2016 progress in two-loop electroweak pseudoobservables and further prospects [S-matrix approach]

Janusz Gluza, Tord Riemann

1st FCC Physics Workshop

CERN, 16 January 2017

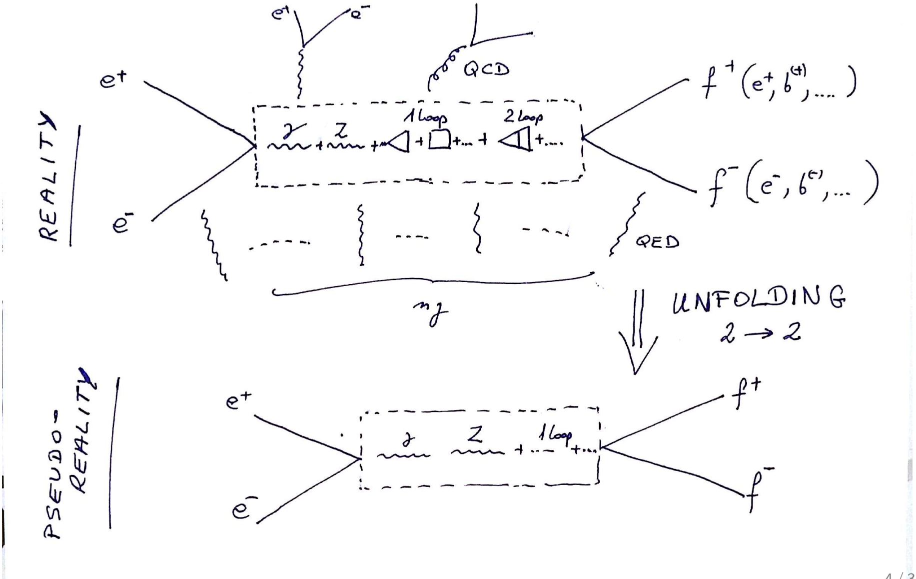
Based on collaboration with: Ievgen Dubovyk, Ayres Freitas, Johann Usovitsch

- Ievgen Dubovyk, Ayres Freitas, JG, Tord Riemann, Johann Usovitsch
  "The two-loop electroweak bosonic corrections to $\sin^2 \theta_{\mathrm{eff}}^{\mathrm{b}}$"
  Phys.Lett. B762 (2016) 184
- TR LL16 talk, PoS LL2016 (2016) 075:
  "30 years, some 700 integrals, and 1 dessert, or:
  Electroweak two-loop corrections to the Z$\bar{b}$b vertex",
  arXiv:1610.07059;
- JG LL16 talk, PoS LL2016 (2016) 034:
  "Numerical integration of massive two-loop Mellin-Barnes integrals in Minkowskian regions",
  arXiv:1607.07538

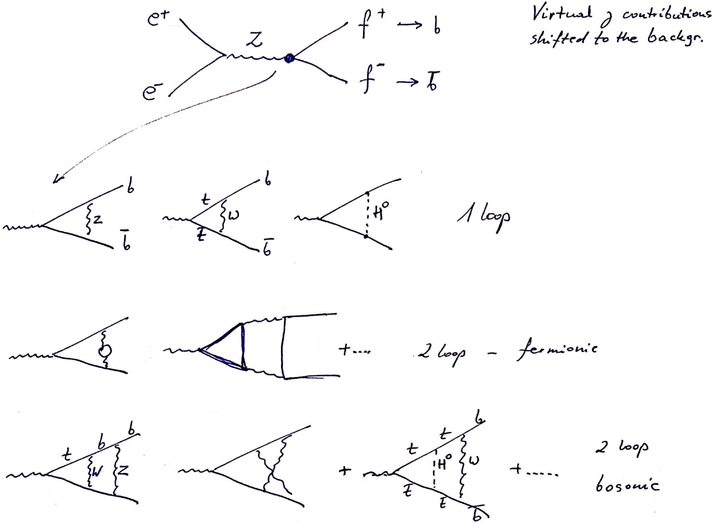
Outline

1. Introduction
    - Electroweak Pseudo-observables (EWPOs)
    - The effective weak mixing angle $\sin^2\theta_{\rm eff}^{\rm b}$
2. Fresh rolls: 2-loop EW bosonic corrections to $\sin^2\theta_{\rm eff}^{\rm b}$
3. Numerical 2-loop calculations
    - Mellin-Barnes ~~versus~~ and sector decomposition methods
4. Pseudo-observables, S-matrix and $\gamma - Z$ interferences
5. Summary and Outlook: To be or not to be (optimistic)
6. Backup slides: details of numerical methods
    - Construction of MB integrals
    - Avoiding numerical instabilities

Road to the $Z\bar{b}b$ vertex and 2-loop EW corrections (1)



Road to the $Z\bar{b}b$ vertex and 2-loop EW corrections (2)



Virtual γ contributions shifted to the backgr.

