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Resummation for observables at TeV colliders

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Abstract. I review the status of, and discuss recent progress in the field of resummation, concentrating on QCD effects for hadron collider observables.

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1. Introduction

'Resummation' is shorthand for all-order summation of classes of potentially large terms in quantum field perturbation theory. To review status and progress in a field defined so generally is an impossibly wide scope, and I will restrict myself to certain resummation flavors in QCD, related to observables at high-energy hadron colliders.

Let us, in this introductory section, form an impression of what resummation is and does. Let $\mathrm{d}\sigma$ be a quantity with the schematic perturbative expansion

$$\mathrm{d}\sigma = 1 + \alpha_{\mathrm{s}}(L^2 + L + 1) + \alpha_{\mathrm{s}}^2(L^4 + L^3 + L^2 + L + 1) + \cdots, \tag{1}$$

where α_{s} is the coupling, also serving as expansion parameter, L is some logarithm that is potentially large. In our discussion we focus on gauge theories, and on the case with at most two extra powers of L per order, as eq. (1) illustrates. An extra order corresponds to an extra emission of a gauge boson, the two ('Sudakov') logs resulting from the situation where the emission is simultaneously soft and collinear to the parent particle direction.

Denoting $L = \ln A$, we next ask what A is. In fact, A depends on the quantity $\mathrm{d}\sigma$. For example, for a thrust (T) distribution $A = 1 - T$, while for $\mathrm{d}\sigma(p\bar{p} \to Z + X)/\mathrm{d}p_{\mathrm{T}}^Z$ $A = M_Z/p_{\mathrm{T}}^Z$. It should be pointed out already here that A is not *necessarily* constructed out of measured variables but can also be a function of unobservable partonic momenta to be integrated over. Example for inclusive heavy quark production A could be $1 - 4m^2/x_1x_2S$ in hadron collisions with energy $\sqrt{S}$, where x_1, x_2 are partonic momentum fractions. When L is numerically large, so that even for small α_{s}, the convergent behavior of the series is endangered, resummation of the problematic terms into an analytic form might provide a remedy, and thereby extend the theory's predictive power to the range of large L.

The resummed form of $\mathrm{d}\sigma$ may be written schematically as

$$\mathrm{d}\sigma_{\mathrm{res}} = C(\alpha_{\mathrm{s}}) \exp\left[Lg_1(\alpha_{\mathrm{s}}L) + g_2(\alpha_{\mathrm{s}}L) + \alpha_{\mathrm{s}}g_3(\alpha_{\mathrm{s}}L) + \cdots\right] + R(\alpha_{\mathrm{s}}), \tag{2}$$

where $g_{1,2,\ldots}$ are computable functions. The series $C(\alpha_{\mathrm{s}})$ multiplies the exponential, and $R(\alpha_{\mathrm{s}})$ denotes the remainder.

Let us make some remarks about eq. (2). First, the residual series $C(\alpha_{\mathrm{s}}), R(\alpha_{\mathrm{s}})$ are without logs, and therefore (presumably) better-behaved. The dependence on the logarithm has moved into the exponent, which is now a series in α_{s}, and under better control. This is the main merit of resummation. The formula also has a certain predictivity, in the sense that the one-loop leading logarithmic term, after exponentiation, predicts the leading logarithmic terms at all orders, and similarly for subleading terms. Second, the exponential in the resummed form reflects, in a certain sense, the Poisson statistics of independent gauge boson emissions. Third, to implement energy or momentum conservation in a convenient way, the L in (2) is often not the logarithm of a (observed or unobserved) momentum space variable (say, p_{T}), but rather of a conjugate variable (impact parameter b) resulting from a Fourier or other integral transform. An expression like (2) may then be evaluated numerically and used phenomenologically, after an appropriate inverse transform, but it should be mentioned that this is not always an unambiguous procedure, in particular for QCD: the all-order resummation can introduce severely singular infrared behavior into $\mathrm{d}\sigma_{\mathrm{res}}$ that is not present in finite order approximations. Therefore, a resummed result must, in such cases, be specified together with a prescription on how to handle this singular behavior numerically. Although a nuisance in this sense, such ambiguities can in fact be interesting in themselves because they take the form of power corrections. The study of such ambiguities thereby provides access to these important and insufficiently studied corrections.

Specifying the theoretical accuracy for a perturbative series such as eq. (1) involves stating whether only (leading order (LO)) the lowest order term has been kept, or in addition (next-to-leading order (NLO)) the $\mathcal{O}(\alpha_{\mathrm{s}})$ term, etc. The analogue for the resummed form (2) involves stating whether only g_1 is kept (leading logarithmic (LL) approximation), or in addition g_2 (next-to-leading logarithmic (NLL)) is kept, etc. Note that an increase in the logarithmic accuracy must go along with the inclusion, without double counting, of more terms in the $C(\alpha_{\mathrm{s}})$ series. This procedure is called matching. Just as one may parametrically and systematically increase the accuracy of the perturbative approximation (1) by including ever higher order terms, one may do so for the resummed expression by including ever more terms in the exponent, together with appropriate matching.

Having obtained a first impression of what resummation is and does, we can now turn to a more detailed discussion, and specific cases. Before we do so however, let us, at the end of the introduction, summarize the general 'benefits' and present research directions in resummation. The benefits of resummation are: (i) enhancement of predictive power, (ii) an increase in theoretical accuracy (e.g. reduction of scale uncertainty), and (iii) guidance from QCD resummation ambiguities toward non-perturbative effects. Present research strives (i) to increase accuracy by including higher order terms in exponent, and match accordingly (ii) to extend resummation to more differential cross-sections, and new classes of terms (other logs, constants,...), and (iii) to further develop resummation as a precise and practical tool in phenomenology.

2. Threshold, k_{T} and joint resummation

In this section I introduce the resummation flavors in the section title, which will be featured in subsequent sections. This section begins with a review of factorization properties of the observables to be resummed, because such properties are closely tied to their resummability.

2.1 *Factorization basics*

To illustrate the relevance of factorization for resummation let us consider a single-logarithmic example, involving the perturbative scale dependence of parton distributions.

Moments of the deep-inelastic proton (P) structure function $F_{2,P}$ factorize as (assuming for simplicity that only one non-singlet quark combination q contributes)

$$F_{2,P}(N,Q) \equiv \int_0^1 \mathrm{d}x\, x^{N-1} F_{2,P}(x,Q) = C_q(N,Q/\mu)\, \phi_{q/P}(N,\mu)\,. \tag{3}$$

C_q is an infrared safe coefficient function, and $\phi_{q/P}$ is the distribution function for quark q in the proton. The scale (μ) dependence is distributed among two factors, but cancels between them. This fact allows us to derive the DGLAP evolution equation. Because

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \ln F_{2,P}(N,Q) = 0 \tag{4}$$

we have

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \ln \phi_{q/P}(N, \alpha_{\mathrm{s}}(\mu)) = -\mu \frac{\mathrm{d}}{\mathrm{d}\mu} C_q(N, Q/\mu, \alpha_{\mathrm{s}}(\mu)) \equiv \gamma_q(N, \alpha_{\mathrm{s}}(\mu)). \tag{5}$$

The first equality implies that the scale dependence of the quark distribution function can be computed in perturbation theory. The second shows that it can only depend on the common variables N and $\alpha_{\mathrm{s}}(\mu)$. Integrating over μ from Q_0 to Q leads to

$$\phi_{q/P}(N,Q) = \phi_{q/P}(N,Q_0) \exp\left[\int_{Q_0}^{Q} \frac{\mathrm{d}\mu}{\mu}\, \gamma_q(N, \alpha_{\mathrm{s}}(\mu))\right]. \tag{6}$$

We have now performed a resummation of *single* logarithms (one logarithm per order), because the exponential on the right, when approximating $\gamma_q(N,\alpha_{\mathrm{s}}(\mu)) \simeq \alpha_{\mathrm{s}}(Q)\gamma_q^{(1)}(N)$ reads

$$\exp\left[\alpha_{\rm s}(Q)\gamma_q^{(1)}(N)\ln(Q/Q_0)\right]. \tag{7}$$

The function γ_q is in fact the anomalous dimension of an operator whose expectation value in a proton state is $\phi_{q/P}$. From this perspective, the resummation in eq. (6) is a consequence of the multiplicative renormalization property of quantum field theory, so that such single-log resummation amounts to solving renormalization group equations.

