BFKL Approach to Helicity Amplitudes

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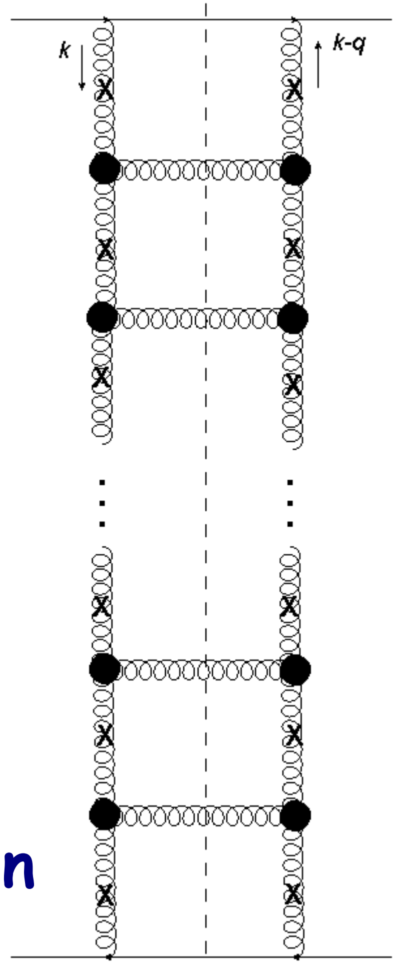
BFKL equation

$$\left(\frac{\partial}{\partial \ln s/s_0} - \omega(-\mathbf{k}^2) - \omega(-(\mathbf{k}-\mathbf{q})^2)\right) G_{BFKL}(\mathbf{k},\mathbf{q}) = \left(K_{real} \otimes G_{BFKL}\right)(\mathbf{k},\mathbf{q})$$



Gluon Regge trajectory

$$\omega(-\mathbf{q}^2) = -\frac{\alpha_s N_c \mu^{2\epsilon}}{(2\pi)^{2-2\epsilon}} \int \frac{d^{2-2\epsilon}\mathbf{k}\ \mathbf{q}^2}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2}$$

$$\simeq -\frac{\alpha_s N_c (4\pi e^{-\gamma})^{\epsilon}}{2\pi}\left(-\frac{1}{\epsilon} + \ln\frac{\mathbf{q}^2}{\mu^2}\right)$$

Two interesting cases: color singlet-BFKL Pomeron
color adjoint-Reggeized gluon

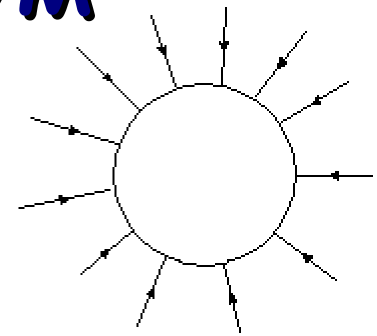
Yes, but not only !!!

A new structure appears in the context of helicity amplitudes.

Table of Contents

- Regge behaviour and BFKL approach
- Planar Scattering Amplitudes in N=4 SUSY
- Maximal Helicity Violation and beyond
- Collinear and Regge Kinematics
- Singlet-Adjoint transition
- More Loops and Legs

MHV amplitudes in N=4 SYM



Loop expansion $A = \sum_l g^{2l} A^{(l)}$

Color ordered amplitudes

$$A^{(L)} = N^L \sum_{\rho \in S_n/Z_n} [T^{a_{\rho(1)}} \dots T^{a_{\rho(n)}}] A^{(L)}(k_{\rho(1)} \dots k_{\rho(n)}, N) + \text{multi-traces}$$

Maximally helicity violating (MHV) amplitudes

$$A^{\text{tree}}(g^+ \dots g^+) = 0, \qquad A^{\text{tree}}(g^- g^+ \dots g^+) = 0$$

$$A^{\text{tree}}_{MHV}(1^+ \dots i^- \dots j^- \dots n) = \frac{\langle ij \rangle^4}{\prod_{k=1}^n \langle k, k+1 \rangle}$$

$$A^{(L)}_{MHV} = A^{\text{tree}}_{MHV} \, \mathcal{M}^{(L)}(s_{i,i+1}, s_{i \dots i+2}, \dots), \quad s_{i,j} = (k_i + .. + k_j)^2$$

Bern, Dixon and Smirnov'05 (BDS)

$$\mathcal{M}_n = \exp\left[\sum_{l=1}^{\infty} a^l f^{(l)}(\epsilon) \mathcal{M}_n^{(1)}(l\epsilon) + C^{(l)} + \mathcal{O}(\epsilon)\right]$$

