Differential equation approach to perturbative calculations.

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Outline

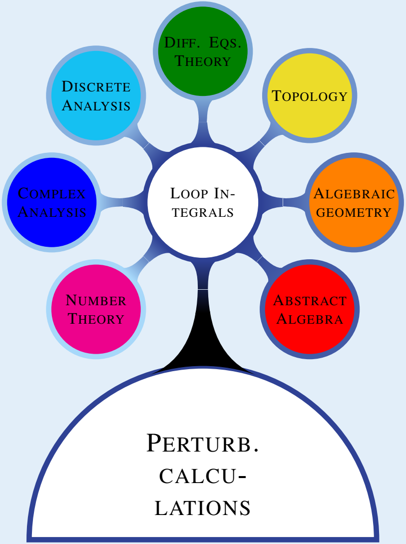
Why multiloop integrals?

Perturbative calculations

Technically, perturbative calculations are reduced to the calculation of (multi)loop integrals.

Multiloop integrals

- Physical applications
- Beautiful mathematics
- Open problems



Multiloop calculations

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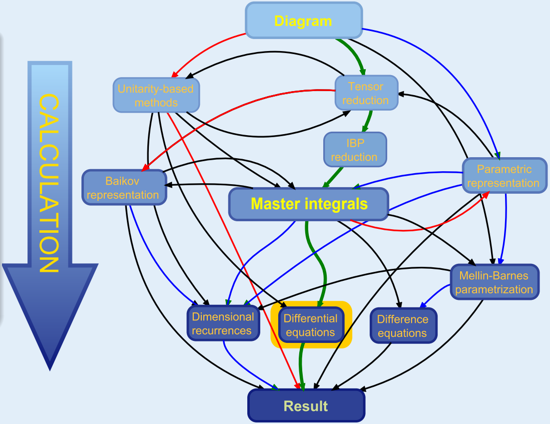
- Multiloop people often prefer **many** loops and **one** scale.
- Phenomenological applications often require **a few** loops but **many** scales.



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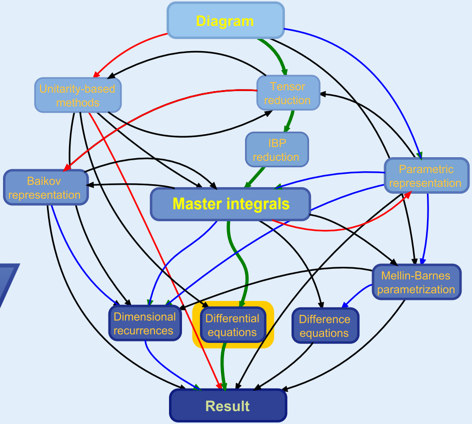


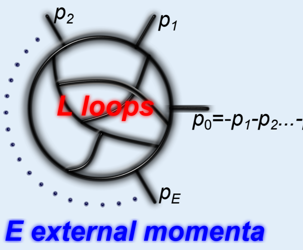
Multiscale problems: IBP+DE approach.

IBP reduction (Chetyrkin&Tkachov 1981)

Scalar integrals family labeled by **n**

$$J(\mathbf{n}) = \int d^d l_1 \ldots d^d l_L j(\mathbf{n}) = \int \frac{d^d l_1 \ldots d^d l_L}{\pi^{\frac{Ld}{2}} D_1^{n_1} \ldots D_N^{n_N}}$$

$D_1, \ldots, D_M$ — denominators of the diagram,
$D_{M+1}, \ldots, D_N$ —numerators ($n_{M+1}, \ldots, n_N \leqslant 0$).



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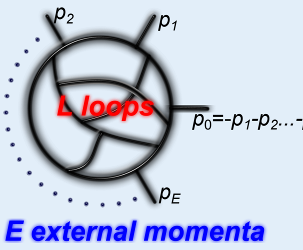

Basic idea of IBP reduction

- Explicit differentiation in (IBP&LI) gives recurrence relations between integrals $J(\mathbf{n})$ with different **n**.
- Using these relations, any integral can be reduced to **finite number** of *master integrals*.