The *double* logarithms in eq. (1) we aim to resum however do not result from ultraviolet momenta in loops, but from soft and collinear momentum configurations in emissions and loops. To resum these, we must consider more involved (re)factorizations than (3). Following the reasoning of ref. [1], we shall focus on the factorization upon which rests threshold resummation, because its features underlie the other resummation flavors I discuss in this report as well.

Soft and collinear logarithmic sensitivities are directly related to the corresponding divergences. Our analysis begins therefore with the identification of all sources of soft and collinear singular behavior in a generic Feynman diagram.

To an L-loop Feynman diagram with I internal lines, external momenta $p_{\rm s}$ and loop momenta l_r there correspond a multi-dimensional complex (indicated by $i\epsilon$) integral

$$G = (I-1)!\prod_{j=1}^{I}\int_0^1 {\rm d}\alpha_j(1-\alpha_1-\cdots-\alpha_I)\prod_{i=1}^{L}{\rm d}^d l_i N\left(k_i,p_{\rm s}\right), D^{-I}, \tag{8}$$

where N is some numerator factor, and k_j is the momentum of internal line j. We have combined all propagator denominators using Feynman parameters α_j, into

$$D = \sum_{j=1}^{I}\alpha_j\left(k_j^2\left(l_i,p_{\rm s}\right)-m^2\right)+i\epsilon\,. \tag{9}$$

Soft and collinear divergences occur when the following conditions are satisfied:

$$D=0, \quad \frac{\partial D}{\partial l_i^\mu}=0 \quad \forall i,\mu. \tag{10}$$

The first condition is not surprising, but is not enough to guarantee a soft or collinear divergence. This is because, given any set of singular points in the space of momenta l_i^μ and α_j for which $D=0$, by Cauchy's theorem one can always deform the integration contours to avoid these points, leading to $D\neq 0$. Now the need for the second condition is clear: since D is at most a quadratic function of l_i^μ, the second condition corresponds to the situation where the two poles in l_i^μ pinch the l_i^μ integration contour on either side, allowing no escape from $D=0$. Furthermore, D is a linear function of α_j so that $D=0$ is only unavoidable if $D=0$ does not depend on α_j, or is equal to zero for α_j near an end-point. The Landau equations [2] summarize this result as

$$\begin{aligned} &k_j^2=m^2 \quad \text{or} \quad \alpha_j=0 \\ &\sum_{j\,\in\,\text{loop }r}\alpha_j k_j\epsilon_{jr}=0 \end{aligned} \qquad \forall j,r \tag{11}$$

where the incidence matrix

$$\epsilon_{jr} = \begin{cases} +1: & k_j \text{ in same direction as } l_r \\ -1: & k_j \text{ in opposite direction to } l_r \\ \ \ 0: & \text{otherwise} \end{cases} \tag{12}$$

ensures that momenta are added vectorially in the sum over all loop lines. The collections of points that satisfy the Landau equations are called pinch surfaces.

Solutions to the Landau equations can be represented by so-called reduced diagrams, which can be constructed out of the original diagram from the Landau equation solution as follows: contract every line i for which α_i is zero on the pinch surface to a point, leave every line j for which $k_j^2 - m^2 = 0$ as a line. Finding all solutions to these algebraic equations is clearly very difficult for more complicated diagrams. Fortunately, a powerful graphical and constructive method exists to do just that, due to Coleman and Norton [3]. We do not discuss this method further here, but see e.g. [4].

Satisfying the Landau equations (11) is only a necessary condition for singular behavior of a diagram; we must decide for each solution, using power counting, if it really corresponds to a singularity. This is a daunting task, but it is in fact possible to classify and power count all infrared and collinear-singular regions in l_i^μ, α_j space in great generality [5]. We take, as is common, for illustration the Drell–Yan cross-section: the cross-section for producing an electroweak boson of measured mass Q in hadronic collisions. Power counting shows that all reduced cut diagrams that correspond to infrared- and/or collinear divergences have the general form shown in figure 1.

Inspecting this figure, we see jet-like lines in parallel to the incoming partons a and b, a hard part H on both sides of the final state cut, consisting purely of

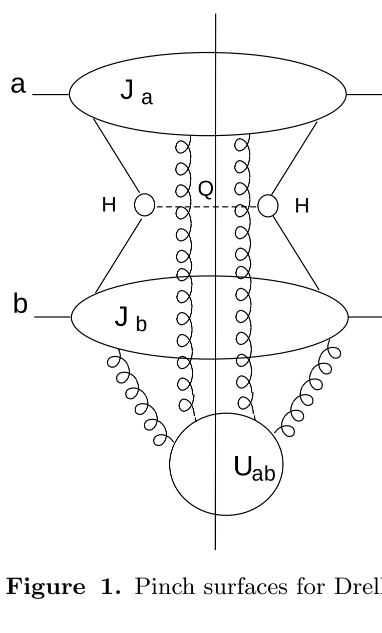

Figure 1. Pinch surfaces for Drell–Yan. Change to include H^* and C.

contracted lines, and soft-gluon lines connecting with the jets, and with each other. The power counting analysis shows that divergences are at worst logarithmic [5].

We may associate the following expression to any reduced diagram representative of the class depicted in figure 1

$$\sum_C \int \frac{\mathrm{d}k_a^+}{2\pi}\frac{\mathrm{d}k_b^-}{2\pi} H^{(C)}\left(k_a^+, k_b^-\right) \prod_\ell \int \frac{\mathrm{d}^4 q_\ell}{(2\pi)^4} \prod_j \int \frac{\mathrm{d}^4 \bar{q}_j}{(2\pi)^4}$$
$$\times J_a^{(C)}\left(k_a^+, q_\ell^\alpha\right)^{\{\mu_1\ldots\mu_n\}} U^{(C)}\left(q_\ell^\alpha, \bar{q}_j^\beta\right)_{\{\mu_1\ldots\nu_1\ldots\}}$$
$$\times J_b^{(C)}\left(k_b^-, \bar{q}_j^\beta\right)^{\{\nu_1\ldots\nu_n\}} . \tag{13}$$

This generic expression closely corresponds to the reduced diagram itself. The functions J_a and J_b gather all terms from vertices and propagators in the jet-like part of the reduced diagram. They are functions of the initial hard momenta k_a and k_b whose dominant components are the plus and minus ones (in light-cone notation), respectively, and of the soft gluon momenta q_α by which they connect. H is built only out of off-shell lines, and does not depend on any q_α, while U is built out of propagators and vertices involving soft-gluons or quark loops only. C labels final states consistent with the definition of the Drell–Yan cross-section. In the above form we have already used the fact that the jet function may be shown to couple only with one parton to H [1]. All further collinear quarks and gluons that might connect $J_{a,b}$ to H either do not lead to singular behavior (in physical gauges), or can be shown to be equivalent to a gauge rotation of the quark field (in covariant gauges).

The many vector indices on the $J_{a,b}$ and U functions that reflect the couplings of the soft gluons prohibit us from deriving much use of this factorization at this stage. Fortunately, expression (13) can be further reduced to a form without such couplings. To this end it is useful to introduce the light-like vectors $v^\mu = \delta_{\mu+}$ and $u^\mu = \delta_{\mu-}$ which point in the directions of the momenta of the incoming partons a, b. The *soft approximation* [1] asserts that we may replace

$$J_a^{(C)}\left(k_a^+, q_\ell^\alpha\right)^{\{\mu_1\ldots\mu_n\}} \to J_a^{(C)}\left(k_a^+, (q_\ell\cdot v)u^\alpha\right)^{\{\xi_1\ldots\xi_n\}}$$
$$\times u_{\xi_1}\ldots u_{\xi_n} v^{\mu_1}\ldots v^{\mu_n},$$
$$J_b^{(C)}\left(k_b^-, \bar{q}_j^\beta\right)^{\{\nu_1\ldots\nu_n\}} \to J_b^{(C)}\left(k_b^-, (\bar{q}_j\cdot u)v^\beta\right)^{\{\lambda_1\ldots\lambda_n\}}$$
$$\times v_{\lambda_1}\ldots v_{\lambda_n} u^{\nu_1}\ldots u^{\nu_n}. \tag{14}$$

In words, the soft approximation amounts to neglecting all dependence on the soft gluon momenta components in the jet functions, in both momenta and polarizations, except the opposite-moving components. Thus, in J_a only the dependence on the minus components of q_ℓ^α is kept. If the jets would be produced at the hard scattering and were moving into the final state the soft approximation would be tantamount to the eikonal approximation, possibly after contour deformation. However, if, as in the present case, the jets are already present in the initial state and are moving into the final state, the validity of eq. (14) is obstructed on a graph-by-graph basis by space-like ('Glauber') gluons. The result (14), now quite non-trivial, nevertheless holds and only emerges after the sum over cuts C consistent with the defined final state. Its proof relies on final state cancellations in the sum over cuts, cancellations that are a consequence of the unitarity properties of jets propagating through a cloud of soft gluons.