$f^{(l)}(\epsilon)$ **is found from cusp anomalous dimension**

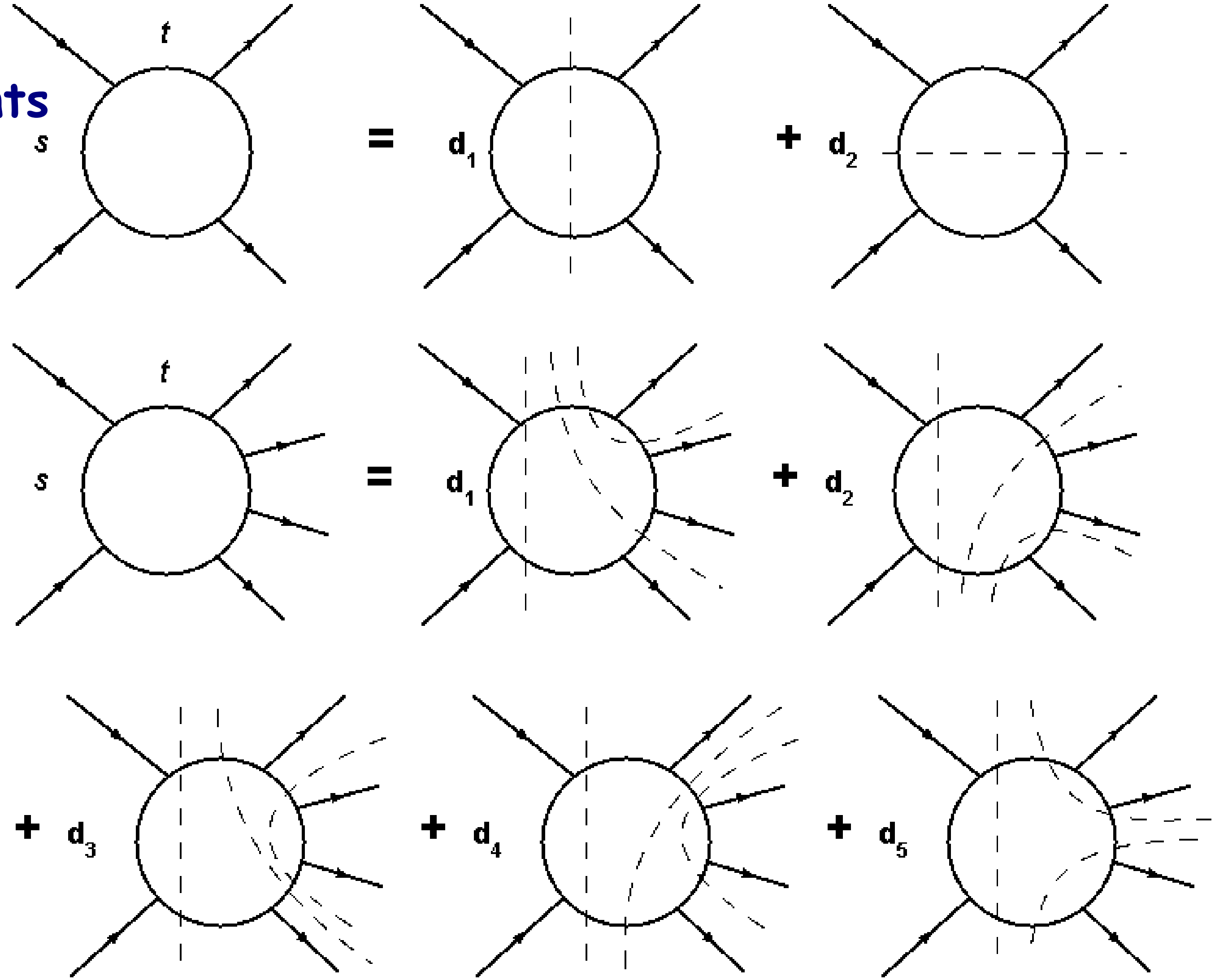
Analytic structure of 2->4 amplitude

Bartels, Lipatov and Sabio Vera'08

In 2->2 the coefficients are fixed by BDS

In 2->4 BDS fixes

$$d_1, d_2, d_3 + d_4, d_5$$

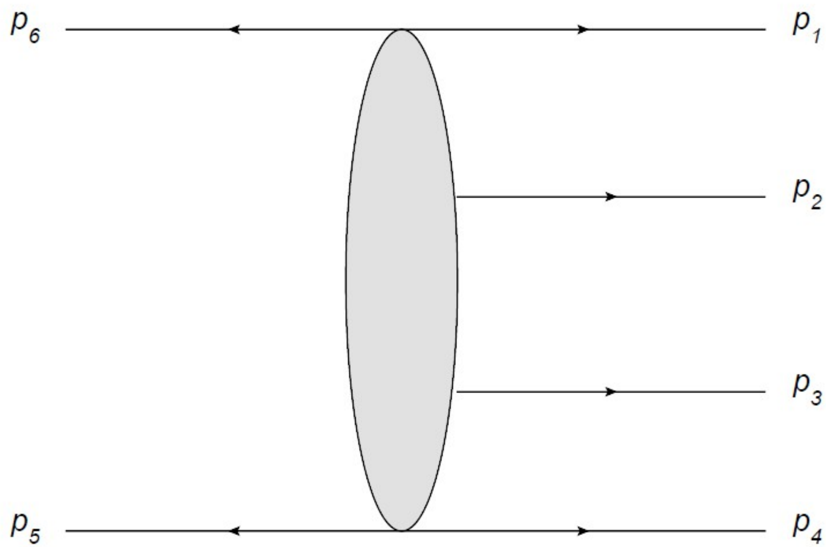


BDS is violated for 6 and more external particles

also Alday and Maldacena'07

Corrections to BDS: Remainder Function

$$\mathcal{A}_6^{MHV}(\epsilon, s_{ij}) = \mathcal{A}_6^{BDS}(\epsilon, s_{ij}) \times R(u_1, u_2, u_3)$$