1 loop

2 loop – fermionic

2 loop bosonic

Pseudo-observables, an example: $d\sigma/d\cos\theta\ (e^+e^-\to\bar{b}b)$

Close to the Z-boson peak and assuming Born-like v, a couplings:

$$\frac{d\sigma}{d\cos\theta} \sim G_F^2\left|\frac{s}{s-M_Z^2+iM_Z\Gamma_Z}\right|^2$$
$$\times\ \left[(a_e^2+v_e^2)(a_b^2+v_b^2)(1+\cos^2\theta)\ \ (2a_ev_e)(2a_bv_b)(2\cos\theta)\right]$$

Factorizations:

Symmetric integration over $\cos\theta$

$$\sigma_T = \int_{-1}^{1} d\cos\theta\frac{d\sigma}{d\cos\theta} \sim \left|\frac{s}{s-M_Z^2+iM_Z\Gamma_Z}\right|^2 G_F(a_e^2+v_e^2)\ \ \mathbf{G_F(a_b^2+v_b^2)}$$

Anti-symmetric integration over $\cos\theta$

$$A_{F-B} = \frac{\left[\int_0^1 d\cos\theta - \int_{-1}^0 d\cos\theta\right]\frac{d\sigma}{d\cos\theta}}{\sigma_T} \sim \overbrace{\frac{2a_ev_e}{a_e^2+v_b^2}}^{A_e}\ \overbrace{\frac{\mathbf{2a_b v_b}}{\mathbf{a_b^2+v_b^2}}}^{A_b}$$

Pseudo-observable A_b

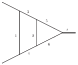
$$A_{\mathrm{b}} = \frac{2\Re e\frac{v_b}{a_b}}{1+\left(\Re e\frac{v_b}{a_b}\right)^2} = \frac{1-4|Q_b|\sin^2\theta^{\mathrm{b}}_{\mathrm{eff}}}{1-4|Q_b|\sin^2\theta^{\mathrm{b}}_{\mathrm{eff}}+8Q_b^2(\sin^2\theta^{\mathrm{b}}_{\mathrm{eff}})^2}$$

Definition of the effective weak mixing angle

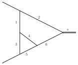
$$\sin^2\theta^{\mathrm{b}}_{\mathrm{eff}} = \frac{1}{4|Q_b|}\left(1-\Re e\frac{v_b}{a_b}\right)$$

- Vertex form factor

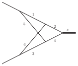
$$V_\mu^{Zb\bar{b}} = \gamma_\mu[v_b(s)-a_b(s)\gamma_5] = \cdots +$$



$+$



$+$



$+\cdots$

$e^+e^- \to Z \to l\bar{l}, b\bar{b}$, status

- 1985 - 1-loop leptonic ($l\bar{l}$) EW and $b\bar{b}$ corrections (Akhundov, Bardin, Riemann)
- 2006 - 2-loop leptonic EW corrections (Awramik, Czakon, Freitas)
- 2008 - 2-loop $b\bar{b}$ EW corrections with fermionic sub-loops (Awramik, Czakon, Freitas, Kniehl)
- 2016 - Completion: 2-loop $b\bar{b}$ bosonic EW corrections - DFGRU

Our project started in 2012 (TR - AF meeting). Basis for success:

1. Ayres Freitas: knowledge of the 2-loop renormalization scheme + experience in previous studies
2. TR, JG, JU, ID: new numerical evaluations based on Mellin-Barnes **(MB)** approach to Feynman integrals — In 2012 we hoped to use known by that time versions of AMBRE/MB tools - completely naive assumption (!)

Our results: Effective weak mixing angle $\sin^2\theta_{\text{eff}}^{\text{b}}$

- The standard model prediction for the effective weak mixing angle can be written as

$$\sin^2\theta_{\text{eff}}^{\text{b}} = \left(1 - \frac{M_W^2}{M_Z^2}\right)(1+\Delta\kappa_{\text{b}})$$

- The bosonic electroweak two-loop corrections amount to

$$\boxed{\Delta\kappa_{\text{b}}^{(\alpha^2,\text{b}os)} = -0.9855 \times 10^{-4}}$$

DFGRU, Phys.Lett. B762 (2016) 184

Collection of radiative corrections: full stabilization at 10^{-4}!

Order	Value $[10^{-4}]$
α	468.945
$\alpha\alpha_s$	-42.655
$\alpha_t\alpha_s^2$	-7.074
$\alpha_t\alpha_s^3$	-1.196
$\alpha_t^2\alpha_s$	1.362
α_t^3	0.123
α^2_{ferm}	3.866
α^2_{bos}	**-0.986**

Table: *Comparison of different orders of radiative corrections to* $\Delta\kappa_{\mathrm{b}}$*.*
Input Parameters: M_Z, Γ_Z, M_W, Γ_W, M_H, m_t, α_s *and* $\Delta\alpha$

