A. Mueller

1. Basics of QCD

1.1 The Lagrangian

$$\mathcal{L} = \bar{q}^{a}_{f}(i\gamma\cdot\partial - m_f)q^{a}_{f} - \frac{1}{4}F^{i}_{\mu\nu}F^{i}_{\mu\nu} - g\,\bar{q}_f\,\gamma\cdot A\,q_f$$

$a=1,2,3$ $\quad q^{af}_{\alpha}(x)$ — $f = u, d, s, c, b, t$

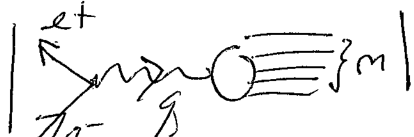
α = Dirac index $\alpha = 1,2,3,4$

$$A_\mu = \frac{\lambda^i}{2}A^i_\mu \qquad \left[\frac{\lambda_i}{2}, \frac{\lambda_j}{2}\right] = i f_{ijk}\frac{\lambda_k}{2}$$

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g f_{ijk} A^j_\mu A^k_\nu$$

$\frac{g^2}{4\pi} = \alpha$ Renormalization gives α_μ

1.2 Example $e^+ + e^- \to$ hadrons; running coupling

$$\sigma_{e^+e^-\to had} = \sum_m \Big|$$  $$\Big|^2$$

Convenient to use dimensionless quantity $R = \frac{\sigma_{e^+e^-\to had}}{\sigma_{e^+e^-\to\mu^+\mu^-}}$

Take $m_f = 0$, then

$$R = R(Q^2/\mu^2, \alpha_\mu) \qquad \mu \text{ necessary for renormalization}$$

μ not physical so

$$\mu^2\frac{d}{d\mu^2}R = 0 = \mu^2\frac{\partial}{\partial\mu^2}R + \underbrace{\mu^2\frac{\partial}{\partial\mu^2}\alpha_\mu}_{\beta(\alpha_\mu)}\frac{\partial}{\partial\alpha_\mu}R$$

$$\left(\mu^2\frac{\partial}{\partial\mu^2} + \beta\frac{\partial}{\partial\alpha_\mu}\right)R(Q^2/\mu^2, \alpha_\mu) = 0 = \left(Q^2\frac{\partial}{\partial Q^2} - \beta\frac{\partial}{\partial\alpha_\mu}\right)R$$

Renormalization group equation for physical, dimensionless, quantity.

In QCD $\beta(\alpha) = -b\alpha^2$ with $b = \frac{11N_c - 2N_f}{12\pi}$

To solve for R introduce running coupling

$\alpha(Q^2) = \alpha(Q^2/\mu^2; \alpha_\mu)$ such that

(i) $\alpha(Q^2)$ obeys Ren. group eq.

(ii) $\alpha(\mu^2) = \alpha_\mu$

Define $\rho(\alpha_\mu) = \int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\beta(\alpha')}$ then,

$$\underline{\alpha(Q^2) = \rho^{-1}[\ln Q^2/\mu^2 + \rho(\alpha_\mu)]}$$

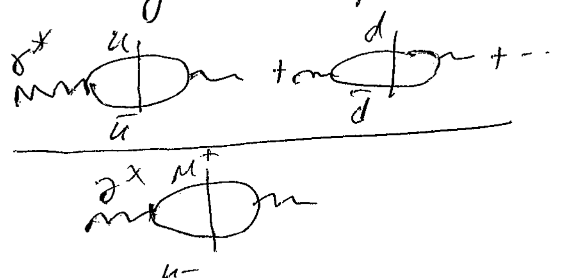
Leading order $\rho(\alpha_\mu) = -\frac{1}{b}\int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\alpha'^2} = \frac{1}{b}\left[\frac{1}{\alpha_\mu} - \frac{1}{\alpha_0}\right]$

Then $\rho^{-1}[z] = \frac{1}{bz + 1/\alpha_0}$ and

$$\alpha(Q^2) = \frac{1}{b\left(\ln Q^2/\mu^2 + \frac{1}{b}\left(\frac{1}{\alpha_\mu} - \frac{1}{\alpha_0}\right)\right) + 1/\alpha_0} = \underline{\frac{\alpha_\mu}{1 + b\alpha_\mu \ln Q^2/\mu^2}}$$

$$\alpha(Q^2) \xrightarrow[Q^2\text{ large}]{} \frac{1}{b \ln Q^2/\mu^2}$$

$R_{Q^2\text{ large}} =$



$= 3\left[\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right]$

(colors, u, d, s, c, b)

$$R = \frac{11}{3}\left[1 + \frac{\alpha(Q^2)}{\pi} + \cdots\right]$$

1.3 The Λ-parameter (Problems)

Problem 1 (E) Suppose $m_f = 0$. Show that if M = proton mass then $\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta \frac{\partial}{\partial \alpha_\mu}\right) M = 0$ implies

$$M = c\,\mu\, e^{-\frac{1}{2}\int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\beta(\alpha')}}$$

where c is independent of μ and α_μ and is such that M has no α_0 dependence.

In general we may trade α_μ dependence for Λ-dependence

$$\alpha(Q^2) = \frac{1}{1/\alpha_\mu + b \ln Q^2/\mu^2} = \frac{1}{b \ln Q^2/\Lambda^2}.$$

Problem 2 (E) Show that Λ, as defined above, satisfies the renormalization group equation when $\beta = -b\alpha^2$.

Problem 3 (H) This is a problem dealing with schemes. If we have defined α_μ in some way say in $\overline{MS}$, we are said to have defined a scheme. We can define a second scheme $\bar{\alpha}_\mu$ in terms of α_μ by

$$\bar{\alpha}_\mu = \alpha_\mu + c_2 \alpha_\mu^2 + c_3 \alpha_\mu^3 + \cdots$$

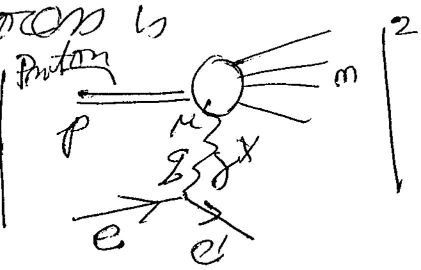
Use $\bar{\beta} = \mu^2 \frac{\partial}{\partial \mu^2} \bar{\alpha}_\mu = -\bar{b}\bar{\alpha}^2 - \bar{b}_1 \bar{\alpha}_1^3 + \cdots$ and $\beta = -b\alpha^2 - b_1\alpha^3 + \cdots$ as well as the above equation relating $\bar{\alpha}_\mu$ to α_μ to show that $b = \bar{b}$ and $b_1 = \bar{b}_1$. Can you argue, but not give a real proof, that one can choose the c_i such that $\bar{b}_i = 0$ for $i \geq 2$

Suppose we define $\Lambda^2 = \mu^2 e^{-\int_{\alpha_0}^{\alpha_\mu} d\alpha'/\beta(\alpha')}$ with α_0 chosen so that $\Lambda^2 \xrightarrow[\alpha_\mu \to 0]{} \mu^2 e^{-\frac{1}{b\alpha_\mu}} (\alpha_\mu)^{-b_1/b^2} (1 + O(\alpha_\mu))$ and similarly for $\bar{\Lambda}^2$. Can you show that $\bar{\Lambda}^2 = \Lambda^2 e^{c_2/b}$ is an exact result.