Once the soft approximation is implemented, the sum over all attachments of soft gluons to the jets can be replaced, after use of Ward identities, by their attachments to eikonal lines [6–8] which only remember the directions of the jets. These eikonal propagators and all their attachments can then be included into the soft function U. At this stage, all dependence on the soft gluons has been removed from the jet functions. If we are well above the threshold, the sum over final state cuts can be carried out for U separately, in an unweighted fashion, and one finds [6] $U = 1$, by unitarity. This then leads to the well-known factorization theorem

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = J_a \otimes J_b \otimes H_{ab} = \int \mathrm{d}k_a^+ \, \mathrm{d}k_b^- \, J_a(k_a^+) J_b(k_b^-) H_{ab}(k_a^+, k_b^-) \,. \qquad (15)$$

The functions $J_{a,b}$ contain virtual as well collinear radiation effects for the initial partons a, b, and constitute parton distribution functions at fixed light cone momentum. The function H_{ab} is then simply the partonic cross-section, with initial state collinear divergences removed. The convolution symbol $\otimes$ links collinear momenta in the usual way.

When we are near threshold ($z \simeq 1$, $z = Q^2/s$) all radiation is soft. The nearness of the threshold and the consequent strong sensitivity of the jet function to changes in the amount of soft radiation they contribute, implies that the sum over final state cuts for U becomes effectively weighted. As a result, although singularities cancel, logarithms remain. Note also that, because all radiation is soft, after factorizing the soft gluons from the jets near threshold, all that is left of the initial state jets are purely virtual contributions. The function gathering all soft gluons is then in fact nothing but the eikonal cross-section $\sigma^{(\mathrm{eik})}$, so that, up to small corrections of order $1 - z$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = H_{ab} \, J_a^V \otimes J_b^V \otimes \sigma^{(\mathrm{eik})}. \qquad (16)$$

A similar refactorization as described above can now be carried out for the eikonal cross-section

$$\sigma^{(\mathrm{eik})} = J_a^{(\mathrm{eik})} \otimes J_b^{(\mathrm{eik})} \otimes U_{ab} \,. \qquad (17)$$

This formula indicates that purely collinear divergences may be factored out of the eikonal cross-section, and serves to define the function U in the near-threshold refactorization. The eikonal jet functions can be recombined with the virtual jet functions to form $J_a = J_a^V J_a^{(\mathrm{eik})}$. The function J_a describes the distribution of initial state parton a in parent parton a at fixed *energy* $\xi_i Q/2$. We are now left with the following factorized form for the Drell–Yan partonic cross-section near threshold

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = H_{ab} \, J_a \otimes J_b \otimes U_{ab}. \qquad (18)$$

Note that after the soft-gluon decoupling, the jet and soft functions are now no longer linked via multiple soft-gluon propagators, but only via constraints expressing, via the convolution symbol $\otimes$, energy conservation accurate up to corrections of order $1-z$

$$\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}Q^2}(Q,z) &= [H_{ab}\ J_{a/a}\otimes J_{b/b}\otimes U_{ab}](Q,z) \\ &= H_{ab}(Q)\int \mathrm{d}\xi_1\,\mathrm{d}\xi_2 J_{a/a}(Q,\xi_1)J_{b/b}(Q,\xi_2) \\ &\quad\times\int \mathrm{d}w\,U_{ab}(wQ)\,\delta(z-(1-w)\xi_1\xi_2)+Y_{\mathrm{th}}, \end{aligned} \tag{19}$$

where ξ_i are energy fractions. Note that radiation that changes the parton flavor is suppressed in near-threshold factorization. Hence the flavors of the parent partons of a and b are also a, b. The remainder Y_{th}, analogous to R in eq. (2), does not diverge, and is even suppressed when $z \to 1$ [9].

For the factorization that underlies recoil (aka. $Q_{\boldsymbol{T}}$) resummation, the arguments are very similar to the above [10]. The observable of interest here is the cross-section for producing an electroweak boson of mass Q at measured small transverse momentum $\mathbf{Q_T}$ in hadronic collisions. It factorizes into jet-, soft and hard functions linked by collinear momentum fractions $x_{a,b}$ and soft transverse momenta

$$\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}Q^2\,\mathrm{d}^2\mathbf{Q_T}}(Q,\mathbf{Q_T}) &= [H_{cd}\ P_{c/a}\otimes P_{d/b}\otimes U_{cd}](Q^2,\mathbf{Q_T})+Y_{\mathrm{rec}} \\ &= H_{cd}(Q)\int \mathrm{d}x_a \int \mathrm{d}x_b \int \mathrm{d}^2\mathbf{k_a}\,\mathrm{d}^2\mathbf{k_b} P_{c/a}(x_a,\mathbf{k_a}) \\ &\quad\times P_{d/b}(x_b,\mathbf{k_b})\int \mathrm{d}^2\mathbf{q}\,U_{cd}(\mathbf{q})\,\delta(Q^2-x_a x_b S) \\ &\quad\times\delta^{(2)}(\mathbf{Q_T}+\mathbf{k_a}+\mathbf{k_b}+\mathbf{q})+Y_{\mathrm{rec}}, \end{aligned} \tag{20}$$

where the second δ function states that the measured transverse momentum results from recoil against radiation from the soft and jet functions. The remainder Y_{rec} does not diverge when $Q_{\mathrm{T}} \to 0$. For recoil refactorization, initial state radiation that changes the parton flavor is not suppressed. The functions $P_{c/a}(x,\mathbf{k})$ represent the distribution of parton c in parton a at momentum fraction x and transverse momentum $\mathbf{k}$.

Joint resummation [11–13] combines threshold and recoil resummation into one consistent formalism. For this case, our observable of interest is electroweak boson production in hadronic collisions at measured mass Q, near threshold, and measured transverse momentum $\mathbf{Q_T}$. Its refactorization reads, schematically

$$\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}Q^2\,\mathrm{d}^2\mathbf{Q_T}}(Q,\mathbf{Q_T},z) &= [H_{ab}\ R_{a/a}\otimes R_{b/b}\otimes U_{ab}](Q,\mathbf{Q_T},z)+Y_{\mathrm{joint}} \\ &= H_{ab}(Q)\int \mathrm{d}\xi_a \int \mathrm{d}\xi_b \int \mathrm{d}^2\mathbf{k_a}\,\mathrm{d}^2\mathbf{k_b} R_{a/a}(\xi_a,\mathbf{k_a})\,R_{b/b}(\xi_b,\mathbf{k_b}) \\ &\quad\times\int \mathrm{d}w \int \mathrm{d}^2\mathbf{q}\,U_{cd}(wQ,\mathbf{q})\,\delta(z-(1-w)\xi_1\xi_2) \\ &\quad\times\delta^{(2)}(\mathbf{Q_T}+\mathbf{k_a}+\mathbf{k_b}+\mathbf{q})+Y_{\mathrm{joint}}. \end{aligned} \tag{21}$$

The functions $R_{a/a}(\xi, \mathbf{k})$ represent the distribution of parton a in parton a at energy fraction ξ and transverse momentum $\mathbf{k}$.

For simplicity we have used the same symbol H and U for the hard and soft functions in the various factorizations, but in practice these functions can be different.

Admittedly, the above discussion is rather sketchy, and may (or may not) leave the reader wishing for a more explicit exposition. The topic of factorization is however rather technical, and does not lend itself easily to a simple yet thorough synopsis. To clarify the discussion somewhat, we shall revisit some of the above issues in the next subsection, where we discuss threshold resummation.

2.2 *Threshold resummation*

Before continuing the tale of threshold resummation for the Drell–Yan cross-section, let us first review some more general aspects of threshold resummation.