Cross ratios in dual coordinates

$$u_i = \frac{x_{i,i+4}^2 x_{i+1,i+3}^2}{x_{i,i+3}^2 x_{i+1,i+4}^2}; \qquad i = 1, 2, 3$$

$$p_5 + p_6 \to p_1 + p_2 + p_3 + p_4$$

$$p_i = x_i - x_{i+1}$$

Multi-Regge kinematics

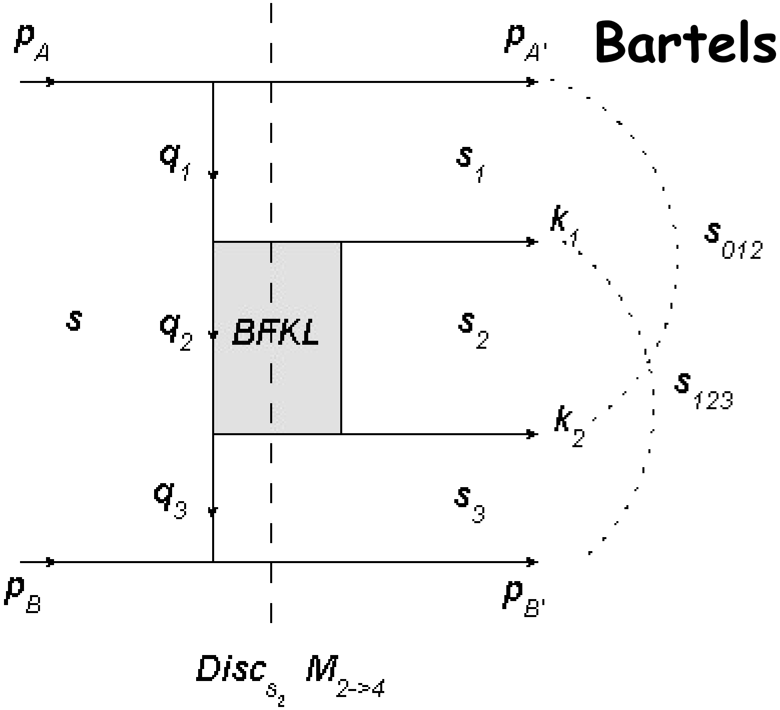
$$-p_6^+ \simeq p_1^+ \gg p_2^+ \gg p_3^+ \gg p_4^+ \simeq -p_5^+$$

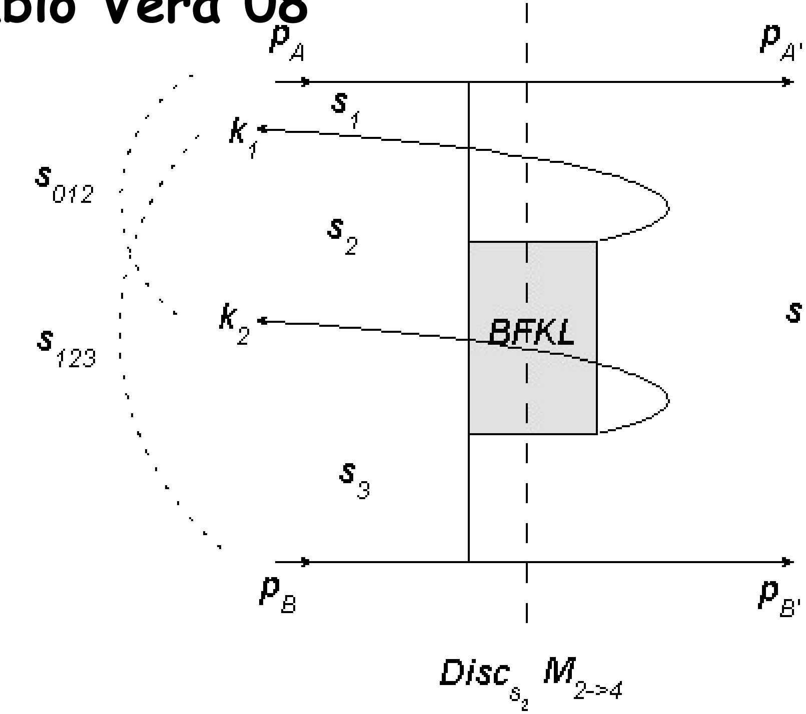
$$u_1 \to 1^-,\ u_2 \to 0^+,\ u_3 \to 0^+$$

$$\tilde{u}_3 = \frac{u_3}{1-u_1} \simeq \mathcal{O}(1) \qquad \tilde{u}_2 = \frac{u_2}{1-u_1} \simeq \mathcal{O}(1) \qquad R(1,0,0) = 1$$

LLA Remainder Function for 2->4 amplitude

Bartels, Lipatov and Sabio Vera'08





$$\Im_{s_2} M_{2\to 4} \propto \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{d\omega}{2\pi i} \left(\frac{s_2}{\mu^2}\right)^{\omega} f_2(\omega), \qquad f_2(\omega) = \Phi_1 \otimes G_{BFKL}^{(8A)} \otimes \Phi_2$$

$$M_{2\to 4} = M_{2\to 4}^{BDS}(1 + i\Delta_{2\to 4})$$

$$\Delta_{2\to 4} = \frac{a}{2} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} \frac{d\nu}{\nu^2 + \frac{n^2}{4}} \left(\frac{q_3^* k_1^*}{k_2^* q_1^*}\right)^{i\nu - \frac{n}{2}} \left(\frac{q_3 k_1}{k_2 q_1}\right)^{i\nu + \frac{n}{2}} \left(s_2^{\omega(\nu,n)} - 1\right)$$