2: Deep inelastic scattering; the parton model

2.1 DIS

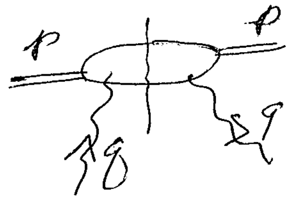
The process is

$$\sum_n \left| \right.$$  $$\left.\right|^2 = L_{\mu\nu}\, W'_{\mu\nu}$$

lepton tensor

$$W'_{\mu\nu} = \frac{4\pi^2 E_p}{m} \sum_n (n|j_\nu(0)|p)^* (n|j_\mu(0)|p)\,(2\pi)^4\delta^4(q+p-p_n)$$

It is convenient to define

$$W_{\mu\nu}(p,q) = \frac{4\pi^2 E_p}{m}\int d^4x\,(p|j_\nu(x)\,j_\mu(0)|p)\,e^{iqx} =$$ 

Problem 4 (E) | Show $W_{\mu\nu} = W'_{\mu\nu}$

Problem 5 (M-H) | Show that

$$W_{\mu\nu} = \frac{4\pi^2 E_p}{m}\, 2\,\mathrm{Im}\, i \underbrace{\int d^4x\, e^{iqx}\,(p|T(j_\nu(x)\,j_\mu(0))|p)}_{\text{Given by Feynman graphs}}$$

2.2 The parton picture

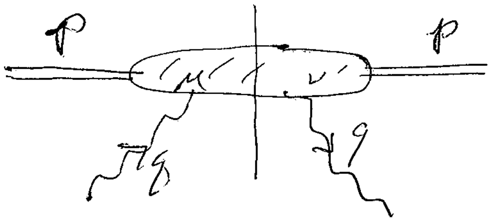
Use the Bjorken frame $p = (p + \frac{m^2}{2p}, 0, 0, p)$ p very large

$q = (q_0, q, 0)$ $\quad q_\mu q_\mu = -Q^2$

$$x = \frac{Q^2}{2p\cdot q} = \frac{q^2}{2pq_0}$$

Write $W_{\mu\nu}(p,q) = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)W_1 + \frac{W_2}{m^2}\left[p_\mu p_\nu - \frac{p\cdot q}{q^2}(p_\mu q_\nu + p_\nu q_\mu) + \frac{(p\cdot q)^2}{(q^2)^2} q_\mu q_\nu\right]$

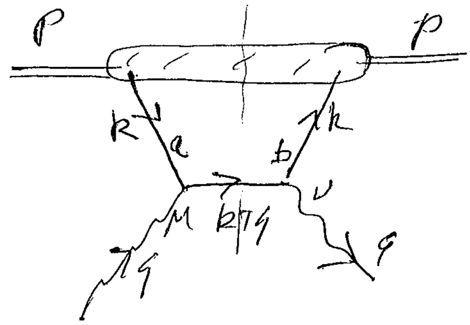
Problem 6 (E) | Use $W_{\mu\nu} = W_{\nu\mu}$ and $q_\mu W_{\mu\nu}(p,q) = 0$ to get above.



$=$

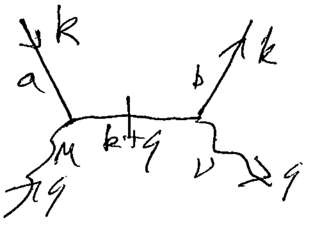
Coulomb gauge

light-cone gauge, $A_+ = 0$



$$v_\pm = \frac{1}{\sqrt{2}}(q_0 \pm q_3)$$

p_+ large, p_- very small



$$= 2\pi\,\underbrace{\delta((k+q)^2)}\,\underbrace{(\gamma_\nu\,\gamma\cdot(q+k)\,\gamma_\mu)_{ba}}\,e_f^2$$

$$\delta(2k_+q_- = Q^2) \qquad q_-(\gamma_\nu\gamma_+\gamma_\mu)_{ba}$$

$$\delta\left(\frac{k_+}{p_+}2p_+q_- - Q^2\right)$$

$$W_{\mu\nu} = \underbrace{\frac{(2\pi)^3E_p}{m}\frac{xq_-}{Q^2}}_{\frac{(2\pi)^3E_p}{2mp_+}}\;\delta\left(\frac{k_+}{p_+}-x\right)\sum_f e_f^2\int d^4k\,\delta\left(x-\frac{k_+}{p_+}\right)A^b_{a}(p,k)(\gamma_\nu\gamma_+\gamma_\mu)_{ba}$$

Take $\mu\nu = ij$ $\qquad W_{ij} = -g_{ij}W_1 + \frac{q_iq_j}{q^2}\left[W_1 + \frac{(p\cdot q)^2}{m^2q^2}W_2\right]$

$$(\gamma_i\gamma_+\gamma_j)_{ba} \longrightarrow \frac{1}{2}(\gamma_i\gamma_+\gamma_j + \gamma_j\gamma_+\gamma_i) = -\frac{1}{2}\{\gamma_i,\gamma_j\}(\gamma_+)_{ba} = -g_{ij}(\gamma_+)_{ba}$$

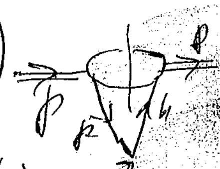
$$\Rightarrow \quad W_1 + \frac{(p\cdot q)^2}{m^2q^2}W_2 = 0 \qquad p\cdot q = m\nu$$

$$W_1 + \overbrace{\frac{2m\nu}{q^2}}^{-1/x}\frac{\nu W_2}{2m} \qquad \boxed{2mxW_1 = \nu W_2} \text{ Callan-Gross relation}$$

$$\nu W_2 = \frac{(2\pi)^3E_p}{p_+}\int d^4k\sum_f A^b_{ba}(p,k)(\gamma_+)_{ba}\,\delta(x - k_+/p_+) \times e_f^2$$

Write $\nu W_2 = \sum_f e_f^2\left[x\,q_f(x,Q^2) + x\,\bar{q}_f(x,Q^2)\right]$

$$q_f(x,Q^2) + \bar{q}_f(x,Q^2) = \frac{(2\pi)^3E_p}{p_+}\int d^4k\,A^b_{ab}(p,k)(\gamma_+)_{ba}\,\delta(x - k_+/p_+) \Rightarrow$$



$\gamma_+\delta(x - \frac{k_+}{p_+})$

Problem 7 (M-H) | Show that

$$\int_0^1 dx\, x^{n-1}\left[q_f(x,Q^2)+\bar{q}_f(x,Q^2)\right] = \frac{(2\pi)^3 F_P}{P_+^{\,n}}\langle p|\overbrace{\bar{q}_f(x)\gamma_{\mu_1}\, i\partial_{\mu_2}\cdots i\partial_{\mu_n}\, q_f(x)}^{\mathcal{O}_{\mu_1\cdots\mu_n}}|p\rangle\Big|_{\mu_i=+}$$

$i\partial_{\mu_2}\to iD_{\mu_2}$, $\quad i\partial_{\mu_n}\to iD_{\mu_n}$

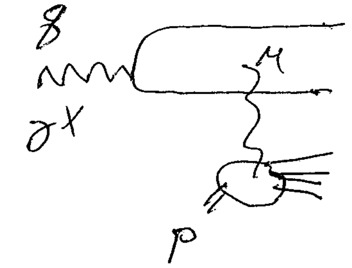
$D_\mu=\partial_\mu+igA_\mu$

scale of renormalization

2.3 The Dipole picture of DIS

In case x is small there is another frame, and picture, of DIS. $p=(m,0,0,0)$

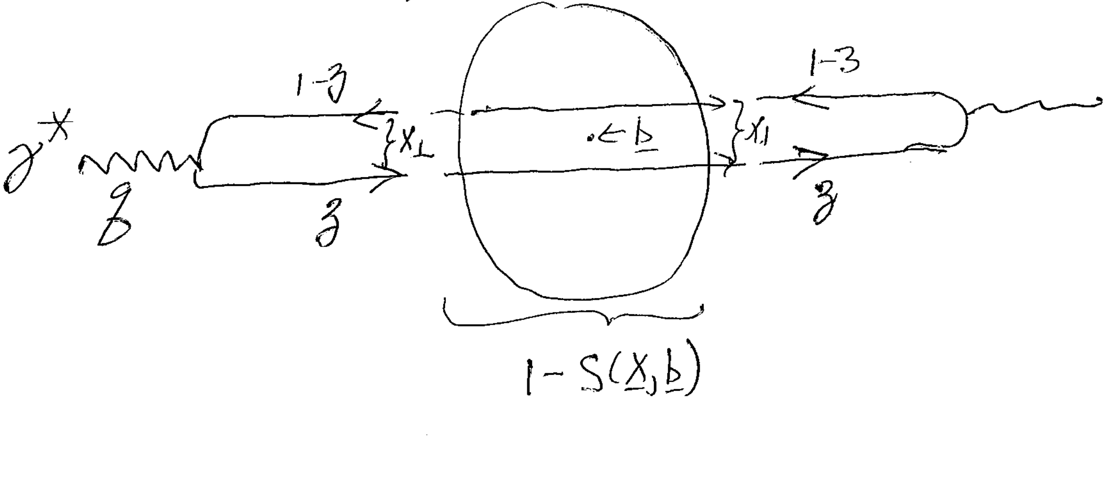
$$q=(q_0,0,0,q)=\left(q-\frac{Q^2}{2q},0,0,q\right)$$

