Euler-Lagrange equations for effective high energy actions in QCD and in gravity

L. N. Lipatov

Petersburg Nuclear Physics Institute

Content

1 Gluon reggeization

QCD Born amplitude at high energies $s \gg t$

$$M_{|_{Born}} = 2\, g\, T^c_{A'A}\, \delta_{\lambda_{A'}\lambda_A}\, \frac{s}{t}\, g\, T^c_{B'B}\, \delta_{\lambda_{B'}\lambda_B}\;,\;\; [T^a, T^b] = i f_{abc} T^c$$

Regge behavior in Leading Logarithmic Approximation

$$M(s,t) = M_{|_{Born}}\, s^{\omega(t)}\,,\;\; \alpha_s \ln s \sim 1\,,\;\; \alpha_s = \frac{g^2}{4\pi} \ll 1$$

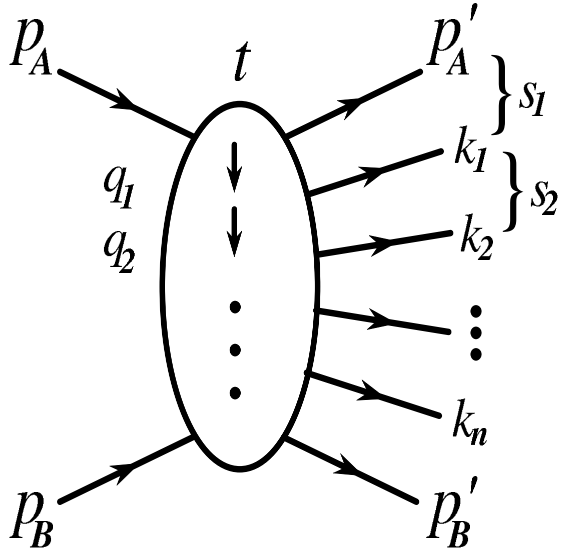
Gluon scattering vertex ($n^+ = 2p_A/\sqrt{s}\,,\;\; n^- = 2p_B/\sqrt{s}$)

$$\gamma^B_{\mu'\mu} = -\delta_{\mu\mu'} + p_{\mu'}\frac{n^+_\mu}{p^+} + p'_\mu \frac{n^+_{\mu'}}{p^+} + q^2\, \frac{n^+_\mu n^+_{\mu'}}{2(p^+)^2} \to \delta_{\lambda'\lambda}$$

Gluon Regge trajectory in LLA

$$\omega(-|q|^2) = -\frac{\alpha_s N_c}{4\pi^2} \int d^2k\, \frac{|q|^2}{|k|^2|q-k|^2} \approx -\frac{\alpha_s N_c}{2\pi}\, \ln \frac{|q^2|}{\lambda^2}$$

2 Production in multi-Regge kinematics



$$M^{FKL}_{2\to 2+n} \sim \frac{s_1^{\omega_1}}{|q_1|^2}\, gT^{d_1}_{c_2c_1} C(q_2,q_1)\, \frac{s_2^{\omega_2}}{|q_2|^2} ... gT^{d_n}_{c_{n+1}c_n}\, C(q_{n+1},q_n)\, \frac{s_{n+1}^{\omega_{n+1}}}{|q_{n+1}|^2}\,,$$

$$C_\mu = -q^\perp_{1\mu} - q^\perp_{2\mu} + n^+_\mu \left(k_1^- + \frac{q_1^2}{k_1^+}\right) - n^-_\mu \left(k_1^+ + \frac{q_2^2}{k_1^-}\right) \,\to\, C(q_2,q_1) = \frac{q_2^\perp\, q_1^{\perp *}}{k_1^{\perp *}}$$

3 BFKL Pomeron

Balitsky-Fadin-Kuraev-Lipatov equation (1975)

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\;,\;\; \sigma_t \sim s^{\Delta}\,,\;\; \Delta = \frac{4\alpha N_c}{\pi}\,\ln 2$$

BFKL Hamiltonian in the operator form

$$H_{12} = \frac{1}{p_1 p_2^*}\ln|\rho_{12}|^2\, p_1 p_2^* + \frac{1}{p_1^* p_2}\ln|\rho_{12}|^2\, p_1^* p_2 + \ln\,|p_1 p_2|^2 - 4\psi(1)\,,$$

