Symmetries of massive and massless Neutrinos

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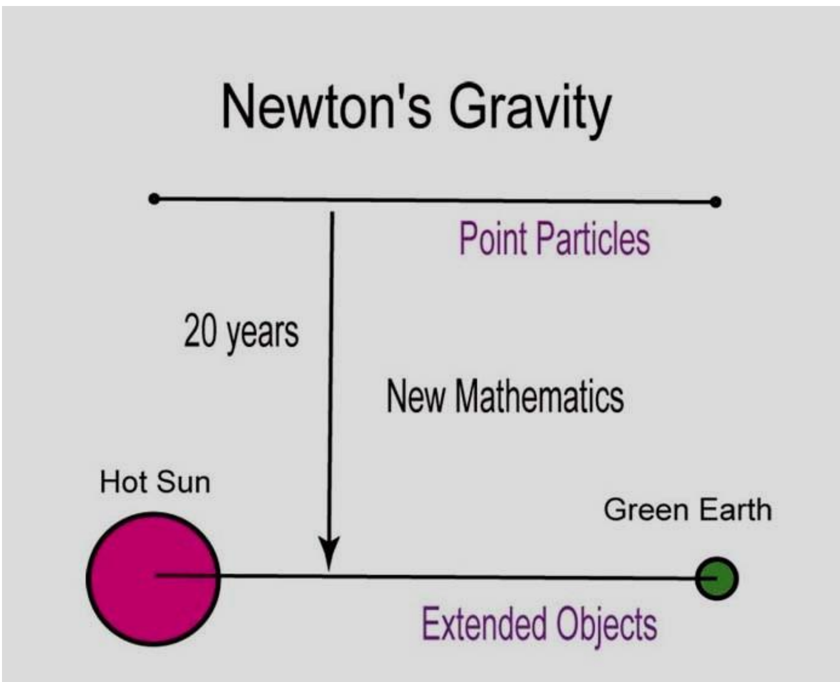
Einstein invented Lorentz transformations for point particles, but never worried about the

orbit of the hydrogen atom,

even though he talked with Bohr before and after 1927.

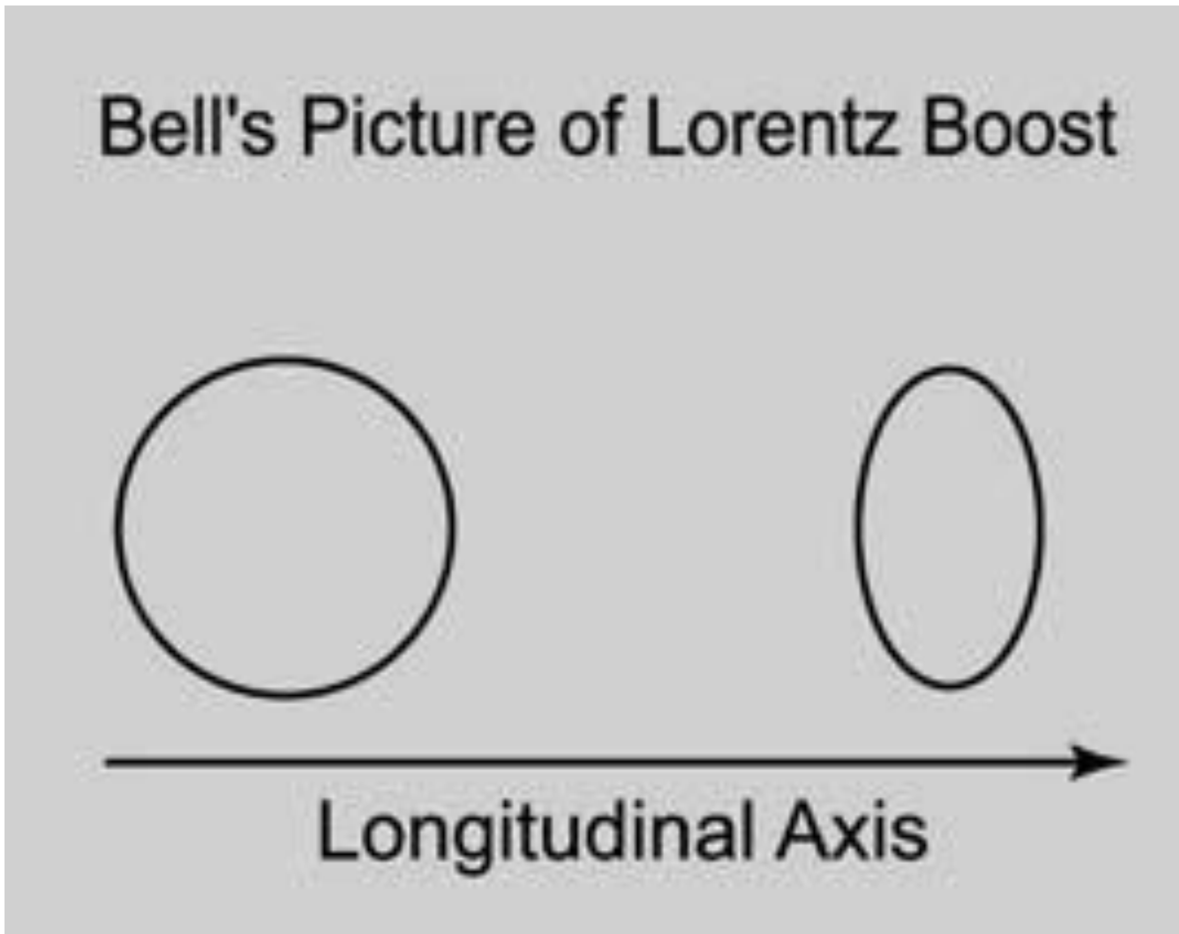
Newton



New Mathematics

Before and after 1927

People seem to have this picture of the Lorentz-boosted orbit.



