Quantum field theoretical approach to the shear and bulk relaxation times

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Outline

- Introduction
- Kubo formula for the shear and bulk relaxation times
- Derivation of shear relaxation time within the real-time formalism

Relativistic viscous hydrodynamics

$$\partial_\mu T^{\mu\nu} = 0$$

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - \Delta^{\mu\nu}(P + \Pi) + \pi^{\mu\nu}$$

ϵ - energy density

P - pressure

u^μ - four-velocity

$\pi^{\mu\nu}$ - stress tensor

Π - bulk pressure

First-order hydrodynamics (in the local rest frame)

$$\Pi_{\rm NS} = -\gamma \partial_i T^{i0}$$

$$\pi^{ij}_{\rm NS} = D_T \left(\partial^i T^{j0} + \partial^j T^{i0} - \frac{2}{3} g^{ij} \partial_k T^{k0} \right)$$

$$\gamma = \frac{\zeta}{\epsilon + P}$$

$$D_T = \frac{\eta}{\epsilon + P}$$

ζ - bulk viscosity

η - shear viscosity

Second-order hydrodynamics (in the local rest frame)

$$\partial_t \Pi = -\frac{\Pi - \Pi_{\rm NS}}{\tau_\Pi}$$

$$\partial_t \pi^{ij} = -\frac{\pi^{ij} - \pi^{ij}_{\rm NS}}{\tau_\pi}$$

τ_Π - bulk relaxation time

τ_π - shear relaxation time

Hydrodynamic modes

no other currents coupled to the energy-momentum tensor → two hydrodynamic modes

The respective dispersion relations are obtained as a consequence of the energy-momentum conservation and relaxation equations

shear mode:

$$0 = -\omega^2\tau_\pi - i\omega + D_T\mathbf{k}^2$$

sound mode:

$$0 = -\omega^2 + v_s^2\mathbf{k}^2 + i\omega(\tau_\pi + \tau_\Pi) - i\left(\frac{4D_T}{3} + \gamma + v_s^2(\tau_\pi + \tau_\Pi)\right)\omega\mathbf{k}^2$$
$$+\tau_\pi\tau_\Pi\omega^4 - \left(\tau_\pi\tau_\Pi v_s^2 + \tau_\Pi\frac{4D_T}{3} + \tau_\pi\gamma\right)\omega^2\mathbf{k}^2 + \mathcal{O}(\mathbf{k}^4)$$

How to find transport coefficients?

Transport coefficients are parameters in hydrodynamics.
They have to be obtained from the respective microscopic theory.

- **Kinetic theory**

  - solving Boltzman equation

    Jeon, Yaffe, arXiv:9512263
    Arnold, Moore, Yaffe, arXiv:0010177, 0209353, 0302165
    York, Moore, arXiv:0811.0729

  - methods of moments

    Denicol et al, arXiv: 1004.5013, 1206.1554, 1202.4551, 1403.0962

- **Quantum field theory**

  - Kubo relations

    Jeon, arXiv: 9409250
    Moore, Sohrabi, arXiv: 1007.5333, 1210.3340

The question: how the shear and bulk relaxation times are related to microscopic quantities?

Linear response theory

$$\delta\langle\hat{A}(t,\mathbf{x})\rangle = \int d^4x' G_R(t-t',\mathbf{x}-\mathbf{x}')\theta(-t')f(t,\mathbf{x})$$

deviation of an observable A from equilibrium

equilibrium retarded response function

Viscous hydrodynamics is a perfect realization of the linear response theory

Linear response to transverse fluctuations:

$$\delta\langle\hat{T}^{x0}(t,k_y)\rangle = \beta_x(k_y)\int dt'\bar{G}_R^{x0,x0}(t-t',k_y)\theta(-t')e^{\varepsilon t'}$$

Linear response to longitudinal fluctuations:

$$\delta\langle\hat{T}^{00}(t,\mathbf{k})\rangle = \beta_0(\mathbf{k})\int dt'\bar{G}_R^{00,00}(t-t',\mathbf{k})\theta(-t')e^{\varepsilon t'}$$