Threshold resummation begins with defining a threshold (elastic limit), $w = 0$. The threshold depends on the observable, e.g. for heavy quark production, for the inclusive cross-section one can use $w = s - 4m^2 = 0$, for p_{T} distribution one can use $w = s - 4(m^2 + p_{\mathrm{T}}^2) = 0$, and for the double-differential distribution $w = s + t + u = 0$. The large logarithm to resum is $\ln w$. However, to resum $\ln(w)$ effects, it is best to first pass to a conjugate space via a Laplace (or Mellin) transform $\int_0 \mathrm{d}w \exp(-w\,N)$. Notice that the limit $w \to 0$ corresponds to $N \to \infty$. The large log is then, in conjugate space, $\ln N$. After resummation, one must undo the transform via

$$\int_C \mathrm{d}N \exp(w\,N) f(N) \tag{22}$$

with C an appropriately chosen contour, such as indicated in figure 2.

For hadronic cross-sections, threshold resummation in general enhances the cross-section. This may be counterintuitive to those who associate radiation from systems

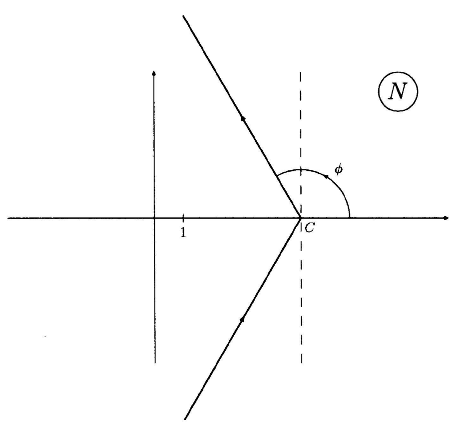

Figure 2. Contour for inverse Laplace or Mellin transform.

near their elastic limits to be (Sudakov) suppressed. In N space Sudakov suppression takes the form

$$\lim_{N\to\infty} \sigma_{\rm H}(N) \sim \exp(-\ln^2 N)\,. \tag{23}$$

This is indeed true for the hadronic cross-section $\sigma_{\rm H}$, but recall that we are resumming the corrections to the *partonic* cross-section, which is the ratio of the hadronic section and the squared parton distribution function

$$\sigma(N) = \frac{\sigma_{\rm H}(N)}{\phi^2(N)}\,. \tag{24}$$

The parton distribution themselves are also Sudakov suppressed: $\phi(N) \sim \exp(-\ln^2 N)$, so that

$$\lim_{N\to\infty} \hat{\sigma}(N) \sim \exp(+\ln^2 N)\,. \tag{25}$$

In words: parton distributions are too stingy with gluon radiation, from including too many virtual gluons, so that in the ratio (24) the net effect is enhancing. This is an important motivation for studying threshold resummation effects in predictions for observables. Note however that jets produced at the hard scattering are also Sudakov suppressed near threshold. A sufficiently strong suppression from such final state jets can overcome the enhancement from initial state jets.

We now return to threshold resummation for the Drell–Yan process. To conform with literature, we shall henceforth denote the initial quark jets at fixed energy by ψ.

Taking Laplace moments with respect to $1-z$ of the convolution form for the Drell–Yan partonic cross-section in eq. (19) leads to the product (suppressing dependence on Q)

$$\sigma\left(N,\epsilon\right) = \psi\left(N,\epsilon\right)\psi\left(N,\epsilon\right)U\left(N\right)H + \mathcal{O}\left(1/N\right), \tag{26}$$

with two initial state quark jets ψ and a soft function U. The function ψ, and therefore also the unsubtracted cross-section σ contains initial state collinear divergences at all orders, which is indicated by the dependence on the dimensional regularization parameter ϵ. In addition, at $\mathcal{O}(\alpha_{\rm s}^n)$ ψ contains terms of order $\ln^{2n} N$ and lower powers of $\ln N$. The soft function U at $\mathcal{O}(\alpha_{\rm s}^n)$ contains terms of order $\ln^n N$ and lower powers, and is furthermore free of divergences. Explicit expressions for U and ψ will be given below. To render the moment space cross-section in eq. (26) finite, the divergences can be subtracted according to

$$\sigma\left(N\right) = \sigma\left(N,\epsilon\right)/\phi^2\left(N,\epsilon\right). \tag{27}$$

Let us now discuss the resummation of $\ln N$ dependence for the individual functions in eq. (26). Since, as remarked earlier, U has only single logarithmic enhancements near threshold, its exponentiation follows quite simply from renormalization group techniques, as shown in the beginning of §2. The soft function depends on the renormalization scale μ directly and through the running coupling $\alpha_{\rm s}$. Under scale variations it behaves as

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}U\left(\frac{Q}{N\mu},\alpha_{\mathrm{s}}\left(\mu\right)\right)=-\nu\left(\alpha_{\mathrm{s}}\left(\mu\right)\right)U\left(\frac{Q}{N\mu},\alpha_{\mathrm{s}}\left(\mu\right)\right). \tag{28}$$

The function ν is associated with the anomalous dimension of the composite operator to be introduced in (31). The exclusive dependence on the ratio Q/N indicates that U depends only on soft kinematic scales. Solving (28) gives

$$U\left(\frac{Q}{N\mu},\alpha_{\mathrm{s}}\left(\mu\right)\right)=U\left(\frac{Q}{\mu},\alpha_{\mathrm{s}}\left(\mu\right)\right)\exp\left[\int_{\mu}^{\mu/N}\frac{\mathrm{d}\lambda}{\lambda}\nu\left(\alpha_{\mathrm{s}}\left(\lambda\right)\right)\right], \tag{29}$$

with all N-dependence exponentiating.

Because ψ contains both collinear divergences (to be canceled by ϕ as in (27)) and double logarithms, its exponentiation is a little more involved. Well-developed arguments exist for the exponentiation properties of the Drell–Yan cross-section near threshold [9,14]. Some are based on identifying further evolution equations [9,15] based on the refactorizations in the previous section, in the spirit of §2.2.1. Others rely on the fact that the refactorizations isolate the eikonal part of the observable in a well-defined way, and then use the property that moments of the eikonal DY cross-section exponentiate at the level of integrands [14,16–18], with exponents consisting of so-called *webs*. These are selections of cut diagrams under criteria defined by graphical topology (irreducibility under cuts of the eikonal lines) and with possibly modified color weights. For this discussion we follow the latter approach. Each web is a cut diagram, and can be integrated over the momentum k that it contributes to the final state.

The definition of eikonal cross-sections requires the concept of Wilson lines

$$\Phi_{\beta}^{(f)}(\lambda_2,\lambda_1;X)=\mathrm{P}\exp\left(-ig\int_{\lambda_1}^{\lambda_2}\mathrm{d}\eta\,\beta\cdot A^{(f)}(\eta\beta+X)\right). \tag{30}$$

These are exponentials of line integrals over the Lie-algebra valued gluon field along direction β^{μ} and passing through space-time point X. The gauge field is a matrix in the representation of parton f. Such Wilson lines summarize the coupling of soft gluons to a single parton, or an entire jet [1,19,20], neglecting recoil, and are therefore the natural building blocks for defining eikonal cross-sections [21]. For DY, the product

$$\mathcal{W}(X)=\Phi_{\beta_2}^{(\bar{q})}\left(0,-\infty;X\right)\ \Phi_{\beta_1}^{(q)}\left(0,-\infty;X\right) \tag{31}$$

represents the QCD radiation generated by the annihilation of the incoming quark and antiquark color sources, along directions β_1 and β_2 respectively, again neglecting recoil. In terms of these operators, the DY eikonal cross-section at fixed gluon radiation energy $w\,Q/2$ can be defined as

$$\sigma^{(\mathrm{eik})}(wQ,\epsilon)=\frac{Q}{6}\int\frac{\mathrm{d}\lambda}{2\pi}\,\mathrm{e}^{-i\lambda wQ/2}\\ \times\ \mathrm{Tr}\ \langle 0|\,\bar{T}\ \left[\mathcal{W}(0)^{\dagger}\right]\ T\left[\mathcal{W}(\lambda\hat{n})\right]\,|0\rangle\ , \tag{32}$$

with $\hat{n}=(1,\vec{0})$. The trace is over colors and $T,\bar{T}$ represent time ordering and anti-time ordering, respectively. The expression is normalized to $\delta(w)$ at lowest order.