$W_{\mu\nu}=$  $\times\,(CC)_{\mu\to\nu}$; γ^* off shell

$$-E_{\gamma^*}+E_\gamma=\frac{Q^2}{2q}\Rightarrow \tau_{\gamma^*}=\frac{2qm}{Q^2}\frac{1}{m}=\frac{1}{mx}$$

now

$$x\,q_f(x,Q^2)+x\,\bar{q}_f(x,Q^2)=\frac{Q^2}{\alpha_{em}e_f^2}\sum_\lambda\int d^2b\int\frac{d^2x}{4\pi^2}\int_0^1 dz\,|\Psi^f_{T\lambda}(x,z)|^2\,\mathrm{Re}\left(1-S(x,b)\right)$$



3. Perturbation Theory

3.1 Interaction picture; "old-fashioned" perturbation theory

$$\mathcal{L} = \underbrace{\tfrac{1}{2}\partial_\mu\phi\,\partial_\mu\phi - \tfrac{1}{2}m^2\phi^2}_{\mathcal{L}_0} - \underbrace{\tfrac{\lambda}{4}\phi^4}_{\mathcal{L}_I = \mathcal{H}_I}$$

Schrödinger picture $\quad |\psi_s(t)\rangle = e^{-iHt}|\psi\rangle, \quad \phi(\vec{x})$

Heisenberg $\quad |\psi\rangle, \quad \phi(\vec{x},t) = e^{iHt}\phi(\vec{x})e^{-iHt}$

Interaction $\quad |\psi_I(t)\rangle = e^{+iH_0t}|\psi_s(t)\rangle$

$$\phi_I(\vec{x},t) = e^{iH_0t}\phi(\vec{x})e^{-iH_0t}$$

Problem 8(E) | Show

$$\langle\psi_I(t)|\mathcal{O}_I(t)|\psi_I(t)\rangle = \langle\psi_s(t)|\mathcal{O}(0)|\psi_s(t)\rangle = \langle\psi|\mathcal{O}(t)|\psi\rangle$$

Get formula for $|\psi_{\vec{p}}\rangle$ where $H|\psi_{\vec{p}}\rangle = E_p|\psi_{\vec{p}}\rangle$

$$|\psi_{\vec{p}}(t)\rangle_I = e^{iH_0t}e^{-iHt}|\psi_{\vec{p}}\rangle$$

$$i\frac{\partial}{\partial t}|\psi_{\vec{p}}(t)\rangle_I = e^{iH_0t}(H-H_0)e^{-iHt}|\psi_{\vec{p}}\rangle = \mathcal{H}_I(t)|\psi_{\vec{p}}(t)\rangle_I$$

$$|\psi_{\vec{p}}(t)\rangle_I = \underbrace{|\psi_{\vec{p}}(-\infty)\rangle_I}_{|\vec{p}\rangle} - i\int_{-\infty}^{t} dt'\,\mathcal{H}_I(t')|\psi_{\vec{p}}(t')\rangle_I\, e^{\epsilon t'}$$

$$\boxed{|\psi_{\vec{p}}\rangle = |\vec{p}\rangle - i\int_{-\infty}^{0} dt'\, e^{\epsilon t'}\mathcal{H}_I(t')|\psi_{\vec{p}}(t')\rangle_I}$$

3.2 Lightcone time

Real time

$t\,,\quad \vec{x}$

$E_{\vec{p}}\,,\quad \vec{p}$

$H|\vec{p}\rangle = E_{\vec{p}}|\vec{p}\rangle$

Lightcone time

$x_+ = \frac{x_0+z}{\sqrt{2}} = \tau\,,\quad x_- = \frac{x_0-z}{\sqrt{2}}\,,\quad \underline{x} = (x,y)$

$P_- = \frac{\underline{p}^2}{2p_+}\,,\quad p_+\ \underline{p}$

$P_-|\underline{p}, p_+\rangle = \frac{\underline{p}^2+m^2}{2p_+}|\underline{p}, p_+\rangle$

3.3 Lightcone gauge

Scalar field propagator $G(k) = \frac{i}{k^2-m^2+i\epsilon}$

Covariant field vector propagator $G_{\mu\nu}(k) = \frac{-i}{k^2+i\epsilon}\left(g_{\mu\nu} - \xi\frac{k_\mu k_\nu}{k^2}\right)$

Lightcone gauge propagator

$$G_{\mu\nu}(k,\eta) = \frac{-i}{k^2+i\epsilon}\left[g_{\mu\nu} - \frac{\eta_\mu k_\nu + \eta_\nu k_\mu}{\eta\cdot k}\right]$$

$\eta\cdot v = v_+$

$\eta_\mu G_{\mu\nu} = 0$

$\Rightarrow A_+ = 0$

Call $\bar{k}_- = \underline{k}^2/2k_+$

Causal — Instantaneous

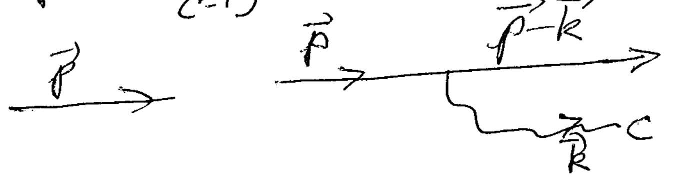
$$G_{\mu\nu} = \frac{-i}{\underbrace{2k_+\left(k_- - \frac{\underline{k}^2}{2k_+} + i\epsilon\right)}_{k_- - \bar{k}_- + i\epsilon}}\left[\underbrace{g_{\mu\nu} - \frac{\eta_\mu \bar{k}_\nu + \eta_\nu \bar{k}_\mu}{k_+}}_{-\sum_{\lambda=\pm}\epsilon_\mu^{(\lambda)}(k)\,\epsilon_\nu^{(\lambda)*}(k)} - \frac{2\,\eta_\mu\eta_\nu(k_- - \bar{k}_-)}{k_+}\right]$$

$$\epsilon_\mu^\lambda(k) = \left(\overset{+}{0}, \frac{\underline{\epsilon}^\lambda\cdot\underline{k}}{k_+}, \underline{\epsilon}^\lambda\right)$$

$$G_{\mu\nu} = \frac{i}{k^2+i\epsilon}\sum_\lambda \epsilon_\mu^\lambda(k)\,\epsilon_\nu^{\lambda*}(k) + i\frac{\eta_\mu\eta_\nu}{k_+^2}$$

3.4 Gluon distribution of (dressed) quark

$$|\Psi_{\vec p}\rangle = |\vec p\rangle + \sum_{c=1}^{N_c^2-1}\int |\vec p-\vec k, \vec k\,\lambda c\rangle\, \psi^c_{\lambda}(\vec k)\, d^3k$$



$$H_I = g\int d^2\underline{x}\,dx_-\ \bar q(x)\frac{\lambda_c}{2}\gamma_\mu q(x)\, A^c_\mu(x)$$

$$A^c_\mu(x) = \sum_{\lambda=\pm}\int \frac{d^2k\,dk_+}{\sqrt{(2\pi)^3\, 2k_+}}\left[\epsilon^\lambda_\mu(k)\, a^c_\lambda(k)\, e^{i\underline{k}\cdot\underline{x} - ik_+x_- - ik_-x_+} + c.c.\right]$$

$$\frac{(\vec p-\vec k', \vec k\,(\lambda,c)|\,H_I\,|p)}{\underbrace{p_-}_{p^2/2p_+} - \underbrace{(p-k)_-}_{(p-k)^2/2(p-k)_+} - \underbrace{k_-}_{k^2/2k_+} + i\epsilon} = \underbrace{\delta^3(k-k')}_{\delta(\underline{k}-\underline{k}')\delta(k_+-k'_+)}\,\psi^c_\lambda(k)$$

Problem 9(M-H) Show that in the limit $k_+/p_+ \ll 1$

$$\psi^c_\lambda(k) = \frac{\lambda^c}{2}\, 2g\, \frac{\epsilon^{(\lambda)*}\cdot \underline{k}}{\underline{k}^2}\,\frac{1}{\sqrt{(2\pi)^3\, 2k_+}}$$

Hint: use $q(x) = \int\frac{d^3p}{\sqrt{(2\pi)^3\, 2p_+}}\sum_{r=1}^{2}\left[u_r(p)\, b_r(\vec p)\, e^{i\underline{p}\cdot\underline{x} - ip_+x_- - ip_-x_+} + v_r(p)\, d^\dagger_r(p)\, e^{-i\underline{p}\cdot\underline{x} + ip_+x_- + ip_-x_+}\right]$

where $\bar u(p-k)\gamma_\mu u(p) \simeq 2p_+\, g_{\mu -}$

Problem 10(M) If one defines the gluon distribution of a dressed quark as

$$xG(x,Q^2) = \sum_{\lambda,c}\int \underbrace{d^3k}_{d^2k\,dk_+}\ \delta(x-k_+/p_+)\,\theta(Q^2-\underline{k}^2)\,(\Psi_p|\,a^{c\dagger}_\lambda(\vec k)\, a^c_\lambda(\vec k)\,|\Psi_p)$$