Holomorphic separability of H_{12} (L. (1986))

$$H_{12} = h_{12} + h_{12}^*\,,\;\; [h_{12}, h_{12}^*] = 0\,,\;\; \rho_{12} = \rho_1 - \rho_2\,,\;\; \rho_r = x_r + iy_r$$

Its Möbius invariance (L. (1986))

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d}\,,\;\; \Psi = \left(\frac{\rho_{12}}{\rho_{10}\rho_{20}}\right)^m \left(\frac{\rho_{12}^*}{\rho_{10}^*\rho_{20}^*}\right)^{\widetilde{m}}\,,\;\; m = \gamma + \frac{n}{2}\,,\;\; \gamma = \frac{1}{2} + i\nu$$

4 BKP equation in LLA

Bartels-Kwiecinski-Praszalowicz equation (1980)

$$E\,\Psi(\vec{\rho}_1,...,\vec{\rho}_n) = H\,\Psi(\vec{\rho}_1,...,\vec{\rho}_n)\;,\;\; H = \sum_{k<l} \frac{\vec{T}_k \vec{T}_l}{-N_c}\, H_{kl}$$

Holomorphic separability at large N_c (L. (1988))

$$H = \frac{1}{2}\,(h + h^*)\,,\;\; [h, h^*] = 0\,,\;\; h = \sum_{k=1}^{n} h_{k,k+1}$$

Monodromy matrix and integrability (L. (1993))

$$\prod_{k=1}^{n} \begin{pmatrix} u + \rho_k\, p_k & p_k \\ -\rho_k^2\, p_k & u - \rho_k\, p_k \end{pmatrix} = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},\;\; [h, A(u)+D(u)] = 0$$

Integrability of equations for QPOs in N=4 SYM (L. (1996))

5 Integrability for adjoint BKP states

Physical channels with n-reggeon composite states

$$s_1, s_2, ..., s_{n-1}, s_{n+1}, ..., s_{2n-1} < 0\,,\; s, s_n > 0$$

Integrals of motion for open spin chain (L. (2009))

$$\hat{D}(u) = \sum_{k=0}^{n-1} u^{n-1-k} \sum_{0<r_1<...<r_k<n} \rho_{r_1} \prod_{s=1}^{k-1} \rho_{r_s, r_{s+1}} \prod_{t=1}^{k} i\partial_{r_t}$$

Sklyanin ansatz for eigenfunctions of hamiltonian h

$$\Omega = \prod_k Q(\hat{u}_k)\, \Omega_0\,,\;\; B(\hat{u}_k) = 0\,,\;\; \Omega_0 = \prod_{r=1}^{n} \frac{1}{\rho_r^2}$$

Baxter equation (L. (2009))

$$D(u)Q(u) = (u+i)^{n-1} Q(u+i)$$

6 Gluon and reggeized gluon fields

Locality of interactions in the rapidity space

$$y = \frac{1}{2}\ln\frac{\epsilon_k + |k|}{\epsilon_k - |k|}\,,\ \ |y - y_0| < \eta\,,\ \ \eta \ll \ln\, s$$

Gauge transformations for quark and gluon fields

$$\psi(x) \to U\psi(x)\,,\ \ v_\mu(x) \to \frac{1}{g} U D_\mu U^{-1}\,,\ \ D_\mu = \partial_\mu + g v_\mu(x)$$

Gauge invariance of reggeized gluons

$$\delta A^{\pm}(x) = 0\,,\ \ U(\infty) = 1\,,\ \ A^{\pm} = A^0 \pm A^3$$

Kinematical constraints for reggeon fields

$$\partial_{\pm} A^{\pm}(x) = 0$$

7 Effective theory for high energy QCD

Lagrangian for reggeized gluon interactions (L. (1995))

$$L_{eff} = L_{QCD} + Tr(V_+ \partial^2_\mu A^+ + V_- \partial^2_\mu A^-) + 2Tr\, \partial^\perp_\sigma A_+ \partial^\perp_\sigma A_-$$

Eikonal representation for effective currents

$$V_\pm = \frac{1}{g} \partial_\pm O(x^\pm)\,,\ O(x^\pm) = -\frac{1}{D_\pm} \overleftarrow{\partial}_\pm$$