Analytic continuation $u_1 \to |u_1| e^{-i2\pi}$

Two loop remainder function from Wilson loops

Drummond, Henn, Korchemsky and Sokatchev'07

In terms of Goncharov polylogarithms Del Duca, Duhr and Smirnov'09

In terms of classical polylogarithms Goncharov, Spradlin, Vergu and Volovich'09

$$R_6^{(2)}(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}$$

$$x_i^{\pm} = u_i x^{\pm},\ x^{\pm} = \frac{u_1+u_2+u_3-1\pm\sqrt{\Delta}}{2u_1u_2u_3}$$

$$\Delta = (u_1+u_2+u_3-1)^2 - 4u_1u_2u_3$$

$$\ell_n(x) = \frac{1}{2}\left(\mathrm{Li}_n(x) - (-1)^n \mathrm{Li}_n(1/x)\right)$$

$$L_4(x^+,x^-) = \frac{1}{8!!}\log(x^+x^-)^4 + \sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}\log(x^+x^-)^m(\ell_{4-m}(x^+)+\ell_{4-m}(x^-))$$

$$J = \sum_{i=1}^{3}(\ell_1(x_i^+) - \ell_1(x_i^-))$$

Analytic continuation $u_1 \to |u_1| e^{-i2\pi}$ Lipatov, A.P.,'11

Multi-Regge kinematics $u_1 \to 1^-,\ u_2 \to 0^+,\ u_3 \to 0^+,\ \tilde{u}_{2,3} \sim \mathcal{O}(1)$

$$R_6^{(2)}\left(|u_1|e^{-i2\pi}, u_2, u_3\right) \simeq \frac{i\pi}{2}\ln(1-u_1)\ln\tilde{u}_2\ln\tilde{u}_3 + \text{subleading terms}$$

Sub-leading (NLLA) terms are some function of

$$B^{\pm} = \frac{1-\tilde{u}_2-\tilde{u}_3\pm\sqrt{(1-\tilde{u}_2-\tilde{u}_3)^2-4\tilde{u}_2\tilde{u}_3}}{2}$$

Complex variable w

$$\Delta_{2\to4}=\frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n\int_{-\infty}^{\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\left(\frac{q_3^*k_1^*}{k_2^*q_1^*}\right)^{i\nu-\frac{n}{2}}\left(\frac{q_3k_1}{k_2q_1}\right)^{i\nu+\frac{n}{2}}\left(s_2^{\omega(\nu,n)}-1\right)$$

Complex in our region
$$B^{\pm}=\frac{1-\tilde{u}_2-\tilde{u}_3\pm\sqrt{(1-\tilde{u}_2-\tilde{u}_3)^2-4\tilde{u}_2\tilde{u}_3}}{2}$$

Convenient variables
$$w=\frac{k_2q_1}{q_3k_1}=\frac{B^+}{\tilde{u}_2},\ w^*=\frac{k_2^*q_1^*}{q_3^*k_1^*}=\frac{B^-}{\tilde{u}_2}$$

Remainder function after continuation in the multi-Regge kinematics

L. Lipatov and A.P.,'11

$$R_6^{(2)\ LLA+NLLA}\left(|u_1|e^{-i2\pi},\frac{1}{|1+w|^2},\frac{|w|^2}{|1+w|^2}\right)\simeq\frac{i\pi}{2}\ln(1-u_1)\ln|1+w|^2\ln\left|1+\frac{1}{w}\right|^2$$
$$+\frac{i\pi}{2}\ln|w|^2\ln^2|1+w|^2-\frac{i\pi}{3}\ln^3|1+w|^2+i\pi\ln|w|^2\left(\mathrm{Li}_2(-w)+\mathrm{Li}_2(-w^*)\right)$$
$$-i2\pi\left(\mathrm{Li}_3(-w)+\mathrm{Li}_3(-w^*)\right)$$

Vanishes for $w\to0$ Symmetric under $w\to1/w$

What can we learn from sub-leading NLLA terms ?

Next-to-leading(NLO) corrections to adjoint BFKL

Lipatov, A.P., '11, Fadin, Lipatov, '12

Eigenvalue

$$\omega(\nu,n) = -a\left(E_{\nu n} + a\,\epsilon_{\nu n}\right)\,,\quad a = \frac{\alpha N_c}{2\pi}$$

$$E_{\nu n} = -\frac{1}{2}\frac{|n|}{\nu^2+\frac{n^2}{4}} + \psi(1+i\nu+\frac{|n|}{2}) + \psi(1-i\nu+\frac{|n|}{2}) - 2\psi(1)$$

$$\epsilon_{\nu n} = -\frac{1}{4}\left(\psi''(1+i\nu+\frac{|n|}{2}) + \psi''(1-i\nu+\frac{|n|}{2}) + \frac{2i\nu\left(\psi'(1-i\nu+\frac{|n|}{2}) - \psi'(1+i\nu+\frac{|n|}{2})\right)}{\nu^2+\frac{n^2}{4}}\right)$$
$$-\zeta(2)\,E_{\nu n} - 3\zeta(3) - \frac{1}{4}\frac{|n|\left(\nu^2-\frac{n^2}{4}\right)}{\left(\nu^2+\frac{n^2}{4}\right)^3}$$

