Gell-Mann–Low for Nonabelian Gauge Theories

Grigorii Pivovarov

INR RAS, Moscow

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- H & M
- A Result
- New PT (How it's done)
- Conclusions

The Gell-Mann–Low Idea

Gell-Mann & Low, 1954

Effective Coupling

$g(Q) = G_Q[a]$

G_Q is a linear form, $a[\phi]$ generating functional of amplitudes

Q momentum used to define effective coupling

Evolution Equation

Amplitudes expanded in $g(Q)$ should be independent of Q

$$\frac{dg(\lambda Q)}{d\log\lambda} = \psi\left(g, \frac{m^2}{\lambda^2 Q^2}\right)$$

ψ is Gell-Mann—Low function

Expansion

$$\psi(g, \lambda) = b_2(\lambda)g^2 + b_3(\lambda)g^3 + \mathcal{O}(g^4)$$

Weinberg's Reform

Weinberg, 1973

Liberation

$A(p) = G(p, \mu, g_\mu)$

g_μ is a renormalized coupling (superseded effective coupling of Gell-Mann–Low)

't Hooft–Weinberg equation

$\frac{dg_\mu}{d\log\mu} = \beta(g_\mu)$

β does not depend on scale

Comparison

Relation to Renormalization

- Gell-Mann–Low ⇒MOM scheme
- ’t Hoof-Weinberg ⇒MS scheme

Technical efficiency

’t Hooft–Weinberg equation is simpler than Gell-Mann–Low

Conceptual Clarity

In Gell-Mann–Low formulation, no need for extra objects

Getting an alternative to MS-scheme

Feasibility

Expansion in terms of g_μ

$$g(Q) = g_\mu + \mathcal{O}(g_\mu^3)$$

Inversion of the expansion

$$g_\mu = g(Q) + \mathcal{O}\left(g^3(Q)\right)$$

Any expansion in g_μ can be recast as expansion in $g(Q)$

(ST identities for renormalization constants are respected)

Relation

$$g(Q) = g_\mu + c_2\left(\frac{Q}{\mu}, \frac{m^2}{Q^2}\right) g_\mu^2 + \mathcal{O}(g_\mu^3)$$

If $Q \sim \mu$ and
$\lim_{Q^2\to\infty} c_n\left(\frac{m^2}{Q^2}\right) = c_n$

$\psi(g) \approx \beta(g)$ at small g

How many scales are introduced by UV renormalization?

Flexibility of Gell-Mann–Low

Within Gell-Mann–Low it is possible to use different scales for normalization (renormalization) of different “elementary” amplitudes

Multiscale renormalization. How many scales are there in QCD?

Six scales of QCD:
Normalization of fields—3 scales
Normalization of a three-particle amplitude—3 scales

Simplest genralization: two scales
One scale to normalize fields, and another, to normalize a three-particle amplitude

Changes are to be Expected

$$g(Q,P) = g_\mu + c_2\left(\frac{Q}{P}\right)g_\mu^2 + \dots$$

Compare to quark masses: $g(Q,m) = g_\mu + c_m\left(\frac{m^2}{Q^2}\right)g_\mu^2 + \dots$

Crucial difference:

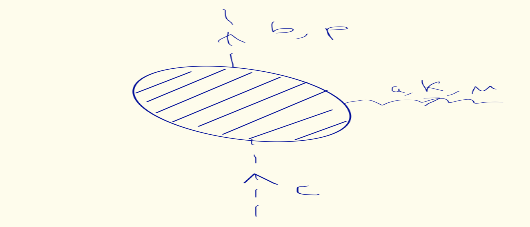
$$\lim_{m\to 0} \frac{dc_m}{d\log m} = 0$$