1-loop contributions	Akhundov:1985
fermionic EW 2-loop corrections	Awramik:2008
$\mathcal{O}(\alpha\alpha_{\mathrm{s}})$ QCD corrections	Djouadi:1987,Djouadi:1987,Kniehl:1989,Kniehl: 1991, Fleischer:1992,Buchalla:1992,Czarnecki:1996
partial higher-order corrections	Avdeev:1994,Chetyrkin:1995
of orders $\mathcal{O}(\alpha_{\mathrm{t}}\alpha_{\mathrm{s}}^2)$	
$\mathcal{O}(\alpha_{\mathrm{t}}\alpha_{\mathrm{s}}^3)$	Schroder:2005,Chetyrkin:2006,Boughezal:2006
$\mathcal{O}(\alpha^2\alpha_{\mathrm{t}})$ and $\mathcal{O}(\alpha_{\mathrm{t}}^3)$	vanderBij:2000,Faisst:2003

Simple fitting formula

$$\Delta\kappa_{\mathrm{b}}^{(\alpha^2,\mathrm{bos})} = k_0 + k_1\, c_{\mathrm{H}} + k_2\, c_{\mathrm{t}} + k_3\, c_{\mathrm{t}}^2 + k_4\, c_{\mathrm{H}}\, c_{\mathrm{t}} + k_5\, c_{\mathrm{W}} \tag{1}$$

$$c_{\mathrm{H}} = \log\left(\frac{M_{\mathrm{H}}}{M_{\mathrm{Z}}} \times \frac{91.1876\mathrm{GeV}}{125.1\mathrm{GeV}}\right)$$

$$c_{\mathrm{t}} = \left(\frac{m_{\mathrm{t}}}{M_{\mathrm{Z}}} \times \frac{91.1876\mathrm{GeV}}{173.2\mathrm{GeV}}\right)^2 - 1 \tag{2}$$

$$c_{\mathrm{W}} = \left(\frac{M_{\mathrm{W}}}{M_{\mathrm{Z}}} \times \frac{91.1876\mathrm{GeV}}{80.385\mathrm{GeV}}\right)^2 - 1$$

$$\begin{aligned} k_0 &= -0.98605 \times 10^{-4}, & k_1 &= 0.3342 \times 10^{-4}, & k_2 &= 1.3882 \times 10^{-4}, \\ k_3 &= -1.7497 \times 10^{-4}, & k_4 &= -0.4934 \times 10^{-4}, & k_5 &= -9.930 \times 10^{-4} \end{aligned} \tag{3}$$

The deviations to the full calculation amount to average (maximal) 5×10^{-8} (1.2×10^{-7}), in the input parameter ranges.

DFGRU, Phys.Lett. B762 (2016) 184

Currently most precise prediction for $\sin^2\theta^{\mathrm{b}}_{\mathrm{eff}}$

$$\begin{aligned}\sin^2\theta^{\mathrm{b}}_{\mathrm{eff}} =& s_0 + d_1\,L_H + d_2\,L_H^2 + d_3\,\Delta_\alpha + d_4\,\Delta_t + d_5\,\Delta_t^2\\ &+ d_6\,\Delta_t\,L_H + d_7\,\Delta_{\alpha_{\mathrm{s}}} + d_8\,\Delta_t\,\Delta_{\alpha_{\mathrm{s}}} + d_9\,\Delta_Z\end{aligned} \tag{4}$$

$$\begin{aligned}&L_H = \log\left(\frac{M_{\mathrm{H}}}{125.7\mathrm{Ge}V}\right), \quad \Delta_t = \left(\frac{m_{\mathrm{t}}}{173.2\mathrm{Ge}V}\right)^2 - 1, \quad \Delta_Z = \frac{M_{\mathrm{Z}}}{91.1876\mathrm{Ge}V} - 1,\\ &\Delta_\alpha = \frac{\Delta\alpha}{0.0059} - 1, \qquad \Delta_{\alpha_{\mathrm{s}}} = \frac{\alpha_{\mathrm{s}}}{0.1184} - 1.\end{aligned} \tag{5}$$

$$\begin{aligned}s_0 = 0.232704, \quad & d_1 = 4.723\times10^{-4}, && d_2 = 1.97\times10^{-4}, && d_3 = 2.07\times10^{-2},\\ & d_4 = -9.733\times10^{-4}, && d_5 = 3.93\times10^{-4}, && d_6 = -1.38\times10^{-4},\\ & d_7 = 2.42\times10^{-4}, && d_8 = -8.10\times10^{-4}, && d_9 = -0.664.\end{aligned} \tag{6}$$

- M_{W} is calculated from the Fermi constant G_μ [Awramik, et al., 2004]
- The deviations to the full calculation amount to average (maximal) 2×10^{-7} (1.3×10^{-6}), in the input parameter ranges.