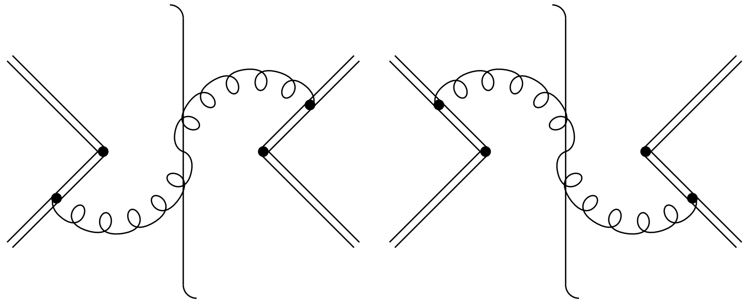

Figure 3. The two webs of the DY eikonal cross-section at order α_s.

As discussed earlier, the eikonal cross-section is related to the full cross-section by

$$\sigma(N) = H\,\psi^V\ \psi^V \sigma^{(\mathrm{eik})}(N)\ . \tag{33}$$

The purely virtual ψ^V contains poles but no N-dependent threshold enhancements since these require also real contributions. The threshold enhancements reside entirely in the eikonal cross-section, which, as stated, exponentiates as:

$$\sigma^{(\mathrm{eik})}(N,\epsilon) = \exp\left\{2\int \frac{\mathrm{d}^{4-2\epsilon}k}{(2\pi)^{4-2\epsilon}}\,\theta\left(\frac{Q}{\sqrt{2}}-k^+\right)\ \theta\left(\frac{Q}{\sqrt{2}}-k^-\right)\right.$$
$$\times w_{\mathrm{DY}}\left(k^2,\frac{k\cdot\beta_1 k\cdot\beta_2}{\beta_1\cdot\beta_2},\mu^2,\alpha_s(\mu),\epsilon\right)$$
$$\left.\times\left(\mathrm{e}^{-N(2\,k^0/Q)}\ -1\right)\right\}, \tag{34}$$

where w_{DY} represents the web for DY at fixed total momentum k^μ, and β_1 and β_2 are the velocities of the colliding eikonal lines. The θ functions serve to limit the plus and minus momenta of total gluon radiation, so that the maximum value of k_{T}^2 is Q^2, and one stays within the exponentiation conditions.

Note that these cross-sections are defined in $d = 4-2\epsilon$ dimensions, indicated by the argument ϵ, and that collinear singularities are still present.

The one-loop radiative contribution to the web is readily computed from the eikonal Feynman diagrams in figure 3 to be

$$w_{\mathrm{DY}}^{(1)(\mathrm{real})}(k) = \frac{2C_F\alpha_s}{\pi}\ \left(4\pi\mu^2\mathrm{e}^{-\gamma_E}\right)^\epsilon\ \delta_+(k^2)\ \frac{\beta_1\cdot\beta_2}{\beta_1\cdot k\,\beta_2\cdot k}\ . \tag{35}$$

Substituting eq. (35) into eq. (34), and expressing the measure of the momentum integration as $\mathrm{d}k_0\mathrm{d}k_3\mathrm{d}^{2-2\epsilon}k_\perp$ one finds

$$\ln\sigma^{(\mathrm{eik})}(N,\epsilon) = \frac{4C_F\alpha_s}{\pi}\left(4\pi^2\mu^2\right)^\epsilon\int\frac{\mathrm{d}k_0\mathrm{d}k_3\mathrm{d}^{2-2\epsilon}k_\perp}{2k_3k_\perp^2}$$
$$\times\delta\left(k_3-\sqrt{k_0^2-k_\perp^2}\right)\left(\mathrm{e}^{-\frac{Nk_0}{2Q}}-1\right). \tag{36}$$

Expanding now in $k_\perp/k_0$ and using the delta function one obtains

$$\ln\sigma^{(\mathrm{eik})}(N,\epsilon) = \frac{4C_F\alpha_\mathrm{s}}{\pi\Gamma(1-\epsilon)}(4\pi)^\epsilon\left(\frac{\mu^2}{Q^2}\right)^\epsilon \times\int_0^1\frac{\mathrm{d}w}{w}\int_0^w\frac{\mathrm{d}\lambda}{\lambda^{1+2\epsilon}}\left(\mathrm{e}^{-Nw/2}-1\right) = \frac{2C_F\alpha_\mathrm{s}}{\pi\Gamma(1-\epsilon)}(4\pi)^\epsilon\left(\frac{\mu^2}{Q^2}\right)^\epsilon\left(-\frac{1}{\epsilon}+\ln^2 N+\cdots\right), \tag{37}$$

where in the last line only the poles and the highest power of $\ln N$ have been displayed. The collinear pole arising from the λ integration will be cancelled by the distribution ϕ (27). The remaining term is the expected double logarithm in N, obtained upon carrying out the w integration. One can now compare the two expressions (26) and (33) for $\sigma^{(\mathrm{eik})}(N,\epsilon)$ to derive

$$\ln\psi(N,\epsilon) = \frac{2C_F\alpha_\mathrm{s}}{\pi\Gamma(1-\epsilon)}(4\pi)^\epsilon\left(\frac{\mu^2}{Q^2}\right)^\epsilon\int_0^1\frac{\mathrm{d}w}{w}\int_0^w\frac{\mathrm{d}\lambda}{\lambda^{1+2\epsilon}}\left(\mathrm{e}^{-Nw}-1\right) -\frac{1}{2}\int_0^1\mathrm{d}z\frac{z^{N-1}-1}{1-z}\nu\left(\alpha_\mathrm{s}\left((1-z)Q\right)\right), \tag{38}$$

the exponentiated version of ψ.

We hope the reader, after this sketchy discussion, finds it not implausible that the threshold-resummed Drell–Yan cross-section, factorized in the $\overline{\mathrm{MS}}$ scheme takes the following general form [9,14]:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2}(Q,z) = H(Q)\int\frac{\mathrm{d}N}{2\pi i}z^{-N}\exp\left[\int_0^1\frac{\mathrm{d}w}{w}\left(\mathrm{e}^{-Nw}-1\right)\times\int_1^w\frac{\mathrm{d}\lambda}{\lambda}A(\alpha_\mathrm{s}(\lambda Q))+D(\alpha_\mathrm{s}(wQ))\right]+Y_\mathrm{th}, \tag{39}$$

with an appropriate contour for the N integral.

Without further derivation we give a generic, rather imprecise but illustrative form for the recoil-resummed partonic cross-section for electroweak boson production in hadronic collisions, in terms of an inverse transform over the impact parameter b [22,23]

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2\,\mathrm{d}^2\mathbf{Q_T}}(Q,\mathbf{Q_T}) = H_{cd}\int\frac{\mathrm{d}^2b}{(2\pi)^2}\mathrm{e}^{i\mathbf{Q_T}\cdot\mathbf{b}}(Q)C_{c/a}(b)\otimes C_{d/b}(b) \otimes\exp\left[-\int_{1/b}^{Q}(\mathrm{d}q/q)A(\alpha_\mathrm{s}(q))\ln(Q/q)+B(\alpha_\mathrm{s}(q))\right]+Y_\mathrm{rec}. \tag{40}$$

The convolution is in collinear momentum fractions, as in eq. (20). The large logarithm that is resummed here is of course $\ln b$. Notice the Sudakov *suppression* in the exponential, which will appear in various figures below as a suppression of the cross-section at very small Q_{T}. Note also that carrying out the inverse b transform requires a choice how to handle the large b part of the integral, as mentioned in the introduction. A number of popular choices for such non-perturbative input have emerged over the years. For the precise form of the resummed cross-section used for each recoil-resummation study discussed in the following sections, as well as the choice of non-perturbative input, one should consult the relevant reference. For example, a number of studies use a form similar to (40), one that is valid in momentum space [24] and does not require an inverse b transform.