Parametrization of the longitudinal fluctuations response function

1. Gravitational Ward identity

- consequence of local symmetry which surfaces as the energy-momentum conservation law
- expresses independence of the response function of gravitational anomalies

$$k_\alpha \left(\bar{G}^{\alpha\beta,\mu\nu}(k) - g^{\beta\mu}\langle \hat{T}^{\alpha\nu}\rangle - g^{\beta\nu}\langle \hat{T}^{\alpha\mu}\rangle + g^{\alpha\beta}\langle \hat{T}^{\mu\nu}\rangle \right) = 0$$

From Ward identity: $$\omega^4 \bar{G}_R^{00,00}(\omega, \mathbf{k}) = \omega^4 \epsilon - \omega^2 \mathbf{k}^2 (\epsilon + P) + \mathbf{k}^4 \bar{G}_L(\omega, \mathbf{k})$$

2. Hydrodynamic limit $\omega \to 0$

$$\bar{G}_L(\omega, \mathbf{k}) \approx \frac{\omega^2}{\mathbf{k}^2}(\epsilon + P) + \frac{\omega^4}{\mathbf{k}^4}\left(\bar{G}_R^{00,00}(0, \mathbf{k}) - \epsilon\right)$$

Parametrization of the longitudinal fluctuations response function

3. General properties of the retarded Green function

$$\mathrm{Re}\, G_R(\omega, \mathbf{k}) = \mathrm{Re}\, G_R(-\omega, \mathbf{k}) \qquad \mathrm{Im}\, G_R(\omega, \mathbf{k}) = -\mathrm{Im}\, G_R(-\omega, \mathbf{k})$$

Most general form of the function:

$$\bar{G}_L(\omega, \mathbf{k}) = \frac{\omega^2(\epsilon + P + i\omega^3 Q(\omega, \mathbf{k}))}{\mathbf{k}^2 - \frac{\omega^2}{Z(\omega, \mathbf{k})} + i\omega^3 R(\omega, \mathbf{k})}$$

$$Q(\omega, \mathbf{k}) = Q_R(\omega, \mathbf{k}) - i\omega Q_I(\omega, \mathbf{k})$$
$$R(\omega, \mathbf{k}) = R_R(\omega, \mathbf{k}) - i\omega R_I(\omega, \mathbf{k})$$
$$Z(\omega, \mathbf{k}) = Z_R(\omega, \mathbf{k}) - i\omega Z_I(\omega, \mathbf{k}) \qquad Z_R(\omega, \mathbf{k}) = Z_{R1}(0, \mathbf{k}) + \omega^2 Z_{R2}(\omega, \mathbf{k})$$

all components are real-valued even functions of frequency and wavevector

Z_R and R_R have non-zero limits when $\omega \to 0, \quad \mathbf{k} \to 0$

all other parts of *Z*, *R*, and Q have finite limits when $\omega \to 0, \quad \mathbf{k} \to 0$

Kubo formula for shear and bulk relaxation times

pole structure of $\bar{G}_L$ $\longleftrightarrow$ dispersion relation of the sound mode

$$\frac{4}{3}\eta + \zeta = \lim_{\omega,\mathbf{k}\to 0} \frac{1}{\omega} \mathrm{Im}\, \bar{G}_L(\omega, \mathbf{k})$$

$$\boxed{\frac{4}{3}\eta\tau_\pi + \zeta\tau_\Pi = -\frac{1}{2} \lim_{\omega,\mathbf{k}\to 0} \partial_\omega^2 \mathrm{Re}\, \bar{G}_L(\omega, \mathbf{k})}$$

Relation between the longitudinal Green function and the retarded one is given by the Ward identity

$$\bar{G}_L(\omega, \mathbf{k}) = \frac{k_i k_j k_m k_n}{\mathbf{k}^4} \bar{G}_R^{ij,mn}(\omega, \mathbf{k}) + P$$

Diagrammatic computation of shear relaxation time

massless scalar field theory $\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{\lambda}{4!}\phi^4$

$2 \leftrightarrow 2$ scatterings – leading order – no scale – only shear viscosity matters

$$\frac{4}{3}\eta\tau_\pi = -\frac{1}{2}\lim_{\omega,\mathbf{k}\to 0}\partial^2_\omega \mathrm{Re}\left(\frac{k_i k_j k_m k_n}{\mathbf{k}^4}\bar{G}_R^{ij,mn}(\omega,\mathbf{k}) + P\right)$$

$$\frac{4}{3}\eta = \lim_{\omega,\mathbf{k}\to 0}\frac{1}{\omega}\mathrm{Im}\left(\frac{k_i k_j k_m k_n}{\mathbf{k}^4}\bar{G}_R^{ij,mn}(\omega,\mathbf{k})\right)$$