Included in the symmetries of particles are

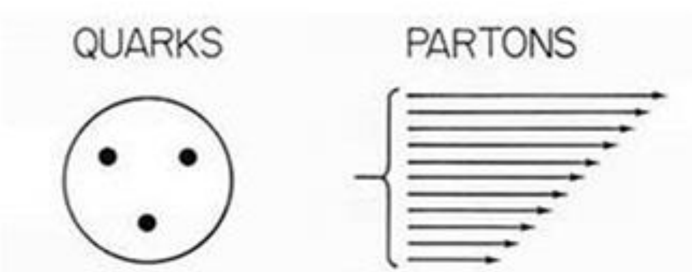


photo of Gell-Mann by Y.S.Kim,
photo of Feynman from AIP Viasual Archieves.

Massless particles.

2. Are they Lorentz-boosted massive particles?

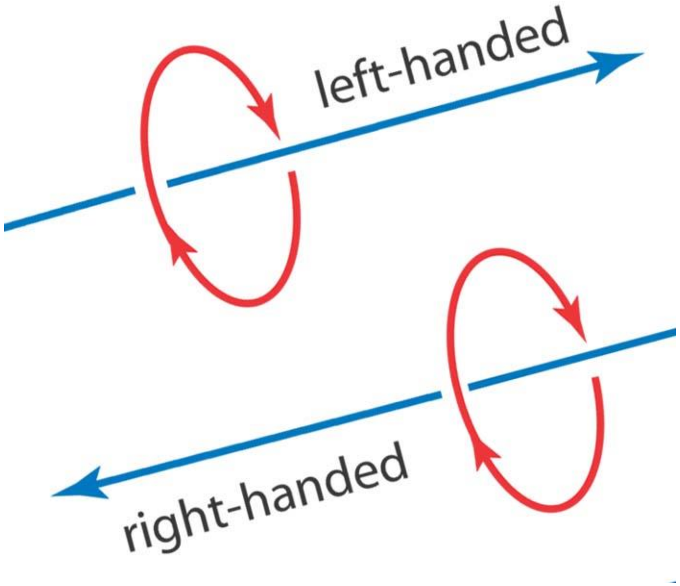
3. Why are the spins parallel to momentum?

4. Why are neutrinos polarized?

Among the massless particles

Photon can be both right-handed and left-handed, while neutrinos are polarized.

Photons have a gauge degree of freedom, while neutrinos do not.



Neutrino polarization as a consequence of gauge invariance?

Internal Space-time symmetries

Eugene Paul Wigner

On his little groups 1939

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ON UNITARY REPRESENTATIONS OF THE INHOMOGENEOUS LORENTZ GROUP*

BY E. WIGNER

(Received December 22, 1937)

1. ORIGIN AND CHARACTERIZATION OF THE PROBLEM

It is perhaps the most fundamental principle of Quantum Mechanics that the system of states forms a *linear manifold*,[1] in which a unitary *scalar product* is defined.[2] The states are generally represented by wave functions[3] in such a way that φ and constant multiples of φ represent the same physical state. It is possible, therefore, to normalize the wave function, i.e., to multiply it by a constant factor such that its scalar product with itself becomes 1. Then, only a constant factor of modulus 1, the so-called phase, will be left undetermined in the wave function. The linear character of the wave function is called the superposition principle. The square of the modulus of the unitary scalar product (ψ, φ) of two normalized wave functions ψ and φ is called the transition probability from the state ψ into φ, or conversely. This is supposed to give the probability that an experiment performed on a system in the state φ, to see whether or not the state is ψ, gives the result that it is ψ. If there are two or more different experiments to decide this (e.g., essentially the same experiment,

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* Parts of the present paper were presented at the Pittsburgh Symposium on Group Theory and Quantum Mechanics. Cf. Bull. Amer. Math. Soc., 41, p. 306, 1935.

[1] The possibility of a future non linear character of the quantum mechanics must be admitted, of course. An indication in this direction is given by the theory of the positron, as developed by P. A. M. Dirac (Proc. Camb. Phil. Soc. *30*, 150, 1934, cf. also W. Heisenberg, Zeits. f. Phys. *90*, 209, 1934; *92*, 623, 1934; W. Heisenberg and H. Euler, ibid. *98*, 714, 1936 and R. Serber, Phys. Rev. *48*, 49, 1935; *49*, 545, 1936) which does not use wave functions and is a non linear theory.

[2] Cf. P. A. M. Dirac, *The Principles of Quantum Mechanics*, Oxford 1935, Chapters I and II; J. v. Neumann, *Mathematische Grundlagen der Quantenmechanik*, Berlin 1932, pages 19–24.

[3] The wave functions represent throughout this paper states in the sense of the "Heisenberg picture," i.e. a single wave function represents the state for all past and future. On the other hand, the operator which refers to a measurement at a certain time t contains this t as a parameter. (Cf. e.g. Dirac, l.c. ref. 2, pages 115–123). One obtains the wave function $\varphi_s(t)$ of the Schrödinger picture from the wave function φ_H of the Heisenberg picture by $\varphi_s(t) = \exp(-iHt/\hbar)\varphi_H$ The operator of the Heisenberg picture is $Q(t) = \exp(iHt/\hbar)Q\exp(-iHt/\hbar)$, where Q is the operator in the Schrödinger picture which does not depend on time. Cf. also E. Schrödinger, Sitz. d. Kön. Preuss. Akad. p. 418, 1930.

The wave functions are complex quantities and the undetermined factors in them are complex also. Recently attempts have been made toward a theory with real wave functions. Cf. E. Majorana, Nuovo Cim. *14*, 171, 1937 and P. A. M. Dirac, in print.

Wigner got his Nobel in 1963, but not for his 1939 paper addressing issues on internal space-time symmetries.

It is generally agreed that he deserved the prize for his 1939 paper on the little groups.

What is wrong with the 1939 paper?

This paper contains the matrix

$$\Lambda_e(\gamma) = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \gamma & \gamma \\ 0 & -\gamma & 1 - \frac{1}{2}\gamma^2 & -\frac{1}{2}\gamma^2 \\ 0 & \gamma & \frac{1}{2}\gamma^2 & 1 + \frac{1}{2}\gamma^2 \end{Vmatrix}$$

Wigner did not explain the physics of this matrix. This is why he did not get the prize for his 1939 paper.