Principal value prescription for propagators

$$\frac{1}{D_\pm} = P\, \frac{e^{-\frac{g}{4}\int_{-\infty}^{x^\pm} dx^\pm v_\pm}}{e^{-\frac{g}{4}\int_{x^\pm}^{\infty} dx^\pm v_\pm}}\, \frac{1}{\partial_\pm}\, \bar{P}\, \frac{e^{-\frac{g}{4}\int_{x^\pm}^{\infty} dx^\pm v_\pm}}{e^{-\frac{g}{4}\int_{-\infty}^{x^\pm} dx^\pm v_\pm}}$$

Effective currents in terms of P-exponents

$$O(x^\pm) = P \frac{e^{-\frac{g}{4}\int_{-\infty}^{x^\pm} d\widetilde{x}^\pm v_\pm}}{e^{-\frac{g}{4}\int_{x^\pm}^{\infty} d\widetilde{x}^\pm v_\pm}}\, \frac{P e^{\frac{g}{4}\int_{-\infty}^{\infty} d\widetilde{x}^\pm v_\pm} + \bar{P} e^{-\frac{g}{4}\int_{-\infty}^{\infty} d\widetilde{x}^\pm v_\pm}}{2}$$

8 Classical equations for effective QCD

Euler-Lagrange equation for high energy QCD

$$[D_\mu, G^{\mu\nu}] = j^\nu \,, \; j^\pm = O(x^\pm)(\partial_\sigma^2 A^\pm) O^+(x^\pm) \,, \; j_\perp^\nu = 0$$

Classical equation for a quasi-elastic kinematics

$$[D_\mu, G^{\mu\nu}] = O(x^+)\, \partial^2_{\perp\sigma}\, A^+(x^-, x_\perp)\; O^+(x^+)\, \delta^\nu_+$$

Transformation to the light-cone gauge $v'_+ = 0$

$$v'_\mu = V^{-1}(v_+)(v_\mu + \frac{\partial_\mu}{g}) V(v_+) \,, \; V(v_+) = P \frac{e^{-\frac{g}{4}\int_{-\infty}^{x^\pm} d\widetilde{x}^\pm v_\pm}}{e^{-\frac{g}{4}\int_{x^\pm}^{\infty} d\widetilde{x}^\pm v_\pm}}$$

Classical solution as a superposition of shock waves

$$\tilde{v}_\nu(x) = \delta_\nu^- \int \frac{d^2 z}{4\pi}\, \ln(|x-z|^2)\, \partial_\sigma^{\perp 2}\, A^+(x^-, z^\perp) = \delta_\nu^-\, A^+(x^-, x_\perp)$$

9 BFKL equation in $N = 4$ SUSY

BFKL kernel eigenvalue in two loops (F., L. (1998))

$$\omega = 4\,\hat{a}\;\chi(n,\gamma) + 4\;\hat{a}^2\,\Delta(n,\gamma)\,,\;\; \hat{a} = g^2 N_c/(16\pi^2)\,,\;\; \gamma = i\nu + 1/2$$

Hermitian separability in $N = 4$ SUSY (K.,L. (2000))

$$\Delta(n,\gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M) + \rho(M^*)}{2\hat{a}/\omega}\,,\; M = \gamma + \frac{|n|}{2}\,,$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2)\,,\; \beta'(z) = \frac{1}{4}\left[\Psi'\Big(\frac{z+1}{2}\Big) - \Psi'\Big(\frac{z}{2}\Big)\right]$$

Maximal transcendentality (K.,L. (2002))

$$\phi(M) = 3\zeta(3) + \Psi^{''}(M) - 2\Phi(M) + 2\beta^{'}(M)\Big(\Psi(1) - \Psi(M)\Big)\,,$$

$$\Phi(M) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+M}\left(\Psi'(k+1) \;-\; \frac{\Psi(k+1) - \Psi(1)}{k+M}\right)$$

10 Pomeron and graviton in N=4 SUSY

Diffusion approximation for the BFKL kernel

$$j = 2 - \Delta - \Delta\,\nu^2\,,\ \gamma = \frac{j}{2} + i\nu$$

AdS/CFT relation with the graviton Regge trajectry

$$j = 2 + \frac{\alpha'}{2}\,t\,,\ t = E^2/R^2\,,\ \alpha' = \frac{R^2}{2}\,\Delta\,,\ \lambda = g^2 N_c$$