IBP

$$\int d^d l_1 \dots d^d l_L \frac{\partial}{\partial l_i} \cdot q_j j(\mathbf{n}) = 0$$

LI

$$p_{1\mu} p_{2\nu} \sum_e p_e^{[\mu} \partial_e^{\nu]} J = 0$$

Differential equations(Kotikov,1991;Remiddi, 1997)

Differentiating the column-vector $\mathbf{J}$ of master integrals with respect to mass or some invariant and performing IBP reduction we obtain differential equations

$$\partial \mathbf{J}(x)/\partial x = \mathbb{M}(x,\varepsilon)\mathbf{J}(x)$$

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Recent remarkable observation (Henn, 2013)

By a suitable change of functions $\mathbf{J}(x) \to \mathbb{T}(x,\varepsilon)\tilde{\mathbf{J}}(x)$, it is possible to transform equation to ε-factorized form (ε-form)

$$\partial \tilde{\mathbf{J}}(x)/\partial x = \varepsilon \mathbb{S}(x)\tilde{\mathbf{J}}(x)$$

Moreover, rational matrix $\mathbb{S}(x)$ has only simple poles and falls off at infinity ($\mathbb{S}(x) = \sum_i \mathbb{S}_i/(x-x_i)$), i.e., the system is globally *Fuchsian*.

Benefits of ε-form

- Given the equation

$$\partial \mathbf{J}(\varepsilon, x)/\partial x = \varepsilon \mathbb{S}(x)\mathbf{J}(\varepsilon, x)$$

it is easy to find coefficients of expansion $\mathbf{J}(\varepsilon, x) = \sum \mathbf{J}_n(x)\varepsilon^n$ 1 by 1:

$$\mathbf{J}_n(x) = \int dx \mathbb{S}(x)\mathbf{J}_{n-1}(x) .$$

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- For many variables if we secure simultaneous ε-form $\partial\mathbf{J}(\varepsilon,\vec{x})/\partial x_i=\varepsilon\mathbb{S}_i(\vec{x})\mathbf{J}(\varepsilon,\vec{x})$, the integrability condition splits into two $\partial_j\mathbb{S}_i=\partial_i\mathbb{S}_j$ and $\mathbb{S}_j\mathbb{S}_i=\mathbb{S}_i\mathbb{S}_j$ and the system can be rewritten as

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$$\mathrm{d}\mathbf{J} = \varepsilon \mathrm{d}\mathbb{A}\mathbf{J}\,.$$

- Usually the form of the system is drastically simplified.

Example

Original matrix:

$$
\begin{pmatrix}
\frac{\left(x^2+1\right)(2\epsilon-1)}{(x-1)x(x+1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2(\epsilon-1)\left(\epsilon x^4-x^4+10\epsilon x^2-8x^2+\epsilon-1\right)}{(x-1)x(x+1)\left(x^2+1\right)(3\epsilon-2)} & \dots & \dots & 0 & 0 & 0 & 0 \\
0 & -\frac{(4\epsilon-3)\left(\epsilon x^4-2\epsilon x^2+4x^2+\epsilon\right)}{(x-1)x(x+1)\left(x^2+1\right)} & \dots & \dots & 0 & 0 & 0 & 0 \\
0 & \frac{2\epsilon^3x^8-3\epsilon^2x^8+\epsilon x^8+16\epsilon^3x^6-84\epsilon^2x^6+102\epsilon x^6-36x^6-36\epsilon^3x^4-18\epsilon^2x^4+\dots}{6(x-1)x^3(x+1)\left(x^2+1\right)(\epsilon-1)(3\epsilon-2)} & \dots & \dots & 0 & 0 & 0 & 0 \\
\frac{2(x-1)(x+1)\epsilon(2\epsilon-1)}{x^2(4\epsilon-1)} & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\
0 & \frac{\epsilon^2x^4-\epsilon x^4-2\epsilon^2x^2+18\epsilon x^2-12x^2+\epsilon^2-\epsilon}{2(x-1)x(x+1)\left(x^2+1\right)(2\epsilon-1)} & \dots & \dots & 0 & 0 & \dots & \dots \\
0 & -\frac{(x-1)(x+1)\left(\epsilon^2x^4-\epsilon x^4-26\epsilon^2x^2+34\epsilon x^2-12x^2+\epsilon^2-\epsilon\right)}{2x^3\left(x^2+1\right)(3\epsilon-2)} & \dots & \dots & 0 & 0 & \dots & \dots
\end{pmatrix}
$$