This matrix remained as the ugliest matrix in physics.

Lorentz group consists of three rotational and thee boost degrees of freedom. The rotation group O(3) is a subgroup of the Lorentz group.

Wigner's Little Groups: Subgroups of the Lorentz group whose transformations keep the given momentum of a particle invariant.

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A massive particle can be brought to its rest frame. The momentum is invariant under rotations, but its spin can be rotated. Thus the little group is like O(3).

A massless particle cannot be brought to the rest frame. Rotations around the momentum leaves it invariant . The dynamical quantity associated with this rotation is called the helicity.

What happens when the particle has a small mass?

Wigner's little groups

For a massive particle at rest, it is O(3). If the particle gains Its momentum, it is a Lorentz-boosted O(3). Thus, if the momentum is not zero, the Little group is a Lorentz-boosted O(3) or O(3)-like little group.

If a particle is massless, the little group consists of rotations around the momentum.

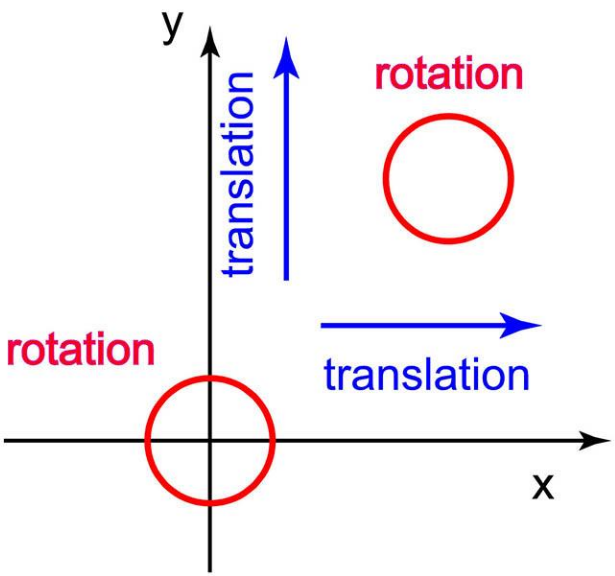
In addition, Wigner showed that this ugly matrix leaves the four-momentum of the massless particle invariant.

$$\Lambda_e(\gamma) = \left\| \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \gamma & \gamma \\ 0 & -\gamma & 1 - \frac{1}{2}\gamma^2 & -\frac{1}{2}\gamma^2 \\ 0 & \gamma & \frac{1}{2}\gamma^2 & 1 + \frac{1}{2}\gamma^2 \end{matrix} \right\|$$

Wigner showed that

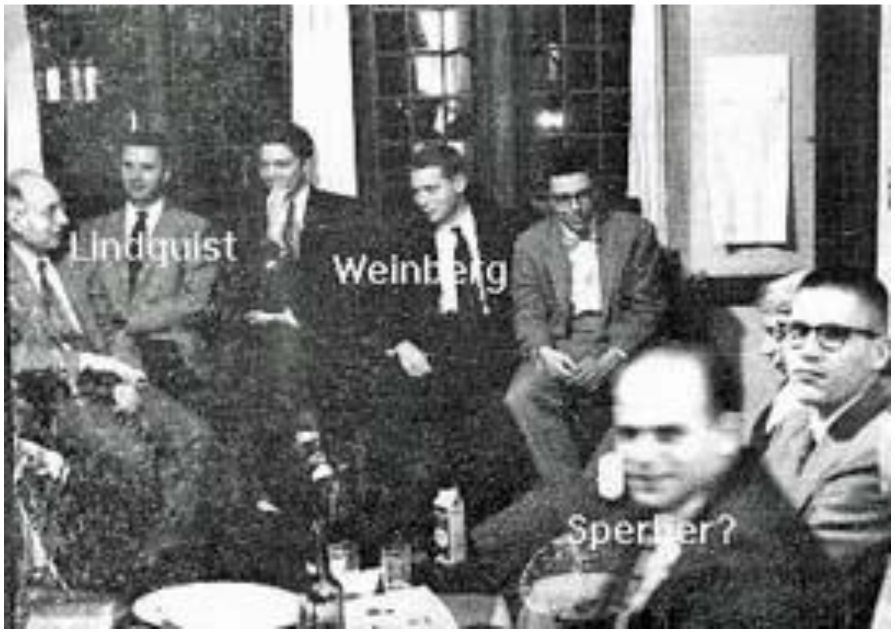
The little group for a massless particle is isomorphic to two-dimensional Euclidean group consisting of one rotation and two translations.

Wigner noted that the rotational degree of freedom corresponds to helicity of the massless particle. He showed further that the ugly matrix is for the translational degree of freedom.



Wigner did not tell anything about the physics of this ugly matrix.

I am not the first one to do something about it. After Wigner's Nobel in 1963,



1957. Steven Weinberg was a student at Princeton. He had and still has had a great admiration for Wigner

S. Weinberg, [Phys. Rev. 133, B1318 (1964).]

S. Weinberg, [Phys. Rev. 134, B882 (1964).]

S. Weinberg, [Phys. Rev. 135, B1049 (1964).]

S. Weinberg, PRL paper (1967) for his own Nobel Prize.

Photos of 1985 and 1987

Weinberg was Treiman's student, and his degree in 1957

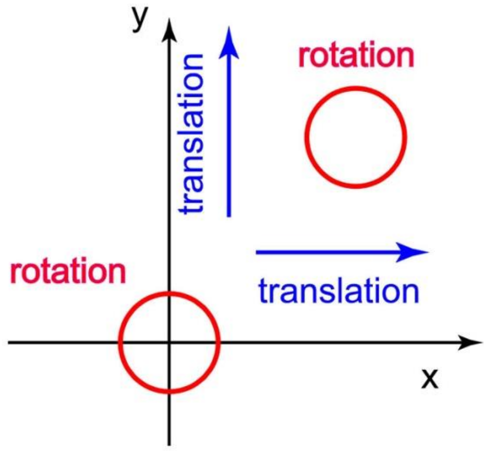
So was I, and I got the degree in 1961. Before writing my PhD thesis, I read Weinberg's thesis. My son was in Treiman's class at Princeton.