DFGRU, Phys.Lett. B762 (2016) 184

Numerical 2-loop calculations

Our approach:

Direct numerical calculations in Minkowskian kinematics

Direct numerical integrations in Minkowskian regions >NLO

Sector decomposition (SD)

FIESTA 3 [A.V.Smirnov, 2014]
SecDec 2 [S. Borowka, G. Heinrich, 2012]
SecDec 3 [S. Borowka, G. Heinrich, P. Jones, M. Kerner, J. Schlenk, T. Zirke, 2013]

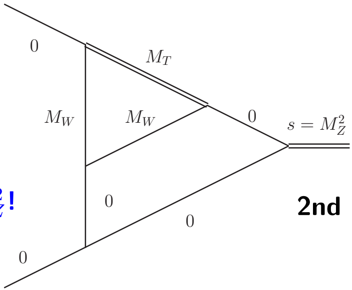
- NICODEMOS, ver 2.0 [A. Freitas]

Now: the Mellin-Barnes (MB) method.
Toolbox: AMBRE/MB/MBnumerics/CUBA

Two steps (automatic construction and numerical evaluation):

1. AMBRE [K.Kajda (planar, ver.2.2), E.Dubovyk (non-planar,ver3.0)]-PlanarityTest[K.Bielas, E.Dubovyk]
2. MBnumerics [J. Usovitsch, E. Dubovyk] - a completely new software !

One of the most difficult IR-divergent integrals with 2 scales



Huge oscillations for $s = M_Z^2$**!**

2nd problem: Γ **'s.**

$$\int dz_1 \int dz_2\, (-s)^{-2-2\epsilon} \left(-\frac{\mathbf{s}}{\mathbf{MT^2}}\right)^{-z_2} \left(-\frac{\mathbf{s}}{\mathbf{MW^2}}\right)^{-z_1} \Gamma[-z_1]\Gamma[-z_2]\Gamma[-z_3]$$

$$\times \quad \frac{\Gamma[-1-2\epsilon-z_1-z_2]\Gamma[-\epsilon-z_1-z_2]\Gamma[2+2\epsilon+z_1+z_2]}{\Gamma[-3\epsilon-z_1-z_2]\Gamma[1-2\epsilon-z_1-z_2]\Gamma[1-z_3]\Gamma[-2\epsilon-2z_1-2z_2-z_3]}$$

$$\times \quad \Gamma[-2\epsilon-z_1-2z_2-z_3]\Gamma[-1-2\epsilon-z_1-z_2-z_3]$$

$$\times \quad \Gamma[-\epsilon-z_1-z_2-z_3]\Gamma[1+z_2+z_3]\Gamma[1+\epsilon+z_1+z_2+z_3]$$

Solutions: see Backup Slides and MBnumerics.m

One of the most difficult IR-divergent integrals with 2 scales, cont’d

MBnumerics.m `2016-04-21     Johann Usovitsch`

```
=
    1.541402128186602 + 0.248804198197504*I
+
   0.12361459942846659 - 1.0610332704387688 *I * eps^-1
+
  -0.33773737955057970 + 3.6*10^-17*I *eps^-2
```

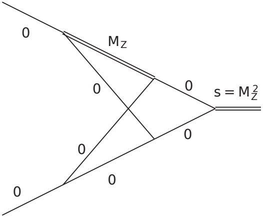
`Time needed` **43 min.**

SecDec

```
=
  1.541 + 0.2487*I
+
  0.123615 - 1.06103*I *eps^-1
+  -0.3377373796 - 5*10^-10*I*eps^-2
```

`Time needed` **24 hours**

The worst case for SD, fine with MBnumerics



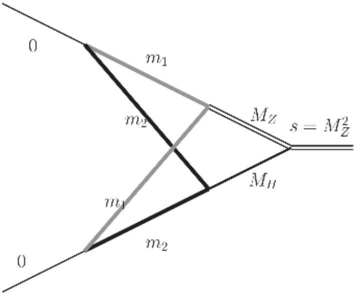
Euclidean results (constant part):

Analytical : -0.4966198306057021
MB(Vegas) : -0.4969417442183914
MB(Cuhre) : -0.4966198313219404
FIESTA : -0.4966184488196595
SecDec : -0.4966192150541896

Minkowskian results (constant part):

Analytical : $-\mathbf{0.778599608}979684 - \mathbf{4.123512}593396311 \cdot i$
MBnumerics : $-\mathbf{0.778599608}324769 - \mathbf{4.123512}600516016 \cdot i$
MB(Vegas) : big error
MB(Cuhre) : NaN
FIESTA : big error
SecDec : big error

8-dim MB integral (less accurate) for the $Z\bar{b}b$ vertex



$$m_1 = M_t, m_2 = M_W$$

The integrals contain up to three dimensionless parameters:

$$\left\{ \frac{M_H^2}{M_Z^2}, \frac{M_W^2}{M_Z^2}, \frac{m_t^2}{M_Z^2}, \frac{(M_Z + i\varepsilon)^2}{M_Z^2} \right\}$$

Important

MB and SD methods are very much complementary!

- MB works well for hard threshold, on-shell cases, not many internal masses, SD is powerful for integrals with internal masses.
- see e.g.: J. Gluza, K. Kajda, T. Riemann and V. Yundin "Numerical Evaluation of Tensor Feynman Integrals in Euclidean Kinematics" Eur. Phys. J. C **71** (2011) 1516; [arXiv:1010.1667 [hep-ph]]

10^{-8} accuracy achieved for **any** self-energy and vertex Feynman integral

with one of the methods.