Finally, the joint-resummed expression for electroweak production may be written, equally schematically, as [25,26]

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2\,\mathrm{d}^2\mathbf{Q_T}}(Q,\mathbf{Q_T},z) = H(Q)\int\frac{\mathrm{d}N}{2\pi\, i}z^{-N}\int\frac{\mathrm{d}^2b}{(2\pi)^2}\mathrm{e}^{i\mathbf{Q_T}\cdot\mathbf{b}} \times\tilde{C}_a(N,b,Q,\mu)\mathrm{e}^{E_{a\bar{a}}(N,b,Q,\mu)}\tilde{C}_{\bar{a}}(N,b,Q,\mu) + Y_{\mathrm{joint}}. \quad (41)$$

Notice the combined N and b dependence in the various functions. This formalism resums $\ln N$ and $\ln b$ logarithms simultaneously. In joint resummation threshold enhancement and recoil suppression compete for dominance. In the limit of small N or large b the function $\tilde{C}$ reduces to C, at least up to a certain accuracy.

After this extensive introduction, let us now turn to various studies involving the resummations mentioned. Since our review merely touches upon the key points of these studies, is certainly neither exhaustive in discussing all their merits, nor in listing all relevant literature, I refer the interested reader to the original papers and references therein for more details.

3. Threshold resummation in Drell–Yan

The Drell–Yan process has a distinguished history as the theoretical laboratory for testing QCD resummation ideas [9,14]. The present level of accuracy in threshold resummation is NNLL [27], achieved by matching to the NNLO calculations of refs [28,29].

The most accurate studies of threshold resummation for Drell–Yan are performed in ref. [27]. Representing the partonic resummed cross-section in moment space as one finds

$$\sigma_{\mathrm{DY}}(N,Q^2) = H(Q)\,\exp\left[G_{\mathrm{DY}}(N,Q)\right], \quad (42)$$

$$G_{\mathrm{DY}} = 2\ln N g_1(2\lambda) + g_2(2\lambda) + \alpha_{\mathrm{s}}g_3(2\lambda) + \cdots, \quad (43)$$

$$\lambda = \beta_0\alpha_{\mathrm{s}}\ln N, \quad (44)$$

which follows from carrying out the integrals in the exponent in eq. (39). Here

$$g_1(\lambda) = \frac{C_F}{\beta_0\lambda}\left[\lambda + (1-\lambda)\ln(1-\lambda)\right]. \quad (45)$$

Results are shown in figure 4 for the convergence properties when increasing the logarithmic accurcay of the exponent, and of the hadronic K factor.

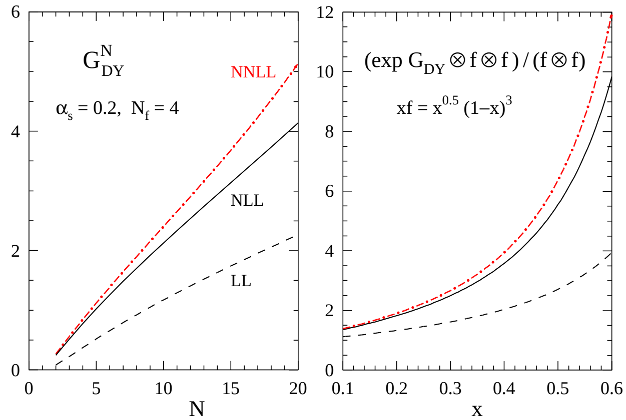

Figure 4. Convergence behavior of Drell–Yan partonic and hadronic cross-sections.

One observes good convergence as the logarithmic accuracy of the resummation is increased. For the inverse Mellin transform, required to compute the hadronic cross-section in momentum space, the minimal prescription [30] was used: the contour in figure 2 is chosen such that the intercept C is to the left of the singularity at $\lambda = 1/2$ in eq. (45).

4. Threshold resummation in deep inelastic scattering

The inclusive non-singlet deep inelastic structure function F_2 is another classic in the history of threshold resummation. Also here NNLO calculations [31,32] enable NNLL accuracy [27,33]. The structure of the threshold-resummed observable is similar

$$\sigma_{\rm DIS}(N,Q) = H(Q)\, \exp\left[G_{\rm DIS}(N,Q)\right], \tag{46}$$

$$G_{\rm DIS} = \ln N g_1(\lambda) + g_2^{\rm DIS}(\lambda) + \alpha_{\rm s} g_3^{\rm DIS}(\lambda) + \cdots, \tag{47}$$

$$\lambda = \beta_0 \alpha_{\rm s} \ln N. \tag{48}$$

The results analogous to those of figure 4 are shown in figure 5.

One observes even better convergence than for Drell–Yan as the logarithmic accuracy of the resummation is increased. The minimal prescription was again used for the inverse Mellin transform.

This observable was also studied [33] using an alternative all-order form, involving tower expansions:

$$\sigma_{\rm DIS}(N,Q) = 1 + \sum_{k=1}^{\infty} \alpha_{\rm s}^k \left(c_{k1} L^{2k} + c_{k2} L^{2k-1} + c_{k3} L^{2k-3} + \cdots\right).$$

Each tower converges, and first four towers in N space seem to provide good estimates at large x. The singularity mentioned in the previous section is effectively mapped to far-subleading towers.

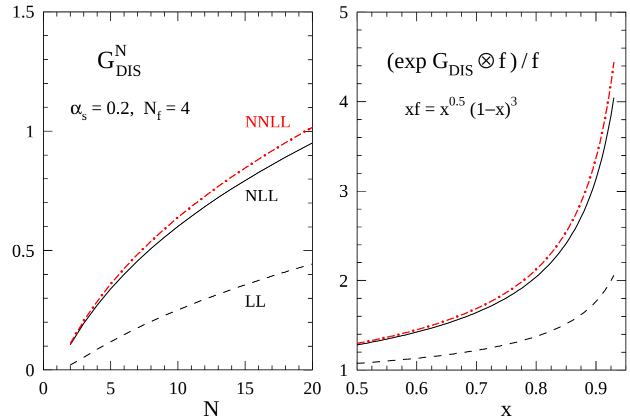

Figure 5. Convergence behavior of DIS partonic and hadronic structure functions.

5. Threshold resummation in Higgs production

Although the Higgs boson is not yet discovered, the QCD corrections to its inclusive production cross-section have received a large amount of attention in recent years, making it now the best-computed cross-section in QCD.

The NLO calculations, after having been done first in the heavy m_t limit [34,35] were carried out with full m_t dependence in ref. [36]. The large size of the corrections prompted a study [37] involving partial NLL threshold resummation for the Higgs cross-section, which also showed that heavy m_t limit is in fact valid for $m_H \gg m_t$. Threshold resummation is relevant even for light Higgs at the LHC, because the gluon flux distribution will produce most Higgs near partonic threshold. This study also noticed the importance of $\ln^i N/N$ terms, of purely collinear origin.

Its exact corrections have now been computed, in the large m_t limit, to NNLO by various groups [38–40]. The threshold-resummed cross-section is similar in form to eq. (42), and all functions $g_{1,2,3}$ are known [41]. In ref. [41] the full NNLL threshold-resummed Higgs cross-section, matched carefully to NNLO, and including the $\ln^i N/N$ terms was constructed and uncertainties estimated. Results are shown in figure 6 for the K factor, defined with respect to the LO cross-section at a fixed scale.

The resummed cross-section has smaller scale uncertainty than the fixed order one, as indeed should happen generically [42]. The NNLO corrections and the threshold resummation together show that the perturbative series for the Higgs production cross-section is considerably better-behaved than at first feared.

6. Recoil resummation in Higgs production

Recoil resummation attempts to predict the *shape* of the Higgs boson transverse momentum spectrum. A number of studies [25,43–48] have now been performed for this observable. In the 2003 Les Houches workshop [49] a comparison was made

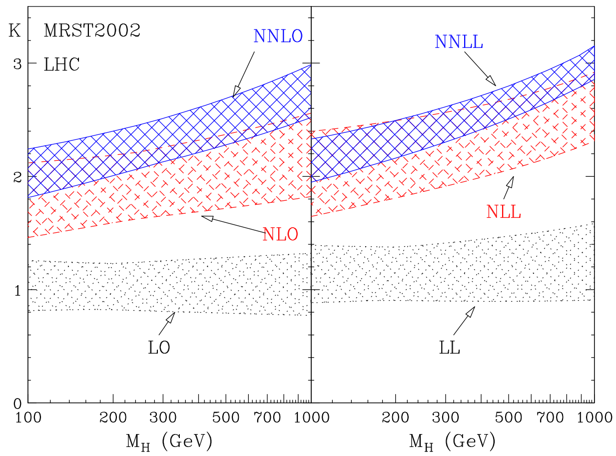

Figure 6. *K*-factor for Higgs production at the LHC at various orders.

between the various calculations, including the comparison of the spectra produced by PYTHIA [50] and HERWIG [51]. The results of the various studies are shown in figure 7.