In 1964, Weinberg was famous, but I was a struggling young PhD. I also had and still have a great respect for Wigner.

I studied Wigner's paper and became interested in the ugly matrix . I also studied the following three papers.

A.Janner and T. Jenssen,
Physica 53, 1 (1971);
ibid. 60, 292 (1972).

K. Kuperzstych,
Nuovo Cimento B31, 1 (1976)



I then came to the conclusion that these translational degrees correspond to the gauge transformation applicable to the photon four vector or four-potential

Generators of the Lorentz Group

For the space-time coordinate (x, y, z, t), Lorentz transformations are generated by the following three rotation generator,

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and by the following thee boost generators.

$$K_1 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}.$$

They satisfy the commutation relations

$$[J_i, Jj] = i\epsilon_{ijk} J_k, \quad [J_i, Kj] = i\epsilon_{ijk} K_k, \quad [K_i, Kj] = -i\epsilon_{ijk} J_k,$$

The rotation group is a subgroup of the Lorentz group.

The rotation sub group is Wigner's little group for a massive particle at rest, leading to the concept of spin.

Two-by-two representation, using Pauli matrices

There is another set of matrices satisfying the same set of commutation relations.

$$J_i = \frac{1}{2}\sigma_i, \qquad K_i = \frac{i}{2}\sigma_i.$$

These matrices also satisfy the same set of commutation relations.

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k, \quad [K_i, K_j] = -i\epsilon_{ijk}J_k.$$

These relations are invariant when we replace K_i by $\dot{K}_i = -K_i$. Thus

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad \left[J_i, \dot{K}_j\right] = i\epsilon_{ijk}\dot{K}_k, \quad \left[\dot{K}_i, \dot{K}j\right] = -i\epsilon_{ijk}J_k.$$

This enables us to define particles and anti-particles in the Dirac theory.

In addition to the rotation subgroup,

Let us consider the follwing three matrices.

$$J_3, \quad N_1 = K_1 + J_2, \qquad N_2 = K_2 - J_1$$

They satisfy the following closed set of commutation relations.

$$[N_1, N_2] = 0, \quad [J_z, N_1] = iN_2, \quad [J_3, N_2] = -iN_1.$$

They are therefore the generators of another subgroup of the Lorentz group. Wigner observed that transformations of this group leaves the four-momentum of the massless particle invariant. Thus this subgroup is the little group for massless particles.

Geometrical interpretation of this group?

Two-dimensional Euclidean group with one rotation and two translations

Let us consider the two-dimensional xy plane. We can rotate around the system around the z axis, with

$$J_3 = -i\left(\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right).$$

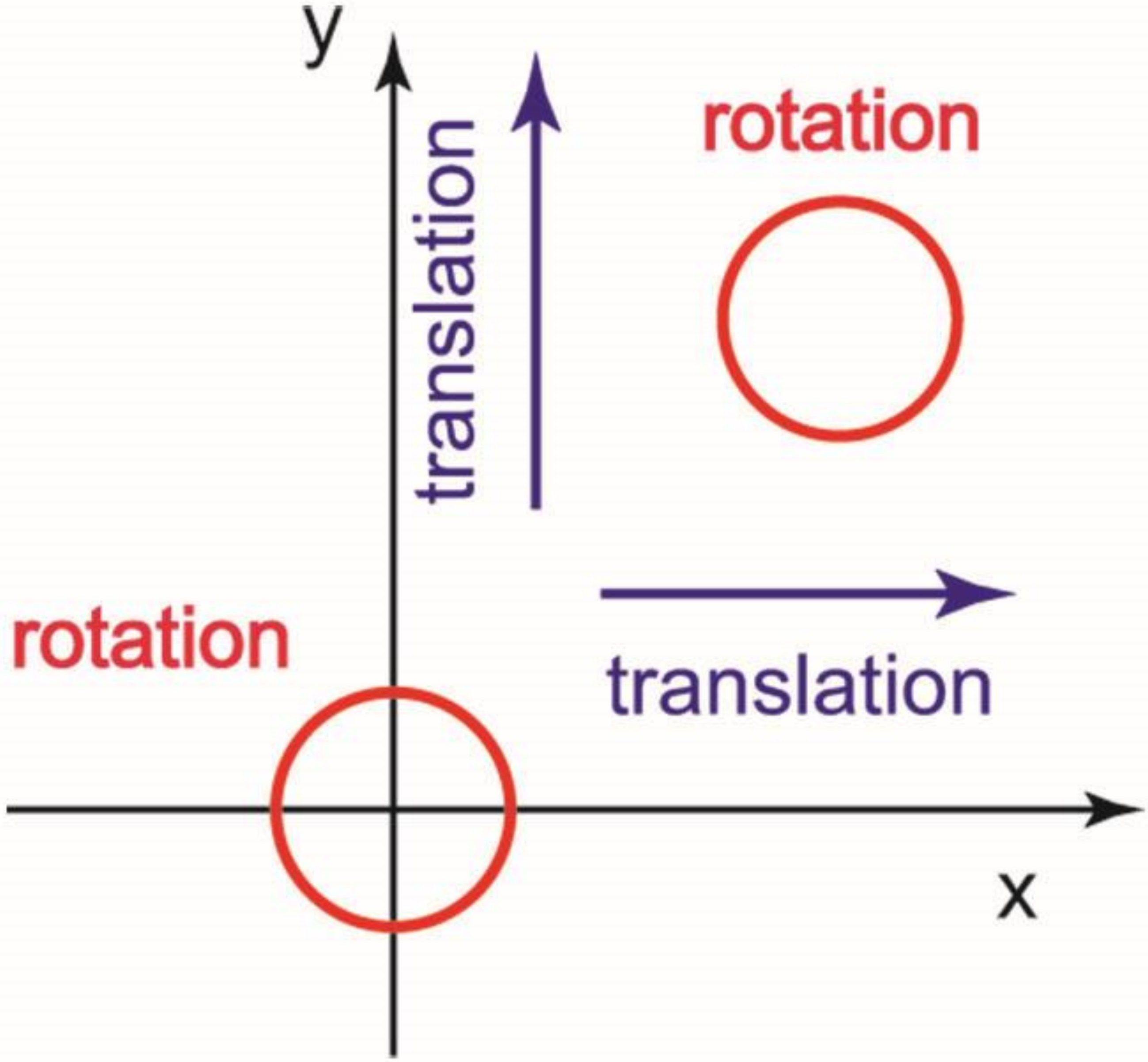
Translations along x and y are generated by

$$P_1 = -i\frac{\partial}{\partial x}, \qquad P_2 = -i\frac{\partial}{\partial y}.$$