Impact Factor

$$\Phi^{(1)}(\nu,n) = E_{\nu n}^2 - \frac{1}{4}\frac{n^2}{\left(\nu^2+\frac{n^2}{4}\right)^2} - \zeta(2)$$

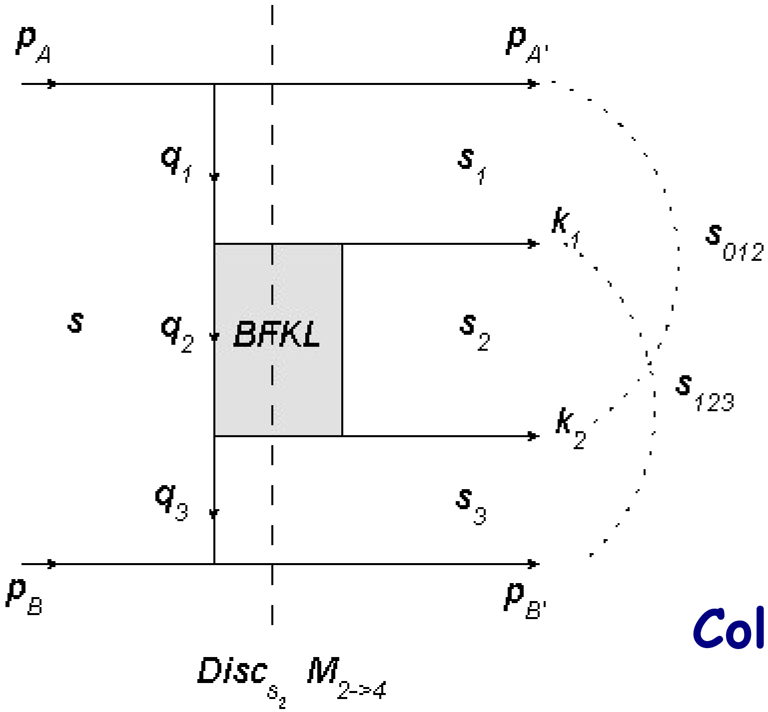
$$\Phi_{Reg}(\nu,n) = 1 + \Phi^{(1)}_{Reg}(\nu,n)\,a + \Phi^{(2)}_{Reg}(\nu,n)\,a^2 + \ldots$$

NNLO and NNNLO corrections calculated by Dixon et al. '13 from corresponding remainder function obtained by symbol technique.

All orders of adjoint BFKL calculated by Basso, Caron-Huot, Sever '15

Collinear and Regge limits of 2->4 amplitude

Bartels, Lipatov and A.P. '12



Is it the same limit?

It is certainly not in terms of momenta, but is it the same in terms of cross ratios?

$$u_1 = \frac{s\ s_2}{s_{012}\ s_{123}}, \quad u_2 = \frac{s_1\ t_3}{s_{012}\ t_2}, \quad u_3 = \frac{s_3\ t_1}{s_{123}\ t_2}$$

Collinear limit $s_3 \to 0 \quad \Rightarrow u_3 \to 0, \quad u_2 \to 1 - u_1$

Multi-Regge kinematics $|s| \gg |s_{012}|, |s_{123}| \gg |s_1|, |s_2|, |s_3| \gg |t_1|, |t_2|, |t_3|$

Regge limit $u_1 \to 1^-,\ u_2 \to 0^+,\ u_3 \to 0^+,\ \tilde{u}_{2,3} \sim \mathcal{O}(1)$

Collinear limit $u_3 \to 0, \quad u_2 \to 1 - u_1, \quad \tilde{u}_3 \to 0$

We can take Regge limit and then collinear limit or vice versa.

Do they commute? In QCD they do, double log limit of BFKL and DGLAP.

Collinear limit of six-particle remainder function

Alday, Gaiotto, Maldacena, Sever and Vieira, '10

$$u_3 = \frac{1}{\cosh^2 \tau}, \quad u_1 = \frac{e^{\sigma} \sinh \tau \tanh \tau}{2(\cos \phi + \cosh \tau \cosh \sigma)}, \quad u_2 = \frac{e^{-\sigma} \sinh \tau \tanh \tau}{2(\cos \phi + \cosh \tau \cosh \sigma)}$$