We see that the analytic resummations, the most accurate (NNLL) one of which is that of [46], followed by that of [52], are quite consistent with each other. Also the shape from HERWIG's prediction is consistent with the analytic shapes, while that of PYTHIA is considerably softer. For a more complete discussion, see the relevant contributions in ref. [49].

All curves for the analytic resummations are based on versions of the recoil-resummed formula (40), except the one labelled Kulesza *et al*, which is based on the joint resummation formula (41). The joint-resummed shape is slightly softer that the recoil shapes.

7. Joint resummation for *W* and *Z* production

The authors of ref. [26] also derived the recoil spectrum for the W and Z bosons at the Tevatron, performing not only the N and but also the b integral with minimal prescription [12], vitiating the need for any non-perturbative input. As the dash–dotted line in figure 8 shows, the result is in strikingly good agreement with CDF (and D0) data.

The agreement is even slightly improved by nevertheless including a small non-perturbative component (solid line).

8. Threshold resummation in prompt photon production

In recent years it has become clear [53] that prompt photon production, which takes place via the LO subprocesses in figure 9 is not such a good probe of the gluon density as some of these diagrams suggest; NLO calculations do not describe the

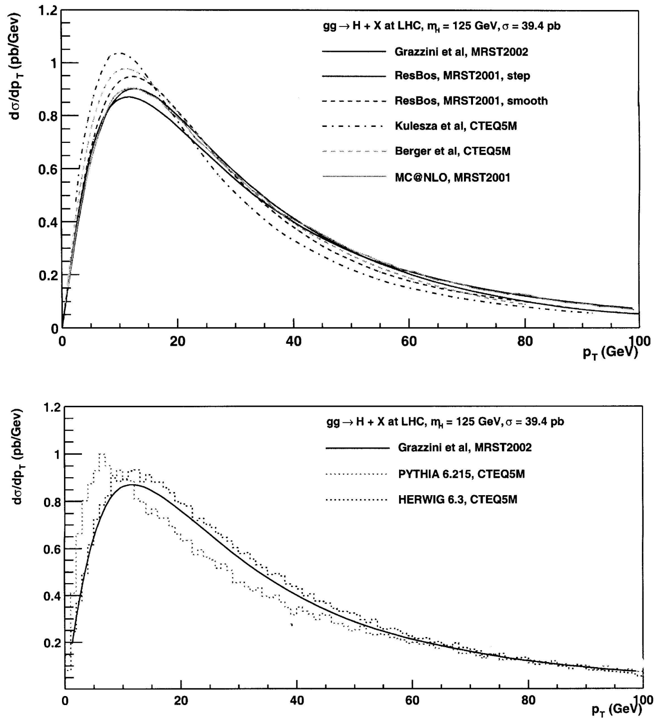

Figure 7. Predictions for the production of a 125 GeV Higgs boson at the LHC, all normalized to the same cross-section.

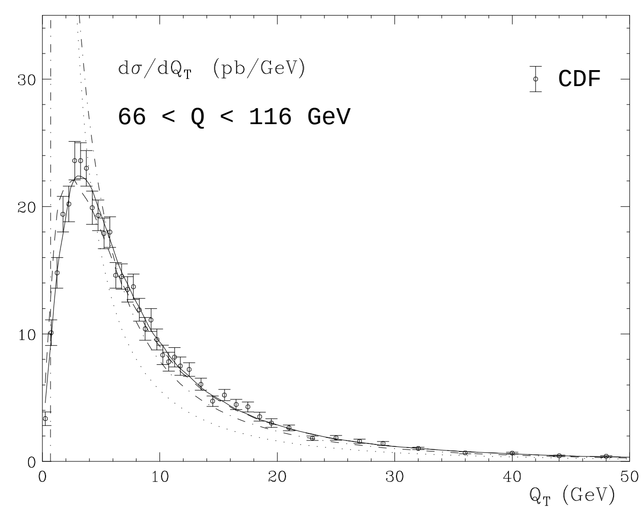

Figure 8. Recoil spectrum from joint resummation for W, Z production at the Tevatron.

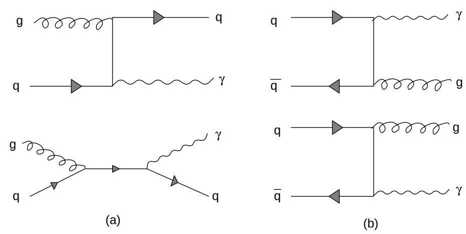

Figure 9. Prompt photon production via the LO subprocesses.

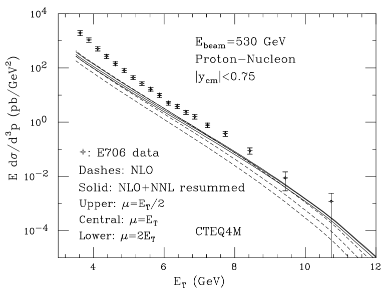

Figure 10. Prompt photon production in fixed target data. Comparison of NLO and threshold-resummed with E706 data.

collective world data well. One might attempt threshold resummation as a possible cure. This is slightly more involved for this process, having $2 \to 2$ kinematics at leading order, unlike the $2 \to 1$ cases we discussed so far. Threshold resummation was carried [54,55] out for the distribution $\mathrm{d}\sigma/\mathrm{d}p_{\mathrm{T}}^{\gamma}$, yielding what is shown in figure 10.

The threshold-resummation for the double-differential distribution [56,57] allows the computation of the photon pseudorapidity distribution in a restricted p_T^{γ} range, see figure 11.

We see that corrections are non-too-large, and that scale dependence is again reduced. Threshold resummation has thus served to sharpen the issue, rather than solve it. A possible cure within the framework of collinear factorization might involve joint resummation [12].

9. Threshold resummation in heavy quark production

Another observable based on $2 \to 2$ kinematics for which threshold resummation has been studied extensively is the inclusive top quark cross-section at the Tevatron. For threshold resummation of this observable it is sufficient to define threshold as $s = 4m^2$, with m the top mass, although other choices are possible as well. The NLL form of the resummed cross-section is an inverse Laplace transform

$$\sigma_{ij}(w) = \int_C \mathrm{d}N \exp(w\,N) \exp\left[\ln N g_1(\alpha_{\mathrm{s}} \ln N) + g_2(\alpha_{\mathrm{s}} \ln N)\right],$$

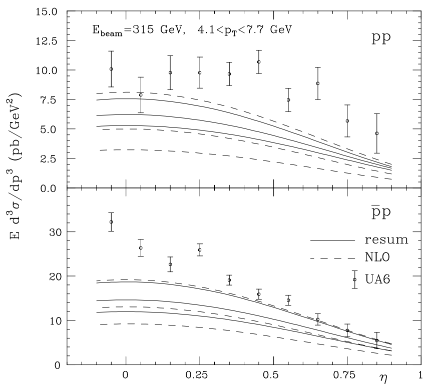

Figure 11. Prompt photon production in fixed target data. Comparison of NLO and threshold-resummed with UA6 data.

where ij stands for either the $q\bar{q}$ production subprocess (dominant at the Tevatron) or the gg production subprocess. While earlier studies [30,58,59] gave progressive understanding about how to threshold-resum this observable, the lack of proper understanding of the soft radiation, included in g_2, prevented reaching NLL accuracy. In refs [60,61] it was shown that the resummation of the soft function and the effects of coherent soft gluon emission from multiple underlying color antennas it contained could be controlled by renormalization group and anomalous dimensions for special composite operators constructed out of Wilson lines, as in §2. For the inclusive NLL threshold-resummed top quark cross-section this was implemented in ref. [62] and applied in a recent study providing the best estimates for size and uncertainties of the top quark production cross-section [63].