How to unfold - rough scheme

We have to describe

$$e^+e^- \longrightarrow (\gamma, Z) \longrightarrow f^+f^-(\gamma), \tag{7}$$

S-matrix Ansatz in the complex energy plane

$$\begin{aligned}\mathcal{A}^{e^+e^-\to b\bar{b}} &= \underbrace{\frac{R_Z}{s-s_Z} + \overbrace{\frac{R_\gamma}{s}}^{}}_{\gamma - Z\ interference}\!\!\!\!\!\overbrace{\quad + S + (s-s_Z)S' + \ldots}^{Background}, \\ s_Z &= \overline{M}_Z^2 - i\overline{M}_Z\overline{\Gamma}_Z\end{aligned}$$

- $R, S, S', \ldots$ are individually gauge-invariant and UV-finite - unitarity and analyticity of the S-matrix. IR-finite, when soft and collinear real photon emission is added. [Willenbrock, Valencia,1991] [Sirlin,1991] [Stuart,1991]

  [Riemann, 1991, 1992] [H. Veltman,1994] [Passera, Sirlin, 1998] [Gambino, Grassi, 2000]

  [Awramik, Czakon, Freitas, 2006].

The term $R_\gamma(s)/s$ is part of the the background

- The poles of $\mathcal{A}$ have complex residua R_Z and R_γ.
- There is only ONE pole in mathematics, while in physics we observe two of them: photon excahange at $s = 0$, Z exchange at $s_0 = s_Z$. Mathematicaly, the appearance of the photon pole is result of summing of part of background around Z pole, $s_0 = s_Z$

  [T. Riemann, APPB 2015]

$$\begin{aligned}
\frac{R_\gamma(s)}{s} &= \frac{\sum_{n=0}^{\infty} R_n (s-s_0)^n}{s} \\
&= \frac{\sum_{n=0}^{\infty} R_n (s-s_0)^n}{s_0 - (s_0 - s)} \\
&= \sum_{n=0}^{\infty} R_n (s-s_0)^n \, \frac{1}{s_0} \, \frac{1}{1 - \frac{s_0 - s}{s_0}} \\
&= \sum_{n=0}^{\infty} R_n (s-s_0)^n \, \frac{1}{s_0} \left[1 + \frac{s_0 - s}{s_0} + \left(\frac{s_0 - s}{s_0} \right)^2 \cdots \right];
\end{aligned}$$

Conclusions

- We used numerical approach to $Z \to bb$ based on MB and SD methods.
- The main challenge was the calculation of massive two-loop vertex diagrams, now AMBRE/MB/MBnumerics/CUBA works with 8 digits and for MB integrals of dim < 5.
- New automatized tools AMBRE 3 and MBnumerics for the evaluation of the Mellin-Barnes integrals in Minkowskian kinematics together with sector decomposition programs SecDec 3 and Fiesta 3 are sufficient to calculate all needed integrals for Z resonance physics.
- Continuum physics (cross sections) needs also 2-loop boxes, this has to be studied.
- Final calculation at two-loop order to the electroweak effective weak mixing angle $\sin^2\theta^{\rm b}_{\rm eff}$ is presented as a simple fitting formula

Outlook

Further plans connected also with FCC:

1. Evaluation of other pseudoobservables, Γ_{Zbb}, $\Gamma_{Z_{\mathrm{tot}}}$,...
2. S-matrix theory: exact two-loops description of the Z-physics resonance,
   e.g. A. Leike, T. Riemann, and J. Rose, "S-matrix approach to the Z line shape", Phys. Lett. B273 (1991) 513, [hep-ph/9508390];
   T. Riemann, "S-matrix Approach to the Z Resonance", APPBB46 (2015) 11, 2235;
   DFGRU, PoS LL2016 (2016) 075: "30 years, some 700 integrals, and 1 dessert, or: Electroweak two-loop corrections to the Z$\bar{b}$b vertex", arXiv:1610.07059;
3. Further development of MB tools;
4. Further applications: e.g. including box diagrams (cross sections).
5. Open. Finally, two-loop EW enough for FCC? N^3LO with AMBRE/MB/MBnumerics? (Really exciting prospect!)

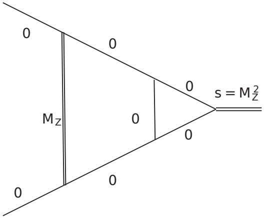
Note recent progress in differential equation method of solving Feynman (master) integrals: Henn 2013 (concept), Prausa 2017 (algorithm)

Backup slides

Basic problem: Steping up from Euclidean to direct calculation in Minkowskian kinematics

$$\frac{1}{(-)p^2 - m^2} \longrightarrow singularities \longrightarrow \frac{1}{(-)p^2 - m^2 + i\delta}$$

Resonance: $s = M_Z^2, \quad s = -M_Z^2$



Step 1

Construction of MB integrals

http://us.edu.pl/∼gluza/ambre/

Mellin-Barnes representations in HEP - method

- "Om definita integraler", R. H. Mellin, Acta Soc. Sci. Fenn. 20(7), 1 (1895), "The theory of the gamma function", E. W. Barnes Messenger Math. 29(2), 64 (1900).