They then satisfy the commutation relations

$$[P_1, P_2] = 0, \quad [J_z, P_1] = iP_2, \quad [J_3, P_2] = -iP_1.$$

This is the Lie algebra for the E(2) group. If we replace P_1 and P_2 by N_1 and N_2 respectively, this algebra becomes the same as the little group for massless particles. Thus, we say that the little group for the massless particle is E(2)-like, or isomorphic to the two-dimensional Euclidean group.


y
rotation
translation
rotation
translation
x

Let us go back to the four-by-four matrices, and construct transformation matrices from those generators.

The ugly matrix is

$$D(\gamma, \phi) = \exp\left\{-i\gamma\left[(\cos\phi)N_1 + (\sin\phi)N_2\right]\right\}.$$

The Taylor expansion of this expression leads to

$$D(\gamma, \phi) = \begin{pmatrix} 1 & 0 & \gamma - \cos\phi & \gamma\cos\phi \\ 0 & 1 & -\gamma\sin\phi & \gamma\sin\phi \\ \gamma\cos\phi & \gamma\cos\phi & 1 - \gamma^2/2 & \gamma^2/2 \\ \gamma\sin\phi & \gamma\sin\phi & -\gamma^2/2 & 1 + \gamma^2/2 \end{pmatrix}$$

If $\phi = 90^o$, this matrix becomes Wigner's ugly matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\gamma & \gamma \\ 0 & 0 & 1 - \gamma^2/2 & \gamma^2/2 \\ \gamma & \gamma & -\gamma^2/2 & 1 + \gamma^2/2 \end{pmatrix}$$

When applied to the four-momentum of a massless particle

If the D matrix is applied to the photon four momentum

$$D(\gamma, \phi) \begin{pmatrix} 0 \\ 0 \\ p_3 \\ p_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ p_3, \\ p_3 \end{pmatrix},$$

nothing happens. This is the definition of the little group

Indeed, this is the definition of Wigner's little group, leaving the four-momentum incariant.

When applied to the photon four-potential,

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_0 \end{pmatrix}$$

satisfying the Lorrentz-condition $A_3 = A_0$,

$$D(\gamma, \phi) \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_3 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_3 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ A_1 \cos\phi + A_2 \sin\phi \\ A_1 \cos\phi + A_2 \sin\phi \end{pmatrix}.$$

The D matrix produces a gauge transformation.

After all these, Wigner’s ugly matrix performs a

gauge transformation

For the two-by-two representaion

$$N_1 = K_1 + J_2 = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, \qquad N_2 = K_2 - J_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

and

$$\dot{N}_1 = -K_1 + J_2 = \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix}, \qquad \dot{N}_2 = -K_2 - J_1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

Thus

$$D(\gamma, \phi) = \begin{pmatrix} 1 & 0 \\ i\gamma e^{i\phi} & 1 \end{pmatrix} \qquad \dot{D}(\gamma, \phi) = \begin{pmatrix} 1 & -i\gamma e^{i\phi} \\ 0 & 1 \end{pmatrix}.$$

These D matrices perform gauge transformations on the spinors.

Applications of the D matrices to the spinors.

$$D(\gamma,\phi)\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\i\gamma e^{-i\phi}\end{pmatrix}, \qquad D(\gamma,\phi)\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}.$$

and

$$\dot{D}(\gamma,\phi)\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}, \qquad \dot{D}(\gamma,\phi)\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}-i\gamma e^{-i\phi}\\1\end{pmatrix}.$$

Two of the spinors remain invariant, but the other two do not.

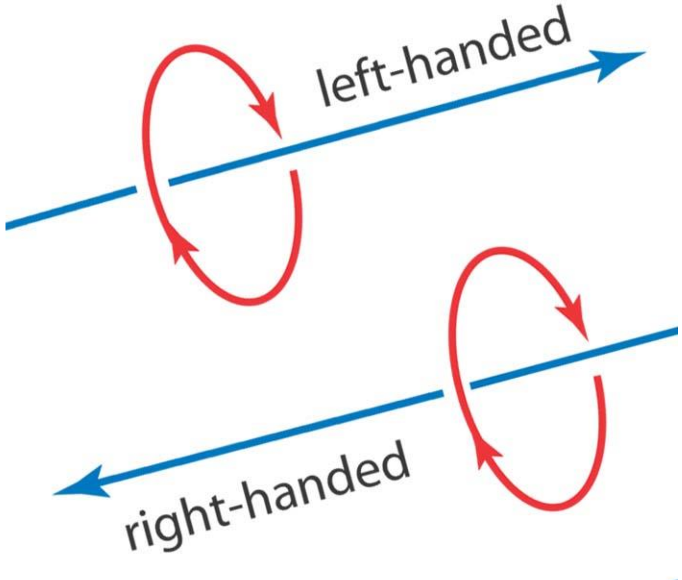
The gauge invariant components are

$$D(\gamma,\phi)\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix},$$

for left-handed neutrinos, and

$$\dot{D}(\gamma,\phi)\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix},$$