Large coupling expansion (KLOV, BPST, KL, GKL)

$$\gamma = 1 - \sqrt{1 + (j-2)/\Delta}\,,\ \Delta = 2\lambda^{-1/2} + \lambda^{-1} - 1/4\,\lambda^{-3/2} - 2(1+3\zeta_3)\lambda^{-2}$$

Exact expression for the slope of γ (KLOV, V., Basso)

$$\gamma'(2) = -\frac{\lambda}{24} + \frac{1}{2}\frac{\lambda^2}{24^2} - \frac{2}{5}\frac{\lambda^3}{24^2} + \frac{7}{20}\frac{\lambda^4}{24^4} - \frac{11}{35}\frac{\lambda^5}{24^5} + ... = -\frac{\sqrt{\lambda}}{4}\,\frac{I_3(\sqrt{\lambda})}{I_2(\sqrt{\lambda})}$$

11 Multi-Regge processes in gravity

Multi-graviton production amplitude (L. (1982))

$$M_{2\to 2+n} \sim s^2 \kappa \delta_{\lambda_A \lambda_{A'}} \frac{s_1^{\omega_1}}{|q_1|^2} \kappa C(q_2, q_1) \ldots \kappa C(q_{n+1}, q_n) \frac{s_{n+1}^{\omega_{n+1}}}{|q_{n+1}|^2} \kappa \delta_{\lambda_B \lambda_{B'}}$$

Graviton-graviton-reggeized graviton vertex

$$\gamma^{++}_{\mu'\nu',\mu\nu} = \gamma^{+}_{\mu'\mu} \gamma^{+}_{\nu'\nu} \to \delta_{\lambda\lambda'}$$

Reggeized graviton-reggeized graviton-graviton vertex

$$\gamma_{\mu\nu} = \gamma_\mu \gamma_\nu - q_1^2 q_2^2 \left(\frac{n^+}{k^+} - \frac{n^-}{k^-} \right)_\mu \left(\frac{n^+}{k^+} - \frac{n^-}{k^-} \right)_\nu \to C(q_2, q_1)$$

Graviton Regge trajectory (L. (1982))

$$j = 2 + \omega\,, \;\; \omega(q^2) = \frac{\alpha}{\pi} \int \frac{q^2\, d^2 k}{k^2 (q-k)^2} f(k,q)\,, \;\; \alpha = \frac{\kappa^2}{8\pi^2}$$

12 Fields in high energy gravity

Locality in the rapidity space

$$y = \frac{1}{2} \ln \frac{\epsilon_k + |k|}{\epsilon_k - |k|}\,,\ |y - y_0| < \eta\,,\ \eta \ll \ln s$$

Metric tensor and its coordinate transformation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,,\ \delta g_{\mu\nu} = D_\mu \chi_\nu + D_\nu \chi_\mu$$

Coordinate invariance of reggeized graviton fields

$$\delta A^{\pm\pm}(x) = 0$$

Kinematical constraints for reggeon fields

$$\partial_\pm A^{\pm\pm}(x) = 0$$

13 Reggeized graviton interactions

Action for the high energy gravity (L. 2011)

$$S = -\frac{1}{2\kappa^2}\int d^4x \left(\sqrt{-g}\, R + L_{ind}\right)$$

Induced lagrangian

$$L_{ind} = j_{++}\, \partial_\sigma^2 A^{++} + j_{--}\, \partial_\sigma^2 A^{--} + \partial_\sigma A^{++} \partial_\sigma A^{--}$$

Perturbative expansion of effective currents

$$j_{\pm\pm} = \partial_\pm j^\mp = h_{\pm\pm} - \left(h_{\rho\pm} - \frac{1}{2}\,\frac{\partial_\rho}{\partial_\pm} h_{\pm\pm}\right)^2 + ...$$

Hamilton-Jacobi equation

$$j^\mp = 2x^\mp - \omega^\mp\,,\;\; g^{\mu\nu}\partial_\mu\omega^\pm\partial_\nu\omega^\pm = 0$$

14 Global light-cone time systems

Coordinate transformation to the global light-cone time system

$$g'^{\pm\pm} = 0\,,\;\; g^{\mu\nu}\partial_\mu x'^{\pm}\,\partial_\nu x'^{\pm} = 0\,,\;\; x'^{\pm} = \omega^{\pm}$$