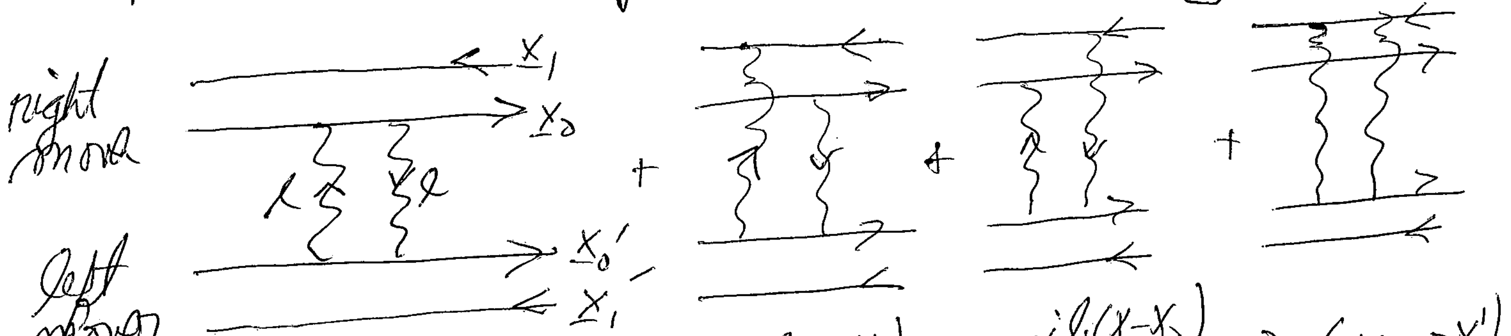
Show that $xG(x,Q^2) = \frac{\alpha C_F}{\pi}\ln Q^2/\mu^2$ where $\sum_{c=1}^{N_c^2-1}\frac{\lambda_c}{2}\frac{\lambda_c}{2} = \frac{N_c^2-1}{2N_c}\, I = C_F\, I$

and where μ is an infrared cutoff.

4. Dipole scattering, evolution, and the BK equation

Small-x DIS can be given in terms of dipole scattering on a target. Let's study dipole scattering and evolution in some detail.

4.1 Lowest order dipole-dipole scattering



Phase factors $\left(1 - e^{i\underline{l}\cdot(\underline{x}_1-\underline{x}_0)} - e^{-i\underline{l}\cdot(\underline{x}_1-\underline{x}_0)} + 1\right)\cdot(x \leftrightarrow x')$

Let $\underline{x}_{01} = \underline{x}_1 - \underline{x}_0$

Phase factor $\left(2 - e^{-i\underline{l}\cdot\underline{x}_{01}} - e^{i\underline{l}\cdot\underline{x}_{01}}\right)\left(2 - e^{-i\underline{l}\cdot\underline{x}_{01}'} - e^{i\underline{l}\cdot\underline{x}_{01}'}\right)$

Other factors (take large N_c)

$$\frac{d^2l}{(2\pi)^2(\underline{l}^2)^2}\underbrace{\left[\frac{1}{N_c}\mathrm{tr}\,T_aT_b\right]^2}_{\frac{N_c^2-1}{4N_c^2}}\frac{g^4}{(4\pi\alpha)^2}.\quad \text{So}$$

$$\sigma(x_{01},x_{01}') = \alpha^2\int\frac{d^2l}{(\underline{l}^2)^2}\left(2 - e^{-i\underline{l}\cdot\underline{x}_{01}} - e^{i\underline{l}\cdot\underline{x}_{01}}\right)\left(2 - e^{-i\underline{l}\cdot\underline{x}_{01}'} - e^{i\underline{l}\cdot\underline{x}_{01}'}\right)$$

Suppose $x_{01}' \gg x_{01}$. Then a log in $\left(\frac{1}{x_{01}'}\right)^2 < l^2 < \frac{1}{x_{01}^2}$

$$\sigma = \alpha^2\cdot\int_{1/x_{01}'^2}^{1/x_{01}^2}\frac{d^2l}{(\underline{l}^2)^2}\underbrace{\left(\underline{l}\cdot\underline{x}_{01}\right)^2}_{\frac{1}{2}l^2x_{01}^2}\cdot 2 \simeq 2\pi x_{01}^2\alpha^2\ln\frac{x_{01}'}{x_{01}}$$

More completely

$$\sigma(x_{01}, x_{01}') = 2\pi\alpha^2 x_<^2 (1 + \ln x_>/x_<)$$

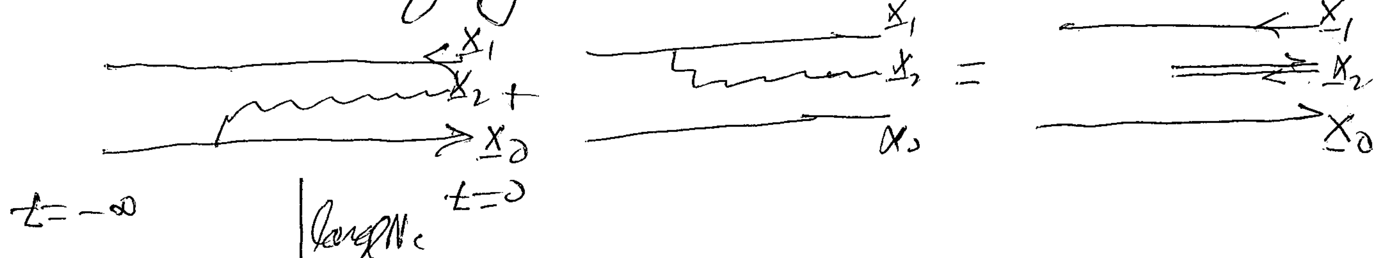
<u>Problem 11(H)</u> Show that, to logarithmic accuracy,

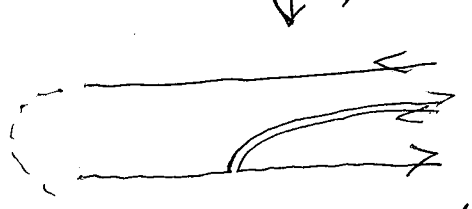
$$\sigma(x_{01}) = \frac{\alpha\pi^2 x_{01}^2}{N_c} \, xG(x, 1/x_{01}^2)$$

for the scattering of a dipole of size x_{01} on a quark. For the quark $xG(x, 1/x_{01}^2) = \frac{\alpha C_F}{\pi} \ln \frac{1}{\mu^2 x_{01}^2}$ in agreement with problem 10. Use the above formula for σ, which is a general formula, to determine the gluon distribution, xG_{dipole}, for a dipole of size x_{01}.

4.2 Dipole evolution, the BFKL and BK Equations

In the large N_c limit a dipole wavefunction can become a two dipole wavefunction with the emission of a gluon. (Put in color later)





We can write the dipole wavefunction as

$$|\Psi\rangle = \sqrt{Z}\int d^2x_0\, d^2x_1\, \Psi(x_0,x_1)\,|x_0,x_1\rangle + \sum_\lambda \int d^2x_0\, d^2x_1\, d^2x_2\, \Psi_\lambda(x_0,x_1,x_2,k_+)\,dk_+\,|x_0,x_1;x_2(k_+,\lambda)\rangle$$

$$\Psi_\lambda = \underbrace{\int \frac{d^2k}{2\pi}\left(e^{-ik\cdot(x_2-x_0)} - e^{-ik\cdot(x_2-x_1)}\right)}_{\text{FT to coordinate space}} \frac{-i}{k^2/2k_+}\,\frac{ig}{\sqrt{(2\pi)^3 2k_+}}\,\frac{\epsilon^{(\lambda)*}\cdot k}{k_+}$$

Use $\int \frac{d^2k}{2\pi}\, e^{ik\cdot x}\,\frac{k}{k^2} = i\, x/x^2$ to get

$$\Psi_\lambda(x_0,x_1,x_2,k_+) = \frac{ig}{\pi\sqrt{2\pi\, 2k_+}}\,\epsilon^{(\lambda)*}\cdot\left(\frac{x_0-x_2}{(x_0-x_2)^2} - \frac{(x_1-x_2)}{(x_1-x_2)^2}\right)$$