$2 \leftrightarrow 4$ scatterings – next-to-leading order – thermal mass appears – bulk viscosity emerges

retarded Green function

$$\begin{aligned}\bar{G}_R^{ij,mn}(x,y) \;&=\; -\delta^{(4)}(x-y)\big(\delta^{jm}\langle\hat{T}^{in}(y)\rangle + \delta^{jn}\langle\hat{T}^{im}(y)\rangle - \delta^{ij}\langle\hat{T}^{mn}(y)\rangle\big)\\ &\quad -i\theta(x_0-y_0)\langle[\hat{T}^{ij}(x),\hat{T}^{mn}(y)]\rangle\end{aligned}$$

components of stress-energy tensor as operators $\hat{T}^{ij}(x) \cong \partial^i\phi(x)\partial^j\phi(x)$

Diagrammatic computation of shear relaxation time

start with the real-time formalism in (1,2) basis: $\mathrm{Re}\,\bar{G}_R \propto G_{1111} - G_{2222}$ $\quad \mathrm{Im}\,\bar{G}_R \propto G_{2211} - G_{1122}$

building blocks: $\Delta_{11},\ \Delta_{22},\ \Delta_{12},\ \Delta_{21}$

efficient description given in (r,a) basis:

(Wang, Heinz, arXiv:0201116)

$$\Delta_{ra}(p) = \frac{1}{(p_0 + i\epsilon)^2 - \mathbf{p}^2} \qquad \Delta_{ar}(p) = \frac{1}{(p_0 - i\epsilon)^2 - \mathbf{p}^2}$$

$$\Delta_{rr}(p) = (1 + 2n(p_0))\,(\Delta_{ra}(p) - \Delta_{ar}(p))$$

$$\Delta_{aa}(p) = 0$$

information on the particle properties,
energy and liftime,
encoded in the spectral function:

$$\rho \propto \delta(p_0 - E) + \delta(p_0 + E)$$

well-defined energy, infinite liftime,
no scatterings, no transport

in mathematical sense:

$$\int \frac{dp_0}{2\pi} \Delta_{ra}(p)\Delta_{ar}(p) \propto \frac{1}{\epsilon}$$

poles pinch the real axis from both sides making the integral infinite

Need for the dressed propagators

Interacting theory - spectral function approximated by the Lorentzians
– propagators get dressed

$$\Delta_{ra}(p) = \frac{1}{(p_0 + i\Gamma_p)^2 - E_p^2} \qquad \Delta_{ar}(p) = \frac{1}{(p_0 - i\Gamma_p)^2 - E_p^2}$$

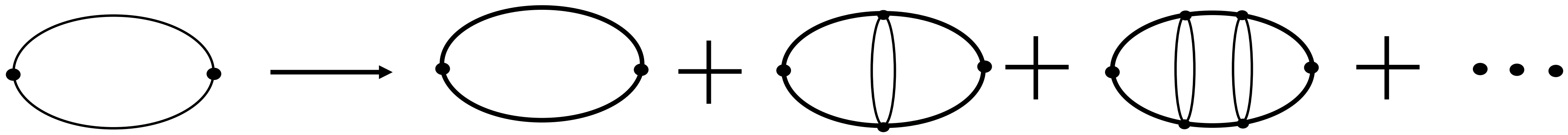
$$E_p^2 = \mathbf{p}^2 + \mathrm{Re}\,\Sigma \qquad \Gamma_p = -\frac{\mathrm{Im}\,\Sigma}{2E_p}$$