There is another use of resummed formulae: providing a controlled estimate of uncalculated higher orders by expansion. We consider the threshold-resummed double-differential cross-section for top quark pair production, which requires the threshold $s + t + u = 0$. Using the formalism of ref. [56], the result has the form [64]

$$\frac{\mathrm{d}^2\sigma_{ij}(w, A, B)}{\mathrm{d}A\,\mathrm{d}B} = \int_C \mathrm{d}N \exp(w\,N) \times \exp\left[\ln N g_1(\alpha_\mathrm{s} \ln N, A, B) + g_2(\alpha_\mathrm{s} \ln N, A, B)\right] \tag{49}$$

with A, B defined either in single-particle inclusive kinematics $A, B = t, u$, or in pair invariant mass kinematics $A, B = M_{t\bar{t}}, \cos\theta$ with θ the c.m. scattering angle. The estimate now proceeds in the following straightforward way: (i) expand to NLO and NNLO with appropriate matching, (ii) use NLO to judge the quality of the NNLO estimate. The result is a set of formulae for all NNLO corrections of order $\alpha_\mathrm{s}^2 \ln^i w$ with $i = 4, 3, 2$.

These formulae have been used to estimate NNLO corrections to the inclusive cross-section [64] (after integration over A, B) for both top quark production at the Tevatron and b quark production for the HERA-B experiment, as well as differential cross-sections [65]. Even though these are only estimates, they contain enough higher order information for a significant reduction of scale dependence.

10. Resummation of constants in Drell–Yan

In these last two sections we turn the readers to a few recent developments in the methodology of resummation.

Here we discuss the extension of the class of terms that can be resummed from logarithms to constants, as represented by the terms labelled '1' in eq. (1). The resummations of such terms, once put on a theoretically sound basis, would have significant phenomenological consequences [37].

The first evidence that the dominant non-logarithmic perturbative contributions could be exponentiated goes back to [66], where it was shown that the partonic Drell–Yan cross-section in the DIS factorization scheme contains the ratio of the time-like to the space-like Sudakov form factor: large perturbative contributions are left over in the exponentiated form of this ratio after the cancellation of IR divergences. This observation was made more precise in ref. [9]. There, the resummation of threshold logarithms for the Drell–Yan process was proven to all logarithmic orders, making use of refactorization: the Mellin transform of the cross-section is expressed near threshold, approached by letting the Mellin variable N grow very large, as a product of functions, each organizing a class of infrared and collinear enhancements; the refactorization is valid up to corrections which are *suppressed* by powers of N at large N, so that terms independent of N (constants) can also be treated by the methods used to resum logarithms of N. In ref. [67] the results of refs [9,68] were exploited to show that for processes which are electroweak at tree level the exponentiation of N independent terms is in fact complete.

For the DIS scheme, defined by using for ϕ in eq. (27) the full DIS non-singlet structure function, one finds

$$\sigma_{\mathrm{DIS}}(N) = \left|\frac{\Gamma(Q^2,\epsilon)}{\Gamma(-Q^2,\epsilon)}\right|^2 \exp\left[\mathcal{F}_{\mathrm{DIS}}(\alpha_{\mathrm{s}})\right] \exp\left[\int_0^1 dz\, \frac{z^{N-1}-1}{1-z} \right. \\ \times\left\{2\int_{(1-z)Q^2}^{(1-z)^2Q^2} \frac{\mathrm{d}\xi^2}{\xi^2} A\left(\alpha_{\mathrm{s}}(\xi^2)\right) - 2B\left(\alpha_{\mathrm{s}}\left((1-z)Q^2\right)\right) \right. \\ \left.\left. +D\left(\alpha_{\mathrm{s}}\left((1-z)^2Q^2\right)\right)\right\}\right]. \tag{50}$$

Similarly, for the $\overline{\mathrm{MS}}$ scheme one has the expression

$$\sigma_{\overline{\mathrm{MS}}}(N) = \left|\frac{\Gamma(Q^2,\epsilon)}{\Gamma(-Q^2,\epsilon)}\right|^2 \left(\frac{\Gamma(-Q^2,\epsilon)}{\phi_v(Q^2,\epsilon)}\right)^2 \exp\left[\mathcal{F}_{\overline{\mathrm{MS}}}(\alpha_{\mathrm{s}})\right] \\ \times\exp\left[\int_0^1 \mathrm{d}z\, \frac{z^{N-1}-1}{1-z}\left\{2\int_{Q^2}^{(1-z)^2Q^2} \frac{\mathrm{d}\xi^2}{\xi^2} A\left(\alpha_{\mathrm{s}}(\xi^2)\right) \right.\right. \\ \left.\left. +D\left(\alpha_{\mathrm{s}}\left((1-z)^2Q^2\right)\right)\right\}\right]. \tag{51}$$

All the functions appearing in eqs (50) and eqs (51) can be explicitly evaluated at two loops by matching with the complete two-loop calculation of ref. [29]. The refactorization analysis at the level of constants offers however an alternative, and often simpler method to determine the two-loop coefficients in these expressions by using the fact that real and virtual contributions in the factorized expressions can be made separately finite.

It should be pointed out that the exponentiation of N independent terms does not have the predictive power of the standard resummation of threshold logarithms. In that case, typically, an entire tower of logarithms can be exactly predicted to all orders by performing just a low order calculation. Here, on the other hand, functions such as $\mathcal{F}_{\mathrm{DIS}}$ and $\mathcal{F}_{\overline{\mathrm{MS}}}$ receive new non-trivial contribution at each perturbative order. The exponentiation pattern is nonetheless non-trivial, new equations emerge, and higher order terms predicted by the exponentiation can be considered representative of the size of the complete higher order correction [67].

11. Automated resummation

A novel approach to resummation is one that removes the theorist from the game [69], an approach at present mostly directed at the resummation of event shape variables.

Event shapes and jet resolution parameters (final-state 'observables') measure the extent to which the energy flow of the final state departs from that of a Born event. Their study has been fundamental for measurements of the strong coupling [70,71] as well as the QCD color factors [72]; final states also provide valuable information on the yet poorly understood transition from parton to hadron level (see [73] for a recent review). In the region where an observable's value v (e.g. $1-T$, where T is the thrust) is small, one should resum logarithmically enhanced contributions that arise at all orders in the perturbative series. For a number of observables such a resummation has been carried out manually at next-to-leading logarithmic (NLL) accuracy [74]. But achieving NLL accuracy requires a detailed analysis of the observable's properties, and is often technically involved. Recently a new approach was proposed [69] based on a general NLL resummed master formula valid for a large class of final-state observables.

Under not-overly restrictive conditions, the NLL resummation for the observable's distribution (the probability $\Sigma(v)$ that the observable's value is less than v) for a fixed Born configuration is given by the 'master' formula [69]:

$$\Sigma(v) = \mathrm{e}^{-R(v)}\mathcal{F}(R'(v)), \quad R'(v) = -v\frac{\mathrm{d}R(v)}{\mathrm{d}v}. \tag{52}$$

The function $R(v)$ is a *Sudakov* exponent that contains all leading (double) logarithms and all NLL (single-log) terms that can be taken into account by exponentiating the contribution to $\Sigma(v)$ from a single emission. This function depends on a small number of parameters [69].

The advantage of having introduced a master formula is that the resummation of the observable can be performed entirely automatically. The master formula and applicability conditions are encoded in a computer program (CAESAR: computer automated expert semi-analytical resummer), which gives only the observable's definition in the form of a computer routine, returns the observable's distribution $\Sigma(v)$ at NLL accuracy (where possible). More details can be found in refs [49,69].

12. Other recent developments

Various other recent developments should be mentioned. In ref. [75] the formalism for threshold resummation was extended, at NLL accuracy, beyond $2 \to 2$ kinematics to an arbitrary number of partons involved in the hard scattering.

Important new classes of logarithms, so-called non-global logarithms, entering at the NLL level, were identified [76] in observables that are defined to be sensitive to only radiation in only parts of phase space.

A Lagrangian for near-elastic dynamics, soft-collinear effective theory, was formulated [77], and used to good effect, mostly in the context of B-decays.

13. Conclusions

In this by no means complete or particularly thorough review, I have attempted to provide some insight into some of the methods used to resum prominent classes of large terms for observables at TeV hadron colliders, as well as to give an impression of a number of recent influential studies in this field. I hope the reader comes away with the impression that our understanding of the organizational structure of high order logarithmic terms in perturbative series has advanced to a high level, but that there is still wide room for improvement of existing resummations, ample opportunity for extensions to new and relevant classes of terms and new observables, and an excellent case for serious use of resummation as a tool for precise phenomenology.

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