Remainder function in collinear limit $\tau \to \infty$ at weak coupling

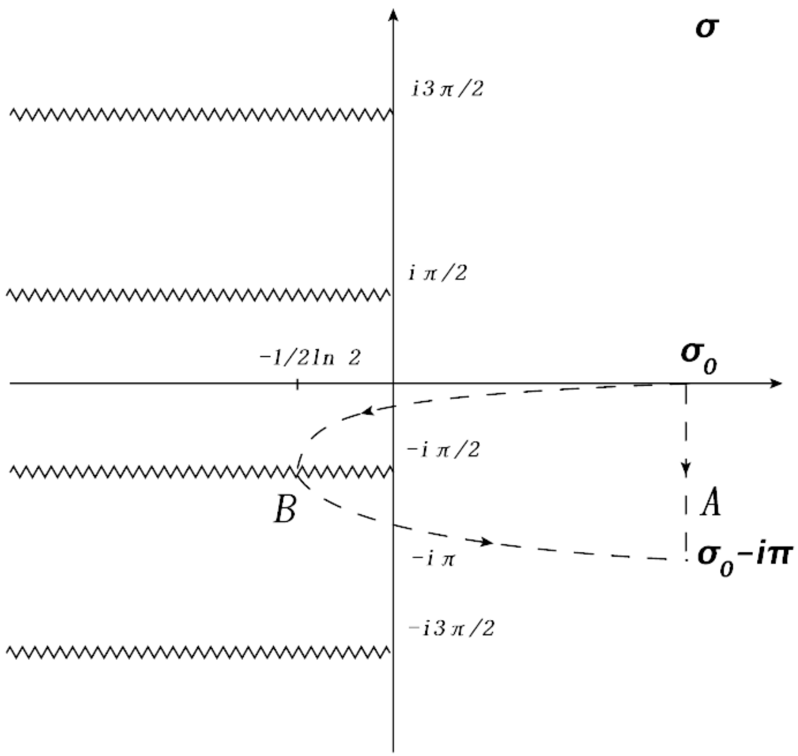
$$R^{(\ell)} \sim \cos \phi e^{-\tau} \frac{(-1)^{\ell-1} \tau^{\ell-1}}{(\ell-1)!} h_\ell \qquad h_\ell \propto \int dp e^{ip\sigma} c^0(p) \left[\gamma_1(p)\right]^{\ell-1}$$

$$c^0(p) \propto \frac{1}{1+p^2} \frac{\pi}{\cos \frac{p\pi}{2}} \qquad \gamma_1(p) = \psi\left(\frac{3}{2} + i\frac{p}{2}\right) + \psi\left(\frac{3}{2} - i\frac{p}{2}\right) - 2\psi(1)$$

In Regge limit $\sigma \to \infty$ it vanishes

Is it possible to perform the analytic continuation to Mandelstam channel?

Bartels, Lipatov and A.P. '12



Path A

$$\frac{u_1}{u_3} = e^{2\sigma} \qquad \sigma \Rightarrow \sigma - i\pi.$$

Path B

$$u_3 \to 1 - u_1 \quad \text{for} \;\; u_2 \;\to\; 0.$$

$$e^{2\sigma} \simeq \frac{u_1}{1-u_1}, \quad \sigma \simeq \frac{1}{2}\ln\frac{u_1}{1-u_1}$$

$$u_1 = |u_1| e^{-i\pi}$$

Two loops

$$h_2 \sim \cosh\sigma\left(2\ln\left(1+e^{2\sigma}\right)\ln\left(1+e^{-2\sigma}\right) - 4\ln\left(2\cosh\sigma\right)\right) + 4\sigma\sinh\sigma$$

This function vanishes at $\sigma \to \infty$ But it diverges after continuation !!!

We fully reproduce the result of Alday et al. also at three loops !

This analytic continuation used by Basso, Caron-Huot, Sever '14 to calculate adjoint BFKL eigenvalue and impact factor at any coupling !!!

Non MHV six-particle remainder function in LLA

Lipatov, A.P., Schnitzer '13 A.P., Spradlin, Vergu, Volovich '13

$$R^{LLA}_{6;NMHV} \simeq -\frac{ia}{2}\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\nu \frac{(-1)^n}{(i\nu+\frac{n}{2})^2} w^{i\nu+\frac{n}{2}}(w^*)^{i\nu-\frac{n}{2}}\left(\left(\frac{s_{23}}{s_0}\right)^{-aE_{n,\nu}}-1\right)$$