Example

After the change of functions:

$$\epsilon\begin{pmatrix}
\frac{2(x^2+1)}{(x^2-1)x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2(x^2+1)}{(x^2-1)x} & \frac{1}{x} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{4}{x} & 0 & -\frac{1}{x} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{12}{x} & -\frac{6(x^2+1)}{(x^2-1)x} & 0 & 0 & 0 & 0 \\
\frac{1}{x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{2}{x} & \frac{2(x^2+1)}{(x^2-1)x} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{2(x^2+1)}{(x^2-1)x} & -\frac{2}{x} \\
0 & \frac{4}{x} & 0 & \frac{1}{x} & 0 & 0 & \frac{2}{x} & -\frac{2(x^2-1)}{(x^2+1)x}
\end{pmatrix}$$

Problem formulation

Given a system

$$\partial \mathbf{J}(x)/\partial x = \mathbb{M}(x,\varepsilon)\mathbf{J}(x)$$

is it possible and how to find a change of functions reducing the system to ε-form

$$\partial \tilde{\mathbf{J}}(x)/\partial x = \varepsilon \sum_k \frac{\mathbb{S}_k}{x-x_k}\tilde{\mathbf{J}}(x),$$

i.e., is it possible and how to find such $\mathbb{T}(x,\varepsilon)$ that

$$\mathbb{T}^{-1}\mathbb{M}\mathbb{T} - \mathbb{T}^{-1}\partial_x\mathbb{T} = \varepsilon \sum_k \frac{\mathbb{S}_k}{x-x_k}?$$

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Approaches so far:

- Using ad hoc arguments, e.g., finding homogeneous integrals from Feynman parametrization (Henn, 2013, 2014).
- Applying a regular procedure when luckily hitting a special form of the initial matrix $\mathbb{M}$ (Gehrmann et al., 2014), (Argeri et al., 2014). E.g., when $\mathbb{M}(x,\varepsilon) = \mathbb{M}_0(x) + \varepsilon\mathbb{M}_1(x)$

Algorithm of reduction to ε-form (R.L. 2015)

Stage 1. Eliminating higher-order poles

Input: Rational matrix $\mathbb{M}(x,\varepsilon)$
Output: Rational matrix with only simple poles (*Fuchsian* singularities) on the extended complex plane, $\mathbb{M}(x,\varepsilon) = \sum_k \frac{\mathbb{M}_k(\varepsilon)}{x-x_k}$.

Stage 2. Normalizing eigenvalues

Input: Matrix from the previous step, $\mathbb{M}(x,\varepsilon) = \sum_k \frac{\mathbb{M}_k(\varepsilon)}{x-x_k}$.
Output: Matrix of the same form, but with the eigenvalues of all $\mathbb{M}_k(\varepsilon)$ confined to unit interval $(-1/2, 1/2]$.

Stage 3. Factoring out ε

Input: Matrix from the previous step.
Output: Matrix in ε-form, $\mathbb{M}(x,\varepsilon) = \varepsilon \sum_k \frac{\mathbb{S}_k}{x-x_k}$

Reduction to ε-form

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- Why not try to do better — get rid of $P(x)$?
- Not always possible due to negative solution of the Riemann-Hilbert problem by Bolibrukh(Bolibrukh,1989).
- **No algorithms for global reduction (including infinity point) so far.**

Idea of reduction

Balance transformation

- Both reduction to Fuchsian form and normalization of the matrix residues are based on the following transformation

$$\mathbb{T}(x) = \mathscr{B}(\mathbb{P}, x_1, x_2 | x) \stackrel{\text{def}}{=} \overline{\mathbb{P}} + \frac{x - x_2}{x - x_1}\mathbb{P},$$

where $\mathbb{P}$ is some projector and $\overline{\mathbb{P}} = \mathbb{I} - \mathbb{P}$. When $x_1 = \infty$ or $x_2 = \infty$ omit denominator or numerator, respectively.

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- Classical algorithms put $x_2 = \infty$ and try to construct $\mathbb{P}$ improving system property at x_1.
- In a long sequence of such transformations the system is reduced in all finite points, but behaviour at $x = \infty$ is **totally spoiled**.