Coordinate transformation to the system with the global time

$$g'^{00} = 1\,,\;\; g^{\mu\nu}\partial_\mu x^0\,\partial_\nu x^0 = 0$$

Schwarzschild metric with the time $\widetilde{t}$

$$dx^2 = (1 - \frac{2M}{r})d\widetilde{t}^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Schwarzschild metric with the global time (Painleve (1921))

$$dx^2 = (1 - \frac{2M}{r})dt^2 + 2\sqrt{\frac{2M}{r}}dtdr - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

15 Classical equation for effective action

Einstein-Hilbert equation for effective gravity

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = \theta_{\mu\nu}\,, \;\; \theta_{\mu\nu} = c_{\mu\nu}\partial^2_\sigma A^{--} + d_{\mu\nu}\partial^2_\sigma A^{++}$$

Coordinate transformation to the metrics $g'^{\pm\nu} = \eta^{\pm\nu}$

$$g'^{\rho\sigma} = g^{\mu\nu}\partial_\mu x'^\rho \partial x'^\sigma\,, \;\; T'^{\rho\sigma} = \theta^{\mu\nu}\partial_\mu x''^\rho \partial_\nu x''^\sigma$$

Stress tensor in an arbitrary coordinate system

$$\theta_{\mu\nu} = \partial_\mu x'^+ \partial_\nu x'^+ \partial^2_\chi A^{--}(x') + \partial_\mu x'^- \partial_\nu x'^- \partial^2_\chi A^{++}(x')$$

Solution of equations for a quasi-elastic kinematics

$$\tilde{g}_{\mu\nu}(x) = \eta_{\mu\nu} + \partial_\mu x'^+ \partial_\nu x'^+ \, A^{--}(x')\,,$$

$$\tilde{g}^{AS}_{\mu\nu} = \eta_{\mu\nu} + a\,\delta^+_\mu \delta^+_\nu \ln|x_\perp| \delta(x^+)$$

16 Graviton trajectory at supergravity

One loop graviton Regge trajectory (L. (1982))

$$j = 2 + \omega\,,\ \omega(q^2) = \frac{\alpha}{\pi} \int \frac{q^2\, d^2k}{k^2 (q-k)^2} f(k,q)\,,\ \alpha = \frac{\kappa^2}{8\pi^2}\,,$$

$$f(k,q) = (k, q-k)^2 \left(\frac{1}{k^2} + \frac{1}{(q-k)^2} \right) - q^2 + \frac{N}{2} (k, q-k)$$

Gravitino action

$$S_{3/2} = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \int d^4x \sum_{r=1}^{N} \bar{\psi}^r_\mu \gamma_5 \gamma_\nu \partial_\rho \psi^r_\sigma$$

Divergencies of the graviton Regge trajectory

$$\omega(q^2) = -\alpha |q|^2 \left(\ln \frac{|q|^2}{\lambda^2} + \frac{N-4}{2} \ln \frac{|\Lambda|^2}{|q|^2} \right)$$

17 Double-logarithms in gravity

Mellin representation for the scattering amplitude

$$A(s,t) = A_{Born}\, s^{-\alpha|q|^2\, \ln \frac{|q|^2}{\lambda^2}}\, \Phi(\xi)\,,\;\; \Phi(\xi) = \int_{a-i\infty}^{a+i\infty} \frac{d\,\omega}{2\pi i\,\omega} \left(\frac{s}{|q|^2}\right)^{\omega} f_\omega$$

Infrared evolution equation for super-gravity (BLS (2012))

$$f_\omega = 1 + \alpha|q|^2 \left(\frac{d}{d\,\omega}\,\frac{f_\omega}{\omega} - \frac{N-6}{2}\,\frac{f_\omega^2}{\omega^2}\right)\,,\;\; \xi = \alpha\,|q|^2\,\ln^2 \frac{s}{|q|^2}$$