$$\boxed{\begin{aligned} mathematics \longrightarrow \frac{1}{(A+B)^{\lambda}} &= \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\Gamma(\lambda+z)\Gamma(-z)\frac{B^{z}}{A^{\lambda+z}} \\ physics \longrightarrow \frac{1}{(p^2-m^2)^{a}} &= \frac{1}{\Gamma(a)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\Gamma(a+z)\Gamma(-z)\frac{(m^2)^{z}}{(p^2)^{a+z}} \end{aligned}}$$

It is recursive $\Longrightarrow$ <u>multidimensional complex integrals</u>.

$$\int\limits_{-\frac{1}{3}-i\infty}^{-\frac{1}{3}+i\infty} dz_1 \int\limits_{-\frac{2}{3}-i\infty}^{-\frac{2}{3}+i\infty} dz_2 \left(\frac{-s}{M_Z^2}\right)^{-z_1} \frac{\Gamma[-z_1]^3\Gamma[1+z_1]\Gamma[z_1-z_2]\Gamma[-z_2]^3\Gamma[1+z_2]\Gamma[1-z_1+z_2]}{s\ \ \Gamma[1-z_1]^2\Gamma[-z_1-z_2]\Gamma[1+z_1-z_2]}$$

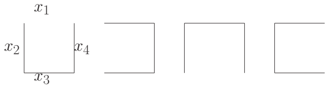
Overlaped integrals

Multiloop Feynman diagrams, general MB integrals

$$\frac{1}{D_1^{n_1} D_2^{n_2} \ldots D_N^{n_N}} \to \int \prod_{j=1}^{N} dx_j \; x_j^{n_j - 1} \delta(1 - \sum_{i=1}^{N} x_i) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

$$N_\nu = n_1 + \ldots + n_N$$

The functions U and F are called graph or Symanzik polynomials.

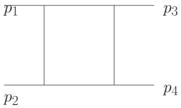


Trees contributing to the polynomial U for the square diagram

$$\mathbf{U} \;=\; \mathbf{x_1 + x_2 + x_3 + x_4}$$! *1-loop* $\longrightarrow 1$

$$\mathbf{F} \;=\; \mathbf{t \cdot x_1 x_3 + s \cdot x_2 x_4}$$

2 – trees contributing to the polynomial F for the square diagram



Dimension of MB integrals depends on factorizations of F and U!

Cuts of internal lines such that:

- U: (i) every vertex is still connected to every other vertex by a sequence of uncut lines; (ii) no further cuts without violating (i)
- F: (iii) divide the graph into two disjoint parts such that within each part (i) and (ii) are obeyed and such that at least one external momentum line is connected to each part;

Step 2

Numerics of MB integrals

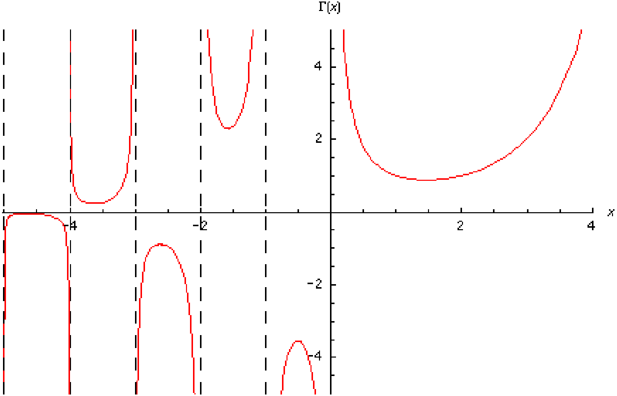
http://mbtools.hepforge.org/

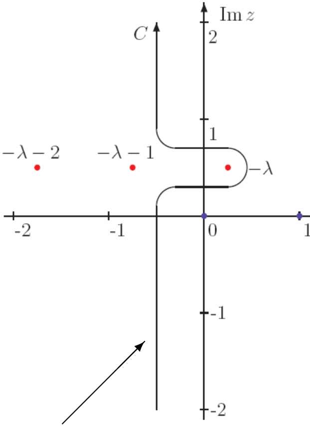
Gamma function: Singularities in the complex plane

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

$$\int dz \Gamma[z+\lambda]$$

SINGULARITIES REGULAR

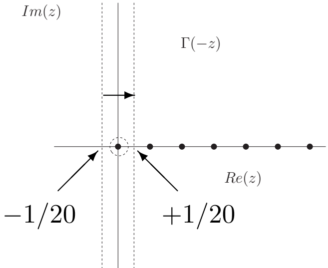




Contours: shifts, deformations

(* shifting contours *)