Problem 12(E-M) Show that

$$\sum_\lambda |\Psi_\lambda|^2 = \frac{\alpha}{\pi^2}\,\frac{1}{k_+}\,\frac{x_{01}^2}{x_{02}^2\, x_{12}^2}$$

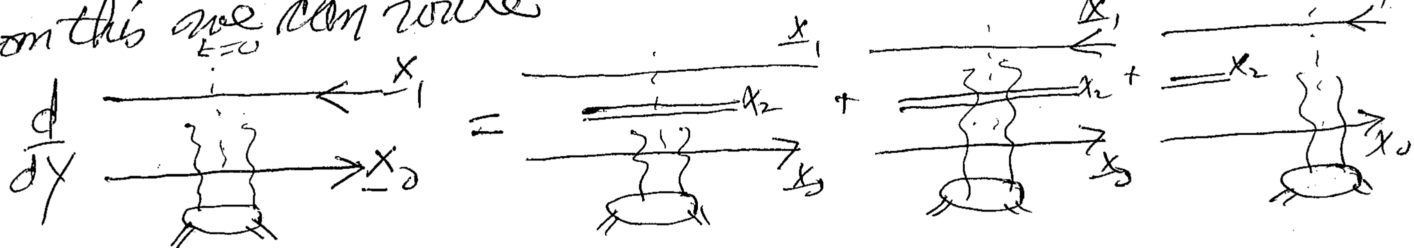
The color factor is $C_F = \frac{N_c}{2}$ at large N_c so

$$\sum_\lambda |\Psi_\lambda|^2 d^2x_2\, dk_+ = \frac{\alpha N_c}{2\pi^2}\,\frac{x_{01}^2\, d^2x_2}{x_{02}^2\, x_{12}^2}\,\underbrace{\frac{dk_+}{k_+}}_{dy} \qquad y = \ln k_+$$

and

$$Z = 1 - \frac{\alpha N_c}{2\pi^2}\int \frac{d^2x_2\, x_{01}^2}{x_{12}^2\, x_{02}^2}\, dy$$

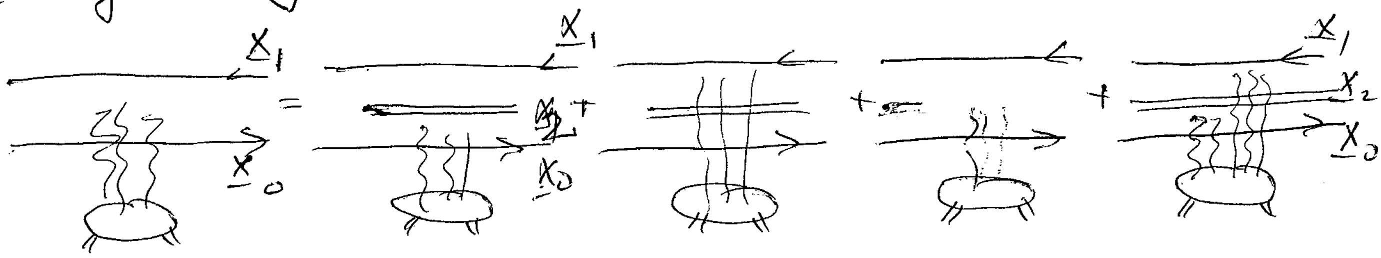
From this we can write



$$\boxed{\frac{dT(x_{01},Y)}{dY} = \frac{\alpha N_c}{2\pi^2}\int \frac{d^2x_2\, x_{01}^2}{x_{02}^2\, x_{12}^2}\left[T(x_{02},Y) + T(x_{12},Y) - T(x_{01},Y)\right]}$$

Dipole version of BFKL eq.

More generally



$$\frac{dT(x_{01},Y)}{dY} = \frac{\alpha N_c}{2\pi^2}\int \frac{d^2x_2\, x_{01}^2}{x_{02}^2\, x_{12}^2}\left[T(x_{02},Y) + T(x_{12},Y) - T(x_{01},Y) + \underbrace{T^{(2)}(x_0,x_1,x_2,Y)}_{\text{M field } T(x_{02},Y)\,T(x_{12},Y)}\right]$$

BK equation

4.3 Solution of the BFKL-Dipole equation

Consider the BFKL-Dipole eigenvalue equation

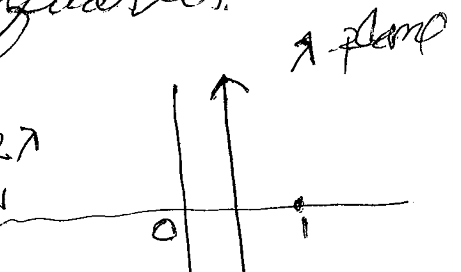
$$\frac{1}{4\pi}\int \frac{d^2x_2\, x_{01}^2}{x_{02}^2\, x_{12}^2}\left[x_{02}^{2\lambda} + x_{12}^{2\lambda} - x_{01}^{2\lambda}\right] = \chi(\lambda)\, x_{01}^{2\lambda}$$

Explicit calculation gives $\chi(\lambda) = \psi(1) - \frac{1}{2}\psi(\lambda) - \frac{1}{2}\psi(1-\lambda)$

$\boxed{\text{Problem 13(M-H)}}$ Show that $\chi(\lambda)$ has the correct pole at $\lambda = 0$, $1/2\lambda$, as required by the eigenvalue equation.

Now one can write

$$T(x_{01},Y) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\lambda}{2\pi i}\, T_\lambda(Y)\, x_{01}^{2\lambda}$$

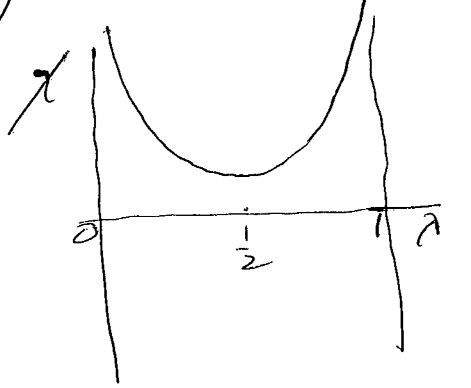


which gives

$$\frac{dT_\lambda}{dY} = \frac{2\alpha N_c}{\pi}\chi(\lambda)\, T_\lambda \Rightarrow T_\lambda(Y) = T_\lambda^{(0)}\, e^{\frac{2\alpha N_c}{\pi}\chi(\lambda) Y}$$

At large Y, $T(x_{01},Y)$ is determined by the saddle point in λ, at $\lambda = 1/2$,

$$\chi(\lambda) \simeq \underbrace{\chi(\tfrac{1}{2})}_{2\ln 2} + \frac{1}{2}\underbrace{\chi''(\tfrac{1}{2})}_{14\zeta(3)}\left(\lambda-\tfrac{1}{2}\right)^2$$



which gives

$$T(x_{01},Y) = T^{(0)}_{\frac{1}{2}}\,\frac{x_{01}\,e^{(\alpha_P-1)Y}}{\sqrt{56\alpha N_c\zeta(3)Y}}$$

$$\alpha_P - 1 = \frac{2\alpha N_c}{\pi}\chi(\tfrac{1}{2}) = \frac{4\alpha N_c}{\pi}\ln 2$$

If one scatters a dipole of size x_{01} on a dipole of size x'_{01},

$x'_{01} > x_{01}$

$$\sigma(x_{01},x'_{01}) = \pi\alpha^2 x_{01}^2\int\frac{d\lambda}{2\pi i}\left(\frac{x_{01}}{x'_{01}}\right)^{2(\lambda-1)}\frac{e^{\frac{2\alpha N_c}{\pi}\chi(\lambda)Y}}{\lambda^2(\lambda-1)^2}$$

$\boxed{\text{Problem 14(M)}}$ Show that

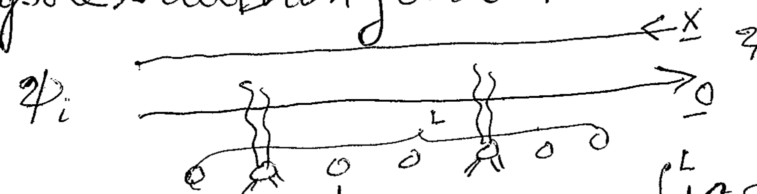
$$\sigma(x_{01},x'_{01},Y)\xrightarrow[Y\ \text{large}]{}\frac{8\pi\alpha^2 x_{01}x'_{01}\,e^{(\alpha_P-1)Y}}{\sqrt{14\alpha N_c\zeta(3)Y}}$$

and also show that at $Y=0$ we get lowest order $\sigma(x_{01},x'_{01})$.