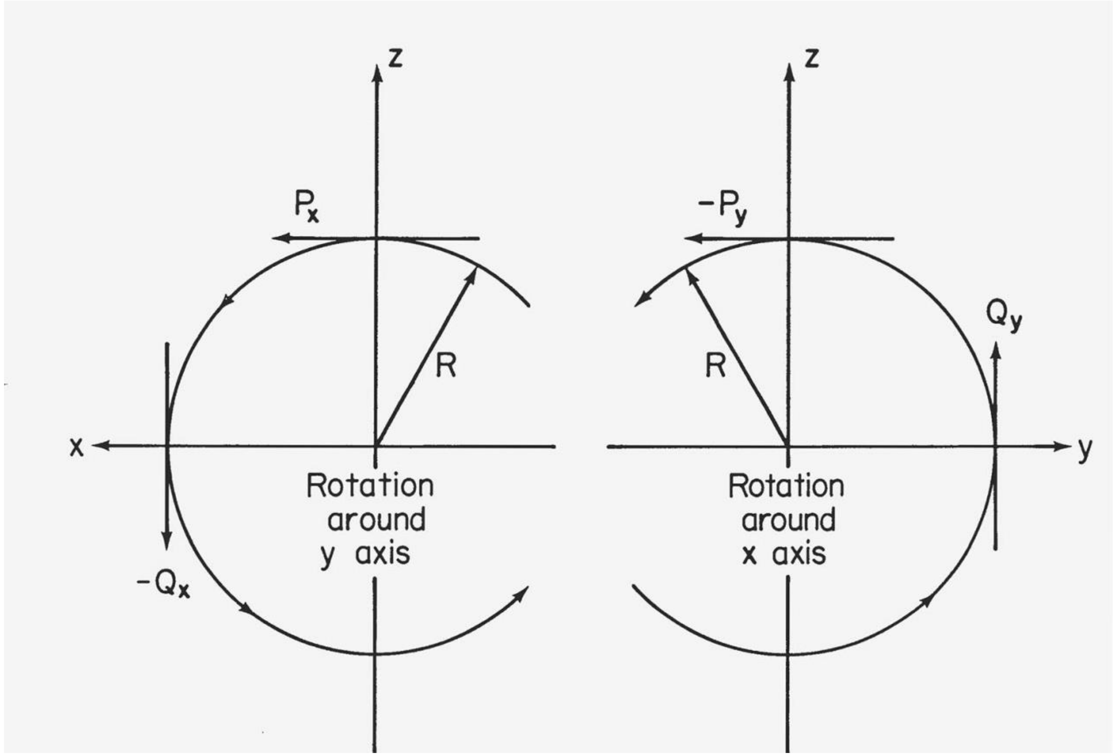
for the right-handed anti-neutrinos.



Neutrino polarization due to Gauge Invariance.

I took these results to Wigner to tell what his ugly matrix is all about.

Wigner was happy. The question then is how the twotranslations can become one gauge transformation. We noted that the E(2) group is on a flat plane tangent to a sphere at its north pole. There also is a cylinder tangent at the equator.



Cylindrical group is isomorphic to the E(2) group.

E(2) group:

$$J_3 = -i\left(\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right).$$

Translations along x and y are generated by

$$P_1 = -i\frac{\partial}{\partial x}, \qquad P_2 = -i\frac{\partial}{\partial y}.$$

They satisfy the commutation relations

$$[P_1, P_2] = 0, \quad [J_z, P_1] = iP_2, \quad [J_3, P_2] = -iP_1.$$

Let us consider another set of generators:

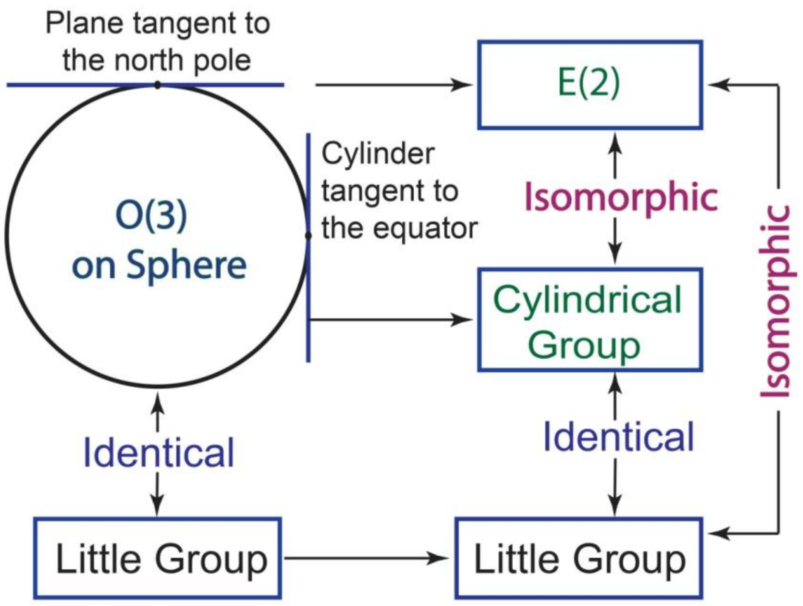
$$Q_1 = ix\frac{\partial}{\partial z}, \qquad Q_2 = -iy\frac{\partial}{\partial z}.$$

$$[Q_1, Q_2] = 0, \quad [J_3, Q_1] = iQ_2, \quad [J_3, Q_2] = -iQ_1.$$

They satisfy the same set of commutation relations.

The Q operators, together with the rotation around the z axis, generate the cylindrical group with one translation and one rotation, as in the case of massless particles with its helicity and gauge degree of freedom.

- We know about the particle at rest and the particle with the speed of light.



