvExpl	1	2	2
		4	2

Discrete (birthday) spacings

- Number the boxes from 1 to k .
 - Let $l_1 \leq l_2 \leq \dots \leq l_n$ the box numbers where the points fall.
 - Compute the spacings $S_j = l_{j+1} - l_j$, $1 \leq j \leq n - 1$.
 - Let $Y =$ number of collisions between the spacings.
- Under \mathcal{H}_0 , Y is approx. Poisson with mean $\lambda = n^3/(4k)$.

$$p^+(y) = P[Y \geq y \mid Y \sim \text{Poisson}(\lambda)].$$

Too much structure \Rightarrow spacings too regular, Y too large.

Ex.: Pts uniformly distrib. in k/b boxes, the other boxes empty.

\Rightarrow Rejection will occur for $n = O((k/b)^{1/3})$.

For the following tests, we took k such that $\lambda = \min(64, n/32)$.

Good LCGs, $s = 2$, $n = K\rho^{1/3}$, suspect p -values $p^+(y)$:

e	$K = 2$	$K = 4$	$K = 8$	$K = 16$
19			6.6E-06	$< 10^{-15}$
20			8.1E-05	$< 10^{-15}$
21				$< 10^{-15}$
22				$< 10^{-15}$
23				$< 10^{-15}$
24				$< 10^{-15}$
25				$< 10^{-15}$
26				$< 10^{-15}$
27				$< 10^{-15}$
28				$< 10^{-15}$
29				$< 10^{-15}$
30				$< 10^{-15}$
31				$< 10^{-15}$
32				$< 10^{-15}$
33				$< 10^{-15}$
34				$< 10^{-15}$
35				$< 10^{-15}$
36				$< 10^{-15}$
37				$< 10^{-15}$
38				$< 10^{-15}$
39				$< 10^{-15}$

$$s = 2, n = K\rho^\gamma.$$

Generators	γ	K^*
GoodLCG	1/3	16
BadLCG2	1/3	4
LFSR2	3/8	64
LFSR3	3/8	64
MRGOrder2	1/3	16
CombMRG2	1/3	16
InvExpl	2/3	16
CombCubic2	2/3	8
TausLCG2	1/2	16
CombCubLCG	1/2	16

Other examples of tests

Nearest pairs of points in $[0, 1)^s$.

Sorting card decks (poker, etc.).

Rank of random binary matrix.

Linear complexity of binary sequence.

Measures of entropy.

Complexity measures based on data compression.

Etc.

For a given class of applications, the most relevant tests would be those that mimic the behavior of what we want to simulate.

Two-level (and more) tests. Beware of approx. errors in p -values!

Can apply filters to output: select certain bits, remove r most significant bits, take bits in reverse order, etc.

The TestU01 software

[L'Ecuyer et Simard, ACM Trans. on Math. Software, 2007].

- ▶ Written as a C library, with 240-page user's guide. Widely used. Link on my web page.
- ▶ Large variety of statistical tests. For both algorithmic and physical RNGs.
- ▶ Also implements a large variety of RNGs.
- ▶ Some predefined batteries of tests:
 - SmallCrush: quick check, 15 seconds;
 - Crush: 96 test statistics, 1 hour;
 - BigCrush: 144 test statistics, 6 hours;
 - Rabbit: for bit strings.
- ▶ Many widely-used generators fail these batteries unequivocally.

Results of test batteries applied to some well-known RNGs

ρ = period length;

t-32 and t-64 gives the CPU time to generate 10^8 random numbers.

Number of failed tests (p -value $< 10^{-10}$ or $> 1 - 10^{-10}$) in each battery.