5. The McLerran Venugopalan model and saturation

5.1 Dipole scattering on a nucleus

$S\psi_i = \psi_f$



$$S(\underline{x},\underline{b}) = 1 - \int_0^L dz\,\rho\underbrace{\frac{\sigma(x_\perp)}{2}}_{T(x_\perp)} + \underbrace{\int_0^L dz_2\,\rho\frac{\sigma(x_\perp)}{2}\int_0^{z_2}dz_1\,\rho\frac{\sigma(x_\perp)}{2}}_{\frac{1}{2!}\left(\int_0^L dz\,\rho\frac{\sigma(x_\perp)}{2}\right)^2} + \cdots$$

$$\frac{dS(\underline{x})}{d^2\underline{b}}$$

$$S(x_\perp, b) = e^{-\frac{\rho \sigma(x_\perp) L}{2}} \qquad L = 2\sqrt{R^2 - b^2}$$

$$S^2(x_\perp, b) = e^{-\rho\sigma L} = e^{-L/\lambda} \qquad \lambda = \text{mean free path}$$

$$\lambda = \frac{1}{\rho\sigma}$$

$$\sigma = \sigma_{\text{dip-nucleon}} = \frac{\alpha \pi^2 x_\perp^2}{N_c} \times G(x, 1/x_\perp^2) = 2\alpha\pi^2 x_\perp^2 \frac{C_F}{N_c^2 - 1} \times G$$

$$\sigma = \frac{2\pi^2 \alpha C_F x_\perp^2 xG}{N_c^2 - 1}$$

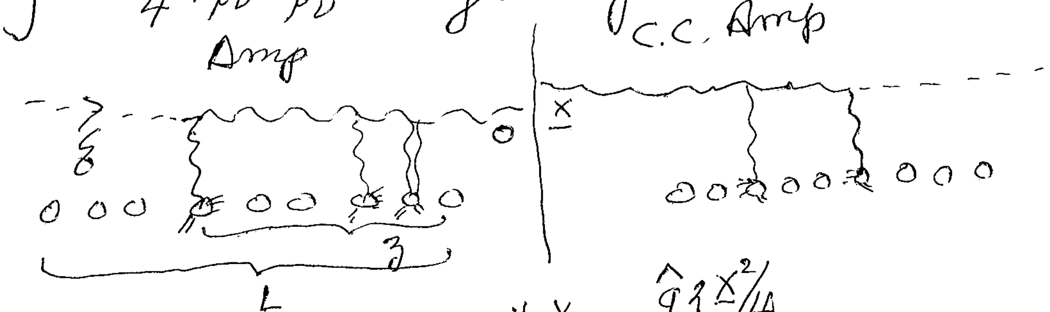
$$\frac{1}{2}\rho\sigma L = \underbrace{\underbrace{\frac{4\pi^2 \alpha C_F}{N^2 - 1} \cdot xG}_{\hat{q}_F} \cdot L}_{Q_s^2(F)} \cdot x_\perp^2/4$$

$$S_F = e^{-Q_s^2(F) x_\perp^2/4} \qquad Q_s^2(G) = \frac{N_c}{C_F} Q_s^2(F)$$

$$\boxed{S = e^{-Q_s^2 x_\perp^2/4}} \qquad \text{Scattering matrix in MV model.}$$

5.2 Gluon distribution of large nucleus in MV model

Use $j = -\frac{1}{4} F^i_{\mu\nu} F^i_{\mu\nu}$ as gluon probe



$$\frac{dxG_A}{d^2b\, d^2k} = \int_0^L dz\, \rho\, xG_N(x, 1/x_\perp^2)\, e^{-ik\cdot x}\, e^{-\hat{q} z\, x^2/4} \frac{d^2x}{4\pi^2}$$

acts like dipole scattering

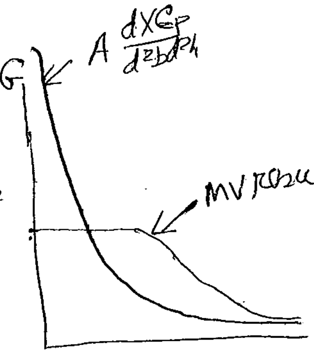
$$\frac{dxG_A}{d^2b\,d^2k} = \frac{N_c^2-1}{4\pi^4\alpha N_c}\int d^2x\, e^{-ik\cdot x}\,\frac{1-e^{-x^2Q_s^2/4}}{x^2} = \text{Weizsäcker-Williams unintegrated gluon dist}$$

Note

(i) $$\int d^2b\, d^2k\; dxG = \int d^2b\; 4\pi^2\,\frac{N_c^2-1}{4\pi^4\alpha N_c}\cdot\underbrace{\frac{Q_s^2}{4}}_{\frac{1}{4}\frac{\cancel{4\pi^2}\alpha N_c}{N_c^2-1}\,2\rho\sqrt{R^2-b^2}\,xG_N}$$

$$= \rho\int \underbrace{d^2b}_{\pi db^2}\, 2\sqrt{R^2-b^2}\; xG = \frac{4}{3}\pi R^3\rho\, xG$$

$$= A\,xG_N \qquad \text{no shadowing}$$



(ii) When $k_\perp^2 \ll Q_s^2$

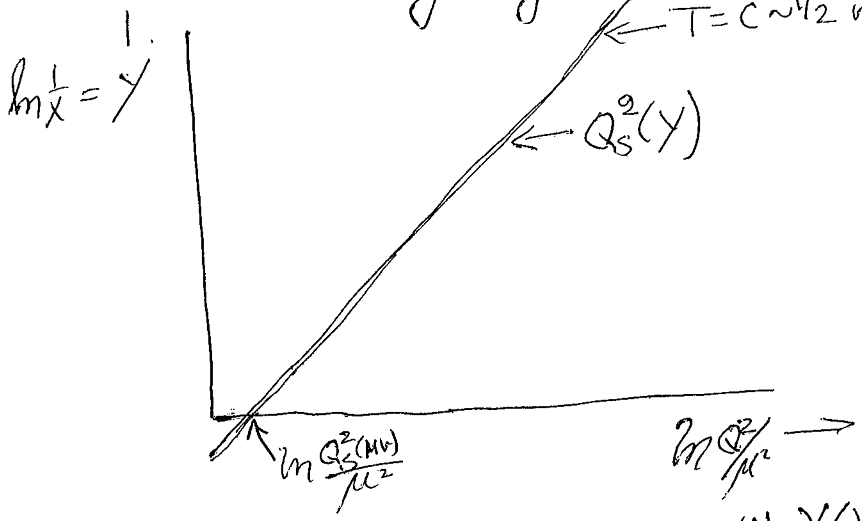
$$\frac{dxG}{d^2b\,d^2k} = \frac{N_c^2-1}{4\pi^4\alpha N_c}\cdot\int_{4/Q_s^2}^{1/k_\perp^2}\frac{\pi\, dx_\perp^2}{x_\perp^2} \simeq \frac{N_c^2-1}{4\pi^3\alpha N_c}\ln Q_s^2/k_\perp^2$$

$$f_g = \frac{(2\pi)^3}{N_c^2-1}\,\frac{1}{2}\,\frac{dxG}{d^2b\,d^2k} \simeq \frac{\ln Q_s^2/k_\perp^2}{\alpha N_c}$$

Picture in light cone gauge

Multiple scattering in covariant gauge looks like gauge rotations in light cone gauge. In light cone gauge $\frac{dxG}{d^2b\,d^2k_\perp}$ is a description of gluons in a nuclear wavefunction