BUT

$$\lim_{P\to 0} \frac{dc_2}{d\log P} \neq 0$$

Extracting a coupling

Connected Amplitude



$$= A_\mu^{abc}(p,k)$$

Linear Form

$$G_{p,k}[A] = \frac{f^{abc} i p^\mu A_\mu^{abc}(p,k)}{C_G D(A) p^2} \equiv g(p,k)$$

for $SU(N)$, $C_G = N, D(A) = N^2 - 1$

$$g(p,k) = g_\mu + \mathcal{O}(g_\mu^3)$$

Evolution

Scale Dependence

$$g_\lambda \equiv g(\lambda p, \lambda k)$$

Evolution Equation

$$\frac{dg_\lambda}{d\log\lambda} = \psi(g_\lambda, p, k)$$

Gell-Mann–Low Function

For nonabelian gauge theory with a simple compact group and N_F massless Dirac fields

$$\psi(g,p,k) = \frac{g^3}{(4\pi)^2}[3C_G + \alpha C_G(r-1) + \left(\frac{8}{3}T_R N_F - \frac{13}{3}C_G\right)(r-1)] + \ldots$$

$$f^{adc}f^{bdc} = C_G\delta_{ab},\ \alpha = \text{gauge parameter},\ \mathrm{Tr}T^aT^b = \delta^{ab}T_R(=\tfrac{1}{2})$$

$$r = \frac{(pk)^2}{p^2k^2}$$

At $r = 1$

$$\psi(g) = \frac{g^3}{(4\pi)^2}3C_G$$

Generating Functional of Connected Amplitudes

$$\mathrm{A}(\phi) = \mathrm{C_T V}(\phi)$$

$$\mathrm{C_T} = -\mathrm{i} \log \mathrm{T} \exp \mathrm{i} \mathrm{T}^{-1}$$

$$\mathrm{C_T} = (\exp \mathrm{i}, \mathrm{T}^{-1})$$

$$(\mathrm{A}, \mathrm{B}) \equiv (\mathrm{BA})^{-1} \mathrm{AB}$$

$$\mathrm{C_T V} = \mathrm{V} + \mathcal{O}(\mathrm{V}^2)$$

Projectors

A model is T and $\mathcal{V}$ (V $\in \mathcal{V}$)

A $\in \mathcal{A}$; $\mathcal{A}$ and $\mathcal{V}$ share the tangent space at zero—the space $\mathcal{P}$ of parameters

Decomposition $\mathcal{L} = \mathcal{P} \oplus \mathcal{M}$

Projectors

$$\mathrm{P}\mathcal{P} = \mathcal{P}, \mathrm{P}\mathcal{M} = 0$$
$$\overline{\mathrm{P}}\mathcal{P} = 0, \overline{\mathrm{P}}\mathcal{M} = \mathcal{M}$$
$$\mathrm{P} + \overline{\mathrm{P}} = \mathrm{E}$$

Factors

At given decomposition $\mathcal{L} = \mathcal{P} \oplus \mathcal{M}$

Any CV $\approx$ V can be factored

$$\mathrm{C} = \mathrm{C_l C_r},\ \mathrm{C_r F} - \mathrm{F} \in \mathcal{P},\ \mathrm{C_l F} - \mathrm{F} \in \mathcal{M}$$

$$\mathrm{C_r F} \approx \mathrm{F},\ \mathrm{C_l F} \approx \mathrm{F}$$

$$\mathrm{C_l} = \mathrm{PC} + \overline{\mathrm{P}}$$

Factors are invertible

Explicit Formula

$$\mathcal{V} = \mathrm{I}\mathcal{P}$$

$$\mathrm{A}(\mathrm{A}^{\mathrm{a}}_{\mu}, \overline{\mathrm{c}}^{\mathrm{b}}, \mathrm{c}^{\mathrm{c}}) = \mathrm{C}_{\mathrm{T}}\mathrm{I}(\mathrm{C}_{\mathrm{T}}\mathrm{I})^{-1}_{\mathrm{l}}\mathrm{g}_{\lambda}\mathrm{e}_{\mathrm{g}}$$

$$\mathrm{e}_{\mathrm{g}} = \mathrm{f}^{\mathrm{abc}}\mathrm{A}^{\mathrm{a}\mu}\left(\partial_{\mu}\overline{\mathrm{c}}^{\mathrm{b}}\mathrm{c}^{\mathrm{c}} - \mathrm{A}^{\mathrm{b}\nu}\partial_{\mu}\mathrm{A}^{\mathrm{c}}_{\nu}\right)$$

Conclusions

- New perturbation theory over a coupling extracted from amplitudes at reference momenta is constructed
- Gell-Mann–Low function ψ is computed in the leading nontrivial order for nonabelian gauge theories
- For colinear reference momenta, the leading order of ψ does not depend on gauge and on the number of flavors
- No asymptotic freedom
- Expectation of a nontrivial ultraviolet fixed point
- NLO of ψ should be computed