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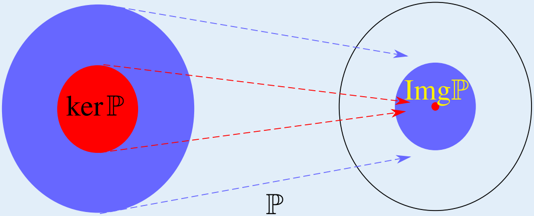

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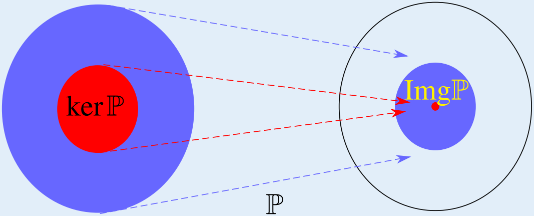

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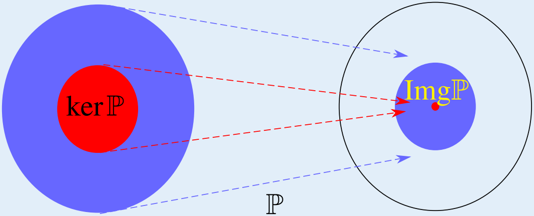
- Idea of global reduction: use the above freedom to keep system properties at x_2 under control.



Stages 2 & 3: Factoring out ε

After Stage 2 we should have

$$\mathbb{M}(x,\varepsilon) = \sum_k \frac{\mathbb{M}_k(\varepsilon)}{x - x_k},$$

such that all eigenvalues of all "residues" $\mathbb{M}_k(\varepsilon)$ are $\propto \varepsilon$. We need to find an x-independent transformation $\mathbb{T}(\varepsilon)$, such that

$$\mathbb{T}^{-1}(\varepsilon)\,\mathbb{M}_k(\varepsilon)\,\mathbb{T}(\varepsilon) = \varepsilon\mathbb{S}_k$$

How can we do it without knowing $\mathbb{S}_k$ in r.h.s.?

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Stage 3. Trick:write it twice

$$\mathbb{T}^{-1}(\varepsilon)\,\frac{\mathbb{M}_k(\varepsilon)}{\varepsilon}\mathbb{T}(\varepsilon)=\mathbb{S}_k=\mathbb{T}^{-1}(\mu)\,\frac{\mathbb{M}_k(\mu)}{\mu}\mathbb{T}(\mu)$$

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$$\mathbb{T}(\varepsilon) \times \left(\mathbb{T}^{-1}(\varepsilon)\,\frac{\mathbb{M}_k(\varepsilon)}{\varepsilon}\mathbb{T}(\varepsilon) = \mathbb{S}_k = \mathbb{T}^{-1}(\mu)\,\frac{\mathbb{M}_k(\mu)}{\mu}\mathbb{T}(\mu)\right) \times \mathbb{T}^{-1}(\mu)$$

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Linear system for matrix elements of $\mathbb{T}(\varepsilon,\mu) = \mathbb{T}(\varepsilon)\,\mathbb{T}^{-1}(\mu)$

$$\frac{\mathbb{M}_k(\varepsilon)}{\varepsilon}\mathbb{T}(\varepsilon,\mu) \;=\; \mathbb{T}(\varepsilon,\mu)\,\frac{\mathbb{M}_k(\mu)}{\mu}$$

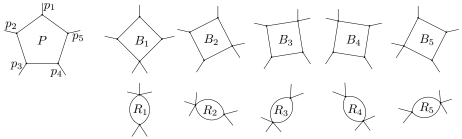
What may go wrong

- At stage 1 and 2 we might fail to construct $\mathbb{P}$ with required properties due to the restriction $\mathrm{Img}\mathbb{P}\cap\ker\mathbb{P}=\{0\}$. This is naturally associated with obstructions to positive solution of Hilbert's 21st problem. In particular, if some monodromy matrix is diagonalizable, we can always balance with the corresponding point.
- Eigenvalues of matrix residues after Stage 1 might be not of the form $n+\alpha\varepsilon$. In particular, it often happens that n is half-integer in a pair of points x_1 and x_2. One can then get rid of half-integer n by passing to $y=\sqrt{(x-x_1)/(x-x_2)}$, so that $x=(x_1-x_2y^2)/(1-y^2)$ is a rational substitution.
- Third step might result in degenerate matrix $\mathbb{T}(\varepsilon,\mu)$ for any μ. E.g. $\mathbb{T}(\varepsilon,\mu)=0$.
- Computational complexity might be overwhelming. One should use block-triangular structure of the equations.