5.3 The scaling region and the unitarity boundary



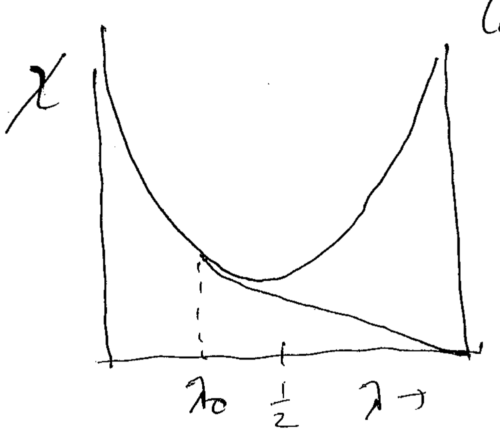
Dipole scattering on target

$Q^2 \equiv 1/x_\perp^2 \qquad \frac{1}{x} = \frac{2 p_+ m}{Q^2}$

Use BFKL $\quad T \sim \int d\lambda \; e^{\frac{2\alpha N_c}{\pi}\chi(\lambda) Y - (1-\lambda)\ln Q^2/\mu^2}$

require (i) $\frac{2\alpha N_c}{\pi}\chi'(\lambda) Y = -\ln Q^2/\mu^2$

(ii) $\frac{2\alpha N_c}{\pi}\chi(\lambda) Y = (1-\lambda_0)\ln Q^2/\mu^2$



$$\Rightarrow \frac{\chi'(\lambda_0)}{\chi(\lambda_0)} = -\frac{1}{1-\lambda_0}$$

$$\boxed{\chi(\lambda_0) = -\chi'(\lambda_0)(1-\lambda_0)}$$

$\lambda_0 \simeq 0.36$

$$\boxed{Q_s^2(Y) = \mu^2 e^{\frac{2\alpha N_c}{\pi}\frac{\chi(\lambda_0)}{1-\lambda_0} Y}}$$

In nucleus $\boxed{Q_s^2(Y) = A^{1/3} e^{\frac{2\alpha N_c}{\pi}\frac{\chi(\lambda_0)}{1-\lambda_0}} \mu^2}$

At given Y $\quad T(Y,Q^2) \sim \left(\frac{Q_s^2(Y)}{Q^2}\right)^{1-\lambda_0}$ scaling law

$$T(Y,Q^2,A) \sim \left(\frac{\mu^2}{Q^2}\right)^{1-\lambda_0} e^{\frac{2\alpha N_c}{\pi}\chi(\lambda_0) Y} \cdot \left[A^{1/3}\right]^{1-\lambda_0}$$

shadowing since $1-\lambda_0 \neq 1$.

5.4 Shadowing; a table

MV model
no evolution

Gluon distribution of A; no shadowing
Quark distribution, νW_2 shadowing

$$\nu W_2 \sim \ln Q^2/Q_s^2(MV) \text{ vs } \ln Q^2/\Lambda^2$$

$$\nu W_2 = \frac{Q^2}{\alpha_{em}} \sum_f \sum_\lambda \int d^2b \underbrace{\int \frac{d^2x}{4\pi} \int_0^1 dz \left| \psi_{T\lambda}^f(x, Q, z) \right|^2}_{\frac{dx_\perp^2}{(x_\perp^2)^2}} (1 - S(x, b)) \quad (1 - e^{-Q_s^2 x_\perp^2/4})$$

$$\nu W_2 \sim \int_{1/Q^2}^{1/Q_s^2} \frac{dx_\perp^2}{x_\perp^2} = \ln Q^2/Q_s^2(MV)$$

$$\ln Q^2/Q_s^2 = \ln Q^2/\Lambda^2 - \underbrace{\ln \frac{Q_s^2(MV)}{\Lambda^2}}_{\text{the shadowing part. leading twist!}}$$

Evolution on top
of MV initial condition

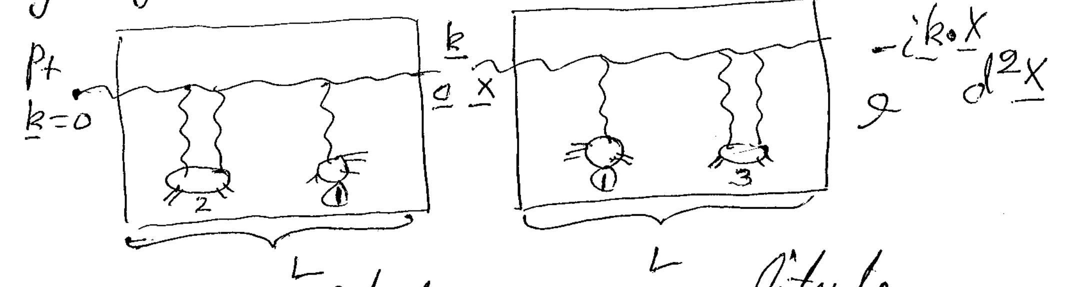
$$\nu W_2 \underset{\substack{\text{scaling}\\\text{region}}}{\sim} (A^{1/3})^{1-\lambda_0}$$

If $\frac{\ln Q^2/\Lambda^2}{\alpha Y} \gg 1$, $\lambda \Rightarrow 0$ and shadowing becomes parametrically small but goes to zero slowly.

6. $p_\perp$-broadening and $\hat{q}$

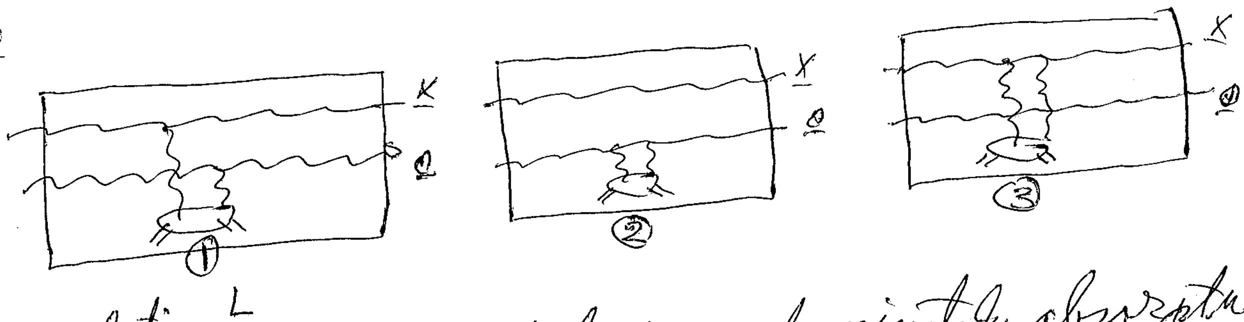
6.1 The MV model

Send a gluon through a hot or cold medium where initially the gluon has no transverse momentum



Amplitude   c.c amplitude

The single scattering is the same as for a dipole



when the elastic scattering amplitude is predominately absorptive, real in the notation I have been using. Thus

$$\frac{dN}{d^2k} = \int \frac{d^2x}{4\pi^2}\, e^{-i\underline{k}\cdot\underline{x}}\, \underbrace{S(\underline{x})}_{\text{dipole S-matrix}}$$

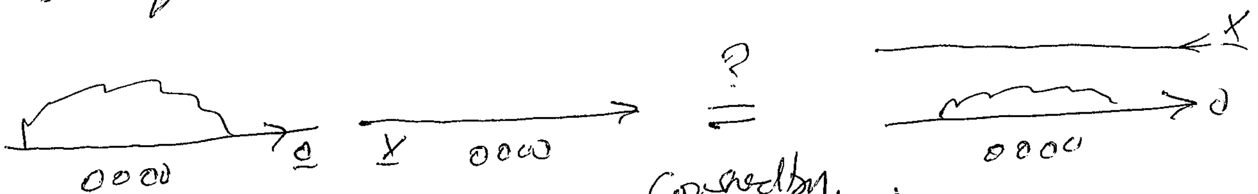
note $\int \frac{dN}{d^2k}\, d^2k = S(\underline{0}) = 1$ corresponding to the single gluon we have sent through the medium.

6.2 Adding small-x evolution. (Long coherence time limit)

The claim is that the above picture, using a dipole S-matrix to get $\frac{dN}{d^2k}$ continues to be ok. Let's see how this works.