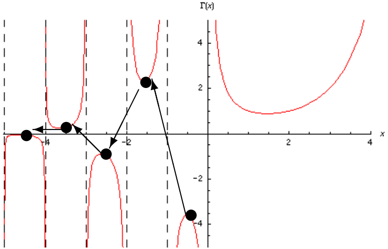
```
In[203]:=
sim = Gamma[-z]
Out[203]=
Gamma[-z]

In[227]:=
Sum[-Residue[Gamma[-z], {z, n}], {n, 0, 100}] // N
Out[227]=
0.367879

In[226]:=
n1 = NIntegrate[
  1 / (2 Pi) sim /. z → -1 / 20 + I y, {y, -10, 10}]
Out[226]=
0.367879 + 0. ⅈ

In[230]:=
n2 = NIntegrate[
  1 / (2 Pi) sim /. z → 1 / 20 + I y, {y, -10, 10}]
Out[230]=
-0.632121 + 0. ⅈ

In[231]:=
n2 - n1
Out[231]=
-1. + 0. ⅈ

In[232]:=
Residue[sim, {z, 0}]
Out[232]=
-1

(* B5l2m2 *)
```



Two basic observations for shifting z follows

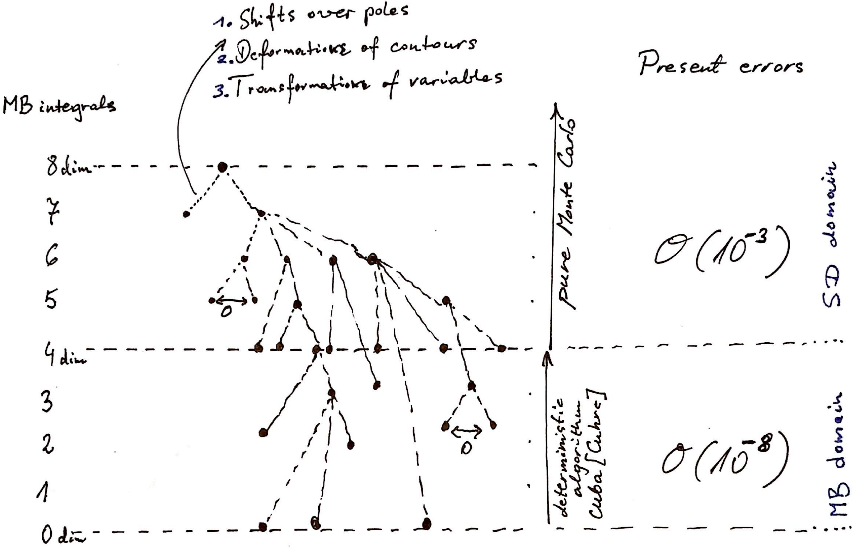
$$
\begin{aligned}
& \int dz_1 \ldots dz_k \ldots I(\ldots, Re[z_k] + n + Im[z_k], \ldots) && I_{orig} \\
= \quad & Residue[\int dz_1 \ldots \cancel{dz_k} \ldots I]_{Re[z_k]+n} && I_{Res} \\
+ \quad & \int dz_1 \ldots dz_k \ldots I(\ldots, Re[z_k] + (n+1) + Im[z_k], \ldots) && I_{new}
\end{aligned}
$$

1. Residues **lower** dimensionality of original MB integrals.
2. Integral after passing a pole (proper shifts) **can be made smaller**.



Top-bottom approach to evaluation of multidimensional MB integrals

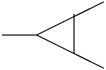
MBnumerics.m - I. Dubovyk, J. Usovitsch, T. Riemann



BASIC PROBLEMS in Minkowski kinematics

I. Bad oscillatory behavior of integrands;

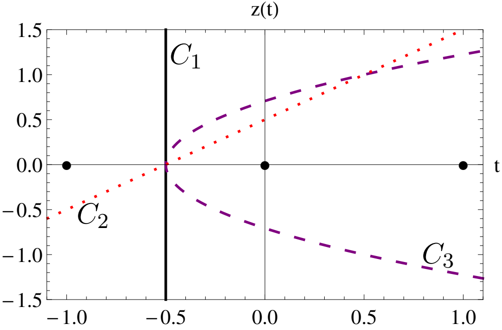
II. Fragile stability for integrations over products and ratios of Γ functions.



$$
\begin{aligned}
V(s) &= \frac{e^{\epsilon\gamma_E}}{i\pi^{(4-2\epsilon)/2}}\int \frac{d^{(4-2\epsilon)}k}{[(k+p_1)^2-m^2][k^2][(k-p_2)^2-m^2]} \\
&= \frac{V_{-1}(s)}{\epsilon} + V_0(s) + \cdots ,
\end{aligned}
$$

$$
\begin{aligned}
V_{-1}(s)|_{m=1} &= -\frac{1}{2s}\int\limits_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\frac{dz}{2\pi i}\quad \underbrace{(-s)^{-z}}_{\textbf{Problem I}}\ \overbrace{\frac{\Gamma^3(-z)\Gamma(1+z)}{\Gamma(-2z)}}^{\textbf{Problem II}} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\frac{s^n}{\binom{2n}{n}(2n+1)} = \frac{2\arcsin(\sqrt{s}/2)}{\sqrt{4-s}\sqrt{s}},
\end{aligned}
$$