Example applications of ε-form

Onshell pentagon in d dimensions (M. Kozlov & R.L., 2016)

Master integrals



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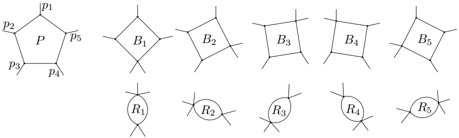

New functions $\tilde{P}$, $\tilde{B}$, $\tilde{R}$

$$P = \frac{\sqrt{\Delta}}{s_1 s_2 s_3 s_4 s_5}\left(\widetilde{P} - \sum_{i=1}^{5}\frac{1}{2}\left(1 - \frac{r_i}{\sqrt{\Delta}}\right)\widetilde{B}_i\right),$$

$$B_i = \frac{1}{s_{i+2}s_{i-2}}\widetilde{B}_i\,, \quad R_i = \frac{\varepsilon}{2(1-2\varepsilon)}\widetilde{R}_i\,.$$

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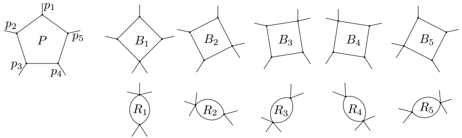

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Equation for new functions (ε-form)

$$d\widetilde{P} = -\varepsilon\left\{\widetilde{P}d(\log S) + \sum_{i=1}^{5}\left[-\widetilde{B}_i d\left(\log\left(1+\frac{r_i}{\sqrt{\Delta}}\right)\right) + \widetilde{R}_i d\left(\log\frac{(\sqrt{\Delta}+r_i)(r_{i+2}+r_{i-2})}{(\sqrt{\Delta}+r_{i+2})(\sqrt{\Delta}+r_{i-2})}\right)\right]\right\},$$

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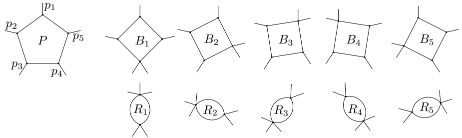

New functions $\tilde{P}$, $\tilde{B}$, $\tilde{R}$

$$P = \frac{\sqrt{\Delta}}{s_1 s_2 s_3 s_4 s_5}\left(\widetilde{P} - \sum_{i=1}^{5}\frac{1}{2}\left(1-\frac{r_i}{\sqrt{\Delta}}\right)\widetilde{B}_i\right),$$

$$B_i = \frac{1}{s_{i+2}s_{i-2}}\widetilde{B}_i\,, \quad R_i = \frac{\varepsilon}{2(1-2\varepsilon)}\widetilde{R}_i\,.$$

Equation for new functions (ε-form)

$$d\widetilde{P} = -\varepsilon\left\{\widetilde{P}d(\log S) + \sum_{i=1}^{5}\left[-\widetilde{B}_i d\left(\log\left(1+\frac{r_i}{\sqrt{\Delta}}\right)\right) + \widetilde{R}_i d\left(\log\frac{(\sqrt{\Delta}+r_i)(r_{i+2}+r_{i-2})}{(\sqrt{\Delta}+r_{i+2})(\sqrt{\Delta}+r_{i-2})}\right)\right]\right\},$$

Result for $s_i < 0$

$$P^{(6-2\varepsilon)}(s_1,s_2,s_3,s_4,s_5) = \frac{2\Gamma(1-\varepsilon)^2\Gamma(1+\varepsilon)}{\varepsilon\Gamma(1-2\varepsilon)}\left[\frac{2\pi^{3/2}\Gamma[1/2-\varepsilon]}{\Gamma[1-\varepsilon]\sqrt{\Delta}}(-S)^{-\varepsilon}\right.$$

$$\left.+\sum_{i=1}^{5}(-s_i)^{-\varepsilon}\int_1^\infty\frac{dt}{t}t^{\varepsilon}\mathrm{Re}\,\frac{1}{b_i(t)}\left\{\arctan\frac{b_i(t)}{r_i} - \arctan\frac{b_i(t)}{r_{i+2}} - \arctan\frac{b_i(t)}{r_{i-2}} + \frac{\pi}{2}\left[\mathrm{sign}r_{i+2}+\mathrm{sign}r_{i-2}-\mathrm{sign}r_i-\mathrm{sign}\left(r_{i+2}+r_{i-2}\right)\right]\right\}\right].$$