self-energy introduces additional powers of coupling constant

$$\mathrm{Im}\,\Sigma \sim \mathcal{O}(\lambda^{-2})$$

→

emergence of rungs: coupling constants in vertices cancel these from self-energy



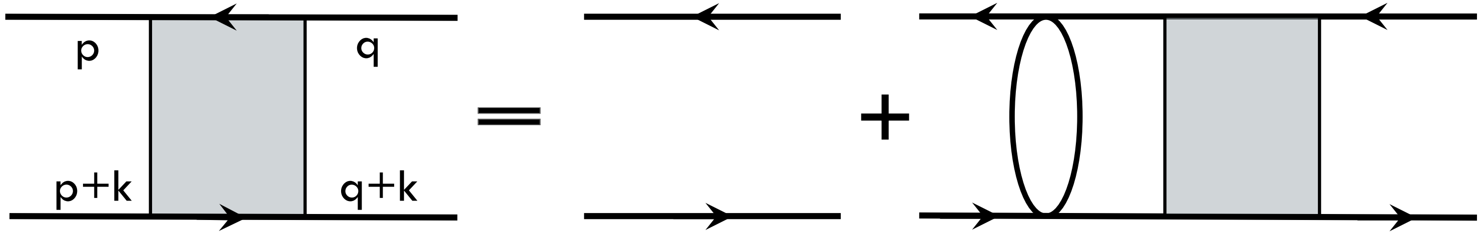
single loop description not valid

full leading order - infinite numer of diagrams must be summed over

Bethe-Salpeter equation

$$\mathrm{Re}\,\bar{G}_R \propto G_{rrra} + G_{rrar} + G_{rarr} + G_{arrr} + G_{raaa} + G_{araa} + G_{aara} + G_{aaar}$$

$$\mathrm{Im}\,\bar{G}_R \propto G_{rrra} + G_{rrar} - G_{rarr} - G_{arrr} - G_{raaa} - G_{araa} + G_{aara} + G_{aaar}$$



vanishing of Δ_{aa}
and
KMS conditions for 4-point Green functions

Wang, Heinz, arXiv:9809016
Carrington et al, arXiv:0608298

→

$$\mathrm{Re}\,\bar{G}_R \propto (n(p_0) - n(p_0+\omega))\,(G_{aarr} + G^*_{aarr})$$

$$\mathrm{Im}\,\bar{G}_R \propto (n(p_0) - n(p_0+\omega))\,(G_{aarr} - G^*_{aarr})$$

pinching poles approximation:

$\Delta_{ra}(p)\Delta_{ar}(p)$ taken into account

$\Delta_{ra}(p)\Delta_{ra}(p)$ and $\Delta_{ar}(p)\Delta_{ar}(p)$ omitted

Shear relaxation time

$$\eta \propto \frac{1}{\omega}\mathrm{Im}\,\bar{G}_R\Big|_{\omega,\mathbf{k}\to 0} \approx \partial_\omega \mathrm{Im}\,\bar{G}_R\Big|_{\omega,\mathbf{k}\to 0} \qquad \eta\tau_\pi \propto \partial_\omega^2 \mathrm{Re}\,\bar{G}_R\Big|_{\omega,\mathbf{k}\to 0}$$

+contour integration + pinching poles approximation

$$\frac{4}{3}\eta = \frac{\beta}{10}\int \frac{d^3p}{(2\pi)^3}\frac{n(E_p)(1+n(E_p))}{E_p^2\Gamma_p} I(\mathbf{p}) \left(D(p)\big|_{p_0=p_1} + D(p)\big|_{p_0=p_2} \right)$$

$$\frac{4}{3}\eta\tau_\pi = \frac{\beta}{20}\int \frac{d^3p}{(2\pi)^3}\frac{n(E_p)(1+n(E_p))}{E_p^2\Gamma_p^2} I(\mathbf{p}) \left(D(p)\big|_{p_0=p_1} + D(p)\big|_{p_0=p_2} \right)$$

$$\boxed{\tau_\pi = \frac{1}{2\Gamma} \equiv \lambda_{\mathrm{mfp}}}$$

$D(p)$ - effective vertex

Relaxation time of shear viscosity is related to the intrinsic microscopic time scale of many-body system. It comes from the imaginary part of the pole of a respective Green function.