Contour deformations



$$
\begin{aligned}
z(t) = x_0 + it \ : \quad V_{-1}^{C_1}(s) &= \int_{-\infty}^{+\infty} (i)\ dt\ J[z(t)]; \\
z(t) = x_0 + \theta t + it \ : \quad V_{-1}^{C_2}(s) &= \int_{-\infty}^{+\infty} (\theta + i)\ dt\ J[z(t)] \\
z(t) = x_0 + at^2 + it \ : \quad V_{-1}^{C_3}(s) &= \int_{-\infty}^{+\infty} (2at + i)\ dt\ J[z(t)];.
\end{aligned}
$$

$s = 2,\ z(t) = \Re[-1/2] + i\,y, \quad y \in (-a, +a)$

$$V_{-1}(2)\big|_{\text{analyt.}} = \mathbf{0.785398163397448309 62} = \frac{\pi}{\mathbf{4}}$$

$$V_{-1}(2)\big|^{MB.m}_{\text{Pantis}} = 0.7925 - \cancel{0.0225\,i}$$

$$V_{-1}(2)\big|_{C_1,\ a=15} = 0.7548660085063523 - \cancel{0.229985258820015\,i}$$

$$V_{-1}(2)\big|_{C_1,\ a=10^2} = 0.734793130888525378 44 + \cancel{0.074901423602937676597\,i}$$

$$V_{-1}(2)\big|_{C_1,\ a=10^3} = 0.847181850735310769 15 - \cancel{0.094865760649354977853\,i}$$

$$V_{-1}(2)\big|_{C_1,\ a=10^4} = 4.457455498513997718 8 + \cancel{4.513981236464512227 5\,i}$$

✓ $$V_{-1}(2)\big|_{\mathbf{C_2}} = \mathbf{0.785398163}3859819 - 5.420140575251864 \cdot 10^{-15\,\checkmark}\ i$$

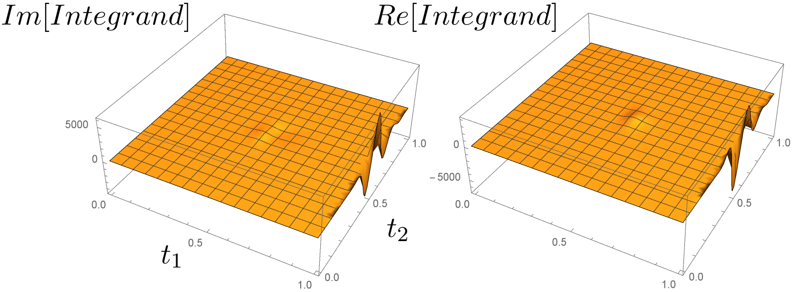
✓ $$V_{-1}(2)\big|_{\mathbf{C_3}} = \mathbf{0.785398163}2958756 + 2.435551760271437 \cdot 10^{-15\,\checkmark}\ i$$

Transformations of integration variables (Mappings)

$$\int_{-\frac{1}{3}-i\infty}^{-\frac{1}{3}+i\infty} dz_1 \int_{-\frac{2}{3}-i\infty}^{-\frac{2}{3}+i\infty} dz_2 \left(\frac{-s}{M_Z^2}\right)^{-z_1} \frac{\Gamma[-z_1]^3\Gamma[1+z_1]\Gamma[z_1-z_2]\Gamma[-z_2]^3\Gamma[1+z_2]\Gamma[1-z_1+z_2]}{s\ \ \Gamma[1-z_1]^2\Gamma[-z_1-z_2]\Gamma[1+z_1-z_2]}$$

Logarithmic (in MB.m, M. Czakon, CPC 2006):

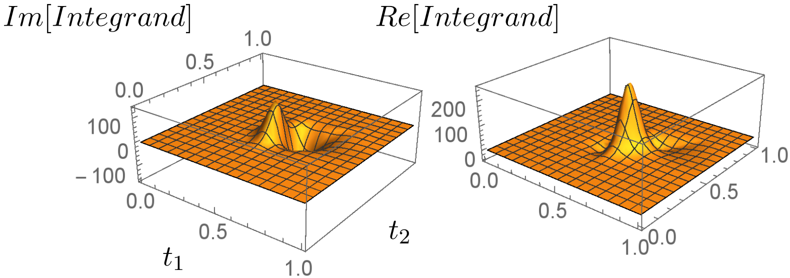
$$z_k \quad = \quad x_k + i\ln\left(\frac{t_k}{1-t_k}\right), \quad t_k \in (0,1), \quad \text{the Jacobians :} \quad J_k(t_k) = \frac{1}{t_k(1-t_k)}.$$



Transformations of variables (Mappings)

Tangent (in MBnumerics.m, ID, JU, TR, 2016):

$$z_k \quad = \quad x_k + i\,\frac{1}{\tan(-\pi t_k)}, \quad t_k \in (0,1), \quad \text{the Jacobians}: \quad J_k = \frac{\pi}{\sin^2\left[(\pi t_k)\right]}$$



In addition, $\Gamma \to e^{\ln\Gamma}$ improves numerical stability considerable, either.

The most difficult cases (for SD)