Onshell pentagon in d dimensions

A few words about analytical continuation

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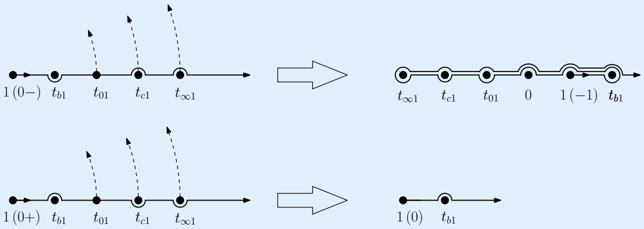
4. $P(s_1, s_2, s_3, s_4, s_5)$ is given by one-fold integral with branching integrand. We should track movement of branching points with changing of s_i.



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Result of analytical continuation

$$P^{(6-2\varepsilon)}(s_1,s_2,s_3,s_4,s_5) = \frac{2\Gamma(1-\varepsilon)^2\Gamma(1+\varepsilon)}{\varepsilon\Gamma(1-2\varepsilon)}\left[\Theta\left(s_is_j>0\right)\frac{2\pi^{3/2}\Gamma[1/2-\varepsilon]}{\Gamma[1-\varepsilon]\sqrt{\Delta}}(-S-i0)^{-\varepsilon}\right.$$

$$\left.+\sum_{i=1}^{5}(-s_i-i0)^{-\varepsilon}\int_1^\infty\frac{dt}{t}t^\varepsilon\mathrm{Re}\,\frac{1}{b_i(t)}\left\{\arctan\frac{b_i(t)}{r_i}-\arctan\frac{b_i(t)}{r_{i+2}}-\arctan\frac{b_i(t)}{r_{i-2}}+\frac{\pi}{2}\left[\mathrm{sign}r_{i+2}+\mathrm{sign}r_{i-2}-\mathrm{sign}r_i-\mathrm{sign}\left(r_{i+2}+r_{i-2}\right)\right]\right\}\right],$$

$$r_n=\sum_{i=0}^{4}(-1)^i s_{n+i}s_{n+i+1}\,,\quad \Delta=\det\left(2p_i\cdot p_j|_{i,j=1,\dots 4}\right)=\sum_{i=1}^{5}r_ir_{i+2}\,,\quad S=4s_1s_2s_3s_4s_5/\Delta,\quad b_i(t)=\sqrt{(St/s_i-1)\Delta+i0}.$$

Note 1: Analytical continuation is **not** reduced to the replacement $s_i\to s_i+i0$.
Note 2: arbitrary order of ε-expansion is one-fold integral of elementary functions.

Electromagnetic e^+e^--pair production in ion collisions

Total Born cross section (R.L.& K. Mingulov, 2016)

At $\gamma = 1/\sqrt{1-\beta^2} \gg 1$ up to power suppressed (w.r.t. $1/\gamma$) terms Racah obtained in 1936 (At that time — heroic deed!)

$$\sigma = \frac{(Z_1\alpha)^2(Z_2\alpha)^2}{\pi m^2}\left[\frac{28L_0^3}{27} - \frac{178L_0^2}{27} + \left(\frac{370}{27} + \frac{7\pi^2}{27}\right)L_0 + \frac{7\zeta_3}{9} - \frac{13\pi^2}{54} - \frac{116}{9}\right], \quad L_0 = \log(2\gamma)\,.$$

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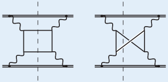
Simple application of presented approach: exact in γ calculation.

- Three-loop cut diagrams:  .