Conclusions

- Kubo formula consisting of the shear and the bulk relaxation times was found
- Shear relaxation time was obtained within the real-time formalism in (r,a) basis
- Shear relaxation time is related to the mean free path (lifetime) of the thermal excitation
- First step on the way to compute the bulk relaxation time

Kubo formula for shear relaxation time

By studying metric perturbations:

$$\eta\tau_\pi + \kappa = \frac{1}{2}\lim_{\omega,k_z\to 0}\partial_\omega^2 \mathrm{Re}\, G_R^{xy,xy}(\omega,k_z)$$

$$\kappa = \lim_{\omega,k_z\to 0}\partial_{k_z}^2 \mathrm{Re}\, G_R^{xy,xy}(\omega,k_z)$$

Moore, Sohrabi, arXiv:1007.5333, 1210.3340

By studying transverse fuctuations response function:

$$\eta\tau_\pi + \kappa = -\frac{1}{2}\lim_{\omega,k_y\to 0}\partial_\omega^2 \mathrm{Re}\, \bar{G}_R^{xy,xy}(\omega,k_y)$$

$G_R^{xy,xy}(\omega,k_z)$ and $\bar{G}_R^{xy,xy}(\omega,k_y)$ have different momentum dependence

It is hard to find analytic structure of $G_R^{xy,xy}(\omega,k_z)$

It is unclear how to relate kappa coefficient to the function $\bar{G}_R^{xy,xy}(\omega,k_y)$

To derive the shear relaxation time it is safe and sufficient to rely on longitudinal fluctuations response function $\bar{G}_L$

QFT vs. kinetic theory result

Comparison in classical massless gas limit

Kinetic theory

$$\frac{\eta}{\tau_\pi} = \frac{\langle \epsilon + P \rangle}{5} = \frac{4}{450}\pi^2 T^4$$

QFT

$$\eta = \lim_{\omega, k_y \to 0} \frac{1}{\omega} \mathrm{Im}\, \bar{G}_R^{xy,xy}(\omega, k_y)$$

$$\bar{G}_R^{xy,xy}(x,y) = -\delta^{(4)}(x-y)\langle \hat{T}^{xx}(y)\rangle - i\theta(x_0 - y_0)\langle [\hat{T}^{xy}(x), \hat{T}^{xy}(y)]\rangle$$

$$\eta = \frac{\beta}{2}\int \frac{d^3p}{(2\pi)^3}\frac{p_x^2 p_y^2}{E_p^2 \Gamma_p} n(E_p)(n(E_p)+1)$$

$$\tau_\pi = \frac{1}{2\Gamma}$$

→

$$\frac{\eta}{\tau_\pi} = \frac{4}{450}\pi^2 T^4$$

Bethe-Salpeter equations

$$G_{aarr}(p+k,-p,-q-k,q) = -\Delta_{ar}(p+k)\,\Delta_{ra}(p)\Big[i(2\pi)^4\delta^4(p-q)$$
$$+\int\frac{d^4l}{(2\pi)^4}(K_{rraa}-N_{l+k}K_{rrra}+N_lK_{rrar})(p+k,-p,-l-k,l)G_{aarr}(l+k,-l,-q-k,q)\Big]$$

$$G^*_{aarr}(p+k,-p,-q-k,q) = -\Delta_{ra}(p+k)\,\Delta_{ar}(p)\Big[-i(2\pi)^4\delta^4(p-q)$$
$$+\int\frac{d^4l}{(2\pi)^4}(K_{rraa}+N_{l+k}K_{rrar}-N_lK_{rrra})(p+k,-p,-l-k,l)G^*_{aarr}(l+k,-l,-q-k,q)\Big]$$

kernel:

$$K_{\beta_1\gamma_1\beta_4\gamma_4}(p+k,-p,-l-k,l) = \frac{1}{2}\int\frac{d^4s}{(2\pi)^4}\lambda_{\gamma_1\gamma_2\gamma_3\gamma_4}\lambda_{\beta_1\beta_2\beta_3\beta_4}\Delta_{\gamma_2\beta_2}(s)\Delta_{\gamma_3\beta_3}(s-l+p)$$

bare 4-point vertex:

$$\lambda_{\gamma_1\gamma_2\gamma_3\gamma_4} = \frac{\lambda}{4}[1-(-1)^{n_a}]$$

Evaluation of derivatives

$$\eta \propto \partial_\omega \mathrm{Im}\,\bar{G}_R\Big|_{\omega,\mathbf{k}\to 0} \propto \partial_\omega[(n(p_0)-n(p_0+\omega))(G_{aarr}-G^*_{aarr})]\Big|_{\omega,\mathbf{k}\to 0}$$