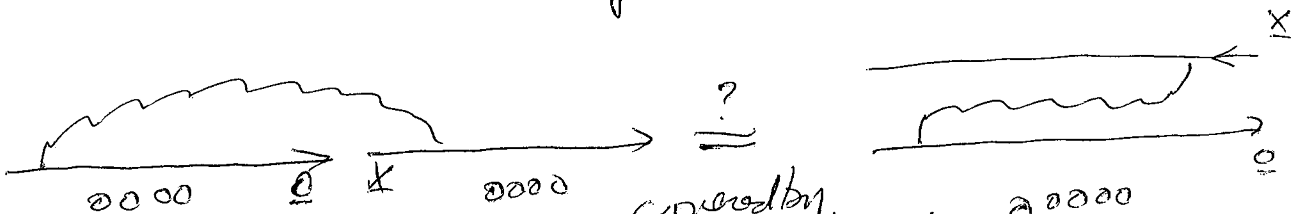
Types of graphs

(a)



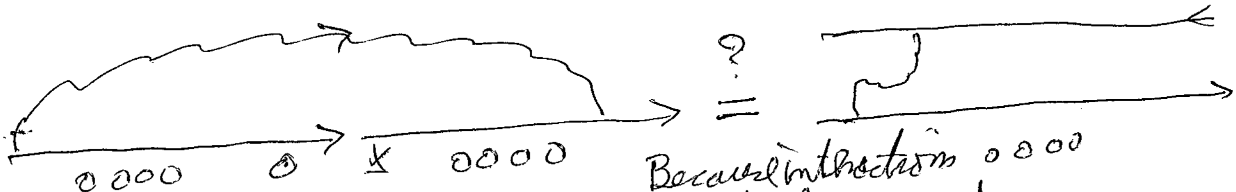
Covered by previous discussion

(b)



covered by previous discussion

(c)



Because interactions with given conserved probability conservation

Thus

$$\frac{dN}{d^2b\,d^2p} = \int \frac{d^2x}{4\pi^2}\, e^{-ip\cdot x}\, S_{BK}(x, b, Y)$$

$$\frac{dS}{dY} = \frac{\alpha N_c}{2\pi^2}\int \frac{d^2z\, x^2}{z^2(z-x)^2}\left[S(z,Y)\,S(x-z,Y) - S(x,Y)\right]$$

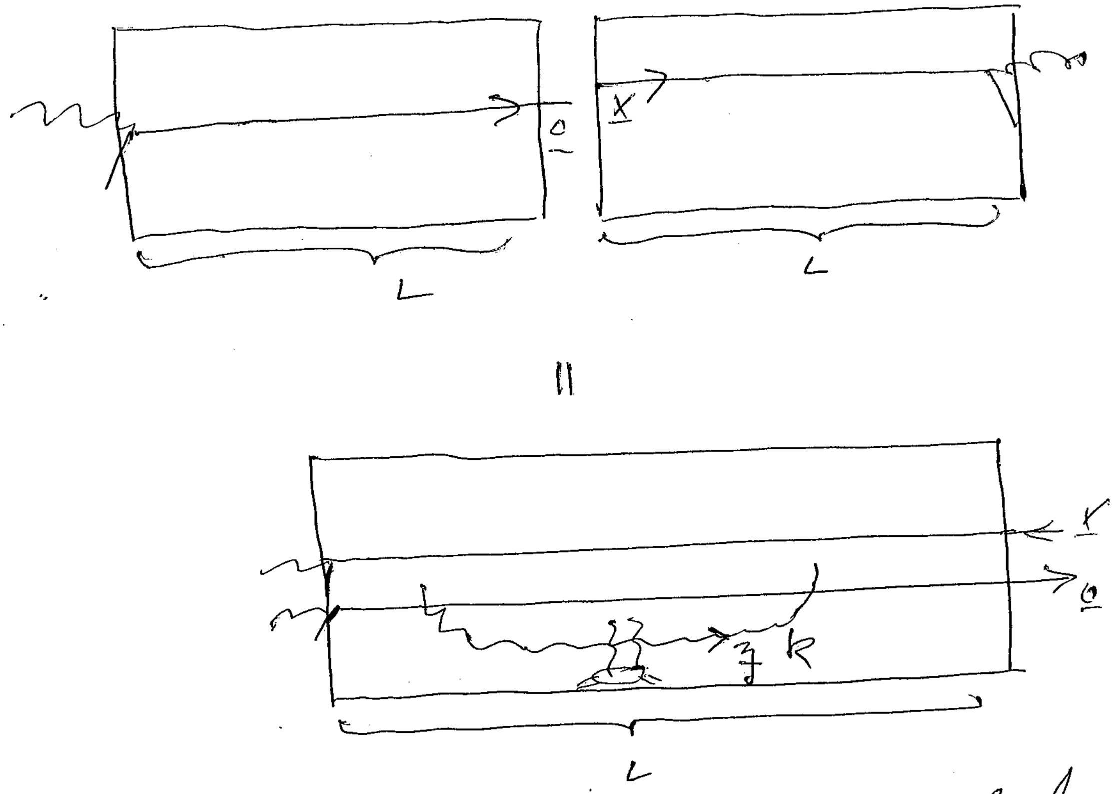
Problem 15(M) Show (naively) that

$$\langle p_\perp^2\rangle = -\nabla_x^2\, S(x,Y)\Big|_{x=0}$$

6.3 Higher order terms in $\hat{q}$ via $p_\perp$-broadening

Produce a quark jet locally in a medium and calculate $p_\perp$-broadening due (i) multiple scattering, (ii) ~~radiation~~ radiation (Sudakov) and (iii) evolution in the medium

? not medium induced



Take $\underline{x}^2 \sim 1/Q_s^2(MV)$. Then dipole $\underline{x}$ typically has one scattering in the medium. We are looking for double logarithms dk^2/k^2, $\frac{dk_+}{k_+}$, $\frac{2k_+}{k^2} \lesssim L$. $|\underline{z}| \gg |\underline{x}|$ to get a scattering in medium

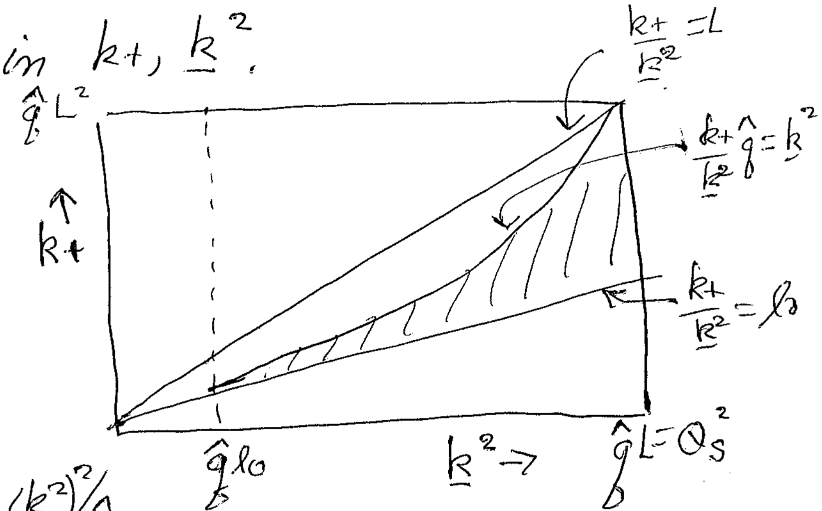
Prob of emission of z $\left(\frac{d^2z}{(z^2)^2}\right)$ $\cdot$ cross section (z^2) will give log if only a single scattering occurs. Thus

$$\underbrace{\tau}_{k_+/\underline{k}^2} \hat{q} < \underline{k}^2 \Rightarrow k_+ < (\underline{k}^2)^2/\hat{q}$$

Contribution to S looks like

$$S = 2\frac{\alpha N_c}{2\pi^2} \, d^2 z \, \frac{\underline{x}^2}{(z^2)^2} \, \frac{dk_+}{k_+} \, \frac{-\hat{q}L}{4} z^2 .$$

Easier to put in limits in $k_+, \underline{k}^2$.



Total is (single scattering)

$$\ln S = -\frac{\hat{q}L\underline{x}^2}{4}\Big(1 + \frac{\alpha N_c}{\pi} \underbrace{\int_{\hat{q}l_0}^{\hat{q}L} \frac{dk^2}{\underline{k}^2} \underbrace{\int_{\underline{k}^2 l_0}^{(\underline{k}^2)^2/\hat{q}} \frac{dk_+}{k_+}}_{\ln \underline{k}^2/\hat{q}l_0}}_{\frac{1}{2}\ln^2 L/l_0}\Big) = -\frac{\hat{q}L\underline{x}^2}{4}\Big(1 + \frac{\alpha N_c}{8\pi}\ln^2 L^2/l_0^2\Big)$$

Resumming

$$\ln S = -\frac{\hat{q}L}{4}\underline{x}^2\Big(1 + \sqrt{\frac{4\pi}{\alpha N_c}}\frac{1}{\ln L^2/l_0^2} I_1\Big[\sqrt{\frac{\alpha N_c}{\pi}\ln L^2/l_0^2}\,\Big]\Big)$$

$$\langle p_\perp^2 \rangle = \hat{q}L\Big(1 + \sqrt{\frac{4\pi}{\alpha N_c}}\frac{1}{\ln L^2/l_0^2} I_1\Big[\sqrt{\frac{\alpha N_c}{\pi}\ln L^2/l_0^2}\Big]\Big)$$