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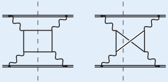
- Three-loop cut diagrams:  .
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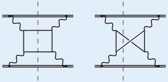
- Three-loop cut diagrams:  .
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- DE reduction:

$$\frac{\partial}{\partial x}\widetilde{\mathbf{J}} = \varepsilon\left[\frac{1}{x}M_0 + \frac{1}{x-1}M_1 + \frac{1}{x+1}M_2\right]\widetilde{\mathbf{J}}, \quad x = \frac{1-\beta}{1+\beta},$$

$$M_0 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 2 & 1 & 0 & 1 & 1 \end{bmatrix}, \quad M_1 = \mathrm{diag}(2,0,2,2,-6,0,2,0), \quad M_2 = \mathrm{diag}(0,0,0,0,0,0,0,-2).$$

Electromagnetic e^+e^--pair production in ion collisions

Exact in γ result

$$\sigma = \frac{(Z_1\alpha)^2(Z_2\alpha)^2}{\pi m^2}\left\{-\frac{1-\beta^2}{12\beta^2}L^4 + \frac{2\left(23\beta^2-37\right)S_{3a}}{9\beta^2} + \frac{2\left(11\beta^2-25\right)S_{3b}}{9\beta^2} - \frac{26S_2}{9\beta} - \frac{\left(\beta^6+217\beta^4-135\beta^2+45\right)L^2}{54\beta^6} + \frac{5\left(67\beta^4-48\beta^2+18\right)L}{27\beta^5} - \frac{2\left(78\beta^4-35\beta^2+15\right)}{9\beta^4}\right\},$$

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$$S_{3a} = \mathrm{Li}_3\left(\frac{1-\beta}{1+\beta}\right) + L\,\mathrm{Li}_2\left(\frac{1-\beta}{1+\beta}\right) - \frac{L^2}{2}\log\left(\frac{2\beta}{1+\beta}\right) - \frac{L^3}{12} - \zeta_3\,,$$

$$S_{3b} = \mathrm{Li}_3\left(-\frac{1-\beta}{1+\beta}\right) + \frac{L}{2}\mathrm{Li}_2\left(-\frac{1-\beta}{1+\beta}\right) + \frac{L^3}{24} - \frac{\pi^2 L}{24} + \frac{3\zeta_3}{4}\,,$$

$$S_2 = \mathrm{Li}_2\left(-\frac{1-\beta}{1+\beta}\right) + L\log\left(\frac{\beta+1}{2}\right) - \frac{L^2}{4} + \frac{\pi^2}{12}\,, \quad L = \log\left(\frac{1+\beta}{1-\beta}\right).$$

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Exact in γ result

$$\sigma=\frac{(Z_1\alpha)^2(Z_2\alpha)^2}{\pi m^2}\left\{-\frac{1-\beta^2}{12\beta^2}L^4+\frac{2\left(23\beta^2-37\right)S_{3a}}{9\beta^2}+\frac{2\left(11\beta^2-25\right)S_{3b}}{9\beta^2}-\frac{26S_2}{9\beta}\right.$$
$$\left.-\frac{\left(\beta^6+217\beta^4-135\beta^2+45\right)L^2}{54\beta^6}+\frac{5\left(67\beta^4-48\beta^2+18\right)L}{27\beta^5}-\frac{2\left(78\beta^4-35\beta^2+15\right)}{9\beta^4}\right\},$$

High-energy asymptotics:

$$\sigma=\frac{(Z_1\alpha)^2(Z_2\alpha)^2}{\pi m^2}\left\{\frac{28L_0^3}{27}-\frac{178L_0^2}{27}+\left(\frac{370}{27}+\frac{7\pi^2}{27}\right)L_0+\frac{7\zeta_3}{9}-\frac{13\pi^2}{54}-\frac{116}{9}\right. \Longleftarrow \text{Racah results}$$

$$\text{First correction} \Longrightarrow \left.-\frac{1}{\gamma^2}\left[\frac{4L_0^4}{3}-\frac{98L_0^3}{27}+\frac{188L_0^2}{27}-\left(\frac{172}{27}+\frac{25\pi^2}{54}\right)L_0-\frac{73\zeta_3}{18}+\frac{5\pi^2}{27}+\frac{43}{27}\right]+\ldots\right\},$$

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Low-energy asymptotics:

$$\sigma = \frac{296(Z_1\alpha)^2(Z_2\alpha)^2\beta^8}{55125\pi m^2}\left(1+\frac{7708\beta^2}{3663}+\ldots\right).$$

Note: highly suppressed as β^8!

Summary

- IBP reduction +DE reduction to ε-form is the most powerful approach to multiscale (multi)loop problems.
- An algorithm of finding ε-form of the differential systems for multiloop integrals is developed.
- Some applications of this algorithm already appeared. Applications to perturbative QCD calculations are ongoing. Suggestions are welcome!

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Thank you!