$$\frac{1}{-i\nu+\frac{n}{2}} \to -\frac{1}{i\nu+\frac{n}{2}}$$

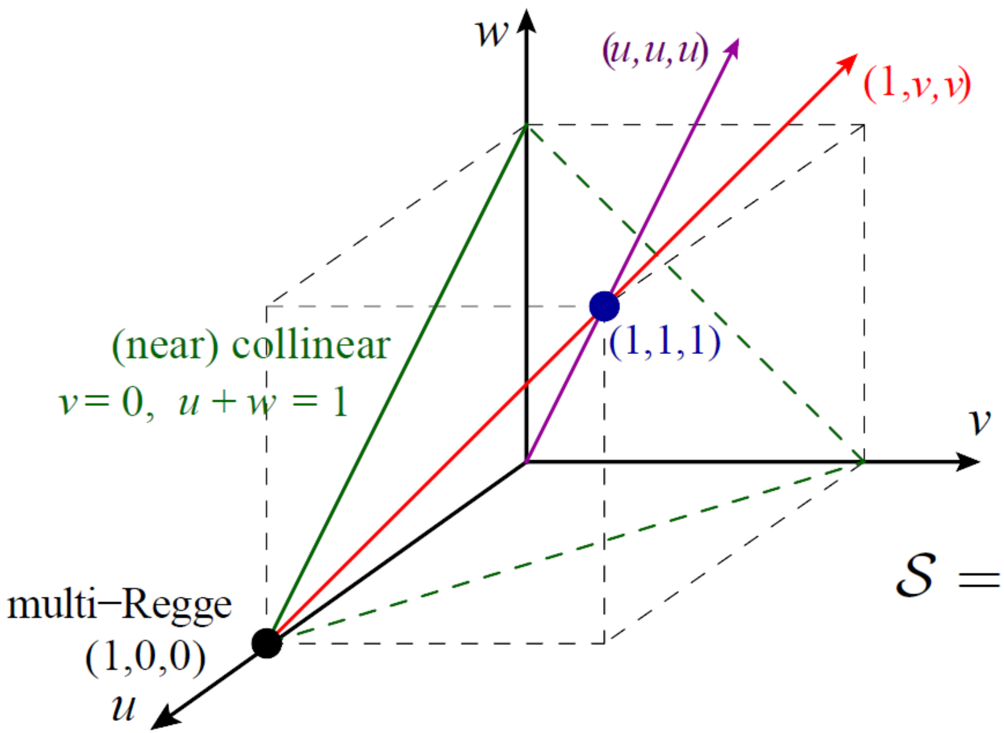
In LLA simply related to MHV

$$\int dw^* \frac{w}{w^*}\frac{\partial}{\partial w} R_{NMHV} = -R_{MHV}$$

Explicit relation at two loops

$$R^{(2)\,LLA}_{NMHV} \simeq \frac{i\pi}{2}\log\left(\frac{s_{23}}{s_0}\right)\left\{\frac{1}{1+w^*}\Big(\log|w|^2\log(1+w^*)-\mathrm{Li}_2(-w)+\mathrm{Li}_2(-w^*)\Big)\right.$$
$$\left.+\frac{1}{1+\frac{1}{w^*}}\left(\log\frac{1}{|w|^2}\log\left(1+\frac{1}{w^*}\right)-\mathrm{Li}_2\left(-\frac{1}{w}\right)+\mathrm{Li}_2\left(-\frac{1}{w^*}\right)\right)\right\}$$

BFKL calculations as input for symbol ansaetze



Symbol Technique

$$\mathcal{S}(\ln x) = x, \quad \mathcal{S}(\mathrm{Li}_2(x)) = -x \otimes (1-x)$$

$$\mathcal{S} = \{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$$

$$y_u = \frac{u - z_+}{u - z_-}, \qquad y_v = \frac{v - z_+}{v - z_-}, \qquad y_w = \frac{w - z_+}{w - z_-}.$$

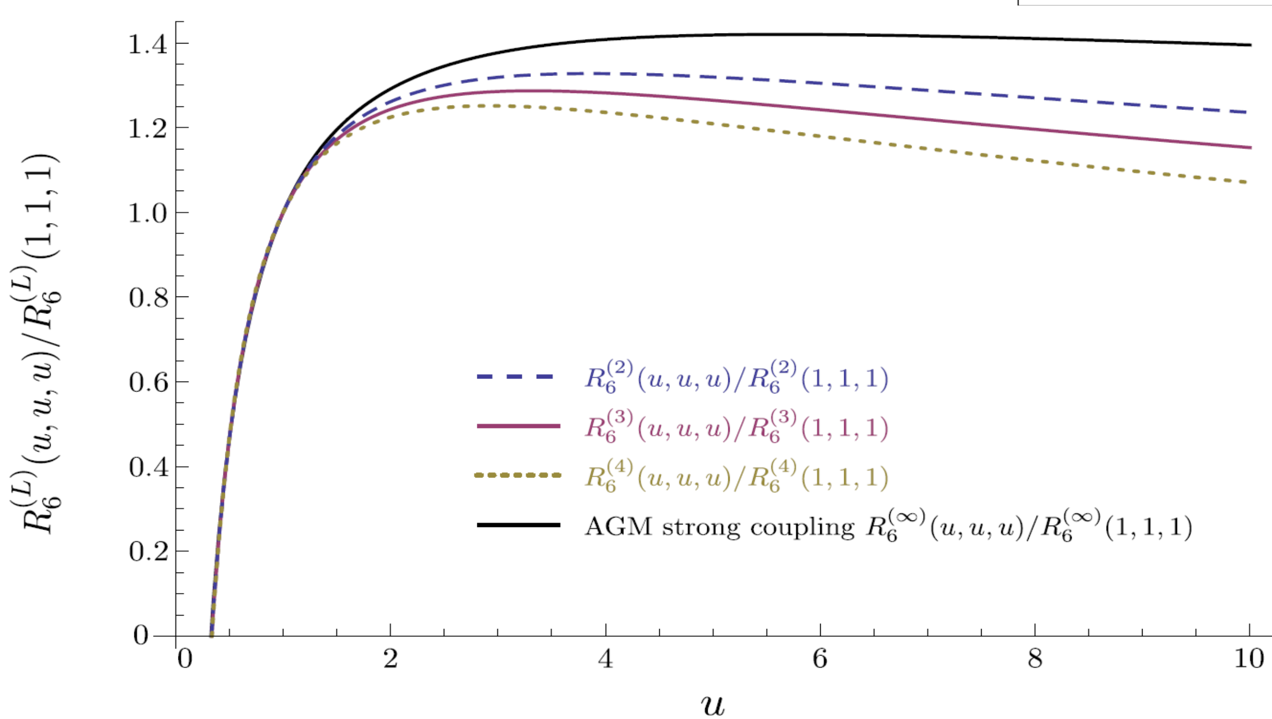
Dixon up to 10 loops , any order 6-particle, three loops Non-MHV

$$z_\pm = \frac{1}{2}\left[-1 + u + v + w \pm \sqrt{\Delta}\right], \qquad \Delta = (1-u-v-w)^2 - 4uvw$$