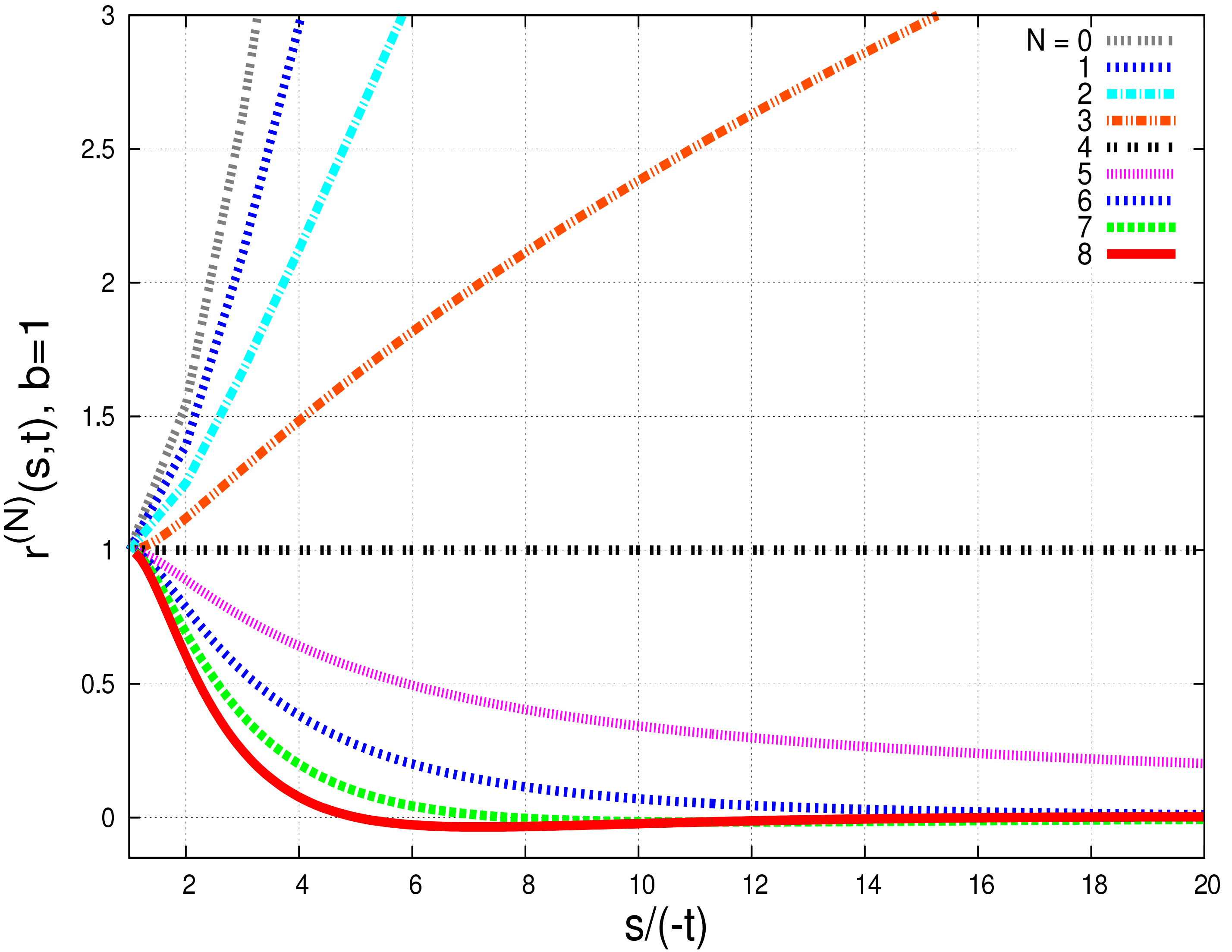
Its solution in terms of the parabolic cylinder function

$$\frac{f_\omega}{\omega} = \frac{2}{6-N}\,\frac{1}{\sqrt{b}}\,\frac{d}{d\,x}\,\ln\left(e^{\frac{x^2}{4}}\, D_{\frac{6-N}{2}}(x)\right)\,,\;\; x = \frac{\omega}{\sqrt{b}}$$

Perturbative expansion

$$\Phi(\xi) = 1 - \frac{N-4}{2}\,\frac{\xi}{2} + \frac{(N-4)(N-3)}{2}\,\frac{\xi^2}{4!} - \frac{N-4}{8}\,(5N^2 - 26N + 36)\,\frac{\xi^3}{6!} + \ldots$$

18 Amplitudes in DL approximation



19 Discussion

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