Generator	$\log_2 \rho$	t-32	t-64	S-Crush	Crush	B-Crush
LCG in Microsoft VisualBasic	24	3.9	0.66	14	—	—
LCG(2^{32} , 69069, 1), VAX	32	3.2	0.67	11	106	—
LCG(2^{32} , 1099087573, 0) Fishman	30	3.2	0.66	13	110	—
LCG(2^{48} , 25214903917, 11), Unix	48	4.1	0.65	4	21	—
Java.util.Random	47	6.3	0.76	1	9	21
LCG(2^{48} , 44485709377909, 0), Cray	46	4.1	0.65	5	24	—
LCG(2^{59} , 13^{13} , 0), NAG	57	4.2	0.76	1	10	17
LCG($2^{31}-1$, 16807, 0), Wide use	31	3.8	3.6	3	42	—
LCG($2^{31}-1$, 397204094, 0), SAS	31	19.0	4.0	2	38	—
LCG($2^{31}-1$, 950706376, 0), IMSL	31	20.0	4.0	2	42	—
LCG($10^{12}-11$, ..., 0), Maple	39.9	87.0	25.0	1	22	34

Generator	$\log_2 \rho$	t-32	t-64	S-Crush	Crush	B-Crush
Wichmann-Hill, MS-Excel	42.7	10.0	11.2	1	12	22
CombLec88 , boost	61	7.0	1.2		1	
Knuth(38)	56	7.9	7.4		1	2
ran2 , in Numerical Recipes	61	7.5	2.5			
CombMRG96	185	9.4	2.0			
MRG31k3p	185	7.3	2.0			
MRG32k3a SSJ + others	191	10.0	2.1			
MRG63k3a	377	—	4.3			
LFib(2^{31} , 55, 24, +), Knuth	85	3.8	1.1	2	9	14
LFib(2^{31} , 55, 24, -), Matpack	85	3.9	1.5	2	11	19
ran3 , in Numerical Recipes		2.2	0.9		11	17
LFib(2^{48} , 607, 273, +), boost	638	2.4	1.4		2	2
Unix-random-32	37	4.7	1.6	5	101	—
Unix-random-64	45	4.7	1.5	4	57	—
Unix-random-128	61	4.7	1.5	2	13	19

Generator	$\log_2 \rho$	t-32	t-64	S-Crush	Crush	B-Crush
Knuth-ran_array2	129	5.0	2.6		3	4
Knuth-ranf_array2	129	11.0	4.5			
SWB(2^{24} , 10, 24)	567	9.4	3.4	2	30	46
SWB($2^{32} - 5$, 22, 43)	1376	3.9	1.5		8	17
Mathematica-SWB	1479	—	—	1	15	—
GFSR(250, 103)	250	3.6	0.9	1	8	14
TT800	800	4.0	1.1		12	14
MT19937, widely used	19937	4.3	1.6		2	2
WELL19937a	19937	4.3	1.3		2	2
LFSR113	113	4.0	1.0		6	6
LFSR258	258	6.0	1.2		6	6
Marsaglia-xorshift	32	3.2	0.7	5	59	—

Generator	$\log_2 \rho$	t-32	t-64	S-Crush	Crush	B-Crush
Matlab-rand, (until 2008)	1492	27.0	8.4		5	8
Matlab in randn (normal)	64	3.7	0.8		3	5
SuperDuper-73, in S-Plus	62	3.3	0.8	1	25	—
R-MultiCarry, (changed)	60	3.9	0.8	2	40	—
KISS93	95	3.8	0.9		1	1
KISS99	123	4.0	1.1			
AES (OFB)		10.8	5.8			
AES (CTR)	130	10.3	5.4			
AES (KTR)	130	10.2	5.2			
SHA-1 (OFB)		65.9	22.4			
SHA-1 (CTR)	442	30.9	10.0			

Conclusion

- ▶ A flurry of computer applications require RNGs.
A poor generator can severely bias simulation results, or permit one to cheat in computer lotteries or games, or cause important security flaws.
- ▶ Don't trust blindly the RNGs of commercial or other widely-used software, especially if they hide the algorithm (proprietary software...).
- ▶ Some software products have good RNGs; check what it is.
- ▶ RNGs with multiple streams are available from my web page in Java, C, and C++. Also OpenCL library, mostly for GPUs.
- ▶ Examples of recent proposals or work in progress:
Fast nonlinear RNGs with provably good uniformity;
RNGs based on multiplicative recurrences;
Counter-based RNGs. RNGs with multiple streams for GPUs.