$$= \partial_\omega[(n(p_0)-n(p_0+\omega))]\mathrm{Im}\,G_{aarr}\Big|_{\omega,\mathbf{k}\to 0}$$

$$\eta\tau_\pi \propto \partial^2_\omega \mathrm{Re}\,\bar{G}_R\Big|_{\omega,\mathbf{k}\to 0} \propto \partial^2_\omega[(n(p_0)-n(p_0+\omega))(G_{aarr}+G^*_{aarr})]\Big|_{\omega,\mathbf{k}\to 0}$$

$$= 2\partial_\omega[(n(p_0)-n(p_0+\omega))]\partial_\omega[G_{aarr}+G^*_{aarr}]\Big|_{\omega,\mathbf{k}\to 0}$$

Other terms vanish due to

$$n(-p_0) = -1 - n(p_0)$$

$$\Delta_{ra}(-p) = \Delta_{ar}(p)$$

$$K_{\dots\alpha_i\dots\alpha_j\dots}(k_1,\dots,k_i,\dots,k_j,\dots,k_n) = K_{\dots\alpha_j\dots\alpha_i\dots}(k_1,\dots,k_j,\dots,k_i,\dots,k_n)$$

Effective vertex

$$\mathrm{Im}G_{aarr}(p,-p,-q,q) \quad = \quad -|\Delta_{ra}(p)|^2\left[(2\pi)^4\delta^4(p-q)+\int\frac{d^4l}{(2\pi)^4}\mathcal{K}(p,-p,-l,l)\ \mathrm{Im}G_{aarr}(l,-l,-q,q)\right]$$

$$G_{aarr}(p+k,-p,-q-k,q)=i\Delta_{ar}(p+k)i\Delta_{ar}(-p)M_{rrrr}(p+k,-p,-q-k,q)$$

$$\mathrm{Im}M_{rrrr}(p,-p,-q,q) \quad = \quad (2\pi)^4\delta^4(p-q)+\int\frac{d^4l}{(2\pi)^4}\mathcal{K}(p,-p,-l,l)|\Delta_{ra}(l)|^2\ \mathrm{Im}M_{rrrr}(l,-l,-q,q)\Big]$$

Effective vertex defined as

$$D(p)=\int\frac{d^4q}{(2\pi)^4}I(\mathbf{q})\mathrm{Im}M_{rrrr}(p,-p,-q,q)$$

It satisfies the integral equation

$$D(p)=I(\mathbf{p})-\int\frac{d^4l}{(2\pi)^4}\mathcal{K}(p,-p,-l,l)|\Delta_{ra}(l)|^2D(l)$$

Contour integration

$$\int \frac{dp_0}{2\pi} n(p_0)(n(p_0)+1) \frac{1}{((p_0+i\Gamma_p)^2 - E_p^2)((p_0-i\Gamma_p)^2 - E_p^2)} D(p)$$

Four poles: $p_1 = i\Gamma_p + E_p, \quad p_2 = i\Gamma_p - E_p, \quad p_3 = -i\Gamma_p + E_p, \quad p_4 = -i\Gamma_p - E_p$

By the residue theorem: $\frac{n(E_p)(n(E_p)+1)}{4E_p^2\Gamma_p}\left(D(p)\big|_{p_0=p_1} + D(p)\big|_{p_0=p_2}\right)$

$$\int \frac{dp_0}{2\pi} n(p_0)(n(p_0)+1) \frac{\Gamma_p(p_0^2+\Gamma_p^2+E_p^2)}{((p_0+i\Gamma_p)^2 - E_p^2)^2((p_0-i\Gamma_p)^2 - E_p^2)^2} D(p)$$

The same poles but they are of the second order $\quad \mathrm{Res}(f(p_0), p_i) = \lim_{p_0 \to p_i} \partial_{p_0}(p_0 - p_i)f(p_0)$

$$\frac{n(E_p)(n(E_p)+1)}{16E_p^2\Gamma_p^2}\left[\left(D(p)\big|_{p_0=p_1} + D(p)\big|_{p_0=p_2}\right) + \Gamma_p\left(\partial_{p_0} D(p)\big|_{p_0=p_1} + \partial_{p_0} D(p)\big|_{p_0=p_2}\right)\right]$$