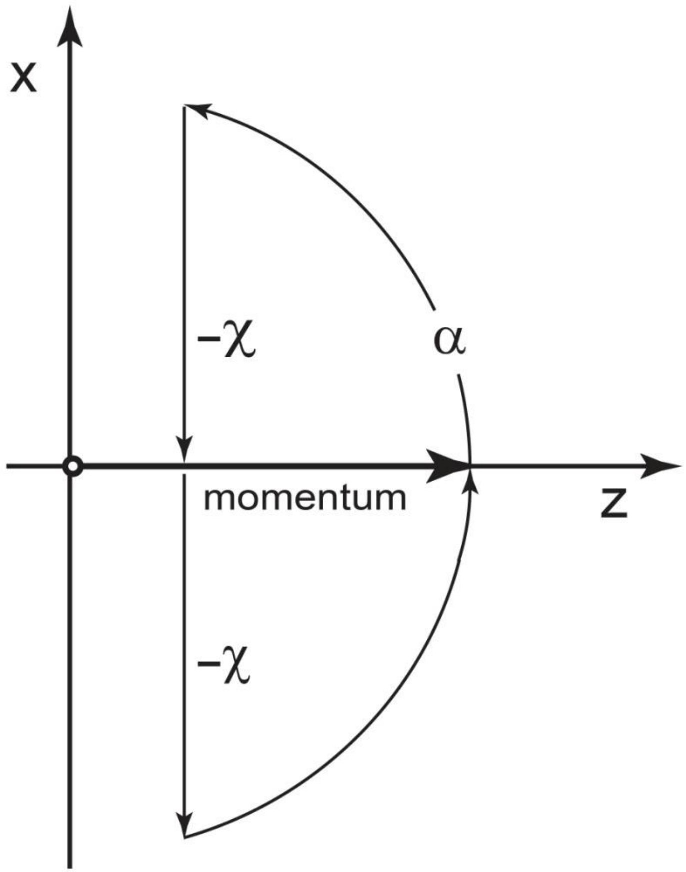
As the momentum/mass becomes infinite

How about in between?
We can boost the rotation matrix.
Does this become a gauge transformation in the infinite momentum limit?

Further contents of Einstein's energy-momentum relation

	Massive Slow	← between →	Massless Fast
Energy Momentum	$E = \frac{p^2}{2m}$	← $E = \sqrt{m^2 + p^2}$ →	$E = p$
Spin, Gauge Helicity	S_3 S_1 S_2	← Little Groups →	S_3 Gauge Trans.
Quarks Partons	Quark Model	← (Covariant Phase Space) →	Parton Model

Little group for non-zero momentum



$$R(\alpha) = \exp(-i\alpha J_2)$$

$$S(-\chi) = \exp(i\chi K_1)$$

$$R(\alpha) = \exp(-i\alpha J_2)$$

Thus the closed loop,

$$R(\alpha)S(-2\chi)R(\alpha) =$$

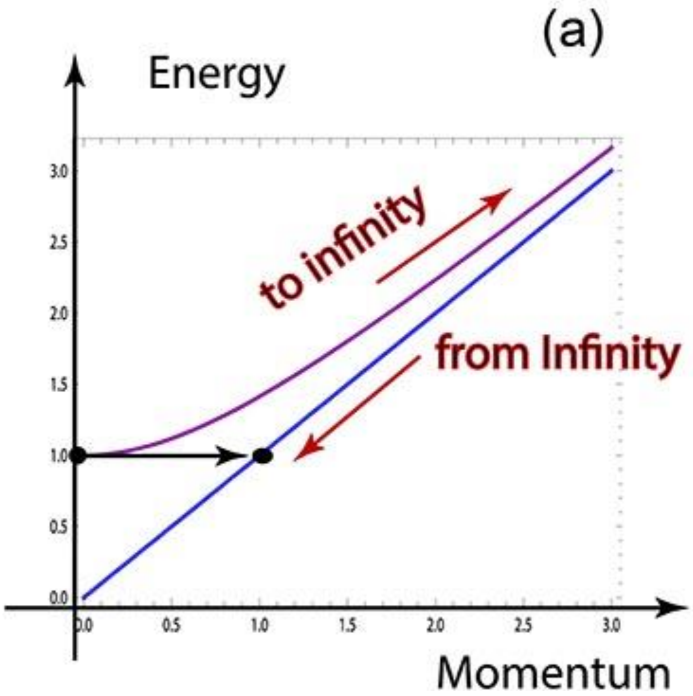
$$\begin{pmatrix} (\cos\alpha)\cosh\chi & -\sinh\chi - (\sin\alpha)\cosh\chi \\ -\sinh\chi + (\sin\alpha)\cosh\chi & (\cos\alpha)\cosh\chi \end{pmatrix}$$

which becomes

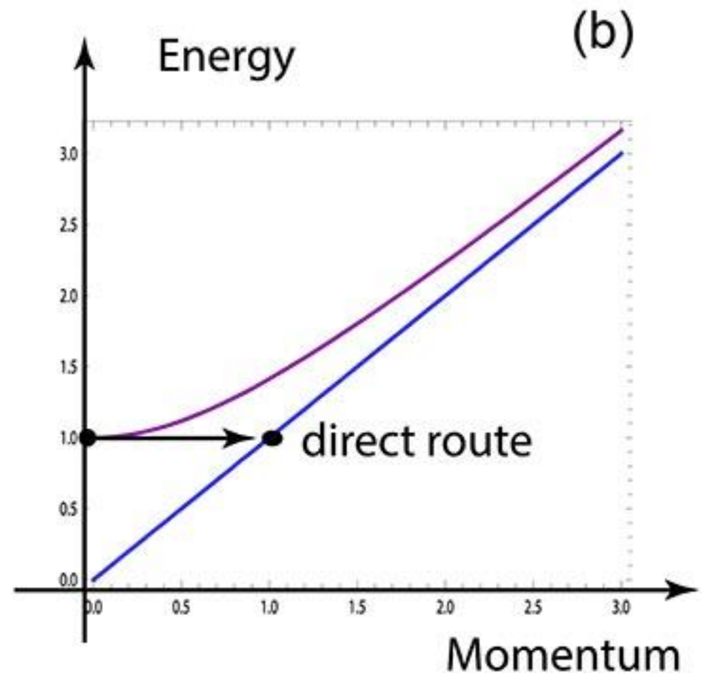
$$\begin{pmatrix} 1 & -2\sinh\chi \\ 0 & 1 \end{pmatrix},$$