Dixon et al. '11-'15

Symbol technique allows to calculate many loops and legs using some assumptions about symbol entry. 10 loops for MHV and 3 loops NMHV for 2->4 amplitude.
BFKL calculations is one of basic boundary conditions used for symbol calculations.

Constraint	$L=2$	$L=3$	$L=4$
1. Integrability	75	643	5897
2. Total S_3 symmetry	20	151	1224
3. Parity invariance	18	120	874
4. Collinear vanishing (T^0)	4	59	622
5. OPE leading discontinuity	0	26	482
6. Final entry	0	2	113
7. Multi-Regge limit	0	2	80
8. Near-collinear OPE (T^1)	0	0	4
9. Near-collinear OPE (T^2)	0	0	0



Transition from singlet to adjoint BFKL

Leading order

Singlet

$$\chi(n,\gamma) = -\frac{1}{2}\left(\psi\left(\frac{1}{2}+i\nu+\frac{n}{2}\right)+\psi\left(\frac{1}{2}-i\nu+\frac{n}{2}\right)+\psi\left(\frac{1}{2}+i\nu-\frac{n}{2}\right)+\psi\left(\frac{1}{2}-i\nu-\frac{n}{2}\right)\right)+2\psi(1)$$

Adjoint

$$E_{\nu,n} = -\frac{1}{2}\frac{|n|}{\nu^2+\frac{n^2}{4}}+\psi\left(1+i\nu+\frac{|n|}{2}\right)+\psi\left(1-i\nu+\frac{|n|}{2}\right)-2\psi\left(1\right)$$

$$=\frac{1}{2}\left(\psi\left(i\nu+\frac{n}{2}\right)+\psi\left(-i\nu+\frac{n}{2}\right)+\psi\left(+i\nu-\frac{n}{2}\right)+\psi\left(-i\nu-\frac{n}{2}\right)\right)-2\psi(1).$$

Can be also written as

$$E_{\nu,n} = -\frac{1}{2}\frac{|n|}{\nu^2+\frac{n^2}{4}}+\psi\left(1+i\nu+\frac{|n|}{2}\right)+\psi\left(1-i\nu+\frac{|n|}{2}\right)-2\psi(1) = E^+_{\nu,n}+E^-_{\nu,n}$$

$$E^{\pm}_{\nu,n} \equiv -\frac{1}{2}\frac{1}{\pm i\nu+\frac{|n|}{2}}+\psi\left(1\pm i\nu+\frac{|n|}{2}\right)-\psi(1).$$

NLO Adjoint

$$E^{(1)}_{\nu,n} = -\frac{1}{4}D^2_\nu E_{\nu,n} + \frac{1}{2}V D_\nu E_{\nu,n} - \zeta_2 E_{\nu,n} - 3\zeta_3$$

$$V \equiv -\frac{1}{2}\left[\frac{1}{i\nu + \frac{|n|}{2}} - \frac{1}{-i\nu + \frac{|n|}{2}}\right] = \frac{i\nu}{\nu^2 + \frac{n^2}{4}}$$

Identity

$$\psi(z) = \frac{1}{2}\left[\psi\left(\frac{z}{2}\right) + \psi\left(\frac{z+1}{2}\right)\right] + \ln(2)$$

NLO Singlet

$$\delta(n,\gamma) = \psi''(M) + \psi''(1-\tilde{M}) + 6\zeta_3 - 2\zeta_2\chi(n,\gamma) - 2\Phi(|n|,\gamma) - 2\Phi(|n|,1-\gamma).$$

$$\Phi(|n|,\gamma) + \Phi(|n|,1-\gamma)$$

$$= \Phi_2(M) + \beta'(M)\left[\psi(1) - \psi(1-\tilde{M})\right] + \Phi_2(1-\tilde{M}) + \beta'(1-\tilde{M})\left[\psi(1) - \psi(M)\right]$$

$$\Phi_2(z) = \sum_{k=0}^{\infty} \frac{\beta'(k+1) + (-1)^k \psi'(k+1)}{k+z} - \sum_{k=0}^{\infty} \frac{(-1)^k(\psi(k+1) - \psi(1))}{(k+z)^2}$$

$$\beta'(z) = -\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+z)^2}$$

Summary

- BFKL approach allows to calculate log(s) corrections to BDS amplitude in the Minkowski space

- Unexpected cross talk between Regge and collinear kinematics makes it possible to pass from one to another without all order resummation.

- BFKL is useful for calculations of non-MHV amplitude, which are extensively complicated in general kinematics.

- BFKL results are directly accessible from symbol of the remainder function.

- More Loops and Legs in Regge kinematics.

- Very unexpected connection between singlet and adjoint BFKL eigenvalues