when $\sinh\chi = (\sin\alpha)\cosh\chi$ **for the massless particles,**

Routes to massless particles



Traditional route



New route via D-loop

Wigner's Little group for small-mass neutrinos

$$\begin{pmatrix} (\cos\alpha)\cosh\chi & -\sinh\chi - (\sin\alpha)\cosh\chi \\ -\sinh\chi + (\sin\alpha)\cosh\chi & (\cos\alpha)\cosh\chi \end{pmatrix}$$

can also be written as

$$\begin{pmatrix} (\cos\theta) & -e^{\eta}(\sin\theta) \\ e^{-\eta}(\sin\theta) & (\cos\theta) \end{pmatrix}$$

which is a Lorentz-boosted rotation matrix. If we let

$$e^{\eta}\sin\theta = \gamma,$$

then

$$\begin{pmatrix} \cos\theta & -\gamma \\ e^{-2\eta}\gamma & \cos\theta \end{pmatrix}$$

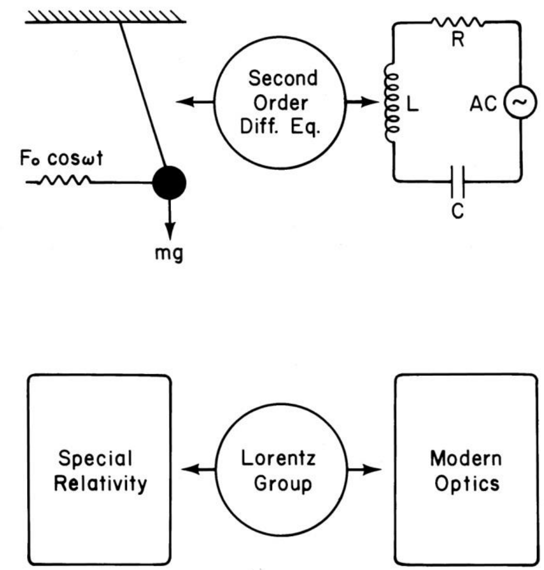
If η becomes very large, θ becomes very small. Thus, the matrix becomes

$$\begin{pmatrix} 1 - \sqrt{\gamma\epsilon}/2 & -\gamma \\ \epsilon & 1 - \sqrt{\gamma\epsilon}/2 \end{pmatrix},$$

where $\epsilon = \gamma e^{-2\eta}$.

This is the little group matrix for small-mass neutrinos, but it is still a Lorentz-boosted rotation matrix.

All the detailed stories are in my latest book.
The co-authors are Sibel Baskal and Marilyn Noz.



The Lorentz group can speak both high-energy physics and modern optics including entanglement.

Toyotomi Hideyoshi (1536-98)
unified Japan, appointed himself as the
Emperor of China, and invaded Korea in 1591.

To humans, he looks like a monkey.

To monkeys, he looks like a human.

This person has been writing papers since 1961. People seem to have difficulties in understanding his papers.

To optics people, I look like a particle physicist.

To particle people, I look like an optical physicist.

In this talk, however, I talked like a high-energy physicist.

Entanglement is a contagious word.

This lunch took place at a seafood restaurant near Princeton in 1991. In 33 AD, there was a much more elegant meeting somewhere in Jerusalem. We do not have a photo of this dinner meeting, but Leonardo da Vinci painted the scene.

This word came from Optics. Sooner or later you will be infected.

In 1990, I published a paper with Eugene Wigner on space-time entanglement, but optics people are using the same formula.

Lorentz boosts entangles space and time.

- **Einstein formulated his special relativity in this house in Bern, Switzerland.**
- **For this, he had to use the Lorentz group formulated by Lorentz and Poincare.**

In order to tell this news to Einstein, I went to his office and his house in Princeton. He was not there.

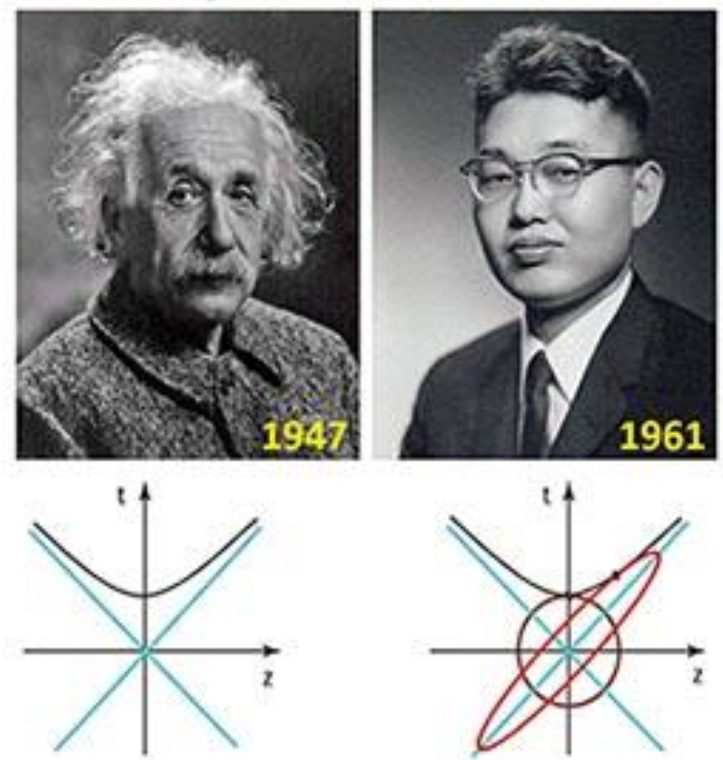
- **Moses talked to God by writing Five Books about God – in the Old Testament.**
- **I am trying to talk to Einstein by making webpages.**

I like to tell this to Einstein, but I will not be able to see him.

- **Moses talked to God by writing five books about God. They are of course the first five books of the Old Testament.**

- **How to talk to Einstein? Construct webpages about Einstein. OK?**

Photos by O.J.Turner of Princeton



To Einstein's hyperbola of 1905 (left), I have added a circle and squeezed it.